UNIT – III

NUMERICAL DIFFERENTIATION AND INTEGRATION

Numerical Differentiation and Integration

Numerical Differentiation	Numerical Integration
Newton's forward difference formula to compute derivatives (Equal interval)	Trapezoidal rule [$n = 1$ in Quadrature formula]
Newton's backward difference formula to compute derivatives(Equal interval)	Simpson's one third rule [$n = 2$ in Quadrature formula]
Lagrange's Interpolation formula (Equal or unequal intervals)	Simpson's three eighth rule [$n = 3$ in Quadrature formula]
Newton's divided difference Interpolation formula (Equal or unequal intervals)	Romberg's method
Maxima and minima of a tabulated function	Two point Gaussian's quadrature formula
	Three point Gaussian's quadrature formula
	Double integrals by Trapezoidal rule
	Double integrals by Simpson's 1/3 rule

Approximation of derivatives using interpolation polynomials

Numerical Differentiation

Given $y = y(x) = f(x)$ [in a table]

$\frac{dy}{dx} = y'(x) = f'(x)$ is the first numerical derivative

$\frac{d^2y}{dx^2} = y''(x) = f''(x)$ is the second numerical derivative

$\frac{d^ny}{dx^n} = y^{1\cdots n}(x) = f^{1\cdots n}(x)$ is the nth numerical derivative

Newton's forward difference formula to compute derivative

WKT, Newton's forward difference interpolation formula is

$$y(x) = f(x_0+uh) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots$$

$$= y_0 + \frac{u}{1!}\Delta y_0 + \frac{u^2-u}{2!}\Delta^2 y_0 + \frac{u^3-3u^2+2u}{3!}\Delta^3 y_0 + \frac{u^4-6u^3+11u^2-6u}{4!}\Delta^4 y_0 + \cdots$$

where $u = \frac{x-x_0}{h}$

First derivative

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\left(\frac{1}{h}\right) = \frac{1}{h}\frac{dy}{du}$$

$$\text{i.e., } \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \left(\frac{2u-1}{2}\right)\Delta^2 y_0 + \left(\frac{3u^2-6u+2}{6}\right)\Delta^3 y_0 + \left(\frac{4u^3-18u^2+22u-6}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{dy}{dx}\right]_{\substack{\text{at } x = x_0 \\ \Rightarrow u = 0}} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

Second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{d}{dx}\left[\frac{dy}{du}\frac{du}{dx}\right] = \frac{d}{dx}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{d}{du}\frac{du}{dx}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{d}{du}\frac{1}{h}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{1}{h}\frac{d}{du}\left[\frac{1}{h}\frac{dy}{du}\right] = \frac{1}{h^2}\frac{d^2y}{du^2}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \left(\frac{12u^2-36u+22}{24}\right)\Delta^4 y_0 + \cdots\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{\substack{\text{at } x = x_0 \\ \Rightarrow u = 0}} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{22}{24}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

Third derivative

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\frac{6}{3!}\nabla^3 y_n + \left(\frac{24v+36}{4!}\right)\nabla^4 y_n + \cdots\right]$$

$$\left[\frac{d^3y}{dx^3}\right]_{\substack{\text{at } x = x_n \\ \Rightarrow v = 0}} = \frac{1}{h^3}\left[\frac{6}{3!}\nabla^3 y_n + \frac{36}{4!}\nabla^4 y_n + \cdots\right]$$

4.8.3 Maxima and Minima of a tabulated function

(If the intervals are same)

WKT, Newton's forward difference interpolation formula is

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

First derivative

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \cdots\right] \quad (1)$$

substitute $h, \Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0, \cdots$ gives the equation

$$\frac{dy}{dx} = \text{an equation in } u \quad (2)$$

Newton's Forward Formula

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \ldots \ldots \ldots \ldots ..$$

$u = \frac{x-x_0}{h}$ *and* $h = x_1 - x_0$

Newton's Backward Formula

$$y(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots \ldots \ldots \ldots \ldots ..$$

$h = x_1 - x_0$ $v = \frac{x-x_n}{h}$

1. ***For the following data.***

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x=1.1

Given :

$x_0 = 1.0$ $y_0 = 7.989$
$x_1 = 1.1$ $y_1 = 8.403$
$x_2 = 1.2$ $y_2 = 8.781$
$x_3 = 1.3$ $y_3 = 9.129$
$x_4 = 1.4$ $y_4 = 9.451$
$x_5 = 1.5$ $y_5 = 9.750$
$x_6 = 1.6$ $y_6 = 10.031$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	7.989						
		0.141 (Δy_0)	−0.036 ($\Delta^2 y_0$)				
1.1	8.403			0.006 ($\Delta^3 y_0$)			
		0.378			-0.02		
1.2	8.781		−0.030			0.002	
		0.348		0.004			
1.3	9.129		-0.026		0.0		
		0.322					
1.4	9.451		-0.023	0.004		-0.001	-0.003
		0.299					
1.5	9.750		-0.018		-0.01		
		0.281		0.005			
1.6	10.031						

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta_{y_0} - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \frac{1}{6}\Delta^6 y_0\right]$$

$h = 0.1$

$$\left(\frac{dy}{dx}\right)_{x_0} = \left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[1.141 - \frac{1}{2}(-0.036) + \frac{1}{3}(0.006) - \frac{1}{4}(-0.02) + \frac{1}{5}(0.002) - \frac{1}{6}(-0.003)\right]$$

=3.946

$$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 y_0\right]$$

$$= \frac{1}{(0.1)^2}\left[(-0.036) - (0.006) + \frac{11}{12}(-0.02) - \frac{5}{6}(0.002) + \frac{137}{180}(-0.003)\right]$$

= -3.545

The following data gives the velocity of a particle for 20 seconds at an interval of 5 seconds.

Time(sec)	0	5	10	15	20
Velocity (m/sec)	0	3	14	69	228

Find (a) Initial acceleration using the entire data (b) Final acceleration.

Solution: The difference table is

Time $t = x$	Velocity $v = y(x)$	$\Delta y(x)$	$\Delta^2 y(x)$	$\Delta^3 y(x)$	$\Delta^4 y(x)$
$0 (= x_0)$	$0 (= y_0)$				
		$3 (= \Delta y_0)$			
5	3		$8 \left(= \Delta^2 y_0\right)$		
		11		$36 \left(= \Delta^3 y_0\right)$	
10	14		44		$24 \left(= \Delta^4 y_0 \text{ or } \nabla^4 y_n\right)$
		55		$60 \left(= \nabla^3 y_n\right)$	
15	69		$104 \left(= \nabla^2 y_n\right)$		
		$159 (= \nabla y_n)$			
20	$228 (= y_n)$				

Here $x_0 = 0, h = \text{interval length} = 5$

(a) WKT, acceleration $= \dfrac{dv}{dt}$ = rate of change of velocity

To find initial acceleration, put $\left(\dfrac{dv}{dt}\right)_{t=t_0} = \left(\dfrac{dv}{dt}\right)_{t=0}$.

i.e., initial acceleration exists at $t = 0 = x_0$ [which is nearer to beginning of the table], so we use

Newton's forward difference formula for first derivative.

$\therefore$ Newton's forward difference interpolation formula is

$$y(x) = f(x_0 + uh) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots$$

Now, Newton's forward difference formula for first derivative at $x = x_0 = 0\,[\Rightarrow u = 0]$

i.e., $y'(x_0) = \left(\frac{dv}{dt}\right)_{t=0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$ [Here $h = 5$]

$$= \frac{1}{5}\left[3 - \frac{1}{2}(8) + \frac{1}{3}(36) - \frac{1}{4}(24)\right] = 1$$

$\Rightarrow y'(0) = 1$

$\therefore$ Initial acceleration (acceleration when $t = 0$) is 1 m/sec^2.

Final acceleration exists at $t = 20 = x_4$ [Nearer to ending of table], so use Newton's backward difference interpolation formula for first derivative, put $\left(\frac{dv}{dt}\right)_{t=t_n} = \left(\frac{dv}{dt}\right)_{t=20}$.

WKT, Newton's backward difference interpolation formula is

$$y(x) = f(x_n + vh) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \frac{v(v+1)(v+2)(v+3)}{4!}\nabla^4 y_n + \cdots$$

$$y'(x_n) = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{2}{3!}\nabla^3 y_n + \frac{6}{4!}\nabla^4 y_n + \cdots\right]$$

$$= \frac{1}{5}\left[159 + \frac{1}{2}(104) + \frac{2}{6}(60) + \frac{6}{24}(24)\right]$$

$$= \frac{1}{5}(237) = 47.2 \text{ m / sec}^2$$

Find the value of $f'(8), f''(9)$, maximum and minimum value from the following data, using an approximate interpolation formula.

x	4	5	7	10	11
$f(x)$	48	100	294	900	1210

Solution: The values of x are unequally spaced.

To find $f(x)$, we use Newton's divided difference formula (or) Lagrange formula.

WKT, Newton's Divided difference formula is

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1)(x - x_2)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) + \cdots \quad (1)$$

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		
		310			
11	1210				

$$\therefore (1) \Rightarrow f(x) = 48 + (x-4)(52) + (x-4)(x-5)(15) + (x-4)(x-5)(x-7)(1)$$

$$= 48 + 52x - 208 + 15\left[x^2 - 9x + 20\right] + x^3 - x^2(16) + x(83) - 140$$

$$= x^3 + x^2(-16+15) + x(83 - 135 + 52) - 208 + 300 - 140 + 48$$

$$= x^3 - x^2 + x(0) + 0$$

$$\therefore f(x) = x^3 - x^2$$

$$f'(x) = 3x^2 - 2x, \quad f'(8) = 176$$

$$f''(x) = 6x - 2, \quad f''(9) = 52$$

<u>To find maximum and minimum</u>

Put $f'(x) = 0 \Rightarrow 3x^2 - 2x = 0 \Rightarrow x = 0, x = \frac{2}{3}$

At $x = 0$, $f''(x=0) = 6(0) - 2 = -2 < 0$

$\Rightarrow x = 0$ is a maximum point & maximum value is $f(x=0) = 0$.

At $x = \frac{2}{3}$, $f''\left(x = \frac{2}{3}\right) = 6\left(\frac{2}{3}\right) - 2 = 2 > 0$.

$\Rightarrow x = \frac{2}{3}$ is a minimum point & minimum value is $f\left(x = \frac{2}{3}\right) = -\frac{4}{27}$

Evaluate y' and y'' at $x = 2$ given

x	0	1	3	6
y	18	10	−18	40

Solution:

$$\left[\text{Ans}: y(x) = x^3 - \frac{70}{9}x^2 - \frac{15}{9}x + 18, y'(x=2) = -\frac{187}{9}, y''(x=2) = -\frac{22}{9}\right]$$

Find the value of $\cos(1.747)$ using the values given in the table below :

x	1.70	1.74	1.78	1.82	1.86
$\sin x$	0.9916	0.9857	0.9781	0.9691	0.9584

Solution: $[\text{Ans}: -0.175]$

Find $\sec 31°$ from the following data :

θ	31	32	33	34
$\tan\theta$	0.6008	0.6249	0.6494	0.6745

Solution: $\left[\text{Ans}: \sec^2 31 = 1.3835 \Rightarrow \sec 31 = 1.174,\ \text{Hint}: 1° = \frac{\pi}{180} = 0.017453292\right]$

$2.$ *Find the value of y when* $x = 2$ *for the following data.*

x	0	5	10	15
y	14	379	1444	3584

Given :

$x_0 = 0 \qquad y_0 = 14$

$x_1 = 5 \qquad y_1 = 379$

$x_2 = 10 \qquad y_2 = 1444$

$x_3 = 15 \qquad y_3 = 3584$

$$h = x_1 - x_0 = 5 - 0 = 5$$

$$u = \frac{x - x_0}{h} = \frac{x - 0}{5} = \frac{2 - 0}{5} = \frac{2}{5} = 0.4$$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	14			
		$379 - 14 = 365$ (Δy_0)		
5	379		$1065 - 365 = 700$ $(\Delta^2 y_0)$	
10	1444	$1444 - 379 = 1065$		$1075 - 700 = 375$ $(\Delta^3 y_0)$
			$2140 - 1065 = 1075$	
15	3584	$3584 - 1444 = 2140$		

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$y(2) = 14 + \frac{0.4}{1!}(365) + \frac{0.4(0.4-1)}{2!}(700) + \frac{0.4(0.4-1)(0.4-2)}{3!}(375)$$

$$y(2) = 14 + 146 + \frac{(-0.24)}{2}(700) + \frac{(-0.384)}{6}(375)$$

$$y(2) = 14 + 146 - 84 + 24 = 100$$

Newton's Backward Formula

$$y(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots \ldots \ldots \ldots \ldots ..$$

$h = x_1 - x_0$ $\quad v = \frac{x - x_n}{h}$

3. ***Find the value of y when and $x = 43$ & $x = 84$ for the following data.***

x	40	50	60	70	80	90
y	184	204	226	250	276	304

***Solution*:**

$x_0 = 40$ $y_0 = 184$
$x_1 = 50$ $y_1 = 204$
$x_2 = 60$ $y_2 = 226$
$x_3 = 70$ $y_3 = 250$
$x_4 = 80$ $y_4 = 276$
$x_5 = 90$ $y_5 = 304$

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
40	184					
		$204 - 184 = 20$				
50	204		2			
		$226 - 204 = 22$		0		
60	226		2		0	
		$250 - 226 = 24$		0		0
70	250		2		0	
		$276 - 250 = 26$		0		
80	276		2			
		$304 - 276 = 28$				
90	304					

<u>*(i) when x = 43*</u>

$h = x_1 - x_0 = 5 - 40 = 10$

$$u = \frac{x - x_0}{h} = \frac{x - 40}{10} = \frac{43 - 40}{10} = \frac{3}{10} = 0.3$$

$$y(x) = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$y(43) = 184 + \frac{0.3}{1!}(20) + \frac{0.3(0.3-1)}{2!}(2) + \frac{0.3(0.3-1)(0.3-2)}{3!}(0)$$

$$y(43) = 184 + 6 + \frac{0.3(-0.7)}{2}(2) + 0$$

$$y(43) = 184 + 6 - 0.21$$

$$y(43) = 189.79$$

<u>*(ii) when x = 84*</u>

$h = x_1 - x_0 = 50 - 40 = 10$

$$v = \frac{x - x_n}{h} = \frac{x - 90}{10}\; v = \frac{84 - 90}{10} = \frac{-6}{10} = -0.6$$

$$y(x) = y_n + \frac{v}{1!}\nabla y_n + \frac{v(v+1)}{2!}\nabla^2 y_n + \frac{v(v+1)(v+2)}{3!}\nabla^3 y_n + \cdots \ldots\ldots\ldots\ldots$$

$$y(84) = 304 + \frac{(-0.6)}{1!}(28) + \frac{(-0.6)(-0.6+1)}{2!}(2) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}(0) + \cdots$$

$$y(84) = 304 - 16.8 + \frac{(-0.6)(0.4)}{2}(2) + 0$$

$$y(84) = 304 - 16.8 - 0.24$$

$y(84) = 286.96$

Anna University Questions

1. Find the value of $\tan 45°15'$ by using Newton's forward difference interpolation formula for

$x°$	45	46	47	48	49	50
$\tan x°$	1.00000	1.03553	1.07237	1.11061	1.15037	1.19175

(ND10)

Solution : [Ans: $\tan 45°15' = 1.00876$, by Newton's forward difference formula]

2. Derive Newton's backward difference formula by using operator method. (MJ12)

3. Find the value of y when $x = 5$ using Newton's interpolation formular from the following table:

x :	4	6	8	10
y :	1	3	8	16

(ND12)

Solution : [$y(x = 5) = 1.625$, by Newton's forward difference formula]

4. Fit a polynomial, by using Newton's forward interpolation formula, to the data given below. (8)

x :	0	1	2	3
y :	1	2	1	10

(MJ13)

Solution :

$y(x) = 2x^3 - 7x^2 + 6x + 1$, by Newton's forward & backward formula

$y(x = 4) = 41$

UNIT – III

NUMERICAL DIFFERENTIATION AND INTEGRATION

PROBLEMS BASED ON DOUBLE INTEGRAL

TRAPEZOIDAL RULE AND SIMPSON'S RULE

$\boldsymbol{Trapezoidal\ rule\ for\ Double\ Integral}$

$$I = \frac{hk}{4}[(Sum\ of\ four\ corners) + 2(Sum\ of\ nodes\ on\ boundary) + 4(Sum\ of\ interior\ nodes)]$$

$\mathbf{1.}$ *Evaluate* $\int_1^2\int_3^4 \frac{1}{(x+y)^2}dxdy$ with $h = k = 0.5$

Solution :

$Let\ f(x,y) = \frac{1}{(x+y)^2}$

$(i) Range\ for\ x : 3\ to\ 4\ and\ h = 0.5$

$(ii) Range\ for\ y : 1\ to\ 2\ and\ k = 0.5$

x \ y	3	3.5	4
1	0.0625	0.0494	0.04
1.5	0.0494	0.04	0.0331

2	0.04	0.0331	0.0278

$$f(x,y) = \frac{1}{(x+y)^2}$$

$$f(3,1) = \frac{1}{(3+1)^2} = \frac{1}{16} = 0.0625$$

$$f(3.5,1) = \frac{1}{(3.5+1)^2} = \frac{1}{(4.5)^2} = 0.0494$$

$$f(4,1) = \frac{1}{(4+1)^2} = \frac{1}{25} = 0.04$$

$$I = \frac{hk}{4}[(Sum\ of\ four\ corners) + 2(Sum\ of\ nodes\ on\ boundary) + 4(Sum\ of\ interior\ nodes)]$$

$$I = \frac{(0.5)(0.5)}{4}[(0.0625 + 0.04 + 0.04 + 0.0278) + 2(0.0494 + 0.0494 + 0.0331 + 0.0331) + 4(0.04)]$$

$$I = \frac{0.25}{4}[(0.1703) + 0.330 + 0.16]$$

$$I = 0.0413$$

Simpson's $\frac{1}{3}$ rule for Double Integral

Simpson's 1/3 rule $= \frac{hk}{9}[(\textit{Sum of the corner of the boundary})$
$+2(\textit{sum of the odd nodes of the boundary})$
$+ 4(\textit{sum of the even nodes of the boundary})$
$+ 4(\textit{sum of the odd nodes of the odd rows})$
$+ 8(\textit{sum of the even nodes of the odd rows})$
$+ 8(\textit{sum of the odd nodes of the even rows})$
$+ 16(\textit{sum of the even nodes of the even rows})]$

$$I = \frac{(0.5)(0.5)}{9}[(0.0625 + 0.04 + 0.04 + 0.0278) + 4(0.0494 + 0.0494 + 0.0331 + 0.0331) + 16(0.04)]$$

$$I = \frac{0.25}{9}[(0.1703) + 0.660 + 0.64]$$

$$I = 0.0408$$

Evaluate the integral $\int_1^{1.4}\int_2^{2.4} \frac{dx\,dy}{xy}$ using Trapezoidal rule. Verify your results by actual integration.

Solution: $f(x,y) = \frac{1}{xy}$, x varies from $(2, 2.4)$

y varies from $(1, 1.4)$

Divide the range of x and y into 4 equal parts.

$$h = \frac{2.4-2}{4} = 0.1,\ k = \frac{1.4-1}{4} = 0.1$$

The values of $f(x,y)$ at the nodal points are given in the table :

y \ x	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3698	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

By Trapezoidal rule for double integration

$$I = \frac{hk}{4}\left[\begin{array}{l}\text{Sum of values of } f \text{ at the four corners} \\ +2\left(\begin{array}{l}\text{Sum of values of } f \text{ at the nodes} \\ \text{on the boundary except the corners}\end{array}\right) \\ +4\,(\text{Sum of the values at the interior nodes})\end{array}\right]$$

$$= \frac{(0.1)(0.1)}{4}\left[\begin{array}{l}(0.5+0.4167+0.2976+0.3571) \\ +2\left(\begin{array}{l}0.4762+0.4545+0.4348+0.3788+0.3472+0.3205 \\ +0.3106+0.3247+0.3401+0.3846+0.4167+0.4545\end{array}\right) \\ +4\left(\begin{array}{l}0.4329+0.4132+0.3953+0.3623+0.3344 \\ +0.3497+0.3663+0.3698+0.3788\end{array}\right)\end{array}\right]$$

By actual integration

$$\int_1^{1.4}\int_2^{2.4}\frac{1}{xy}dxdy = \int_1^{1.4}\left(\int_2^{2.4}\frac{1}{x}dx\right)\frac{1}{y}dy = \int_1^{1.4}(\log x)_2^{2.4}\frac{1}{y}dy$$

$$= (\log 2.4 - \log 2)(\log y)_1^{1.4}$$

$$= 0.0613$$

Evaluate the integral $\int_1^{1.4}\int_2^{2.4}\frac{dxdy}{xy}$ **using Simpson's rule. Verify your results by actual integration.**

Solution: $f(x,y) = \frac{1}{xy}$, x varies from $(1, 1.4)$

y varies from $(2, 2.4)$

Divide the range of x and y into 4 equal parts.

$$h = \frac{2.4-2}{4} = 0.1,\ k = \frac{1.4-1}{4} = 0.1$$

By Extended Simpson's rule

$$I = \frac{hk}{9}\left[\begin{array}{l} (\text{Sum of the values of } f \text{ at the four corners}) \\ +2\,(\text{Sum of the values of } f \text{ at the odd positions on the boundary except the corners}) \\ +4\,(\text{Sum of the values of } f \text{ at the even positions on the boundary except the corners}) \\ +4\left(\begin{array}{l}\text{Sum of the values of } f \text{ at the odd positions} \\ \text{on the odd rows of the matrix except boundary rows}\end{array}\right) \\ +8\left(\begin{array}{l}\text{Sum of the values of } f \text{ at the even positions} \\ \text{on the odd rows of the matrix except boundary rows}\end{array}\right) \\ +8\left(\begin{array}{l}\text{Sum of the values of } f \text{ at the odd positions} \\ \text{on the even rows of the matrix except boundary rows}\end{array}\right) \\ +16\left(\begin{array}{l}\text{Sum of the values of } f \text{ at the even positions} \\ \text{on the even rows of the matrix except boundary rows}\end{array}\right) \end{array}\right]$$

$$= \frac{(0.1)(0.1)}{9}\left[\begin{array}{l} (0.5 + 0.4167 + 0.2976 + 0.3571) \\ +2\,(0.4545 + 0.3472 + 0.3247 + 0.4167) \\ +4\left(\begin{array}{c} 0.4762 + 0.4348 + 0.3788 + 0.3205 + 0.3106 \\ +0.3401 + 0.3846 + 0.4545 \end{array}\right) \\ +4\,(0.3788) \\ +8\,(0.3968 + 0.3623) \\ +8\,(0.3497 + 0.4132) \\ +16\,(0.3663 + 0.3344 + 0.4329 + 0.3953) \end{array}\right]$$

The values of $f(x,y)$ at the nodal points are given in the table :

y \ x	2	2.1	2.2	2.3	2.4
1	0.5	0.4762	0.4545	0.4348	0.4167
1.1	0.4545	0.4329	0.4132	0.3953	0.3788
1.2	0.4167	0.3698	0.3788	0.3623	0.3472
1.3	0.3846	0.3663	0.3497	0.3344	0.3205
1.4	0.3571	0.3401	0.3247	0.3106	0.2976

Anna University Questions

1. Evaluate $\int_1^5 \left[\int_1^4 \frac{1}{x+y}dx\right] dy$ by Trapezoidal rule in x–direction with h = 1 and Simpson's one-third rule in y–direction with k = 1. (ND10)

   Solution: [By Trap. : I = 2.4053, Simp. : I = 2.122]

2. Evaluate $\int_0^2 \int_0^1 4xy\,dx\,dy$ using Simpson's rule by taking $h = \frac{1}{4}$ and $k = \frac{1}{2}$. (ND12)

   Solution: [3.1111]

3. Evaluate $\int_1^{1.4} \int_2^{2.4} \frac{1}{xy} dx\,dy$ using Simpson's one-third rule. (MJ13)

   Solution: [0.0613]

4. Taking $h = k = \frac{1}{4}$, evaluate $\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \frac{\sin(xy)}{1+xy} dx\,dy$ using Simpson's rule . (AM14)

   Solution: [0.0141]

UNIT – III

NUMERICAL DIFFERENTIATION AND INTEGRATION

Two point Gaussian quadrature formula

Two point Gaussian quadrature formula : (Guass two point formula)

Given $I = \int_a^b f(x)dx$

Case (i) If $a = -1, b = +1$, then

$$I = \int_{-1}^{1} f(x)dx = f\left[-\frac{1}{\sqrt{3}}\right] + f\left[\frac{1}{\sqrt{3}}\right]$$

Case (ii) If $a = 0, b = 1$, then

$$I = \int_0^1 f(x)\,dx = \frac{1}{2}\int_{-1}^{1} f(x)\,dx, \text{ if } f(x) \text{ is an even function}$$

Case (iii) If $(a \neq -1 \& b \neq 1)$, then $x = \frac{b-a}{2}z + \frac{b+a}{2} = mz + c \Rightarrow dx = mdz$

$$I = \int_a^b f(x)dx = \int_{-1}^{1} f(z)\,mdz = m\int_{-1}^{1} f(z)\,dz$$

$$= m\left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)\right]$$

Evaluate $\int_1^2 \frac{dx}{x}$ by using Gaussian two point formula.

Solution: Here $a \neq -1, b = 2 \neq 1,$

so use $x = \frac{(b-a)z + (b+a)}{2}$

$$x = (z+3)/2 \Rightarrow dx = dz/2$$

$\therefore$ The above integral becomes

$$I = \int_1^2 \frac{dx}{x} = \int_{-1}^{1} \frac{dz/2}{(z+3)/2} = \int_{-1}^{1} \frac{1}{z+3} dz = \int_{-1}^{1} f(z)\, dz, \text{ where } f(z) = \frac{1}{z+3}$$

$\therefore$ By Gaussian two point formula

$$\int_{-1}^{1} f(z)\, dz = f\left(z = \frac{1}{\sqrt{3}}\right) + f\left(z = -\frac{1}{\sqrt{3}}\right)$$

$$= \left(\frac{1}{z+3}\right)_{z=\frac{1}{\sqrt{3}}} + \left(\frac{1}{z+3}\right)_{z=-\frac{1}{\sqrt{3}}}$$

$$= 0.693$$

Using Gaussian two point formula evaluate $\int_0^{\pi/2} \log(1+x)dx$

Solution: $I = \int_{-1}^{1} \log\left[1+\frac{\pi}{4}(1+z)\right]\frac{\pi}{4}dz$

$= \frac{\pi}{4}\left[f\left(-\frac{1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)\right]$

$= 0.858$

Example 4.37. Evaluate $\int_1^2 \frac{dx}{1+x^3}$ by using Gaussian two point formula.

Solution: $I = 0.2544.$

Three point Gaussian quadrature formula

Case (i) $\int_{-1}^{1} f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\sqrt{\frac{3}{5}}\right)\right]+\frac{8}{9}f(0)$

Case (ii) $\int_{0}^{1} f(x)\,dx = \frac{1}{2}\int_{-1}^{1} f(x)dx$ [for even function $f(x)$]

$= -\frac{1}{2}\left\{\frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\sqrt{\frac{3}{5}}\right)\right]+\frac{8}{9}f(0)\right\}$

Case (iii) $(a \neq -1 \& b \neq 1)$, then $x = \frac{b-a}{2}z+\frac{b+a}{2} = mz+c \Rightarrow dx = m\,dz$

$I = \int_a^b f(x)dx = \int_{-1}^{1} f(z)\,m\,dz = m\left\{\frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\sqrt{\frac{3}{5}}\right)\right]+\frac{8}{9}f(0)\right\}$

$$= \int_{-1}^{1} \frac{4}{8+(z+3)^3} dz, \qquad \text{where } f(z) = \frac{4}{8+(z+3)^3}$$

$$= \left\{ \frac{5}{9} \left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right) \right] + \frac{8}{9} f(0) \right\}$$

$$= \frac{5}{9} [0.27505] + \frac{8}{9} (0.11429)$$

$$= 0.25439 \qquad \left[\because \frac{5}{9} = 0.5555, \frac{8}{9} = 0.8888 \right]$$

$$I = 0.02544$$

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by two and three point Gaussian quadrature formula & hence find the value of π.

Solution:

$$\text{Now}, I = \int_0^1 \frac{dx}{1+x^2} = \int_{-1}^{1} \frac{1}{2}\frac{1}{1+x^2}$$

$$= \frac{1}{2}\int_{-1}^{1} f(x)dx, \qquad \text{where } f(x) = \frac{1}{2}\frac{1}{1+x^2}$$

By Gaussian two point formula

$$I = \frac{1}{2}\int_{-1}^{1} f(x)dx = \frac{1}{2}\left[f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)\right] = 0.75 \qquad (1)$$

By Guassain three point formula

$$I = \frac{1}{2}\int_{-1}^{1} f(x)dx = \frac{1}{2}\left[\frac{5}{9}\left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right] + \frac{8}{9}f(0)\right] = 0.79166 \qquad (2)$$

By actual integration

$$I = \int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}(x)\right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0) = \frac{\pi}{4} \qquad (3)$$

From (2) & (3),

$$0.79166 = \frac{\pi}{4}$$

$$\Rightarrow \pi = 3.16664$$

Find $\int_0^{\frac{\pi}{2}} \sin x dx$ by two & three point Gaussian quadrature formula.

Solution: $x = \frac{\pi(z+1)}{4}$

$I = 0.9985$ (by two point formula)

$I = 1.0000$ (by three point formula)

Example 4.41. Find $\int_0^1 \frac{1}{t} dt$ by using Gaussian three point formula.

Solution: [$I = 1.6027$].

UNIT – III

NUMERICAL DIFFERENTIATION AND INTEGRATION

Romberg's method

Romberg's method for a given interval $\left(I = \int_a^b f(x)dx\right)$

when $h = \frac{b-a}{2}$, by trapezoidal rule, we get I_1

when $h = \frac{b-a}{4}$, by trapezoidal rule, we get I_2

when $h = \frac{b-a}{8}$, by trapezoidal rule, we get I_3

$$\text{Romberg's formula for } I_1 \& I_2 = I_{RM_{1,2}} = I_2 + \frac{(I_2 - I_1)}{3}$$

$$\text{Romberg's formula for } I_2 \& I_3 = I_{RM_{2,3}} = I_3 + \frac{(I_3 - I_2)}{3}$$

If $I_{RM_{1,2}} = I_{RM_{2,3}}$, then we can equal $I = I_{RM_{1,2}} = I_{RM_{2,3}}$

Note : Check, use actual integration, we get $I_{AI} = I = I_{RM_{1,2}} = I_{RM_{2,3}}$

Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using Romberg's method. Hence obtain an approximate value for π.

Solution: To find I_1

When $h = \frac{2-0}{2} = 1, y = f(x) = \frac{1}{x^2+4}$

Let $I = \int_0^2 \frac{dx}{x^2+4}$

The tabulated values of y are

x	0	1	2
$f(x) = \frac{1}{x^2+4}$	$\frac{1}{0^2+4} = 0.25$	0.2	0.125
	y_0	y_1	y_2

Using Trapezoidal rule,

$$I_1 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}[(y_0+y_2)+2(y_1)]$$

$$= \frac{1}{2}[(0.25+0.125)+2(0.2)]$$

$$= 0.3875$$

To find I_2

$$h = \frac{2-0}{4} = \frac{1}{2} = 0.5$$

The tabulated values of y are

x	0	0.5	1	1.5	2
$f(x) = \frac{1}{x^2+4}$	0.25	0.23529	0.2	0.160	0.125
	y_0	y_1	y_2	y_3	y_4

To find I_3

$$h = \frac{2-0}{8} = \frac{1}{4} = 0.25$$

The tabulated values of y are

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$f(x) = \frac{1}{x^2+4}$	0.25	0.24615	0.23529	0.21918	0.2	0.17918	0.160	0.14159	0.125
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Using Trapezoidal rule,

$$I_3 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}[(y_0+y_8)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)]$$

$$= 0.125\left[(0.25+0.125)+2(0.24615+0.23529+0.21918+0.2+0.17918+0.16+0.14159)\right]$$

$$= 0.39237$$

Romberg's formula for $I_1 \& I_2$ is

$$I_{RM_{1,2}} = I_2 + \frac{(I_2 - I_1)}{3} = 0.39136 + \frac{(0.39136 - 0.39136)}{3}$$

$$= 0.39265$$

Romberg's formula for $I_2 \& I_3$ is

$$I_{RM_{2,3}} = I_3 + \frac{(I_3 - I_2)}{3} = 0.39237 + \frac{(0.39237 - 0.39237)}{3}$$
$$= 0.39271$$

$$\therefore I_{RM_{1,2}} \cong 0.3927$$
$$I_{RM_{2,3}} \cong 0.3927$$

Here $I_{RM_{1,2}} \& I_{RM_{2,3}}$ are almost equal and $I = 0.3927$ (1)

By actual integration,

$$\int_0^2 \frac{dx}{x^2+4} = \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_{x=0}^{x=2} = \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$
$$= \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8} \quad (2)$$

From (1) & (2),

$$\frac{\pi}{8} = 0.3927$$
$$\Rightarrow \pi = 8(0.3927) = 3.1416$$

Using Romberg's method, evaluate $\int_0^1 \frac{1}{1+x} dx$ correct to 3 places of decimals.

Solution: $I_1 = 0.7083$

$I_2 = 0.6970$

$I_3 = 0.6941$

$I_{RM_{1,2}} = 0.6932$

$I_{RM_{2,3}} = 0.6931$

$\therefore I = 0.693$

Anna University Questions

1. Using Romberg's integration to evaluate $\int_0^1 \frac{dx}{1+x^2}$. (AM10)

   Solution: [0.7854]

2. Using Romberg's rule evaluate $\int_0^1 \frac{1}{1+x}dx$ correct to three decimal places by taking $h = 0.5, 0.25,$ and 0.125. (ND10)

   Solution: [0.6931]

3. Evaluate $\int_0^1 \frac{1}{1+x^2}dx$ using Romberg's method. (AM11,AM10)

   Solution: [0.7854]

4. Use Romberg's method to compute $\int_0^1 \frac{1}{1+x^2}dx$ correct to 4 decimal places. Also evaluate the same integral using tree-point Gaussian quadrature formula. Comment on the obtained values by comparing with the exact value of the integral which is equal to $\frac{\pi}{4}$. (MJ12)

   Solution: [By Rom. : $I = 0.7854$, Dir. Int. : $I = 0.7853982$]

5. Evaluate $\int_0^{\frac{1}{2}} \frac{x}{\sin x}dx$ correct to three decimal places using Romberg's method. (AM14)

   Solution: [0.5070676]

6. Evaluate $\int_0^1 \frac{dx}{1+x}$ and correct to 3 decimal places using Romberg's method and hence find the value of $\log_e 2$. (ND14)

MA8491 NUMERICAL METHODS