UNIT – I– SOLUTION OF EQUATIONS AND EIGEN VALUE PROBLEMS

Problems based on Fixed point iteration method

1.Solve the equations $x^3 + x^2 - 100 = 0$ by IterationMethod

Solution

Let f(x)= $x^3 + x^2 - 100$

f(0)= $0^3 + 0 - 100 = -100 < 0(-ve)$

f(1)=$(1)^3 + (1)^2 - 100 = -98 < 0(-ve)$

f(2)= $(2)^3 + (2)^2 - 100 = -88 < 0(-ve)$

f(3)= $(3)^3 + (3)^2 - 100 = -64 < 0(-ve)$

f(4)= $(4)^3 + (4)^2 - 100 = -20 < 0(-ve)$

f(5)= $(5)^3 + (5)^2 - 100 = 50 > 0(+ve)$

f(x) has a root between 4 &5

f(x) has a +ve root

This equation f(x)= $x^3 + x^2 - 100$=0 can be written as

$x^3 + x^2 - 100$=0

$$x^2(x+1) = 100$$

$$x = \frac{10}{\sqrt{x+1}} = 10(x+1)^{\frac{-1}{2}}$$

$$g'(x) = \frac{d}{dx}[g(x)]$$

$=\frac{d}{dx}[10(x+1)^{\frac{-1}{2}}] = 10\,(\frac{-1}{2})\,(x+1)^{\frac{-3}{2}}$

$=-5(x+1)^{\frac{-3}{2}}=\frac{-5}{(x+1)^{\frac{3}{2}}}$

$|g'(x)|=|\frac{-5}{(x+1)^{\frac{3}{2}}}|=\frac{5}{(x+1)^{\frac{3}{2}}}$

At x=4

$|g'(x=4)|=\frac{5}{(4+1)^{\frac{3}{2}}}=\frac{5}{5^{\frac{3}{2}}}=\frac{5}{5\sqrt{5}}=\frac{1}{\sqrt{5}}=\frac{1}{2.23606}$

=0.57735<1

At x=5 $|g'(x=5)|=\frac{5}{(5+1)^{\frac{3}{2}}}=\frac{5}{6^{\frac{3}{2}}}=0.340201 < 1$

$|g'(x)|$**<1 in (4,5)**

$$\begin{bmatrix} 10 & -2 & 3 \\ 2 & 10 & -5 \\ 3 & -4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 23 \\ -33 \\ 41 \end{bmatrix}$$

AX=B

$$[A,B] = \begin{bmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{bmatrix}$$

Now, we will make the matrix A as a upper triangular.

$$[A,B] \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & -34 & 91 & 341 \end{bmatrix}$$

$$R_2 \leftrightarrow 5R_2 - R_1,$$

$$R_3 \leftrightarrow 10R_3 - 3R_1$$

$$\sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} --- (1)$$

$$R_3 \leftrightarrow 52R_3 + 34R_2$$

This is an upper triangular matrix,

Now, using back substitution method.

$$3780z = 11340$$

$$z = \frac{11340}{3780} = 3$$

$$52y - 28z = -188$$

$$52y - 28(3) = -188$$

$$52y - 84 = -188$$

$$52y = -188 + 84$$

$$52y = -104$$

$$y = -\frac{104}{52} = -2$$

$$10x - 2y + 3z = 23$$

$$10x - 2(-2) + 3(3) = 23$$

$$10x + 4 + 9 = 23$$

$$10x + 13 = 23$$

$$10x = 23 - 13$$

$$10x = 10$$

$$x = 1$$

Hence, the solution is , $x = 1, y = -2, z = 3$

Gauss- Jordan method

Take the equation (1)

$$[A,B] \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix}$$

Now, we will make the matrix A is diagonal matrix.

$$\sim \begin{bmatrix} 12600 & -2520 & 0 & 17640 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{bmatrix}$$

$$R_1 \leftrightarrow 1260R_1 - R_3,$$

$$R_2 \leftrightarrow 135R_2 + R_3$$

$$\sim \begin{bmatrix} 88452000 & 0 & 0 & 88452000 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{bmatrix}$$

$$R_1 \leftrightarrow 7020R_1 + 2520R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Hence, the solution is , $x = 1, y = -2, z = 3$

2. Solve the following system of equations by Gauss elimination method.

Solution:

The given system is equivalent to

$$\begin{bmatrix} 5 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \end{bmatrix}$$

AX=B

$$[A,B] = \begin{bmatrix} 5 & 4 & 15 \\ 3 & 7 & 12 \end{bmatrix}$$

Now, we will make the matrix A as a upper triangular.

$$[A,B] \sim \begin{bmatrix} 5 & 4 & 15 \\ 0 & 23 & 15 \end{bmatrix}$$

$$R_2 \leftrightarrow 5R_2 - 3R_1,$$

This is an upper triangular matrix,

Now, using back substitution method.

$$23\,y = 15$$

$$y = \frac{15}{23} = 0.6522$$

$$5x + 4y = 15$$

$$5x + 4(0.6522) = 15$$

$$5x + 2.6088 = 15$$

$$5x = 15 - 2.6088$$

$$5x = 12.3912$$

$$x = 2.4783$$

Hence, the solution is , $x = 2.4783, y = 0.6522$

3. Using the Gauss-Jordan Method solve the following equations

$$\mathbf{10x + y + z = 12,\ \ 2x + 10y + z = 13,\ \ \ x + y + 5z = 7}$$

Solution:

Interchanging the first and the last equation then,

$$[A, B] = \begin{bmatrix} 1 & 1 & 5 & 1 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{bmatrix}$$

$$[A, B] \sim \begin{bmatrix} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{bmatrix}$$

$$R_2 \leftrightarrow R_2 - 2R_1,$$

$$R_3 \leftrightarrow R_3 - 10R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 5 & 7 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & -9 & -49 & -58 \end{bmatrix}$$

$$R_2 \leftrightarrow \frac{R_2}{8}$$

$$\sim \begin{bmatrix} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & -59.125 & -59.125 \end{bmatrix}$$

$$R_1 \leftrightarrow R_1 - R_2,$$

$$R_3 \leftrightarrow R_3 + 9R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$R_3 \leftrightarrow \frac{R_3}{-59.125}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_1 - 6.125R_3,$$

$$R_2 \leftrightarrow R_2 + 1.125R_3$$

Hence, the solution is , $x = 1, y = 1, z = 1$

UNIT – I– SOLUTION OF EQUATIONS AND EIGEN VALUE PROBLEMS

Problems based on Gauss- Elimination Method, Gauss-Jordan Method

1. Solve the system of equations by (i) Gauss- Elimination Method (ii) Gauss-Jordan Method

$$\mathbf{10x - 2y + 3z = 23, \quad 2x + 10y - 5z = -33, \quad 3x - 4y + 10z = 41}$$

Solution:

Gauss- Elimination Method

The given system is equivalent to

$$\begin{bmatrix} 10 & -2 & 3 \\ 2 & 10 & -5 \\ 3 & -4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 23 \\ -33 \\ 41 \end{bmatrix}$$

AX=B

$$[A, B] = \begin{bmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{bmatrix}$$

Now, we will make the matrix A as a upper triangular.

$$[A, B] \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & -34 & 91 & 341 \end{bmatrix}$$

$$R_2 \leftrightarrow 5R_2 - R_1,$$

$$R_3 \leftrightarrow 10R_3 - 3R_1$$

$$\sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} - - - (1)$$

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Now, using back substitution method.

$$3780z = 11340$$

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$$10x + 4 + 9 = 23$$

$$10x + 13 = 23$$

$$10x = 23 - 13$$

$$10x = 10$$

$$x = 1$$

Hence, the solution is , $x = 1, y = -2, z = 3$

Gauss- Jordan method

Take the equation (1)

$$[A, B] \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix}$$

Now, we will make the matrix A is diagonal matrix.

$$\sim \begin{bmatrix} 12600 & -2520 & 0 & 17640 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{bmatrix}$$

$$R_1 \leftrightarrow 1260R_1 - R_3,$$

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$$\sim \begin{bmatrix} 88452000 & 0 & 0 & 88452000 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{bmatrix}$$

$$R_1 \leftrightarrow 7020R_1 + 2520R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Hence, the solution is , $x = 1, y = -2, z = 3$

2. Solve the following system of equations by Gauss elimination method.

Solution:

The given system is equivalent to

$$\begin{bmatrix} 5 & 4 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \end{bmatrix}$$

AX=B

$$[A, B] = \begin{bmatrix} 5 & 4 & 15 \\ 3 & 7 & 12 \end{bmatrix}$$

Now, we will make the matrix A as a upper triangular.

$$[A, B] \sim \begin{bmatrix} 5 & 4 & 15 \\ 0 & 23 & 15 \end{bmatrix}$$

$$R_2 \leftrightarrow 5R_2 - 3R_1,$$

This is an upper triangular matrix,

Now, using back substitution method.

$$23\,y = 15$$

$$y = \frac{15}{23} = 0.6522$$

$$5x + 4y = 15$$

$$5x + 4(0.6522) = 15$$

$$5x + 2.6088 = 15$$

$$5x = 15 - 2.6088$$

$$5x = 12.3912$$

$$x = 2.4783$$

Hence, the solution is , $x = 2.4783, y = 0.6522$

3. Using the Gauss-Jordan Method solve the following equations

$$\mathbf{10x + y + z = 12,\ \ 2x + 10y + z = 13,\ \ \ x + y + 5z = 7}$$

Solution:

Interchanging the first and the last equation then,

$$[A, B] = \begin{bmatrix} 1 & 1 & 5 & 1 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{bmatrix}$$

$$[A, B] \sim \begin{bmatrix} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{bmatrix}$$

$$R_2 \leftrightarrow R_2 - 2R_1,$$

$$R_3 \leftrightarrow R_3 - 10R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 5 & 7 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & -9 & -49 & -58 \end{bmatrix}$$

$$R_2 \leftrightarrow \frac{R_2}{8}$$

$$\sim \begin{bmatrix} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & -59.125 & -59.125 \end{bmatrix}$$

$$R_1 \leftrightarrow R_1 - R_2,$$

$$R_3 \leftrightarrow R_3 + 9R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 6.125 & 7.125 \\ 0 & 1 & -1.125 & -0.125 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$R_3 \leftrightarrow \frac{R_3}{-59.125}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_1 - 6.125R_3,$$

$$R_2 \leftrightarrow R_2 + 1.125R_3$$

Hence, the solution is , $x = 1, y = 1, z = 1$

UNIT – III– SOLUTION OF EQUATIONS AND EIGEN VALUE PROBLEMS

Problems based on Gauss-Jacobi method and Gauss-Seidal method

Iterative methods

The types of iterative methods are

(i) Gauss-Jacobi method

(ii) Gauss-Seidal method

1. Solve the system of equations by (i) Gauss- Jacobi Method (ii) Gauss-Seidal Method $\mathbf{27x + 6y - z = 85,\ \ x + y + 54z = 110,\ \ 6x + 15y + 2z = 72}$

Solution:

As the coefficient matrix is not diagonally dominant we rewrite the equations

$$27x + 6y - z = 85, 6x + 15y + 2z = 72,$$

$x + y + 54z = 110,$

Since the diagonal elements are dominant in the coefficient matrix, we write, x, y, z as follows:

$$x = \frac{1}{27}[85 - 6y + z]$$

$$y = \frac{1}{15}[72 - 6x - 2z]$$

$$z = \frac{1}{54}[110 - x - y]$$

Gauss- Jacobi Method

Let the initial values be $x = 0, y = 0, z = 0$

First iteration

$$x^{(1)} = \frac{1}{27}[85] = 3.148$$

$$y^{(1)} = \frac{1}{15}[72] = 4.8$$

$$z^{(1)} = \frac{1}{54}[110] = 2.037$$

Second iteration

$$x^{(2)} = \frac{1}{27}[85 - 6y^{(1)} + z^{(1)}] = \frac{1}{27}[85 - 6(4.8) + (2.037)] = 2.157$$

$$y^{(2)} = \frac{1}{15}[72 - 6x^{(1)} - 2z^{(1)}] = \frac{1}{15}[72 - 6(3.148) - 2(2.037)] = 3.269$$

$$z^{(2)} = \frac{1}{54}[110 - x^{(1)} - y^{(1)}] = \frac{1}{54}[110 - 3.148 - 4.8] = 1.890$$

Third iteration

$$x^{(3)} = \frac{1}{27}[85 - 6y^{(2)} + z^{(2)}] = \frac{1}{27}[85 - 6(3.269) + (1.890)] = 2.492$$

$$y^{(3)} = \frac{1}{15}[72 - 6x^{(2)} - 2z^{(2)}] = \frac{1}{15}[72 - 6(2.157) - 2(1.890)] = 3.685$$

$$z^{(3)} = \frac{1}{54}[110 - x^{(2)} - y^{(2)}] = \frac{1}{54}[110 - 2.157 - 3.269] = 1.937$$

Fourth iteration

$$x^{(4)} = \frac{1}{27}[85 - 6y^{(3)} + z^{(3)}] = \frac{1}{27}[85 - 6(3.685) + (1.937)] = 2.401$$

$$y^{(4)} = \frac{1}{15}[72 - 6x^{(3)} - 2z^{(3)}] = \frac{1}{15}[72 - 6(2.492) - 2(1.937)] = 3.545$$

$$z^{(4)} = \frac{1}{54}[110 - x^{(3)} - y^{(3)}] = \frac{1}{54}[110 - 2.492 - 3.685] = 1.923$$

Fifth iteration

$$x^{(5)} = \frac{1}{27}[85 - 6y^{(4)} + z^{(4)}] = \frac{1}{27}[85 - 6(3.545) + (1.923)] = 2.432$$

$$y^{(5)} = \frac{1}{15}[72 - 6x^{(4)} - 2z^{(4)}] = \frac{1}{15}[72 - 6(2.401) - 2(1.923)] = 3.583$$

$$z^{(5)} = \frac{1}{54}[110 - x^{(4)} - y^{(4)}] = \frac{1}{54}[110 - 2.401 - 3.545] = 1.927$$

Sixth iteration

$$x^{(6)} = \frac{1}{27}[85 - 6y^{(5)} + z^{(5)}] = \frac{1}{27}[85 - 6(3.583) + (1.927)] = 2.423$$

$$y^{(6)} = \frac{1}{15}[72 - 6x^{(5)} - 2z^{(5)}] = \frac{1}{15}[72 - 6(2.4332) - 2(1.927)] = 3.570$$

$$z^{(6)} = \frac{1}{54}[110 - x^{(5)} - y^{(5)}] = \frac{1}{54}[110 - 2.432 - 3.583] = 1.926$$

Seventh iteration

$$x^{(7)} = \frac{1}{27}[85 - 6y^{(6)} + z^{(6)}] = \frac{1}{27}[85 - 6(3.570) + (1.926)] = 2.426$$

$$y^{(7)} = \frac{1}{15}[72 - 6x^{(6)} - 2z^{(6)}] = \frac{1}{15}[72 - 6(2.423) - 2(1.926)] = 3.574$$

$$z^{(7)} = \frac{1}{54}[110 - x^{(6)} - y^{(6)}] = \frac{1}{54}[110 - 2.423 - 3.570] = 1.926$$

Eighth iteration

$$x^{(8)} = \frac{1}{27}[85 - 6y^{(7)} + z^{(7)}] = \frac{1}{27}[85 - 6(3.574) + (1.926)] = 2.425$$

$$y^{(8)} = \frac{1}{15}[72 - 6x^{(7)} - 2z^{(7)}] = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$$

$$z^{(8)} = \frac{1}{54}[110 - x^{(7)} - y^{(7)}] = \frac{1}{54}[110 - 2.426 - 3.574] = 1.926$$

Ninth iteration

$$x^{(9)} = \frac{1}{27}[85 - 6y^{(8)} + z^{(8)}] = \frac{1}{27}[85 - 6(3.573) + (1.926)] = 2.426$$

$$y^{(9)} = \frac{1}{15}[72 - 6x^{(8)} - 2z^{(8)}] = \frac{1}{15}[72 - 6(2.425) - 2(1.926)] = 3.573$$

$$z^{(9)} = \frac{1}{54}[110 - x^{(8)} - y^{(8)}] = \frac{1}{54}[110 - 2.425 - 3.573] = 1.926$$

Tenth iteration

$$x^{(10)} = \frac{1}{27}[85 - 6y^{(9)} + z^{(9)}] = \frac{1}{27}[85 - 6(3.573) + (1.926)] = 2.426$$

$$y^{(10)} = \frac{1}{15}[72 - 6x^{(9)} - 2z^{(9)}] = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$$

$$z^{(10)} = \frac{1}{54}[110 - x^{(9)} - y^{(9)}] = \frac{1}{54}[110 - 2.426 - 3.573] = 1.926$$

Hence, $x = 2.426, y = 3.573, z = 1.926$,

Correct to three decimal places.

Gauss- Seidal Method

Let the initial values be $y = 0, z = 0$

First iteration

$$x^{(1)} = \frac{1}{27}[85 - 6y^{(0)} + z^{(0)}] = \frac{1}{27}[85 - 6(0) + (0)] = 3.148$$

$$y^{(1)} = \frac{1}{15}[72 - 6x^{(1)} - 2z^{(0)}] = \frac{1}{15}[72 - 6(3.148) - 0] = 3.541$$

$$z^{(1)} = \frac{1}{54}[110 - x^{(1)} - y^{(1)}] = \frac{1}{54}[110 - 3.148 - 3.541] = 1.913$$

Second iteration

$$x^{(2)} = \frac{1}{27}[85 - 6y^{(1)} + z^{(1)}] = \frac{1}{27}[85 - 6(3.541) + (1.913)] = 2.432$$

$$y^{(2)} = \frac{1}{15}[72 - 6x^{(2)} - 2z^{(1)}] = \frac{1}{15}[72 - 6(2.432) - 2(1.913)] = 3.572$$

$$z^{(2)} = \frac{1}{54}[110 - x^{(2)} - y^{(2)}] = \frac{1}{54}[110 - 2.432 - 3.572] = 1.926$$

Third iteration

$$x^{(3)} = \frac{1}{27}[85 - 6y^{(2)} + z^{(2)}] = \frac{1}{27}[85 - 6(3.572) + (1.926)] = 2.426$$

$$y^{(3)} = \frac{1}{15}[72 - 6x^{(3)} - 2z^{(2)}] = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$$

$$z^{(3)} = \frac{1}{54}[110 - x^{(3)} - y^{(3)}] = \frac{1}{54}[110 - 2.426 - 3.573] = 1.926$$

Fourth iteration

$$x^{(4)} = \frac{1}{27}[85 - 6y^{(3)} + z^{(3)}] = \frac{1}{27}[85 - 6(3.573) + (1.926)] = 2.426$$

$$y^{(4)} = \frac{1}{15}[72 - 6x^{(4)} - 2z^{(3)}] = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$$

$$z^{(4)} = \frac{1}{54}[110 - x^{(4)} - y^{(4)}] = \frac{1}{54}[110 - 2.426 - 3.573] = 1.926$$

Hence, $x = 2.426, y = 3.573, z = 1.926,$

2. Solve the system of equations by Gauss-Seidal Method

$$4x + 2y + z = 85, \quad x + 5y - z = 110, \quad x + y + 8z = 20$$

Solution:

As the coefficient matrix is diagonally dominant solving x,y,z, we get

$$x = \frac{1}{4}[14 - 2y - z]$$

$$y = \frac{1}{5}[10 - x + z]$$

$$z = \frac{1}{8}[20 - x - y]$$

Let the initial values be $y = 0, z = 0$

First iteration

$$x^{(1)} = \frac{1}{4}[14 - 2y^{(0)} - z^{(0)}] = \frac{1}{4}[14 - 2(0) - (0)] = 3.5$$

$$y^{(1)} = \frac{1}{5}[10 - x^{(1)} + z^{(0)}] = \frac{1}{5}[10 - 3.5 + 0] = 1.3$$

$$z^{(1)} = \frac{1}{8}[20 - x^{(1)} - y^{(1)}] = \frac{1}{8}[20 - 3.5 - 1.3] = 1.9$$

Second iteration

$$x^{(2)} = \frac{1}{4}[14 - 2y^{(1)} - z^{(1)}] = \frac{1}{4}[14 - 2(1.3) - (1.9)] = 2.375$$

$$y^{(2)} = \frac{1}{5}[10 - x^{(2)} + z^{(1)}] = \frac{1}{5}[10 - 2.375 + 1.9] = 1.905$$

$$z^{(2)} = \frac{1}{8}[20 - x^{(2)} - y^{(2)}] = \frac{1}{8}[20 - 2.375 - 1.905] = 1.965$$

Third iteration

$$x^{(3)} = \frac{1}{4}[14 - 2y^{(2)} - z^{(2)}] = \frac{1}{4}[14 - 2(1.905) - (1.965)] = 2.056$$

$$y^{(3)} = \frac{1}{5}[10 - x^{(3)} + z^{(2)}] = \frac{1}{5}[10 - 2.056 + 1.965] = 1.982$$

$$z^{(3)} = \frac{1}{8}[20 - x^{(3)} - y^{(3)}] = \frac{1}{8}[20 - 2.056 - 1.982] = 1.995$$

Fourth iteration

$$x^{(4)} = \frac{1}{4}[14 - 2y^{(3)} - z^{(3)}] = \frac{1}{4}[14 - 2(1.982) - (1.995)] = 2.010$$

$$y^{(4)} = \frac{1}{5}[10 - x^{(4)} + z^{(3)}] = \frac{1}{5}[10 - 2.010 + 1.995] = 1.997$$

$$z^{(4)} = \frac{1}{8}[20 - x^{(4)} - y^{(4)}] = \frac{1}{8}[20 - 2.010 - 1.997] = 1.999$$

Fifth iteration

$$x^{(5)} = \frac{1}{4}[14 - 2y^{(4)} - z^{(4)}] = \frac{1}{4}[14 - 2(1.997) - (1.999)] = 2.002$$

$$y^{(5)} = \frac{1}{5}[10 - x^{(5)} + z^{(4)}] = \frac{1}{5}[10 - 2.002 + 1.999] = 1.999$$

$$z^{(5)} = \frac{1}{8}[20 - x^{(5)} - y^{(5)}] = \frac{1}{8}[20 - 2.002 - 1.999] = 2$$

Sixth iteration

$$x^{(6)} = \frac{1}{4}[14 - 2y^{(5)} - z^{(5)}] = \frac{1}{4}[14 - 2(1.999) - (2)] = 2.001$$

$$y^{(6)} = \frac{1}{5}[10 - x^{(6)} + z^{(5)}] = \frac{1}{5}[10 - 2.001 + 2] = 2$$

$$z^{(6)} = \frac{1}{8}[20 - x^{(6)} - y^{(6)}] = \frac{1}{8}[20 - 2.001 - 2] = 2$$

Seventh iteration

$$x^{(7)} = \frac{1}{4}[14 - 2y^{(6)} - z^{(6)}] = \frac{1}{4}[14 - 2(2.001) - (2)] = 2$$

$$y^{(7)} = \frac{1}{5}[10 - x^{(7)} + z^{(6)}] = \frac{1}{5}[10 - 2 + 2] = 2$$

$$z^{(7)} = \frac{1}{8}[20 - x^{(7)} - y^{(7)}] = \frac{1}{8}[20 - 2 - 2] = 2$$

Eighth iteration

$$x^{(8)} = \frac{1}{4}[14 - 2y^{(7)} - z^{(7)}] = \frac{1}{4}[14 - 2(2) - (2)] = 2$$

$$y^{(8)} = \frac{1}{5}[10 - x^{(8)} + z^{(7)}] = \frac{1}{5}[10 - 2 + 2] = 2$$

$$z^{(8)} = \frac{1}{8}[20 - x^{(8)} - y^{(8)}] = \frac{1}{8}[20 - 2 - 2] = 2$$

Hence, $x = 2, y = 2, z = 2$.

UNIT – III– SOLUTION OF EQUATIONS AND EIGEN VALUE PROBLEMS

PROBLEMS BASED ON EIGENVALUES OF A MATRIX BY POWER METHOD

1.Find the numerically largest Eigenvalue of $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ by power method.

Solution:

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an arbitrary initial Eigenvector.

$$AX_1 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} = 6\begin{bmatrix} 0.2 \\ 0.7 \\ 1 \end{bmatrix} = 6X_2$$

X2=1−3244−16350.20.71=0.12.68.3=8.3   00.31=8.3X3

$$AX_3 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 0.3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.1 \\ 0.2 \\ 5.9 \end{bmatrix} = 5.9\begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix} = 5.9X_4$$

$$AX_4 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.2 \\ -0.2 \\ 6.2 \end{bmatrix} = 6.2\begin{bmatrix} 0.4 \\ 0 \\ 1 \end{bmatrix} = 6.2X_5$$

$$AX_5 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.4 \\ 0.6 \\ 7.4 \end{bmatrix} = 7.4\begin{bmatrix} 0.3 \\ 0.1 \\ 1 \end{bmatrix} = 7.4X_6$$

$$AX_6 = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}\begin{bmatrix} 0.3 \\ 0.1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0.6 \\ 7.1 \end{bmatrix} = 7.1\begin{bmatrix} 0.3 \\ 0.1 \\ 1 \end{bmatrix} = 7.1X_7$$

$X_6 = X_7$, Hence, the numerically largest Eigenvalue=7 and the corresponding Eigenvector =

$$\begin{bmatrix} 0.3 \\ 0.1 \\ 1 \end{bmatrix}$$

2.Find the dominant Eigenvalue and the corresponding Eigenvector of $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find also the least latent root and hence the third Eigenvalue also.

Solution:

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be an arbitrary initial Eigenvector.

$$AX_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\, X_2$$

$$AX_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.4 \\ 0 \end{bmatrix} = 7\, X_3$$

X3=16112000310.40=3.41.80=3.4 10.50=3.4 X4

$$AX_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 4\, X_5$$

$X_4 = X_5$, Hence, the numerically largest Eigenvalue=7 and the corresponding Eigenvector =

$$\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

∴ Dominant Eigenvalue= 4 ; corresponding Eigenvector is $(1 \quad 0.5 \quad 0)$

To find the least Eigenvalue, let $B = A - 4I$ since $\lambda_1 = 4$

$$\therefore B = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

We will find the dominant Eigenvalue of B

Let $Y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial Eigenvector.

$$BY_1 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -3\, Y_2$$

$$BY_2 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -5\, Y_3$$

$$BY_3 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -5\, Y_4$$

$$Dominant\ Eigenvalue\ of\ B\ is -5.$$

Adding 4, smallest Eigenvalue of $A = -5 + 4 = -1$

Sum of the Eigenvalues=Trace of $A = 1 + 2 + 3 = 6$

$$4 + (-1) + \lambda_3 = 6,$$

$\therefore \lambda_3 = 3.$

All thethree Eigenvalues are $4, 3, -1$

UNIT – III– SOLUTION OF EQUATIONS AND EIGEN VALUE PROBLEMS

Problems based on NEWTON RAPHSON methods

Introduction:

To find a value x_0 such that $f(x_0) = 0$. If $f(x)$ is a polynomial, then the equation $f(x) = 0$ is called an **algebraic equation**.

Equations which involve transcendental functions like $sin\ x$, $cos\ x$, $tan\ x$, $log\ x$, e^x etc are called **transcendental equations**.

$x^2 - 2x + 3 = 0, x^3 - 4x^2 = 5x - 7 = 0, 4x^4 - 5x^2 + 2x - 5 = 0$are some examples of algebraic equations.

$xe^x - 2 = 0, 2x^2 = 4cosx = 0, xlogx - 12 = 0$ are some examples of transcendental equations.

Any value of a for which $f(a) = 0$ is called a **root or solution of the equation**$f(x) = 0$.

Geometrically, the point at which a curve $y = f(x)$ intersects the x – axis is a root of the equation $f(x) = 0$

NEWTON'S METHOD (OR NEWTON-RAPHSON METHOD)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0,1,2, \ldots$$

1. Find the positive root of $x^4 - x = 10$ correct to three decimal places using Newton-Raphson method.

Solution:

Given, $x^4 - x = 10$

$$f(x) = x^4 - x - 10$$

$$f(0) = 0 - 0 - 10 = -10\ (-ive)$$

$$f(1) = 1^4 - 1 - 10 = -10\ (-ive)$$

$$f(2) = 2^4 - 2 - 10 = 4\ (+ive),$$

So, a root lies between 1 and 2.

Here, $|f(1)| > |f(2)|$

Therefore, the root is nearer to 2.

Let us take, $x_0 = 2$

N-R Formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0,1,2, \ldots$

$$f(x) = x^4 - x - 10$$

$$\Rightarrow f'(x) = 4x^3 - 1$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Put , $n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{{x_0}^4 - x_0 - 10}{4{x_0}^3 - 1}$$

$$= 2 - \frac{2^4 - 2 - 10}{4(2)^3 - 1} = 2 - \frac{4}{31} = 1.871$$

$$x_1 = 1.871$$

Put , $n = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{{x_1}^4 - x_1 - 10}{4{x_1}^3 - 1}$$

$$= 1.871 - \frac{(1.871)^4 - 1.871 - 10}{4(1.871)^3 - 1} = 1.871 - \frac{0.383}{25.199} = 1.856$$

$$x_2 = 1.856$$

Put , $n = 2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{{x_2}^4 - x_2 - 10}{4{x_2}^3 - 1}$$

$$= 1.856 - \frac{(1.856)^4 - 1.856 - 10}{4(1.856)^3 - 1} = 1.856 - \frac{0.010}{24.574} = 1.856$$

$$x_3 = 1.856$$

Here, $x_2 = x_3$

Hence, the better approximate root is 1.856

2. Find the real positive root of $3x - \cos x - 1 = 0$, by Newton's method correct to 3 decimal places.

Solution:

Given, $3x - \cos x - 1 = 0$

$$f(x) = 3x - \cos x - 1$$

$$f(0) = 0 - 1 - 1 = -2\ (-ive)$$

$$f(1) = 3 - \cos 1 - 1 = 2 - \cos 1 = 1.459697\ (+ive)$$

So, a root lies between 0 and 1.

Here, $|f(0)| > |f(1)|$

Therefore, the root is nearer to 1.

Let us take, $x_0 = 0.6$

N-R Formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0,1,2, \ldots$

$$f(x) = 3x - \cos x - 1$$

$$\Rightarrow f'(x) = 3 + \sin x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Put , $n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{3x_0 - \cos x_0 - 1}{3 + \sin x_0}$$

$$= 0.6 - \frac{3(0.6) - \cos(0.6) - 1}{3 + \sin(0.6)} = 0.6 - \frac{(-0.025336)}{3.564642} = 0.607$$

$$x_1 = 0.607$$

Put , $n = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{3x_1 - \cos x_1 - 1}{3 + \sin x_1}$$

$$= 0.607 - \frac{3(0.607) - \cos(0.607) - 1}{3 + \sin(0.607)} = 0.607 - \frac{(0.000023)}{3.570495} = 0.607$$

$$x_2 = 0.607$$

Here, $x_1 = x_2$

Hence, the better approximate root is 0.607

3. Find a root of $x \log_{10} x - 1.2 = 0$ by by Newton's method correct to 3 decimal places.

Solution:

Given, $x \log_{10} x - 1.2 = 0$

$$f(x) = x \log_{10} x - 1.2$$

$$f(1) = \log_{10} 1 - 1.2 = -1.2 \ (-ive)$$

$$f(2) = 2 \log_{10} 2 - 1.2 = -598 \ (-ive)$$

$$f(3) = 3 \log_{10} 3 - 1.2 = 0.231 \ (+ive)$$

So, a root lies between 2 and 3.

Here, $|f(2)| > |f(3)|$

Therefore, the root is nearer to 3.

Let us take, $x_0 = 2.7$

N-R Formula, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0,1,2, \ldots$

$$f(x) = x \log_{10} x - 1.2$$

$$\Rightarrow f'(x) = [x * \frac{1}{x} \log_{10} e] + \log_{10} x$$

$= \log_{10} e + \log_{10} x$

Put , $n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0 \log_{10} x_0 - 1.2}{\log_{10} e + \log_{10} x_0}$$

$$= 2.7 - \frac{2.7 \log_{10} 2.7 - 1.2}{\log_{10} e + \log_{10} 2.7} = 2.7 - \frac{(-0.035)}{0.866} = 2.740$$

$$x_1 = 2.740$$

Put , $n = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1 \log_{10} x_1 - 1.2}{\log_{10} e + \log_{10} x_1}$$

$$= 2.74 - \frac{2.74 \log_{10} 2.74 - 1.2}{\log_{10} e + \log_{10} 2.74} = 2.74 - \frac{(-0.001)}{0.872} = 2.741$$

$$x_2 = 2.741$$

Put , $n = 2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{x_2 \log_{10} x_2 - 1.2}{\log_{10} e + \log_{10} x_2}$$

$$= 2.741 - \frac{2.741 \log_{10} 2.741 - 1.2}{\log_{10} e + \log_{10} 2.741} = 2.741 - \frac{(0)}{0.872} = 2.741$$

$$x_3 = 2.741$$

Here, $x_2 = x_3$ Hence, the better approximate root is 2.741

4. Find the iterative formula for finding the value of $\frac{1}{N}$, where N is a real number, using Newton-Raphson method. Hence evaluate $\frac{1}{26}$ correct to 4 decimal places.

Solution:

Let, $x = \frac{1}{N}$

$$ie).,N = \frac{1}{x}$$

Let, $f(x) = \frac{1}{x} - N,$

$$\Rightarrow f'(x) = -\frac{1}{x^2}$$

N-R Formula, $x_{n+1} = x_n - [\frac{f(x_n)}{f'(x_n)}],\ n = 0,1,2, \ldots$

$$x_{n+1} = x_n - [\frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}}]$$

$$= x_n - x_n^2 [\frac{1}{x_n} - N]$$

$$= x_n + x_n - Nx_n^2 = 2x_n - Nx_n^2$$

$x_{n+1} = x_n[2 - Nx_n]$ is the iterative formula.

To find $\frac{1}{26}$, $take\ N = 26$

Let $x_0 = 0.04\ [\because \frac{1}{25} = 0.04]$

$$x_{n+1} = x_n[2 - Nx_n]$$

$$x_1 = x_0[2 - 26x_0]$$

$$= (0.04)[2 - 26(0.04)] = 0.0384$$

$$x_1 = 0.0384$$

$$x_2 = x_1[2 - 26x_1]$$

$$= (0.0384)[2 - 26(0.0384)] = 0.0385$$

$$x_2 = 0.0385$$

$$x_3 = x_2[2 - 26x_2]$$

$$= (0.0385)[2 - 26(0.0385)] = 0.0385$$

$$x_2 = 0.0385$$

Here, $x_2 = x_3$

Hence, the value of $\frac{1}{26} = 0.0385$