

DEPARTMENT OF MATHEMATICS

**NAME OF THE SUBJECT : STATISTICS &
NUMERICAL METHODS**

SUBJECT CODE : MA8452

REGULATION : 2017

**UNIT - V: NUMERICAL SOLUTION OF
ORDINARY DIFFERENTIAL EQUATIONS**

UNIT V - NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS.

01. Obtain y by Taylor series method, given that $y' = xy + 1$, $y(0) = 1$, for $x = 0.1$ and 0.2 correct to 4 decimal places.

$$y_{n+1} = y_n + \frac{h}{1!} y'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''_n + \dots$$

When $n=0$, $y_1 = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \dots$

When $n=1$, $y_2 = y_1 + \frac{h}{1!} y'_1 + \frac{h^2}{2!} y''_1 + \frac{h^3}{3!} y'''_1 + \dots$

$$y'' = xy' + y$$

$$y''' = xy'' + 2y'$$

$$y^{iv} = xy''' + 3y''$$

$$y_1 = 1.1053 \quad \text{and} \quad y_2 = 1.2228$$

02. Solve $(1+x) \frac{dy}{dx} = -y^2$, $y(0)=1$ by Modified Euler's method by choosing $h = 0.1$, find $y(0.1)$ and $y(0.2)$.

Solution: Given $x_0 = 0$, $y_0 = 1$, $h = 0.1$ and $f(x,y) = -\frac{y^2}{1+x}$.

$$y_{n+1} = y_n + h \left[\frac{h}{2} f(x_n, y_n) + \frac{h}{2} f(x_n + h, y_n + h f(x_n, y_n)) \right]$$

Put $n = 0$ we get $y_1 = y_0 + 0.1 \left[\frac{0.1}{2} f(x_0, y_0) + \frac{0.1}{2} f(x_0 + 0.1, y_0 + 0.1 f(x_0, y_0)) \right]$

$$y_1 = y(0.1) = 1 + 0.1 \left[\frac{0.1}{2} f(0, 1) + \frac{0.1}{2} f(0.1, 1 + 0.1 f(0, 1)) \right] = 0.91278$$

Put $n = 1$ we get $y_2 = y(0.2) = y_1 + 0.1 \left[\frac{0.1}{2} f(x_1, y_1) + \frac{0.1}{2} f(x_1 + 0.1, y_1 + 0.1 f(x_1, y_1)) \right] = 0.84550$

3) If $\frac{dy}{dx} = \log_{10}(x + y)$, $y(0) = 2$ by Euler's method by choosing $h = 0.2$, find $y(0.2)$ and $y(0.4)$.

Given Data is: $x_0 = 0, y_0 = 2, h = 0.2$ and $f(x, y) = \log_{10}(x + y)$

Euler's Formula is $y_{n+1} = y_n + hf(x_n, y_n)$, $n = 0, 1, 2, 3, \dots$

Put $n = 0$ we get, $y_1 = 2.0 + 0.2 \log_{10}(0 + 2) = 2.0602$

Put $n = 1$ we get, $y_2 = 2.0656 + 0.2 \log_{10}(0.2 + 2.0656) = 2.1366$

4) If $\frac{dy}{dx} = -\frac{y^2}{1+x}$, $y(0) = 1$ by Euler's method by choosing $h = 0.1$, find $y(0.1)$ and $y(0.2)$.

Given $x_0 = 0, y_0 = 1, h = 0.1$ and $f(x, y) = -\frac{y^2}{1+x}$.

Euler's Formula is $y_{n+1} = y_n + hf(x_n, y_n)$, $n = 0, 1, 2, 3, \dots$

Put $n = 0$ we get, $y_1 = y(0.1) = 1 - 0.1 \left(\frac{(1)^2}{1+0} \right) = 0.9$

Put $n = 1$ we get, $y_2 = y(0.2) = 0.9 - 0.1 \left(\frac{(0.9)^2}{1+0.1} \right) = 0.82636$

5) Obtain the approximate value of y at $x = 0.1$ & 0.2 for the differential equation $\frac{dy}{dx} = 2y + 3e^x$, $y(0) = 0$, by

Taylor's Series method. Compare the numerical solution obtained with the exact solution.

Given $x_0 = 0, y_0 = 0, h = 0.1$ and $y' = 2y + 3e^x$

Taylor's series expansion is $y_{n+1} = y_n + \frac{h}{1!} y'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''_n + \dots$

For $n = 0$, $y_1 = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \dots$ (1)

$$\frac{dy}{dx} = y'(x) = 2y + 3e^x \qquad y_0' = 3$$

Differentiating $y'(x) = 2y + 3e^x$ successively three times and putting $x = y = 0$, we get

$$y''(x) = 2y' + 3e^x \qquad y_0'' = 9$$

$$y'''(x) = 2y'' + 3e^x \qquad y_0''' = 21$$

$$y^{iv}(x) = 2y''' + 3e^x \qquad y_0^{iv} = 45$$

Putting the values in (1)

$$y(x) = 0 + 3h + \frac{9}{2}h^2 + \frac{21}{6}h^3 + \frac{45}{24}h^4 = 3(0.1) + 4.5(0.1)^2 + 3.5(0.1)^3 + 1.875(0.1)^4 = 0.3486875$$

Exact Solution:

The given differential equation can be written as $\frac{dy}{dx} - 2y = 3e^x$ which is Leibnitz's linear differential equation.

Its I.F. is $I.F. = e^{-\int 2dx} = e^{-2x}$

Therefore the general solution is, $ye^{-2x} = \int 3e^x(e^{-2x})dx + c = -3e^{-x} + c \Rightarrow y = -3e^x + ce^{2x}$ -- (2)

Using the given initial condition $y = 0$ when $x = 0$ in (3) we get $c = 3$.

Thus the exact solution is $y = 3(e^{2x} - e^x)$

When $x = 0.1$, the exact solution is $y(0.1) = 0.348695$

6) If $\frac{dy}{dx} = \sin x + \cos y$, $y(2.5) = 0$ by Modified Euler's method by choosing $h = 0.5$, find $y(3.5)$

Given Data is : $x_0 = 2.5, y_0 = 0, h = 0.5$ and $f(x, y) = \sin x + \cos y$

$$y_{n+1} = y_n + h f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

Put $n = 0$ we get $y_1 = y_0 + 0.5 \left[f\left(x_0 + \frac{0.5}{2}, y_0 + \frac{0.5}{2} [f(x_0, y_0)]\right) \right]$

$$y_1 = 0 + 0.5[f(2.5 + 0.25, 0 + 0.25[f(2.5, 0)])] = 0.6354$$

Put n = 1 we get $y_2 = y_1 + h \left[\frac{0.5}{2} f\left(x_1 + \frac{0.5}{2}, y_1 + \frac{0.5}{2} [f(x_1, y_1)]\right) \right]$

$$y_1 = 0.6354 + 0.5[f(3 + 0.25, 0.6354 + 0.25[f(3, 0.6354)])] = 0.93155$$

7) Apply Runge - Kutta method, to find an approximate value of y when x = 0.2 given

that $\frac{dy}{dx} = x + y, \quad y(0) = 1$

Given: $x_0 = 0, y_0 = 1, h = 0.2$ and $f(x, y) = x + y$

Finding $y_1 = y(0.2)$:

R - K method (for n = 0) is: $y_1 = y(0.2) = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$ ----- (1)

$$k_1 = hf(x_0, y_0) = 0.2 \times [0 + 1] = 0.2; \quad k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 \times \left[\left(0 + \frac{0.2}{2}\right) + \left(1 + \frac{0.2}{2}\right) \right] = 0.2400$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \times \left[\left(0 + \frac{0.2}{2}\right) + \left(1 + \frac{0.24}{2}\right) \right] = 0.2440$$

$$k_4 = hf(x_0 + h, y_0 + k_1) = 0.2 \times [(0 + 0.2) + (1 + 0.2440)] = 0.2888$$

Using the values of k_1, k_2, k_3 and k_4 in (1), we get

$$y_1 = y(0.2) = 1 + \frac{1}{6} (0.2 + (0.24 + 0.244)^2 + 0.2888) = 1.2468$$

Hence the required approximate value of y is 1.2468.

8). Apply Runge – Kutta method, to find an approximate value of y when x = 1.2 in steps of 0.1 given that $y' = x^2 + y^2$ and $y(1) = 1.5$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

9) Solve $\frac{dy}{dx} = y - x^2 + 1$, $y(0) = 0.5$ by Modified Euler's method by choosing $h = 0.1$, find $y(0.1)$ and $y(0.2)$.

$$y_{n+1} = y_n + h f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}h f(x_n, y_n)\right)$$

$$x_0 = 0, y_0 = 0.5, f(x, y) = y - x^2 + 1$$

$$y_1 = 0.655 \text{ and } y_2 = 0.826475$$

10) Using Runge - Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$, $y(0) = 1$ at $x = 0.2$ & 0.4 .

$$\text{Given: } x_0 = 0, y_0 = 1, h = 0.2 \text{ and } f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$$

Finding $y_1 = y(0.2)$

$$\text{R - K method (for } n = 0) \text{ is: } y_1 = y(0.2) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \text{----- (2)}$$

$$k_1 = hf(x_0, y_0) = 0.2 \times f(0, 1) = 0.2 ; \quad k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 \times f(0.1, 1.1) = 0.19672$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 \times f(0.1, 1.0936) = 0.1967;$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 \times f(0.2, 1.1967) = 0.1891$$

Using the values of k_1, k_2, k_3 and k_4 in (2), we get

$$y_1 = y(0.2) = 1 + \frac{1}{6}(0.2 + 2(0.19672) + 2(0.1967) + 0.1891) = 1 + 0.19599 = 1.19599$$

Hence the required approximate value of y is 1.19599.

Finding $y_2 = y(0.4)$:

We have $x_1 = 0.1, y_1 = 1.19599$ and $h = 0.2$

$$\text{R - K method (for } n = 1) \text{ is: } y_2 = y(0.4) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \text{----- (3)}$$

$$k_1 = hf(x_1, y_1) = 0.2 \times f(0.2, 1.19599) = 0.1891; \quad k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2 \times f(0.3, 1.2906) = 0.1795$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2 \times f(0.3, 1.2858) = 0.1793$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = 0.2 \times f(0.4, 1.3753) = 0.1688$$

Using the values of k_1, k_2, k_3 and k_4 in (3), we get

$$y_2 = y(0.4) = 1.19599 + \frac{1}{6}(0.1891 + 2(0.1795) + 2(0.1793) + 0.1688) = 1.19599 + 0.1792 = 1.37519$$

Hence the required approximate value of y is 1.37519

11) Use Milne's method to find $y(0.6)$, given $y' = \frac{1}{x+y}$, $y(0) = 2$, $y(0.2) = 2.0933$, $y(0.4) = 2.1755$, $y(0.6) =$

2.2493

$$x_0 = 0, y_0 = 2, x_1 = 0.2, y_1 = 2.0933, x_2 = 0.4, y_2 = 2.1755, x_3 = 0.6, y_3 = 2.2493$$

$$y_{n+1,P} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$$

$$y_{4,P} = 2.3162$$

$$y_{n+1,C} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$$

$$y_{4,C} = 2.3163$$

12) Given $\frac{dy}{dx} = x - y^2$, $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0795$ and $y(0.6) = 0.1762$. Compute $y(0.8)$ using Milne's Method.

Solution:

$$\frac{dy}{dx} = x - y^2 = f(x, y) \quad , \text{first find } y(0.8) \text{ and then } y(1).$$

Finding y(0.8)

Given:

$$x_0 = 0, y_0 = 0 \quad \text{and} \quad f_0 = f(x_0, y_0) = 0 ; \quad x_1 = 0.2, y_1 = 0.02 \quad \text{and} \quad f_1 = f(x_1, y_1) = 0.1996$$

$$x_2 = 0.4, y_2 = 0.0795 \quad \text{and} \quad f_2 = f(x_2, y_2) = 0.3937$$

$$x_3 = 0.6, y_3 = 0.1762 \quad \text{and} \quad f_3 = f(x_3, y_3) = 0.56895$$

To Find : $y_4 = y(x_4) = y(0.8)$

Predictor Method

$$y_4^{(P)} = y(0.8) = y_0 + \frac{4h}{3} [2f_1 - f_2 + 2f_3] = 0 + \frac{4(0.2)}{3} [(2 \times 0.1996) - 0.3937 + 2 \times 0.56895] = 0.30491$$

Now we compute $f_4 = f(0.8, 0.30491) = 0.7070$

Corrector Method

$$y_4^{(C)} = y(0.8) = y_2 + \frac{h}{3} [f_2 + 4f_3 + f_4] = 0.0795 + \frac{0.2}{3} [0.3937 + 4 \times 0.56895 + 0.7070] = 0.3046$$

13) Given $y'' + xy' + y = 0, y(0)=1, y'(0)=0$, find the value of $y(0.1)$ by Runge-Kutta method of fourth order.

Solution:

Given $y'' + xy' + y = 0, y(0)=1, y'(0)=0, h=0.1, y_0 = 1, x_0 = 0, y_1 = y(0.1)$, put $y' = z$. The equation becomes, $y'' = z' = -xz - y$, let $y' = f(x, y, z)$,

$z' = g(x, y, z)$, given $y_0 = 1, z_0 = y'_0 = 0$. By the formula

$$k_1 = hf(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2(k_2 + k_3) + k_4)$$

$$l_1 = hg(x_0, y_0, z_0)$$

$$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$\Delta z = \frac{1}{6}(l_1 + 2(l_2 + l_3) + l_4)$$

$$k_1 = hf(0, 1, 0) = 0, l_1 = hg(0, 1, 0) = -0.1, k_2 = -0.005, l_2 = -0.09975, k_3 = -0.00499,$$

$$l_3 = -0.0995, k_4 = -0.00995, l_4 = -0.0985, \therefore y_1 = y(0.1) = 0.995.$$

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