INTRODUCTION

Random Experiment:

An experiment whose output is uncertain even though all the outcomes are known.

Example: Tossing a coin, Throwing a fair die, Birth of a baby.

Sample Space:

The set of all possible outcomes in a random experiment. It is denoted by S.

Example:

For tossing a fair coin, $S = \{H, T\}$

For throwing a fair die, $S = \{1, 2, 3, 4, 5, 6, \}$

For birth of a baby, $S = \{M, F\}$

Event:

A subset of sample space is event. It is denoted by A.

Mutually Exclusive Events:

Two events A and B are said to be mutually exclusive events if they do not occur simultaneously. If A and B are mutually exclusive, then $A \cap B = \Phi$

Example:

Tossing two unbiased coins

$$S = \{HH, HT, TH, TT\}$$

(i) Let $A = \{HH\}, B = \{HT\}$

$$A \cap B = \{H\} \neq \Phi$$

Then A and B are not mutually exclusive.

(i) Let $A = \{HH\}, B = \{TT\}$

$$A \cap B = \Phi$$

Then A and B are mutually exclusive.

1.1 Probability:

Probability of an event A is $P(A) = \frac{n(A)}{n(S)}$

i.e., $P(A) = \frac{number\ of\ cases\ favourable\ to\ A}{Total\ number\ of\ cases}$

Axioms of Probability:

(i) $0 \leq P(A) \leq 1$

(ii) $P(S) = 1$

(iii) $P(A \cup B) = P(A) + P(B)$, if A and B are mutually exclusive.

Note:

(i) $P(\phi) = 0$

(ii) $P(\bar{A}) = 1 - P(A)$, for any event A

(iii) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, for any two events A and B.

Independent events:

Two events A and B are said to be independent if occurrence of A does not affect the occurrence of B.

Condition for two events and B are independent:

$$P(A \cap B) = P(A)\, P(B)$$

Conditional Probability:

If the probability of the event A provided the event B has already occurred is called the conditional probability and is defined as

$$P(A|B) = \frac{P(A\cap B)}{P(B)}, \text{ provided } P(B) \neq 0$$

The probability of an event B provided A has occurred already is given by

$$P(B|A) = \frac{P(A\cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

RANDOM VARIABLE

Define Random Variables:

A random variable is a function that assign a real number for all the outcomes in the sample space of a random experiment.

Example:

Toss two coins then the sample space $S = \{HH, HT, TH, TT\}$

Now we define a random variable X to denote the number of heads in 2 tosses.

X(HH) = 2

X(HT) = 1

X(TH) = 1

X(TT) = 0

Types of Random Variables:

(i) Discrete Random Variables

(ii) Continuous Random Variables

1.2 Baye's Theorem

If $B_1, B_2, \ldots, B_n$ be a set of exhaustive and mutually exclusive events associated with a random experiment and D is another event associated with B_i, then

$$P(D/B_i) = \frac{P(B_i) \cdot P(D/B_i)}{\sum_{i=1}^{n} P(B_i)\, P(D/B_i)}$$

State and Prove Bayes Theorem

(OR)

State and Prove Theorem of Probability of Causes.

Soln :

Statement

If $B_1, B_2, \ldots B_n$ be a set of exhaustive and mutually exclusive events associated with random experiment and D is another event associated with (or caused)by Bi. Then

$$P(D \mid B_i) = \frac{P(B_i) \cdot P(D/B_i)}{\sum_{i=1}^{n} P(B_i) P(D/B_i)}$$

Proof :

$$P(B_i \cap D) = P(B_i) \cdot P\left(\frac{D}{B_i}\right)$$

$$P(D \cap B_i) = P(D) \cdot P\left(\frac{B_i}{D}\right)$$

$$P(B_i \mid D) = \frac{P(B_i) \cdot P(D/B_i)}{P(D)} \ldots \ldots \ldots (1)$$

The inner circle reprosents the events D.D can occur along with $B_1, B_2, \dots B_n$ that are exhaustive and mutually exclusive

$\therefore DB_1, DB_1, \dots . DB_n$ are also mutually exclusive such that

$$D = DB_1 + DB_2 + \cdots + DB_n$$

$$\therefore D = \sum DB_i$$

$$P[D] = P[\Sigma DB_i]$$

$$= \sum P[DB_i]$$

$$= \sum P[D \cap B_i]$$

$$P[D] = \sum_{i=1}^{n} P(B_i) \cdot P(D/B_i)$$

Substitute $P[D]$ in eqn (1)

$$(1) \Rightarrow P[B_i/D] = \frac{P(B_i) \cdot P(D/B_i)}{\sum_{i=1}^{n} P(B_i)P(D/B_i)}$$

Hence the proof ,

Problem based on Baye's Theorem

1. **Four boxes A,B,C,D contain fuses. The boxes contain 5000, 3000, 2000 and 1000 fuses respectively. The percentages of fuses in boxes which are defective are 3%, 2%, 1% and 5% respectively.one fuse in selected at random arbitrarily from one of the boxes. It is found to be defective fuse. Find the probability that it has come from box D.**

(OR)

Four boxes A,B,C,D contain fuses. Box A contain 5000 fuses , box B contain 3000 fuses, box C contain 2000 fuses and box D contain 1000 fuses. The percentage of fuses in boxes which are defective are 3%, 2%, 1% and 0.5% respectively. One fuse is select at random from one of the boxes. It is found to be defective fuse . What is the probability that it has come from box D.

Soln:

Since selection ratio is not given

Assume selection ratio is 1 : 1 : 1 : 1

Total = 1+1+1+1 = 4

$P(A) = \frac{1}{4}$

$P(B) = \frac{1}{4}$

$P(C) = \frac{1}{4}$

$P(D) = 1/4$

Let E be the event selecting a defective fuse from any one of the machine

$P(E/A) = 3\% = 0.03$

$P(E/B) = 2\% = 0.02$

$P(E/C) = 1\% = 0.01$

$P(E/D) = 5\% = 0.05$

$$P(E) = P(A)P(E/A) + P(B)P(E/B) + P(C)P(F/C) + P(D)(F/D)$$

$$= \frac{1}{4} \times 0.03 + 1/4 \times 0.02 + 1/4 \times 0.01 + 1/4 \times 0.05$$

$$= 0.0275$$

$$P(D/E) = \frac{P(D)P(E/D)}{P(E)}$$

$$= \frac{\frac{1}{4}\times 0.05}{0.0275} = 0.4545$$

$$= 0.4545$$

2. **In a bolt Factory, Machines A,B and C manufacture respectively 25%, 35% and 40% of total output . also out of these output of A,B,C are 5,4,2 percent respectively are defective. A bolt is drawn at random from the total output and it is found to be defective. What is the probability that it was manufactured by the machine B?**

(OR)

In a company machine A, B and C manufactured bolts, 25%, 35% and 40% of total output . also out of these output of A,B,C are 5,4,2 percent respectively are defective. A bolt is taken random from the total output and it is found to be defective. Find the probability that it was manufactured by the machine B?

Soln:

Given ,P(E_1) =P(A) =25% = 0.25

P(E_2) =P(B) =35% = 0.35

P(E_3) =P(C) =40% = 0.40

Let D be the event of drawing defective bolt

$$P(D/E_1) = 5\% = \frac{5}{100} = 0.05$$

$$P(D/E_2) = 4\% = 0.04$$

$$P(D/E_3) = 2\%) = 0 \cdot 02$$

To find $P(E_2/D)$

By Bayes theorem

$$P(E_2/D) = \frac{P(E_2)P(D/E_2)}{P(E_1)P(D/E_1) + P(E_2)P(D/E_2) + P(E_3)P(D/E_3)}$$

$$= \frac{(0.35)(0.04)}{(0.25)(0.05)+(0.35)(0.04)+(0.4)(0.02)}$$

$$= \frac{0.014}{0.0345}$$

$= 0.406$

3. A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls. One ball is drawn at random from one of the bag and is found to be red. Find the Probability that it was drawn from bag B

(OR)

A box A contains 2 white and 3 red balls and a box B contains 4 white and 5 red balls at random one ball is taking and is found to be red. What is the probability that it was drawn from bag B?

Soln:

Let B_1 be the event that the ball is drawn from the bag A.

Let B_2 be the event that the ball is drawn from the bag B.

Lat A be the event that the drawn ball is red

$$P(B_1) = P(B_2) = \frac{1}{2}$$

$$P(A/B_1) = \frac{3C_1}{5C_1} = \frac{3}{5}$$

$$P(A/B_2) = \frac{5C_1}{9C_1} = \frac{5}{9}$$

$$P(B_2/A) = \frac{P(B_2)P(A/B_2)}{P(B_1)P(A/B_1) + P(B_2)P(A\backslash B_2)}$$

$$= \frac{(\frac{1}{2})(\frac{5}{9})}{(\frac{1}{2})(\frac{3}{5})+(\frac{1}{2})(\frac{5}{9})}$$

$$= \frac{\frac{5}{18}}{\frac{52}{90}}$$

$$P(B_2/A)=\frac{25}{52}$$

1.3 MOMENTS

Definition:

The rth moment about origin is $\mu'_r = E[x^r]$

First moment about origin $\mu'_1 = E[X]$

$= E(X^2) - [E(X)]^2$

Variance $\sigma^2 = \mu'_2 - (\mu'_1)^2$

The rth moment about mean is $\mu_r = E[(X-\mu)^r]$, *where* μ *is mean of* X.

$$\mu_1 = E[(X-\mu)^1]$$

$$= E[X] - E[\mu] = \mu - \mu = 0$$

$$\therefore \mu_1 = 0$$

$$\mu_2 = E[(X-\mu)^2]$$

$$= E[X^2 + \mu^2 - 2X\mu]$$

$$= E[X^2] + \mu^2 - 2E[X]\mu$$

$$= E(X^2) + [E(X)]^2 - 2E(X)E(X)$$

$$= E(X^2) + [E(X)]^2 - 2[E(X)]^2$$

$$= E(X^2) - [E(X)]^2 = \sigma^2$$

$$\therefore \mu_2 = \sigma^2$$

If the probability density of X is given $f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & otherwise \end{cases}$

Find its rth **moment about origin. Hence find evaluate** $E[(2X+1)^2]$

Solution:

The rth moment about origin is given by

$$\mu_r' = E[x_r'] = \int_0^1 x^r f(x)dx$$

$$= \int_0^1 x^r \, 2(1-x)dx$$

$$= 2\int_0^1 (x^r - x^{r+1})dx$$

$$= 2\left[\frac{x^{r+1}}{r+1} - \frac{x^{r+1+1}}{r+2}\right]_0^1$$

$$= 2\left[\frac{1}{r+1} - \frac{1}{r+2}\right]$$

$$= 2\left[\frac{(r+2)-(r+1)}{(r+2)(r+1)}\right] = \frac{2}{r^2+3r+2}$$

$$E[(2X+1)^2] = E[4X^2+4X+1]$$

$$= 4E[X^2] + 4E[X] + 1$$

$$= 4\mu_2' + 4\mu_1' + 1$$

$$= 4\frac{2}{2^2+3(2)+2} + 4\frac{2}{2^2+3(2)+2} + 1$$

$$= \frac{8}{12} + \frac{8}{6} + 1 = 3$$

$$\therefore E[(2X+1)^2] = 3$$

1.4 MOMENT GENERATIVG FUNCTION (MGF)

Let X be a random variable. Then the MGF of X is $M_X(t) = E[e^{tx}]$

If X is a discrete random variable, then the MGF is given by

$$M_X(t) = \sum_x e^{tx} p(x)$$

If X is a continuous random variable, then the MGF is given by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

Define MGF and why it is called so?

Sol: Let X be a random variable. Then the MGF of X is $M_X(t) = E[e^{tX}]$ Let X be a continuous random variable. Then

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \int_{-\infty}^{\infty} \left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \cdots \frac{t^rx^r}{r!} + \cdots\right] f(x)dx$$

$$= \int_{-\infty}^{\infty} \left[f(x) + \frac{tx}{1!} f(x) + \frac{t^2x^2}{2!} f(x) + \cdots \frac{t^rx^r}{r!} f(x) + \cdots\right] dx$$

$$= \int_{-\infty}^{\infty} f(x)dx + \frac{t}{1!}\int_{-\infty}^{\infty} xf(x)dx + \frac{t^2}{2!}\int_{-\infty}^{\infty} x^2 f(x)dx \ \ldots + \frac{t}{r!}\int_{-\infty}^{\infty} x^r f(x)dx + \cdots$$

$$M_X(t) = 1 + \frac{t}{1!}\mu'_1 + \frac{t^2}{2!}\mu'_2 + \cdots \frac{t^r}{r!}\mu'_r + \cdots \ldots \ldots$$

$\because M_X(t)$ generates moments therefore it is moment generation function

NOTE

If X is a discrete RV and if $M_X(t)$ is known, then $\mu_r' = \left[\frac{d^r}{dt^r}[M_X(t)]\right]_{t=0}$

If X is a continuous RV and if $M_X(t)$ is known, then μ_r'

$$= r! \times \text{ coeff of } t^r \text{ in } M_X(t)$$

PROBLEMS UNDER MGF OF DISCRETE RANDOM VARIABLE

$$M_X(t) = \sum_x e^{tx} p(x)$$

If X is a discrete RV and if $M_X(t)$ is known, then $\mu_r' = \left[\frac{d^r}{dt^r}[M_X(t)]\right]_{t=}$

1. **Let X be the number occur when a die is thrown. Find the MGF and hence find Mean and Variance of X.**

Solution:

x	1	2	3	4	5	6
$p(x)$	1/6	1/6	1/6	1/6	1/6	1/6

(i) $M_X(t) = \sum_{x=1}^{6} e^{tx} p(x)$

$= e^t P(1) + e^{2t} P(2) + e^{3t} P(3) + e^{4t} P(4) + e^{5t} P(5) + e^{6t} P(6)$

$= e^t \frac{1}{6} + e^{2t} \frac{1}{6} + e^{3t} \frac{1}{6} + e^{4t} \frac{1}{6} + e^{5t} \frac{1}{6} + e^{6t} \frac{1}{6}$

$M_X(t) = \frac{1}{6}[e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t} + e^{6t}]$

(ii) $E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \frac{1}{6}[e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t} + 6e^{6t}]_{t=0}$

$= \frac{1}{6}[1 + 2 + 3 + 4 + 5 + 6] = \frac{21}{6}$

$E(X) = 3.5$

$E(X^2) = [\frac{d^2}{dt^2}[M_X(t)]]_{t=0}$

$= \frac{1}{6}[e^t + 4e^{2t} + 9e^{3t} + 16e^{4t} + 25e^{5t} + 36e^{6t}]$

$= \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) = \frac{91}{6}$

$=15.1$

(iii) Variance of $X = E(X^2) - [E(X)]^2 = 15.1 - 12.25$

$\sigma_X = 2.85$

2. Find the moment generating function for the distribution

where $(X = x) = \begin{cases} \frac{2}{3}; x = 1 \\ \frac{1}{3}; x = 2 \\ 0 \text{ ; otherwise} \end{cases}$ **. Also find its mean & variance,**

Sol: The probability distribution of X is given by

x	1	2
$p(x)$	2/3	1/3

$M_X(t) = E[e^{tx}] \qquad = \sum_{x=1}^{2} e^{tx}p(x)$

$= e^t p(X = 1) + e^{2t} p(X = 2) = e^t \frac{2}{3} + e^{2t} \frac{1}{3}$

$M_X(t) = \frac{1}{3}(2e^t + e^{2t})$

$E(X) = M_X'(0)$

$= [\frac{d}{dt}[\frac{1}{3}(2e^t + e^{2t})]]_{t=0} \qquad = \frac{1}{3}(2e^t + 2e^{2t})$

$E(X) = \frac{4}{3}$

$$E(X^2) = M_X''(0) = \left[\frac{d^2}{dt^2}\left[\frac{1}{3}(2e^t + e^{2t})\right]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left[\frac{1}{3}(2e^t + 2e^{2t})\right]\right]_{t=0}$$

$$= \left[\frac{1}{3}(2e^t + 4e^{2t})_{t=0}\right] = \frac{6}{3} = 2$$

Variance of $X = E(X^2) - [E(X)]^2 = 2 - \left(\frac{4}{3}\right)^2$

$$\text{Var}(X) = \frac{2}{9}$$

3. Let X be a RV with PMF $p(x) = \left(\frac{1}{2}\right)^x$; $x = 1, 2, 3, \ldots$. Find MGF and hence find mean and variance of X.

Sol:

(i) $M_X(t) = E[e^{tX}]$

$$= \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \left(\frac{1}{2}\right)^x \qquad = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \left[\frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \cdots\right] \qquad = \frac{e^t}{2}\left(1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \cdots\right)$$

$$= \frac{e^t}{2}\left(1 - \frac{e^t}{2}\right)^{-1} \qquad = \frac{e^t}{2}\frac{e^t}{2-e^t}$$

$$M_X(t) = \frac{e^t}{2-e^t}$$

(ii) $E(X) = \left[\frac{d}{dt}[M_X(t)]\right]_{t=0}$

$$= \left[\frac{d}{dt}\left(\frac{e^t}{2-e^t}\right)\right]_{t=0}$$

$$= \left[\frac{(2-e^t)e^t - e^t(0-e^t)}{(2-e^t)^2}\right]_{t=0}$$

$$= \left[\frac{2e^t - e^{2t} + e^{2t}}{(2-e^t)^2}\right]_{t=0} \quad \because d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

$$= \left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0}] = \frac{2}{1}$$

$$E(X) = 2$$

$$E(X^2) = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left(\frac{2e^t}{(2-e^t)^2}\right)\right]_{t=0}$$

$$= \left[\frac{(2-e^t)^2 2e^t - 2e^t 2(2-e^t)(-e^t)}{(2-e^t)^4}\right]_{t=0} = \frac{2+4}{1} = 6$$

(iii) Variance $= E(X^2) - [E(X)]^2 = 6 - 4$

$\text{Var}(X) = 2$

PROBLEMS UNDER MGF OF DISCRETE RANDOM VARIABLE

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

If X is a continuous RV and if $M_X(t)$ is known, then μ_r'

$$= r! \times \text{ coeff of } t^r \text{ in } M_X(t)$$

1. If a random variable "X" has the MGF, $M_X(t) = \frac{2}{2-t}$, find the variance of X.

Solution:

Given $M_X(t) = \frac{2}{2-t} = 2(2-t)^{-1}$

$M_X'(t) = -2(2-t)^{-2}(-1)$

$= 2(2-t)^{-2}$

$M_X'(t=0) = 2(2-0)^{-2} = \frac{2}{4} = \frac{1}{2}$

$M_X''(t) = -4(2-t)^{-3}(-1)$

$= 4(2-t)^{-3}$

$M_X''(t=0) = 4(2-0)^{-3} = \frac{4}{8} = \frac{1}{2}$

$Var(X) = E(X^2) - E[(X)]^2$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

2. Let X be a RV with PDF $f(x) = ke^{-2x}, x \geq 0$. Find (i) k, (ii) MGF, (iii) Mean and (iv) variance

Sol: Given: $f(x) = ke^{-2x}; 0 \leq x < \infty$

i) To find k:

$\int_0^\infty f(x)dx = 1 \Rightarrow \int_0^\infty ke^{-2x}dx = 1k\left[\frac{e^{-2x}}{-2}\right]_0^\infty = 1 \Rightarrow \frac{k}{-2}(e^{-\infty} - 1) = 1$

$\frac{k}{-2}(0-1) = 1 \Rightarrow \frac{k}{2} = 1$

$k = 2$

(ii) $M(t) = E[e^{tx}] = \int_0^\infty e^{tx} f(x)dx$

$= 2\int_0^\infty e^{tx}e^{-2x}dx = 2\int_0^\infty e^{tx-2x}dx$

$$= 2\int_0^\infty e^{-(2-t)x}\, dx = 2\left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^\infty = 2\left(0 + \frac{1}{2-t}\right)$$

$$M_z(t) = \frac{2}{2-t}$$

(iii) To find Mean and Variance:

$$M_X(t) = \frac{2}{2\left(1-\frac{t}{2}\right)} = \left(1-\frac{t}{2}\right)^{-1} = 1 + \frac{t}{2} + \frac{t^2}{2^2} + \cdots$$

Coefficient of $t = \frac{1}{2}$ Coefficient of $t^2 = \frac{1}{2^2}$

Mean $E(X) = \mu_1' = 1! \times$ coefficient of $t \Rightarrow E(X) = \frac{1}{2}$

$$E(X^2) = 2! \times \text{coefficient of } t^2 = 2 \times \frac{1}{2^2} = \frac{1}{2}$$

(iv) Variance $= E(X^2) - [E(X)]^2 = \frac{1}{2} - \frac{1}{4} = \frac{2-1}{4}$

$$\text{Var}(X) = \frac{1}{4}$$

3. Let X be a continuous RV with PDF $f(x) = Ae^{\frac{-x}{3}}; x \geq 0$. Find (i) A, (ii) MGF,(iii) Mean and (iv) variance

Sol: Given: $f(x) = Ae^{\frac{-x}{3}}; 0 \leq x \leq \infty$

(i) To find A :

$$\int_0^\infty f(x)dx = 1 \Rightarrow \int_0^\infty Ae^{\frac{-x}{3}}\, dx = 1 A\left[\frac{e^{\frac{-x}{3}}}{-\frac{1}{3}}\right]_0^\infty = 1 \Rightarrow -3A(0-1) = 1$$

$$3A = 1 \Rightarrow A = \frac{1}{3}$$

$$\therefore f(x) = \frac{1}{3}e^{\frac{-x}{3}}; 0 \leq x \leq \infty$$

(ii) $M_X(t) = E[e^{tx}] = \int_0^\infty e^{tx} f(x)dx = \frac{1}{3}\int_0^\infty e^{tx} e^{\frac{-x}{3}} dx$

$$= \frac{1}{3}\int_0^{\infty} e^{tx-\frac{x}{3}}dx = \frac{1}{3}\int_0^{\infty} e^{-\left(\frac{1}{3}-t\right)x}dx = \frac{1}{3}\left[\frac{e^{-\left(\frac{1}{3}-t\right)x}}{-\left(\frac{1}{3}-t\right)}\right]_0^{\infty}$$

$$= \frac{1}{3}\left[0 + \frac{1}{\frac{1}{3}-t}\right] = \frac{1}{3}\frac{1}{\frac{1-3t}{3}}$$

$$= (1-3t)^{-1}$$

(iii) To find mean and variance:

$M_X(t) = (1-3t)^{-1}$

$= 1 + 3t + 9t^2 + 27t^3 + \cdots$

coefficient of $t = 3$ coefficient of $t^2 = 9$

E(X)=1! X coefficient of t in $M_X(t) = 1 \text{ X } 3$

Mean = 3

E(X)=2! X coefficient of t^2 in $M_X(t)$

= 2X9 =18

(iv)Variance = $E(X^2) - [E(X)]^2$ = 18-9

Var (X) =9

4. Let X be a continuous RV with PDF $f(x)$

$$= \begin{cases} x & ; 0 < x < 1 \\ 2-x & ; 1 < x < 2 \\ 0 & ; \text{elsewhere} \end{cases}$$ **. Find (i) MGF, (ii) Mean and variance.**

Sol: Since X is defined in the region $0 < x < 2, X$ is a continuous RV.

$$M_X(t) = E[e^{tX}] = \int_0^2 e^{tx} f(x)dx$$

$$= \int_0^1 e^{tx} x dx + \int_1^2 e^{tx}(2-x)dx = \int_0^1 x e^{tx} dx + \int_1^2 (2-x) e^{tx} dx$$

$$= \left[x\left(\frac{e^{tx}}{t}\right) - 1\left(\frac{e^{tx}}{t^2}\right)\right]_0^1 + \left[(2-x)\frac{e^{tx}}{t} - (-1)\frac{e^{tx}}{t^2}\right]_1^2$$

$$= \left[1\left(\frac{e^t}{t}\right) - 1\left(\frac{e^t}{t^2}\right) - \left(\frac{-1}{t^2}\right)\right] + \left[0 + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}\right]$$

$$= \left[\frac{e^t}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}\right] = \frac{1}{t^2} - \frac{2e^t}{t^2} + \frac{e^{2t}}{t^2}$$

$$M_X(t) = \frac{1 - 2e^t + e^{2t}}{t^2}$$

To find Mean and Variance:

$$M_X(t) = \frac{1}{t^2}\left[1 - 2\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right)\right.$$
$$\left. + \left(1 + \frac{2t}{1!} + \frac{2^2t^2}{2!} + \frac{2^3t^3}{3!} + \frac{2^4t^4}{4!} + \cdots\right)\right]$$

$$\mu_r' = r! \times \text{coefficient of } t^r$$

$$\text{Coefficient of } t = -\frac{2}{3!} + \frac{2^3}{3!} = \frac{-2}{6} + \frac{8}{6} = 1$$

$$\text{Coefficient of } t^2 = -\frac{2}{4!} + \frac{2^4}{4!} = \frac{14}{24} = \frac{7}{12}$$

$$\mu_1' = 1! \times \text{coefficient of } t$$

$$\mu_1' = 1$$

Mean $= 1$

$$\mu_2' = 2! \times \text{coefficient of } t^2; \ \mu_2' = 2 \times \frac{7}{12} = \frac{7}{6}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2 = \frac{7}{6} - 1$$

$$\text{Var(X)} = \frac{1}{6}$$

5. **Let X be a continuous random variable with PDF $f(x) \frac{1}{2a}; -a < x < a$.**

Then find the M.G.F of X.

Sol: X is a continuous random variable defined in $-a < x < a$.

$$M_x(t) = E[e^{tx}]$$

$$= \int_{-a}^{a} e^{tx} f(x) dx$$

$$= \int_{-a}^{a} e^{tx} \frac{1}{2a} dx$$

$$= \frac{1}{2a} \left(\frac{e^{tx}}{t}\right)^{a} \quad e^{x} - e^{-x} = 2\sin hx$$

$$= \frac{1}{2a} \left(\frac{e^{ta} - e^{-ta}}{t}\right)$$

$$= \frac{1}{2a} \frac{2\sinh at}{t}$$

$$M_X(t) = \frac{\sinh at}{at}$$

Discrete Distributions:

(i) Binomial Distribution

(ii) Poisson Distribution

(iii) Geometric Distribution

Continuous Distributions:

(i) Exponential Distribution

(ii) Uniform Distribution

(iii) Normal Distribution

1.5 Binomial Distribution

Let us consider "n" independent trails. If the successes (S) and failures (F) are recorded successively as the trials are repeated we get a result of the type

S S F F S . . . F S

Let "x" be the number of success and hence we have $(n - x)$ number of failures.

$$P(S\ S\ F\ F\ S\ldots F\ S) = P(S)\ P(S)\ P(F)\ P(F)\ P(S)\ldots P(F)\ P(S)$$

$$= p\ p\ q\ q\ p\ldots q\ p$$

$$= p\ p\ldots p \times q\ q\ q\ldots q$$

$$= x \text{ factor} \times (n - x) \text{ factors}$$

$$= p^x \cdot q^{n-x}$$

But "x" success in "n" trials can occur in nC_x ways.

Therefore the probability of "x" successes in "n" trials is given by

$$P(X = x\ successes) = nC_x p^x q^{n-x}, x = 0, 1, 2, \ldots, n$$

Where $p + q = 1$

Assumptions in Binomial Distribution:

(i) There are only two possible outcomes for each trial (success or failure)

(ii) The probability of a success is the same for each trail.

(iii) There are "n" trials where "n" is constant.

(iv) The "n" trails are independent.

Mean and variance of a Binomial Distribution:

(i) Mean(μ) = np

(ii) Variance(σ^2) = npq

The variance of a Binomial Variable is always less than its mean.

$\therefore npq < np$.

Find the moment generating function of binomial distribution and hence find the mean and variance.

Sol: Binomial distribution is $p(x) = nC_x p^x q^{n-x}, x = 0,1,2, \dots \dots, n$

To find Mean and Variance:

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx} P(x)$$

$$= \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x} \qquad \because \sum_{x=0}^{n} nC_x a^x b^{n-x} = (a+b)^n$$

$$M_X(t) = (pe^t + q)^n$$

Mean $E(X) = \left[\frac{d}{dt}[M_*(t)]\right]_{t=0} = \left[\frac{d}{dt}[(pe^t + q)^n]\right]_{t=0}$

$$= [n(pe^t + q)^{n-1}(pe^t + 0)]_{t=0}$$

$$= np[p+q]^{n-1}$$

$$E(X) = np$$

$$E(X^2) = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0}$$

$$= \left[\frac{d}{dt}[n(pe^t + q)^{n-1}(pe^t)]\right]_{t=0} = np\left[\frac{d}{dt}[(pe^t + q)^{(n-1)}e^t]\right]_{t=0}$$

$$= np[(pe^t + q)^{n-1}e^t + e^t(n-1)(pe^t + q)^{n-2}pe^t]_{t=0}$$

$$= np[(p+q)^{n-1} + (n-1)(p+q)^{n-2}p]$$

$$= np[1 + (n-1)p] \qquad = np[1 + np - p]$$

$$= np[1 - p + np] \qquad = np[q + np] \qquad = npq + n^2p^2$$

$$E(X^2) = (np)^2 + npq$$

Variance $= E(X^2) - [E(X)]^2$

$$= (np)^2 + npq - (np)^2 = n^2p^2 - n^2p^2 + npq$$

Variance= $n$pq

Problems based on Binomial Distribution:

Mean = np

Variance = npq

1. Criticize the following statements " The mean of a binomial distribution is 5 and the standard deviation is 3"

Solution:

Given mean = np = 5 . . . (1)

Standard deviation = $\sqrt{npq} = 3$

⇒ Variance = npq = 9 . . . (2)

$$\frac{(2)}{(1)} \Rightarrow \frac{npq}{np} = \frac{9}{5} = 1.8 > 1$$

Which is impossible. Hence, the given statement is wrong.

2. If $M_X(t) = \frac{(2e^t+1)^4}{81}$, then find Mean and Variance.

Solution:

Given $M_X(t) = \frac{(2e^t+1)^4}{81}$

$$\Rightarrow M_X(t) = \left(\frac{2}{3}e^t + \frac{1}{3}\right)^4$$

Comparing with MGF of Binomial Distribution, $M_X(t) = (pe^t + q)^n$, we get

$p = \frac{2}{3}$ and $= \frac{1}{3}$, n = 4

(i) Mean = $np = 4 \times \frac{2}{3} = \frac{8}{3}$

(ii) Variance $= npq = \frac{8}{3} \times \frac{1}{3} = \frac{8}{9}$

3. Six dice are thrown 729 times. How many times do you expect atleast 3 dice to show a five or six.

Solution:

Given n = 6 and N = 729

Probability of getting (5 or 6) $p = \frac{2}{6} = \frac{1}{3}$

and $q = 1 - \frac{1}{3} = \frac{2}{3}$

$$P(X = x) = nC_x p^x q^{n-x}, x = 0, 1, 2, \ldots, n$$

$$= 6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}, x = 0, 1, 2, \ldots, 6$$

P(atleast 3 dice to show a five or six) $= P(X \geq 3) = 1 - P(X < 3)$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - \left[6C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{6-0} + 6C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^{6-1} + 6C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^{6-2}\right]$$

$$= 1 - [0.0877 + 0.2634 + 0.3292]$$

$$= 1 - 0.6803$$

$$= 0.3197$$

Number of times expecting atleast 3 dice to show 5 or 6 $= 729 \times 0.3197$

$= 233$ times

4. A machine manufacturing screw is known to produce 5% defective. In a random sample of 15 screws, what is the probability that there are (i) exactly 3 defectives (ii) not more than 3 defectives.

Solution:

Given n = 15

$p = 5\% = 0.05$

$q = 1 - p = 1 - 0.05 = 0.95$

$P(X = x) = nC_x p^x q^{n-x}, x = 0, 1, 2, ..., n$

$= 15C_x (0.05)^x (0.95)^{15-x}, x = 0, 1, 2, ..., 15$

(i) P(exactly 3 defectives) $= P(X = 3)$

$= 15C_3 (0.05)^3 (0.95)^{15-3}$

$= 0.056(0.95)^{12}$

$= 0.0307$

(ii) P(no More than 3 defectives) $= P(X \leq 3)$

$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$= 15C_0 (0.05)^0 (0.95)^{15-0} + 15C_1 (0.05)^1 (0.95)^{15-1}$

$+ 15C_3 (0.05)^2 (0.95)^{15-2}$

$+ 15C_3 (0.05)^3 (0.95)^{15-3}$

$= 15C_0 (0.05)^0 (0.95)^{15} + 15C_1 (0.05)^1 (0.95)^{14} + 15C_3 (0.05)^2 (0.95)^{13}$

$+ 15C_3 (0.05)^3 (0.95)^{12}$

$= 0.4633 + 0.3658 + 0.1348 + 0.0307$

$= 0.9946$

1.6 Poisson Distribution

Poisson Distribution is a limiting case of Binomial Distribution under the following assumptions.

(i) The number of trails "n" should be independently large. i.e., $n \to \infty$

(ii) The probability of successes "p" for each trail is indefinitely small.

(iii) $np = \lambda$ should be finite where λ is a constant.

Application of Poisson Distribution:

Determining the number of calls received per minute at a call Centre or the number of unbaked cookies in a batch at a bakery, and much more.

Find the MGF for Poisson distribution and hence find the mean and variance.

Sol: Poisson distribution is $p(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $x = 0,1,2, \ldots \ldots,$

$$M_X(t) = E[e^{tx}]$$

$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{-x} e^{-\lambda} \frac{\lambda^x}{x!} = \sum_{x=0}^{\infty} \frac{(e^t)}{0}$$

$$= e^{-\lambda} \left[1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \frac{(\lambda e^t)^3}{3!} + \cdots\right]$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{-\lambda + \lambda \epsilon^t} \because 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = e^x$$

$M_X(t) = e^{\lambda(e^t-1)}$

To find the mean and variance :

Mean $E(X) = \left[\frac{d}{dt}[M_X(t)]\right]_{t=0}$

$= \left[\frac{d}{dt}[e^{\lambda(e^t-1)}]\right]_{t=0} = [e^{\lambda(e^t-1)}\lambda(e^t)]_{t=0} = e^{\lambda(e^0-1)}\lambda e^0 = e^0\lambda$

Mean $= \lambda$

$E(X^2) = \left[\frac{d^2}{dt^2}[M_X(t)]\right]_{t=0} = \left[\frac{d^2}{dt^2}[e^{\lambda(e^t-1)}]\right]_{t=0} = \left[\frac{d}{dt}[e^{\lambda(e^t-1)}\lambda e^t]\right]_{t=0}$

$= \lambda\left[\frac{d}{dt}[e^{\lambda(e^t-1)+t}]\right]_{t=0}$

$= \lambda[e^{\lambda(e^t-1)+t}(\lambda e^t + 1)]_{t=0}$

$= \lambda[e^0(\lambda + 1)]$

$= \lambda(\lambda + 1)$

$E(X^2) = \lambda^2 + \lambda$

Variance $= E(X^2) - [E(X)]^2$

$= \lambda^2 + \lambda - \lambda^2$

Variance $= \lambda$

Problems based on Poisson Distribution:

1. Write down the probability mass function of the Poisson distribution which is approximately equivalent to B(100, 0.02)

Solution:

Given $n = 1000, p = 0.02$

$$\lambda = np = 100 \times 0.02 = 2$$

The probability mass function of the Poisson distribution

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}; x = 0, 1, 2, \ldots, \infty$$

$$= \frac{e^{-2}2^{x}}{x!}; x = 0, 1, 2, \ldots, \infty$$

2. One percent of jobs arriving at a computer system need to wait until weekends for scheduling, owing to core – size limitations. Find the probability that among a sample of 200 jobs there are no jobs hat have to wait until weekends.

Solution:

Given $n = 200$, p = 0.01

$$\lambda = np = 200 \times 0.01 = 2$$

The Poisson distribution is

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}; x = 0, 1, 2, \dots, \infty$$

P(no jobs to wait until weekends) = $P(X = 0)$

$$P(X = 0) = \frac{e^{-2}2^0}{0!} = e^{-2} = 0.1353$$

3. The proofs of a 500 pages book containing 500 misprints. Find the probability that there are atleast 4 misprints in a randomly chosen page.

Solution:

Given $n = 500$

p = P(getting a misprint in a given page) = $\frac{1}{500}$

$$\lambda = np = 500 \times \frac{1}{500} = 1$$

The Poisson distribution is

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}; x = 0, 1, 2, \dots, \infty$$

$$P(X \geq 4) = 1 - P(X < 4)$$

$$= 1 - [P(X = 0) + P(X = 1)] + P(X = 2) + P(X = 3)$$

$$= 1 - [\frac{e^{-1}1^0}{0!} + \frac{e^{-1}1^1}{1!} + \frac{e^{-1}1^2}{2!} + \frac{e^{-1}1^3}{3!}]$$

$$= 1 - e^{-1}[1 + 1 + \frac{1}{2} + \frac{1}{6}]$$

$= 1 - 0.3679[2.666]$

$= 1 - 0.9809$

$= 0.0192$

1.7 Geometric Distribution

A Discrete random variable "X" is said to follow Geometric distribution, if it assumes only non – negative value4s and its probability mass function is given by $P(X = x) = p(x) = q^{x-1}p, x = 1, 2, \ldots, 0 < p \leq 1$ where $q = 1 - p$.

Why $P(X = x) = p(x) = q^{x-1}p$ is called Geometric Distribution.

Solution:

Putting x = 1, 2, 3, . . in $P(X = x) = p(x) = q^{x-1}p$

We get $q^0p, qp, q^2p, q^3p, \ldots$ which are the various terms of Geometric progression. Hence it is known as Geometric distribution.

State another form of Geometric distribution.

Solution:

The another form of Geometric distribution is

$P(X = x) = p(x) = q^x p, x = 0, 1, 2, \ldots 0 < p \leq 1$ where $q = 1 - p$

Find the MGF of geometric distribution and hence find Mean and variance.

Sol: Geometric distribution is $p(x) = q^{x-1}p; x = 1,2, \dots \dots, \infty$

$$M_X(t) = E[e^{tx}] = \sum_{x=1}^{\infty} e^{tx}p(x) = \sum_{x=1}^{\infty} e^{tx}pq^{x-1}$$

$$= pq^{-1}\sum_{x=1}^{\infty} e^{tx}q^x = pq^{-1}\sum_{x=1}^{\infty} (qe^t)^x$$

$$= pq^{-1}[qe^t + (qe^t)^2 + (qe^t)^3 + \cdots]$$

$$= pq^{-1}qe^t[1 + qe^t + (qe^t)^2 + \cdots]$$

$$= pe^t[1 - qe^t]^{-1}$$

$$M_X(t) = \frac{pe^t}{1-qe^t}$$

To find the mean value of :

$$\text{Mean } E(X) = \left[\frac{d}{dt}[M_X(t)]\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{pe^t}{1-qe^t}\right)\right]_{t=0}$$

$$= \left[\frac{(1-qe^t)pe^t - pe^t(0-qe^t)}{(1-qe^t)^2}\right]_{t=0} = \left[\frac{pe^t - pqe^{2t} + pqe^{2t}}{(1-qe^t)^2}\right]_{t=0}$$

$$= \left[\frac{pe^t}{(1-qe^t)^2}\right]_{t=0} \dots \dots \dots (1)$$

$$= \frac{p}{(1-q)^2} = \frac{p}{p^2}$$

$$E(X) = \frac{1}{p}$$

To find variance of :

$$E(X^2) = [\frac{d^2}{dt^2}[M_X(t)]]_{t=0} = [\frac{d^2}{dt^2}(\frac{pe^t}{1-qe^t})]_{t=0}$$

$$= [\frac{d}{dt}(\frac{pe^t}{(1-qe^t)^2})]_{t=0} \quad \text{From (1)}$$

$$= [\frac{(1-qe^t)^2 pe^t - pe^t 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}]_{t=0}$$

$$= (1-qe^t)\,[\frac{(1-qe^t)pe^t + 2pqe^{2t}}{(1-qe^t)^4}]_{t=0}$$

$$= [\frac{(1-qe^t)(pe^t)+2pqe^{2t})}{(1-qe^t)^3}]_{t=0} = (\frac{p[(1-q)+2q]}{(1-q)^3})$$

$$= p\,[\frac{p+2q}{p^3}] = p\,[\frac{p+q+q}{-p^3}] = \frac{(1+q)}{p^2}$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$

$$= \frac{1+q}{p^2} - \frac{1}{p^2}$$

$$= \frac{1+q-1}{p^2} = \frac{q}{p^2}$$

$$\text{Variance} = \frac{q}{p^2}$$

Problems based on Geometric Distribution

1. Suppose that a trainee soldier shoots a target in an independent fashion. If the probability that the target is shot in any one shot is 0.7 what is the (i) Probability that the target would hit on the 10th attempt?(ii) probability that it taken him less than 4 shots? (iii) Probability that it taken him an even number of shots? (iv) Average number of shots needed to hit the target.

Solution:

Given $p = 0.7$

$q = 1 - p = 1 - 0.7 = 0.3$

The Geometric distribution is $P(X = x) = p(x) = q^{x-1}p; x = 1, 2, 3, \ldots$

(i) P(target would hit on the 10th attempt)$= P[x = 10]$

$$= (0.3)^{10-1}(0.7)$$

$$= 0.0000138$$

(ii) P(target would be hit less than 4 shots) $= P(X < 4)$

$$= P(X = 1) + P(X = 2) + P(X = 3)$$

$$= (0.3)^{1-1}(0.7) + (0.3)^{2-1}(0.7) + (0.3)^{3-1}(0.7)$$

$$= 0.9738$$

(iii) P(he would take an even number of shots)

$= P(X = 2) + P(X = 4) + P(X = 6) + \ldots$

$= (0.3)^{2-1}(0.7) + (0.3)^{4-1}(0.7) + (0.3)^{6-1}(0.7) + \ldots$

$= (0.3)^{1}(0.7) + (0.3)^{3}(0.7) + (0.3)^{5}(0.7) + \ldots$

$= (0.3)^{1}(0.7)[1 + (0.3)^{2} + (0.3)^{4} + \ldots]$

$= (0.3)^{1}(0.7)[1 - (0.3)^{2}]^{-1}$

$= 0.21[0.91]^{-1}$

$= 0.2307$

(iv) Average number $= E(X) = \frac{1}{p} = \frac{1}{0.7} = 1.4286$

2. Let one copy of a magazine out of 10 copies bears a special prize following geometric distribution. Determine its mean and variance.

Solution:

Given $p = \frac{1}{10}$ and $q = 1 - \frac{1}{10} = \frac{9}{10}$

Mean of Geometric distribution $= \frac{1}{p} = 10$

Variance $= \frac{q}{p^2} = \frac{9}{10} \times 10^2 = 90$

3. If the probability is0.05 that a certain kind of measuring device will show excessive drift, what is the probability that the sixth of these measuring devices tested will be the first to show excessive drift?

Solution:

Given $p = 0.05$ and $q = 1 - 0.05 = 0.95, x = 6$

The Geometric distribution is $P(X = x) = p(x) = q^{x-1}p; x = 1, 2, 3, \ldots$

$$P(X = 6) = p(6) = (0.95)^{6-1}(0.05)$$

$$= 0.0387$$

1.9 Exponential Distribution

The definition of exponential distribution is the probability distribution of the time between the events in a Poisson Process.

If you think about it, the amount of time until the event occurs means during the waiting period, not a single event has happened.

This is, in other words Poisson (X = 0).

$$\text{Poisson } P(X=k) = \frac{\lambda^k \cdot e^{-\lambda}}{k!}$$

$$\text{"} \quad P(X=0) = \frac{\lambda^0 \cdot e^{-\lambda}}{0!} = e^{-\lambda}$$

Why did we have to invent Exponential Distribution?

To predict the amount of waiting time until the next event (i.e., success, failure, arrival, etc.).

For example, we want to predict the following:

- The amount of time until the customer finishes browsing and actually purchases something in your store (success).
- The amount of time until the hardware on AWS EC2 fails (failure).
- The amount of time you need to wait until the bus arrives (arrival).

Relationship between a Poisson and an Exponential Distribution:

If the number of events per unit time follows a Poisson distribution, then the amount of time between events follows the exponential distribution. Assuming that the time between events is not affected by the times between previous events (i.e., they are independent), then the number of events per unit time follows a Poisson distribution with the rate λ = 1/μ.

Who else has Memoryless property?

The exponential distribution is the only continuous distribution that is memoryless (or with a constant failure rate). Geometric distribution, its discrete counterpart, is the only discrete distribution that is memory less.

Find the MGF of Exponential distribution and hence find Mean and variance.

Sol. Let X follows the exponential distribution.

By definition, $f(x) = \lambda e^{-\lambda x}; x \geq 0$

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} f(x)dx = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - t)x}\, dx = \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^\infty$$

$$= \frac{-\lambda}{\lambda - t}[e^{-\infty} - e^0] = \frac{-\lambda}{\lambda-t}[0-1]$$

$$= \frac{\lambda}{\lambda - t}$$

To find the mean and variance:

$$M_X(t) = \frac{\lambda}{\lambda(1-\frac{t}{\lambda})} = (1-\frac{t}{\lambda})^{-1}$$

$$= 1 + \frac{t}{\lambda} + \frac{t^2}{\lambda^2} + \cdots$$

Coefficient of $t = \frac{1}{\lambda}$; Coefficient of $t^2 = \frac{1}{\lambda^2}$

$$E(X) = 1! \times \text{ coefficient of } t = \frac{1}{\lambda}$$

$$E(X^2) = 2! \times \frac{1}{\lambda^2} = \frac{2}{\lambda^2}$$

Variance $= E(X^2) - [E(X)]^2$

$$= \frac{2}{\lambda^2} - [\frac{1}{\lambda}]^2$$

$$\text{Variance} = \frac{1}{\lambda^2}$$

Problems based on Exponential Distribution:

1. The time in hours required to repair a machine is exponentially distributed with parameter $\lambda = \frac{1}{2}$ (i) What is the probability that the required time exceeds 2 hours. (ii) What is the conditional probability that the repair takes at least 11 hours given that its duration exceeds 8 hours.

Solution:

Given X is exponentially distributed with parameter $\lambda = \frac{1}{2}$

The exponential distribution is given by $f(x) = \lambda e^{-\lambda x};\ x \geq 0$

$$f(x) = \frac{1}{2} e^{-\frac{x}{2}};\ x \geq 0$$

(i) P(repair time exceeds 2 hours) = $P(X > 2)$

$$= \int_2^\infty f(x)dx$$

$$= \frac{1}{2}\int_2^\infty e^{-\frac{x}{2}}dx$$

$$= \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{\frac{-1}{2}}\right]_2^\infty$$

$$= -\left[e^{-\frac{x}{2}}\right]_2^\infty$$

$$= -[0 - e^{-1}]$$

$$= e^{-1}$$

(ii) P(time required atleast 11 hours / exceeds 8 hours)= $P(X \geq 11/X > 8)$

$$= P(X > 3)$$

$$= \int_3^\infty f(x)dx$$

$$= \frac{1}{2}\int_3^\infty e^{-\frac{x}{2}}dx$$

$$= \frac{1}{2}\left[\frac{e^{-\frac{x}{2}}}{-1/2}\right]_3^\infty$$

$$= -\left[e^{-\frac{x}{2}}\right]_3^\infty$$

$$= -\left[0 - e^{-\frac{3}{2}}\right]$$

$$= e^{-\frac{3}{2}}$$

2. The length of time a person speaks over phone follows exponential distribution with mean 6 minutes. What is the probability that the person will talk for (i) more than 8 minutes (ii) between 4 and 8 minutes.

Solution:

Given X is exponentially distributed with parameter $\lambda = \frac{1}{6}$

The exponential distribution is given by $f(x) = \lambda e^{-\lambda x};\ x \geq 0$

$$f(x) = \frac{1}{6} e^{-\frac{x}{6}};\ x \geq 0$$

(i))$P(X > 8)\ = \int_8^{\infty} f(x)dx$

$$= \frac{1}{6}\int_2^{\infty} e^{-\frac{x}{6}} dx$$

$$= \frac{1}{6}\left[\frac{e^{-\frac{x}{6}}}{\frac{-1}{6}}\right]_8^{\infty}$$

$$= -\left[e^{-\frac{x}{6}}\right]_8^{\infty}$$

$$= -\left[0 - e^{-\frac{8}{6}}\right]$$

$$= e^{-\frac{8}{6}}$$

(ii) P(between 4 and 8 minutes) $= P(4 < X < 8)$

$$= \int_4^8 f(x)dx$$

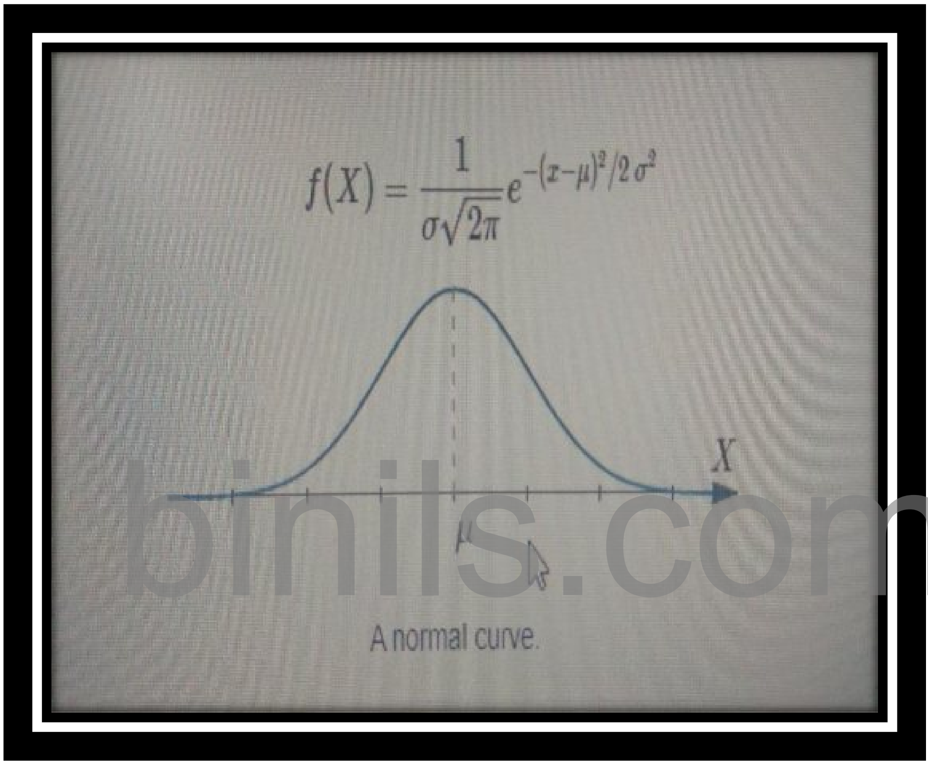
$$= \frac{1}{6}\int_4^8 e^{-\frac{x}{6}} dx$$

$$= \frac{1}{6}\left[\frac{e^{-\frac{x}{6}}}{\frac{-1}{6}}\right]_4^8$$

$$= -\left[e^{-\frac{8}{6}} - e^{-\frac{4}{6}}\right] = 0.25$$

1.10 Normal Distribution

The Normal Probability Distribution is very common in the field of statistics. Whenever you measure things like people's height, weight, salary, opinions or votes, the graph of the results is very often a normal curve.



Properties of a Normal Distribution:

(i) The normal curve is symmetrical about the mean

(ii) The mean is at the middle and divides the area into halves.

(iii) The total area under the curve is equal to 1.

(iv) It is completely determined by its mean and standard deviation σ (or variance σ^2).

Note:

In Normal distribution only two parameters are needed, namely μ and σ^2

Area under the Normal Curve using Integration:

The Probability of a continuous normal variable X found in a particular interval $[a, b]$ is the area under the curve bounded by $x = a$ and $x = b$ is given by $P(a < X < b) = \int_a^b f(X)dx$

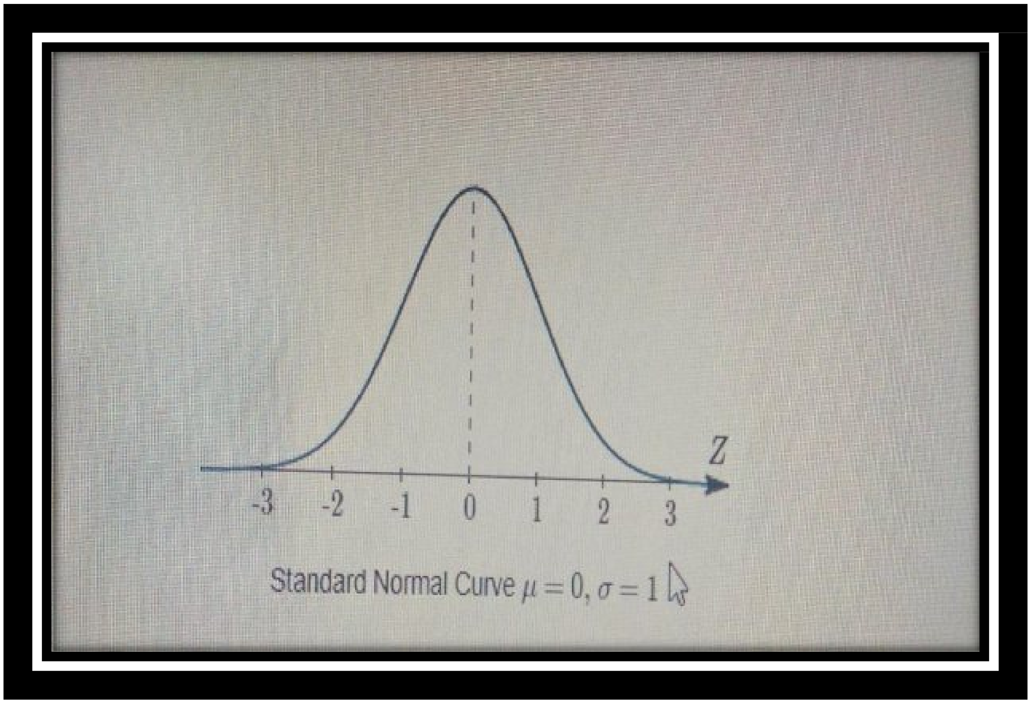
and the area depends upon the values μ and σ.

The standard Normal Distribution:

We standardize our normal curve, with a mean of zero and a standard deviation of 1 unit.

If we have the standardized situation of $\mu = 0$ and $\sigma = 1$ then we have

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



We can transform all the observations of any normal random variable X with mean μ and variance σ to a new set of observations of another normal random variable Z with mean 0 and variance 1 using the following transformation:

$$Z = \frac{X - \mu}{\sigma}$$

The two graphs have different μ and σ, but have the same area.

The new distribution of the normal random variable z with mean 0 and variance 1 (or standard deviation 1) is called a Standard normal distribution.

Formula for the Standardized normal Distribution

If we have mean μ and standard deviation σ , then

$$Z = \frac{X - \mu}{\sigma}$$

Find the moment generating function of Normal distribution

Sol: We first find the M.G.F of the standard normal distribution and hence find mean and variance.

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} ; -\infty < z < \infty$$

where $z = \frac{x-\mu}{\sigma}$

$$M_z(t) = E[e^{tz}]$$

$$= \int_{-\infty}^{\infty} e^{tz} \phi(z) dz$$

$$= \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{(\frac{2tz - z^2}{2})} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\frac{z^2 - 2tz}{2})} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\frac{(z-t)^2 - t^2}{2})} dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\frac{(z-t)^2}{2}) + \frac{t^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\frac{(z-t)^2}{2})} e^{\frac{t^2}{2}} dz$$

$$= \frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\frac{z-t}{\sqrt{2}})^2} dz = \frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-v^2} \sqrt{2} dv$$

$$= \frac{e^{\frac{t^2}{2}}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} dv = \frac{e^{\frac{t^2}{2}}}{\sqrt{\pi}} \sqrt{\pi}$$

$$M_z(t) = e^{\frac{t^2}{2}} \ldots\ldots\ldots (1)$$

$$M_X(t) = M_{\mu + \sigma z}(t); \sum z = \frac{X - \mu}{\sigma}, X = \mu + \sigma z$$

$$= e^{\mu t} M_z(\sigma t)$$

$$= e^{\mu t} \cdot e^{\frac{\sigma^2 t^2}{2}} \text{ From (1)}$$

$$= e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

To find mean and variance:

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$= 1 + \frac{\mu t + \frac{\sigma^2 t^2}{2}}{1!} + \frac{(\mu t + \frac{\sigma^2 t^2}{2})^2}{2!} + \cdots$$

$$= 1 + \frac{\mu t + \frac{\sigma^2 t^2}{2}}{1!} + \frac{(\mu^2 t^2 + \frac{\sigma^4 t^4}{4} + 2\mu t \frac{\sigma^2 t^2}{2})}{2!} + \cdots$$

Coefficient of $t = \mu$ Coefficient of $t^2 = \frac{\sigma^2}{2} + \frac{\mu^2}{2}$

$$E(X) = 1! \times \text{ coefficient of } t$$

$$= \mu$$

$$E(X^2) = 2! \times \text{coefficient of } t^2$$

$$= 2\,(\frac{\sigma^2}{2} + \frac{\mu^2}{2}) \Rightarrow 2\,(\frac{e^2 + u^2}{2})$$

$$= \mu^2 + \sigma^2$$

$$\text{variance} = E(X^2) - [E(X)]^2$$

$$= \mu^2 + \sigma^2 - \mu^2$$

$$= \sigma^2$$

Problems based on Normal distribution

1. **X is normally distributed with mean 12 and SD is 4. Find the probability that (i) $\boldsymbol{X \geq 20}$ $\boldsymbol{(ii) X \leq 20}$ $\boldsymbol{(iii) 0 \leq X \leq 12}$.**

Solution:

Given X follows normally distribution with $\boldsymbol{\mu = 12, \sigma = 4}$

$$P(X \geq 20) = P\,(\frac{X-\mu}{\sigma} \geq \frac{20-\mu}{\sigma}) = P\,(Z \geq \frac{20-12}{4})$$

$= P(Z \geq 2)$

$= 0.5 - P[0 < Z < 2]$

$= 0.5 - 0.4772$ (from the table)

$P(X \geq 20) = 0.0228$

(i) $P(X \leq 20) = 1 - P[\, X > 20] = 1 - 0.0228$

$P(X \leq 20) = 0.9772$

(ii) $P[0 \leq X \leq 12] = P\,(\frac{0-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \geq \frac{12-\mu}{\sigma})$

$= P\,(\frac{0-12}{4} \leq Z \geq \frac{12-12}{4})$

$= P(-3 \leq Z \leq 0)$

$= P(0 \leq Z \leq 3)$ (since the curve is symmetrical)

$P[0 \leq X \leq 12] = 0.4987$ (from the table)

2. **In a normal distribution 31% of the items are under 45 and 8 % are over 64. Find the mean and the standard deviation.**

Solution:

Let the mean and standard deviation of the given normal distribution be μ and .

The area lying to the left of the ordinate at x = 45 is 0.31. The corresponding value of z is negative.

The area lying to the right of the ordinates up to the mean is 0.5 – 0.31 = 0.19

The value of z corresponding to the area 0.19 is 0.5 nearly.

$$\therefore \frac{45-\mu}{\sigma} = -0.5$$

(or) $-0.5\sigma + \mu = 45$ ....(1)

Area to the left of the ordinate at x = 64 is 0.5 – 0.08 = 0.42 and hence the value of z corresponding to this area is 1.4 nearly.

$$\therefore \frac{64-\mu}{\sigma} = 1.4$$

(or) $1.4\sigma + \mu = 64$ ....(2)

Solving (1) and (2) we get

$$-0.5\sigma + \mu = 45$$

$$1.4\sigma + \mu = 64$$

(1) – (2) $\quad - 1.9\sigma = -19$

$$\Rightarrow \sigma = 10$$

Substituting $\sigma = 10$ in (1) we get

$$-0.5\,(10) + \mu = 45$$

$$-5 + \mu = 45$$

$$\Rightarrow \mu = 50$$

3. **The weekly wages of 1000 workmen are normally distributed around a mean of Rs. 70 with a S.D of Rs. 5. Estimate the number of workers whose weekly wages will be (i) between Rs.69 and Rs.72 (ii) less than Rs.69 (iii) more than Rs.72**

Solution :

Let X be the RV denoting the weekly wages of a worker

Given $\mu = 70, \sigma = 5$

The normal variate $z = \frac{X-\mu}{\sigma} = \frac{X-70}{5}$

(i) P(69 < X < 72)

When X = 69, z = $\frac{69-70}{5} = -0.2$

When X = 72, z = $\frac{72-70}{5} = 0.4$

$\therefore P(69 < X < 72\,) = P(\,-0.2 < z < 0.4)$

$= P(-0.2 < z < 0) + P(0 < z < 0.4)$

$= P(0 < z < 0.2) + P(0 < z < 0.4)$

= 0.0793 + 0.1554 (from table)

= 0.2347

Out of 1000 work men , the number of workers whose wages lies between Rs. 69 and Rs.72

= 1000 x P(69 < X < 72)

= 1000 x 0.2347 = 235

(ii) P(less than 69) = P (X < 69)

When x = 69, $z = \frac{X-\mu}{\sigma} = \frac{69-70}{5}$ = -0.2

$\therefore P(X < 69) = P(z < -0.2)$

$= 0.5 - P(0 < z < 0.2)$

= 0.4207

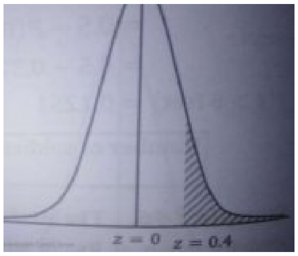
Out of 1000 workmen , the number of workers whose wages are less than Rs. 69

= 1000 x P(z < -0.2)

= 1000 x 0.4207

= 420.7

(iii) P(more than Rs. 72) = P(X > 72)



When x = 72, $z = \frac{X-\mu}{\sigma} = \frac{72-70}{5}$ = -0.2

$$\therefore P(X < 69) = P(z < -0.2)$$

$$= 0.5 - P(0 < z < 0.2)$$

= 0.4207

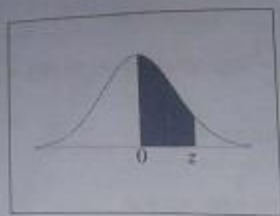
Out of 1000 workmen , the number of workers whose wages are less than Rs. 69

= 1000 x P(z < -0.2)

= 1000 x 0.4207

= 420.7

Standard Normal Distribution Table



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998

1.8 Uniform Distribution (Rectangular Distribution)

A random variable X is said to have a continuous uniform distribution if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & otherwise \end{cases}$$

Find the MGF of uniform distribution and hence find its mean and variance.

Sol: Let X follows uniform distribution in (a, b).

By Definition, $f(x) = \frac{1}{b-a}; a < x < b$.

$$M_X(t) = E[e^{tx}]$$

$$= \int_a^b f(x) e^{tx} dx$$

$$= \int_a^b \frac{1}{b-a} e^{tx} dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t}\right]_a^b$$

$$M_X(t) = \frac{1}{t(b-a)} [e^{bt} - e^{at}]$$

To find mean and variance:

$$M_X(t) = \frac{1}{t(b-a)}\{[1 + \frac{bt}{1!} + \frac{(bt)^2}{2!} + \frac{(bt)^3}{3!} + \cdots]$$

$$- [1 + \frac{at}{1!} + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots]\}$$

$$= \frac{1}{b-a}\{[\frac{1}{t} + \frac{b}{1!} + \frac{b^2t}{2!} + \frac{b^3t^2}{3!} + \cdots] - [\frac{1}{t} + \frac{a}{1!} + \frac{a^2t}{2!} + \frac{a^3t^2}{3!} + \cdots]\}$$

$$\text{Coefficient of } t = \frac{1}{b-a}[\frac{b^2}{2!} - \frac{a^2}{2!}] = \frac{1}{b-a}\frac{b^2-a^2}{2!}$$

$$= \frac{1}{2(b-a)}[(b+a)(b-a)]$$

$$= \frac{a+b}{2}$$

$$\text{Coefficient of } t^2 = \frac{1}{b-a}[\frac{b^3}{3!} - \frac{a^3}{3!}]$$

$$= \frac{1}{6(b-a)}[b^3 - a^3]$$

$$= \frac{1}{6}\frac{1}{(b-a)}(b-a)(b^2 + ba + a^2)$$

$$= \frac{1}{6}(b^2 + ab + a^2)$$

$$= \frac{b^2+ab+a^2}{6}$$

$$\text{Mean} = E(X) = 1! \times \text{ coefficient of } t$$

$$= 1! \times \left(\frac{a+b}{2}\right) = \frac{a+b}{2}$$

$$E(X^2) = 2! \times \text{coefficient of } t^2$$

$$= 2 \times \frac{b^2+ab+a^2}{6}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$\text{Variance } = E(X^2) - [E(X)]^2$$

$$= \frac{b^2+a^2+ab}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{b^2+a^2+ab}{3} - \frac{(a^2+2ab+b^2)}{4}$$

$$= \frac{4b^2+4a^2+4ab-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2+b^2-2ab}{12}$$

$$= \frac{(a-b)^2}{12}$$

$$= \frac{(b-a)^2}{12} \quad \because a < b$$

Problem based on Uniform Distribution

1. A random variable X has a uniform distribution over (0, 10) compute

(i) $P(X < 2)$ (ii) $P(X > 8)$ (iii) $P(3 < X < 9)$

Solution:

The pdf $f(x) = \begin{cases} \frac{1}{b-a}, a < x < b \\ 0, otherwise \end{cases}$

$$f(x) = \begin{cases} \frac{1}{10}, 0 < x < 10 \\ 0, otherwise \end{cases}$$

(i) $P(X < 2) = \int_0^{10} f(x)dx$

$$= \frac{1}{10}\int_0^{10} dx$$

$$= \frac{1}{10}[x]_0^{10}$$

$$= \frac{1}{10}[10 - 0]$$

$$= \frac{1}{5}$$

(ii) $P(X > 8) = \int_8^{10} f(x)dx$

$$= \frac{1}{10}\int_8^{10} dx$$

$$= \frac{1}{10}[x]_8^{10}$$

$$= \frac{1}{10}[10 - 8]$$

$$= \frac{1}{5}$$

(iii) $P(3 < X < 9) = \int_3^9 f(x)dx$

$$= \frac{1}{10}\int_3^9 dx$$

$$= \frac{1}{10}[x]_3^9$$

$$= \frac{1}{10}[9-3]$$

$$= \frac{3}{5}$$

2. A random variable X has a uniform distribution over (-3, 3) compute

(i) $P(X < 2)$ (ii) $P(|X| < 2)$ (iii) $P(|X-2| < 2)$

Solution:

The pdf $f(x) = \begin{cases} \frac{1}{b-a}, a < x < b \\ 0, otherwise \end{cases}$

$$f(x) = \begin{cases} \frac{1}{6}, -3 < x < 3 \\ 0, otherwise \end{cases}$$

(i) $P(X < 2) = \int_{-3}^{2} f(x)dx$

$$= \frac{1}{6}\int_{-3}^{2} dx$$

$$= \frac{1}{6}[x]_{-3}^{2}$$

$$= \frac{1}{6}[2+3]$$

$$= \frac{5}{6}$$

(ii) $P(|X| < 2) = P(-2 < X < 2)$

$$= \frac{1}{6}\int_{-2}^{2} dx$$

$$= \frac{1}{6}[x]_{-2}^{2}$$

$$= \frac{1}{6}[2 + 2]$$

$$= \frac{4}{6}$$

(iii) $P(|X - 2| < 2) = P(-2 < X - 2 < 2)$

$$= P(-2 + 2 < X < 2 + 2)$$

$$= P(0 < X < 4)$$

$$= \frac{1}{6}\int_{0}^{3} dx$$

$$= \frac{1}{6}[x]_{0}^{3}$$

$$= \frac{1}{6}[3 - 0]$$

$$= \frac{3}{6}$$

3. 4 buses arrive at a specified stop at 15 minute intervals starting at 7 am. That is, they arrive at 7, 7.15, 7.30, 7.45 am and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7.30 am. Find the probability that he waits (i) less than 5 minutes for a bus (ii) more than 10 minutes for a bus.

Solution:

The pdf is $f(x) = \begin{cases} \frac{1}{30}, 0 < x < 30 \\ 0, otherwise \end{cases}$

(i) P(a person arrives between 7.10 and 7.15 or 7.25 and 7.30)

$$= P(10 < X < 15) + P(25 < X < 30)$$

$$= \int_{10}^{15} f(x)dx + \int_{25}^{30} f(x)dx$$

$$= \frac{\mathbf{1}}{\mathbf{30}}\int_{10}^{15} dx + \frac{\mathbf{1}}{\mathbf{30}}\int_{25}^{30} dx$$

$$= \frac{1}{30}[x]_{10}^{15} + \frac{1}{30}[x]_{25}^{30}$$

$$= \frac{1}{30}[15 - 10] + \frac{1}{30}[30 - 25]$$

$$= \frac{1}{3}$$

(ii) P(a person arrives between 7.00 and 7.05 or 7.15 and 7.20)

$$= P(0 < X < 5) + P(15 < X < 20)$$

$$= \int_{0}^{5} f(x)dx + \int_{15}^{0} f(x)dx$$

$$= \frac{\mathbf{1}}{\mathbf{30}}\int_{0}^{5} dx + \frac{\mathbf{1}}{\mathbf{30}}\int_{15}^{20} dx$$

$$= \frac{1}{30}[x]_{0}^{5} + \frac{1}{30}[x]_{15}^{20}$$

$$= \frac{1}{30}[5 - 0] + \frac{1}{30}[20 - 15]$$

$= \frac{1}{3}$

4. Subway trains on a certain line run every half an hour between mid-night and six in the morning. What is the probability that a man entering the station at a random time during this period will have to wait atleast twenty minutes.

Solution:

The pdf is $f(x) = \begin{cases} \frac{1}{30-0}, 0 < x < 30 \\ 0, otherwise \end{cases}$

(i) P(a man waiting for atleast 20 minutes) $= P(X \geq 20)$

$$= \int_{20}^{30} f(x)dx$$

$$= \frac{1}{30}\int_{20}^{30} dx$$

$$= \frac{1}{30}[x]_{20}^{30}$$

$$= \frac{1}{30}[30 - 20]$$

$$= \frac{1}{3}$$