

**DEPARTMENT OF MATHEMATICS**

**NAME OF THE SUBJECT : TRANSFORMS & PARTIAL  
DIFFERENTIAL  
EQUATION**

**SUBJECT CODE : MA8353**

**REGULATION : 2017**

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**UNIT - IV : FOURIER TRANSFORMS**

**UNIT – IV FOURIER TRANSFORMS**

<b>IMPORTANT FORMULAE</b>	
1.	<p><b>Fourier transform pair:</b></p> <p>i) The Fourier Transform of <math>f(x)</math> is <math>F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx</math></p> <p>ii) The Inverse Fourier Transform of <math>F(s)</math> is <math>f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)e^{-isx} ds</math></p> <p>Here <math>F(s)</math> &amp; <math>f(x)</math> are called Fourier transform pair.</p>
2.	<p><b>Fourier Cosine transform pair:</b></p> <p>i) The Fourier Cosine Transform of <math>f(x)</math> is <math>F_c[f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx dx</math></p> <p>ii) The Inverse Fourier Cosine Transform of <math>F_c(s)</math> is <math>f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s) \cos sx ds</math></p> <p>Here <math>F_c(s)</math> &amp; <math>f(x)</math> are called Fourier cosine transform pair.</p>
3.	<p><b>Fourier Sine transform pair:</b></p> <p>i) The Fourier Sine Transform of <math>f(x)</math> is <math>F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx dx</math></p> <p>ii) The Inverse Fourier Sine Transform of <math>F_s(s)</math> is <math>f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_s(s) \sin sx ds</math></p> <p>Here <math>F_s(s)</math> &amp; <math>f(x)</math> are called Fourier sine transform pair.</p>
4.	<p>Parsevals Identity for Fourier transform: <math>\int_{-\infty}^{\infty}  F(s) ^2 ds = \int_{-\infty}^{\infty}  f(x) ^2 dx</math></p>
5.	<p>Parsevals Identity for Fourier Cosine transform: <math>\int_0^{\infty}  F_c(s) ^2 ds = \int_0^{\infty}  f(x) ^2 dx</math></p>
6.	<p>Parsevals Identity for Fourier Sine transform: <math>\int_0^{\infty}  F_s(s) ^2 ds = \int_0^{\infty}  f(x) ^2 dx</math></p>
7.	<p>1) <math>\int_0^{\infty} e^{-ax} \cos bx dx = \frac{a}{a^2 + b^2}</math></p> <p>2) <math>\int_0^{\infty} e^{-ax} \sin bx dx = \frac{b}{a^2 + b^2}</math></p> <p>3) <math>\int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}</math></p>

	<p>4) <math>F [xf(x)] = (-i) \frac{d}{ds} \{F [f(x)]\} = (-i) \frac{d}{ds} [F(s)]</math></p> <p>5) <math>F_s [xf(x)] = - \frac{d}{ds} \{F_c [f(x)]\} = - \frac{d}{ds} [F_c(s)]</math></p> <p>6) <math>F_c [xf(x)] = \frac{d}{ds} \{F_s [f(x)]\} = \frac{d}{ds} [F_s(s)]</math></p> <p>7) If <math>f(x)</math> and <math>g(x)</math> are any two functions and <math>F_c(s)</math> &amp; <math>G_c(s)</math> are their Fourier cosine transforms</p> <p style="text-align: center;">then <math>\int_0^{\infty} f(x)g(x)dx = \int_0^{\infty} F_c(s)G_c(s)ds</math> holds.</p> <p>8) If <math>f(x)</math> and <math>g(x)</math> are any two functions and <math>F_s(s)</math> &amp; <math>G_s(s)</math> are their Fourier sine transforms</p> <p style="text-align: center;">then <math>\int_0^{\infty} f(x)g(x)dx = \int_0^{\infty} F_s(s)G_s(s)ds</math> holds.</p>
<b>PART - A</b>	
1.	<p>State Fourier integral theorem.</p> <p>Solution :</p> <p>If <math>f(x)</math> is piecewise continuous, differentiable and absolutely integrable in <math>(-\infty, \infty)</math> then</p> $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{is(x-t)} dt ds$
2.	<p>If <math>F(s)</math> is the Fourier transform of <math>f(x)</math>, then show that <math>F\{f(x-a)\} = e^{ias} F(s)</math></p> <p>Solution :</p> <p>Given <math>F[f(x)] = F(s)</math></p> <p>The Fourier Transform of <math>f(x)</math> is</p> $F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$ $F[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{isx} dx$ <p>Let <math>x-a = t \Rightarrow dx = dt</math></p> $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{is(t+a)} dt$ $= e^{isa} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt$ <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> <math>F[f(x-a)] = e^{ias} F[f(x)]</math> </div>
3.	<p>State Convolution theorem in Fourier Transform.</p> <p>Solution :</p> <p>The Fourier transform of the convolution of <math>f(x)</math> and <math>g(x)</math> is the product of their Fourier transforms .</p> <p>i.e. <math>F[f(x) * g(x)] = F[f(x)] F[g(x)] = F(s).G(s)</math></p>
4.	<p>If <math>F\{f(x)\} = F(s)</math>, then find <math>F\{e^{iax} f(x)\}</math>.</p> <p>Solution :</p>

	$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$ $F[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) e^{isx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx + iax} dx$ $= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i(s+a)x} dx$ <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> <math>F[e^{iax} f(x)] = F(s+a)</math> </div>
5.	<p>State and prove the change of scale property of Fourier Transform.</p> <p>Statement:</p> <p>If <math>F[f(x)] = F(s)</math> then <math>F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx</math></p> <p>Solution :</p> <p>Given <math>F[f(x)] = F(s)</math></p> <p>The Fourier Transform of <math>f(x)</math> is</p> $F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$ $F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax) e^{isx} dx ,$ <p>If <math>a &gt; 0</math> Put <math>ax = t \Rightarrow adx = dt \Rightarrow dx = \frac{dt}{a}</math> when <math>x = -\infty \Rightarrow t = -\infty</math> and <math>x = \infty \Rightarrow t = \infty</math></p> $F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i \frac{s}{a} t} \frac{dt}{a}$ $F[f(ax)] = \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i \frac{s}{a} t} dt = \frac{1}{a} F\left[\frac{s}{a}\right] \quad \text{---(1)}$ <p>If <math>a &lt; 0</math> Put <math>ax = t, adx = dt, dx = \frac{dt}{a}</math></p> <p>when <math>x = -\infty \Rightarrow t = \infty</math> and <math>x = \infty \Rightarrow t = -\infty</math></p> $\Rightarrow F[f(ax)] = \frac{-1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t) e^{i \frac{s}{a} t} \frac{dt}{a} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i \frac{s}{a} t} \frac{dt}{a} = \frac{1}{a} F\left[\frac{s}{a}\right] \quad \text{---(2)}$ <p>From (1) &amp; (2) we get <math>F(f(ax)) = \frac{1}{ a } F\left[\frac{s}{a}\right], a \neq 0</math></p>
6.	<p>Find the Fourier Sine transform of <math>\frac{1}{x}</math></p>
	<p>Solution :</p> <p>The Fourier Sine Transform of <math>f(x)</math> is</p> $F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx dx$

$$F_s \frac{1}{x} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin sx}{x} dx$$

$$F_s \frac{1}{x} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin t}{t} dt = \sqrt{\frac{2}{\pi} \cdot \frac{\pi}{2}} = \sqrt{\frac{\pi}{2}} \quad \therefore \int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}$$

PART-B

1. Find the Fourier transforms of  $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$  and hence evaluate  $\int_0^{\infty} \frac{\sin x}{x} dx$ . Using Parseval's

identity, prove that  $\int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$ .

$$1, \quad -a < x < a$$

**Solution:** Given  $f(x) = \begin{cases} 1, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform  $f(x)$  is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx.$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a} 0 e^{isx} dx + \int_{-a}^a 1 e^{isx} dx + \int_a^{\infty} 0 e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a (\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a \cos sx dx + i \int_{-a}^a \sin sx dx \quad \because \sin sx \text{ is an odd fn. } \therefore \int_{-a}^a \sin sx dx = 0$$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_0^a \cos sx dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[ \frac{\sin sx}{s} \right]_0^a = \frac{2}{\sqrt{\pi}} \frac{\sin as}{s}$$

$$F(s) = \frac{2 \sin as}{\sqrt{\pi} s}$$

**Deduction: 1**

By inverse Fourier transform of  $F(s)$ .

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2 \sin as}{\sqrt{\pi} s} e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\sin as}{s} (\cos sx - i \sin sx) ds$$

$$= \frac{2}{\sqrt{2\pi}} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\sin as}{s} \cos sx ds - i \int_{-\infty}^{\infty} \frac{\sin as}{s} \sin sx ds$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin as}{s} \cos sx \, ds$$

$$\because \frac{\sin as}{s} \sin sx \text{ is an odd function}$$

$$\int_0^{\infty} \frac{\sin as}{s} \cos sx \, ds = \frac{\pi}{2} f(x)$$

Put  $x=0$

$$\int_0^{\infty} \frac{\sin as}{s} \cos 0 \, ds = \frac{\pi}{2} f(0)$$

$$\int_0^{\infty} \frac{\sin as}{s} \, ds = \frac{\pi}{2} (1) \quad \because f(x) = 1 \Rightarrow f(0) = 1$$

Put  $a=1$  and  $s=x$  we get

$$\therefore \int_0^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2 \, ds = \int_{-\infty}^{\infty} [f(x)]^2 \, dx$$

$$\int_{-\infty}^{\infty} \frac{2}{\pi} \sqrt{\frac{\sin sa}{s}} \, ds = \int_{-a}^a 1^2 \, dx$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa}{s} \, ds = [x]_{-a}^a$$

$$\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin sa}{s} \, ds = [a - (-a)]$$

$$\frac{2}{\pi} \int_0^{\infty} \frac{\sin sa}{s} \, ds = 2a$$

$$\int_0^{\infty} \frac{\sin sa}{s} \, ds = \frac{2\pi a}{4}$$

Put  $a=1$  &  $s=t$  we get,

$$\int_0^{\infty} \frac{\sin^2 t}{t^2} \, dt = \int_0^{\infty} \frac{\sin t}{t} \, dt = \frac{\pi}{2}$$

2. Find the Fourier transform of  $f(x) = \begin{cases} x; & \text{if } |x| < a \\ 0; & \text{if } |x| > a \end{cases}$ .

**Solution:** Given  $f(x) = \begin{cases} x, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform  $f(x)$  is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} \, dx.$$

$$\begin{aligned}
 (s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a} 0 e^{isx} dx + \int_{-a}^a x e^{isx} dx + \int_a^{\infty} 0 e^{isx} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a x(\cos sx + i \sin sx) dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a x \cos sx dx + i \int_{-a}^a x \sin sx dx \quad \because x \cos sx \text{ is an odd fn} \therefore \int_{-a}^a x \cos sx dx = 0 \\
 &= i \frac{1}{\sqrt{2\pi}} 2 \int_0^a x \sin sx dx \quad \because x \sin x \text{ is an even function} \therefore \int_{-a}^a x \sin sx dx = 2 \int_0^a x \sin sx dx \\
 &= i \frac{2}{\sqrt{2\pi}} \left[ \frac{x}{s} (-\cos sx) - \frac{1}{s^2} (-\sin sx) \right]_0^a \\
 &= i \sqrt{\frac{2}{\pi}} \left[ \frac{x \cos sx}{s} - \frac{\sin sx}{s^2} \right]_0^a \\
 &= i \sqrt{\frac{2}{\pi}} \left[ \frac{a \cos sa}{s} - \frac{\sin sa}{s^2} - (0) \right] \\
 \boxed{F(s) = i \sqrt{\frac{2}{\pi}} \frac{\sin sa - as \cos sa}{s^2}}
 \end{aligned}$$

3. Find the Fourier transform of  $f(x) = \begin{cases} a - |x| & \text{if } |x| < a \\ 0 & \text{if } |x| > a > 0 \end{cases}$  is  $\frac{\sqrt{2}}{\pi} \frac{1 - \cos as}{s^2}$ . Hence deduce that (i)

$$\int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2} \quad \text{(ii)} \quad \int_0^{\infty} \frac{\sin t}{t^3} dt = \frac{\pi}{3}$$

**Solution:** Given  $f(x) = \begin{cases} a - |x| & -a < x < a \\ 0 & \text{otherwise} \end{cases}$

The Fourier transform  $f(x)$  is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$\begin{aligned}
 (s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a} 0 e^{isx} dx + \int_{-a}^a (a - |x|) e^{isx} dx + \int_a^{\infty} 0 e^{isx} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a (a - |x|) (\cos sx + i \sin sx) dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-a}^a (a - |x|) \cos sx dx + i \int_{-a}^a (a - |x|) \sin sx dx \\
 &\quad \because (a - |x|) \sin sx \text{ is an odd fn} \therefore \int_{-a}^a (a - |x|) \sin sx dx = 0 \\
 &= \frac{1}{\sqrt{2\pi}} 2 \int_0^a (a - x) \cos sx dx
 \end{aligned}$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}} \left[ (a-x) \frac{\sin sx}{s} - (-1) \frac{-\cos sx}{s^2} \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[ \cos sx \right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[ \frac{1}{s^2} (\cos sa - \cos 0) \right]$$

$$F(s) = \sqrt{\frac{2}{\pi}} \frac{1 - \cos sa}{s^2}$$

$$F(s) = 2\sqrt{\frac{2}{\pi}} \frac{\sin^2 \frac{as}{2}}{s^2}$$

$$\because \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \Rightarrow 1 - \cos \theta = 2 \sin^2 \frac{\theta}{2} \text{ here } \theta = \frac{as}{2}$$

Deduction: 1

By inverse Fourier transform of  $F(s)$ .

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin^2 \frac{as}{2}}{s^2} e^{-isx} ds$$

$$= \frac{2}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{as}{2}}{s^2} (\cos sx - i \sin sx) ds$$

$$= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 \frac{as}{2}}{s^2} (\cos sx) ds - i \int_{-\infty}^{\infty} \frac{\sin^2 \frac{as}{2}}{s^2} (\sin sx) ds$$

$$f(x) = \frac{4}{\pi} \int_0^{\infty} \frac{\sin^2 \frac{as}{2}}{s^2} (\cos sx) ds \quad \because \frac{\sin^2 \frac{as}{2}}{s^2} (\sin sx) \text{ is an odd function}$$

$$\int_0^{\infty} \frac{\sin^2 \frac{as}{2}}{s^2} \cos sx ds = \frac{\pi}{4} f(x)$$

Put  $x=0$



$$\int_{-\infty}^{\infty} \frac{\sin \frac{as}{2}}{s} (\cos 0) ds = \frac{\pi}{4} f(0)$$

$$\int_{-\infty}^{\infty} \frac{\sin \frac{as}{2}}{s} ds = \frac{\pi a}{4} \quad \because f(x) = a - |x| \Rightarrow f(0) = a$$

Put  $a=1$  and  $s=t$  get

$$\int_{-\infty}^{\infty} \frac{\sin \frac{s}{2}}{s} ds = \frac{\pi}{4} \quad \text{put } \frac{s}{2} = t \Rightarrow \frac{ds}{2} = dt$$

$$\int_0^{\infty} \frac{\sin t}{2t} 2dt = \frac{\pi}{4}$$

$$\therefore \int_0^{\infty} \frac{\sin t}{t} dt = \frac{\pi}{2}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2 dx = \int_{-\infty}^{\infty} [f(x)]^2 ds$$

$$\int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin^2 \frac{as}{2}}{s^2} ds = \int_{-a}^a (a - |x|)^2 dx$$

$$\frac{8}{\pi^2} \int_0^{\infty} \frac{\sin^2 \frac{as}{2}}{s^2} ds = 2 \int_0^a (a - x)^2 dx \quad \because (a - |x|)^2 \text{ and } \frac{\sin^2 \frac{as}{2}}{s^2} \text{ are even functions}$$

$$\frac{8}{\pi} \int_0^{\infty} \frac{\sin^2 \frac{as}{2}}{s^2} ds = \int_0^a (a - x)^2 dx$$

$$\frac{8}{\pi} \int_0^{\infty} \frac{\sin^2 \frac{as}{2}}{s^2} ds = \frac{(a - x)^3}{-3} \Big|_0^a$$

$$8 \int_0^{\infty} \frac{\sin^2 \frac{as}{2}}{s^2} ds = \frac{a^3}{3}$$

$$\pi \int_0^{\infty} \frac{\sin^2 \frac{as}{2}}{s^2} ds = \frac{a^3 \pi}{3 \times 8}$$

$$\int_0^{\infty} \frac{\sin^2 \frac{as}{2}}{s^2} ds = \frac{a^3 \pi}{3 \times 8}$$

Put  $a=1$  &  $s=t$  we get,

$$\int_0^{\infty} \frac{\sin^2 \frac{s}{2}}{s^2} ds = \frac{\pi}{24} \quad \text{put } \frac{s}{2} = t \Rightarrow \frac{ds}{2} = dt$$

$$\int_0^{\infty} \frac{\sin^2 t}{2t^2} 2dt = \frac{\pi}{24}$$

$$\int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{3}$$

4. Find the Fourier transform of  $f(x) = \begin{cases} 1-|x|, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$  and hence find the value of

(i)  $\int_0^{\infty} \frac{\sin^2 t}{t^2} dt$  . (ii)  $\int_0^{\infty} \frac{\sin^4 t}{t^4} dt$  .

**Solution:**

Hint in the previous problem  $a=1$ .

5. Find the Fourier transform of  $f(x) = \begin{cases} a^2 - x^2, & |x| \leq a \\ 0, & |x| \geq a \end{cases}$  and hence evaluate

(i)  $\int_0^{\infty} \frac{\sin t - t \cos t}{t^3} dt = \frac{\pi}{4}$       (ii)  $\int_0^{\infty} \frac{\sin t - t \cos t}{t^3} dt = \frac{\pi}{15}$

$$a^2 - x^2, \quad -a < x < a$$

**Solution:** Given  $f(x) = \begin{cases} a^2 - x^2, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform of  $f(x)$  is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx .$$

$$(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^a 0 e^{isx} dx + \int_{-a}^a (a^2 - x^2) e^{isx} dx + \int_a^{\infty} 0 e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a (a^2 - x^2) (\cos sx + i \sin sx) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^a (a^2 - x^2) \cos sx \, dx + i \int_{-a}^a (a^2 - x^2) \sin sx \, dx$$

$\therefore (a^2 - x^2) \sin sx$  is an *odd* fn.  $\therefore \int_{-a}^a (a^2 - x^2) \sin sx \, dx = 0$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_0^a (a^2 - x^2) \cos sx \, dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}} \int_0^a (a^2 - x^2) \frac{\sin sx}{s} - (-2x) \frac{-\cos sx}{s} + (-2) \frac{-\sin sx}{s} \Big|_0^a$$

$$= -2 \sqrt{\frac{2}{\pi}} \left[ \frac{x \cos sx}{s^2} - \frac{\sin sx}{s^3} \right]_0^a$$

$$= -2 \sqrt{\frac{2}{\pi}} \left[ \frac{a \cos sa}{s^2} - \frac{\sin sa}{s^3} - (0) \right]$$

$$= -2 \sqrt{\frac{2}{\pi}} \left[ \frac{as \cos sa - \sin sa}{s^3} \right]$$

$$F(s) = 2 \sqrt{\frac{2}{\pi}} \frac{\sin sa - as \cos sa}{s^3}$$

Deduction: 1

By inverse Fourier transform of  $F(s)$ ,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2 \sqrt{\frac{2}{\pi}} \frac{\sin sa - as \cos sa}{s^3} e^{-isx} \, ds$$

$$= \frac{2}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin sa - as \cos sa}{s^3} (\cos sx - i \sin sx) \, ds$$

$$= \frac{\pi}{4} \int_{-\infty}^{\infty} \frac{\sin sa - as \cos sa}{s^3} (\cos sx) \, ds - i \int_{-\infty}^{\infty} \frac{\sin sa - as \cos sa}{s^3} (\sin sx) \, ds$$

$$f(x) = \frac{\pi}{4} \int_0^{\infty} \frac{\sin sa - as \cos sa}{s^3} \cos sx \, ds \quad \because \frac{\sin sa - as \cos sa}{s^3} (\sin sx) \text{ is an odd function}$$

$$\int_0^{\infty} \frac{\sin sa - as \cos sa}{s^3} \cos sx \, ds = \frac{\pi}{4} f(x)$$

Put  $x=0$

$$\int_0^{\infty} \frac{\sin sa - as \cos sa}{s^3} (\cos 0) \, ds = \frac{\pi}{4} f(0)$$

$$\int_0^{\infty} \frac{\sin sa - as \cos sa}{s^3} \, ds = \frac{\pi a^2}{4} \quad \because f(x) = a^2 - x^2 \Rightarrow f(0) = a^2$$

Put  $a=1$  and  $s=t$  get

$$\int_0^{\infty} \frac{\sin t - t \cos t}{t^3} dt = \frac{\pi}{4}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2 ds = \int_{-\infty}^{\infty} [f(x)]^2 dx$$

$$\int_{-\infty}^{\infty} \frac{2\sqrt{\pi}(\sin sa - as \cos sa)}{s^3} ds = \int_a^a (a^2 - x^2)^2 dx$$

$$\frac{8}{\pi} \int_0^{\infty} \frac{(\sin sa - as \cos sa)^2}{s^3} ds = 2 \int_0^a (a^4 - 2a^2x^2 + x^4) dx$$

$\therefore (a^2 - x^2)^2$  and  $\frac{\sin sa - as \cos sa}{s^3}$  are even functions

$$\frac{8}{\pi} \int_0^{\infty} \frac{(\sin sa - as \cos sa)^2}{s^3} ds = a^4 x - \frac{2a^2 x^3}{3} + \frac{x^5}{5}$$

$$\frac{8}{\pi} \int_0^{\infty} \frac{(\sin sa - as \cos sa)^2}{s^3} ds = a^5 - \frac{2a^5}{3} + \frac{a^5}{5}$$

$$\frac{8}{\pi} \int_0^{\infty} \frac{(\sin sa - as \cos sa)^2}{s^3} ds = \frac{15a^5 - 10a^5 + 3a^5}{15}$$

$$\int_0^{\infty} \frac{(\sin sa - as \cos sa)^2}{s^3} ds = \frac{8a^5}{15} \times \frac{\pi}{8}$$

Put  $a=1$  &  $s=t$  we get,

$$\int_0^{\infty} \frac{(\sin t - t \cos t)^2}{t^3} dt = \frac{\pi}{15}$$

6. Find the Fourier transform of  $f(x) = \begin{cases} 1-x^2; & \text{if } |x| < 1 \\ 0; & \text{if } |x| \geq 1 \end{cases}$ .

Hence show that (i)  $\int_0^{\infty} \frac{\sin s - s \cos s}{s^2} \cos \frac{sx}{2} ds = \frac{3\pi}{16}$  and (ii)  $\int_0^{\infty} \frac{(x \cos x - \sin x)^2}{x^6} dx = \frac{\pi}{15}$

**Solution:** Given  $f(x) = \begin{cases} 1-x^2, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform  $f(x)$  is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx.$$

$$\begin{aligned} (s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 0 e^{isx} dx + \int_{-1}^1 (1-x^2) e^{isx} dx + \int_1^{\infty} 0 e^{isx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 (1-x^2)(\cos sx + i \sin sx) dx \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 (1-x^2) \cos sx \, dx + i \int_{-1}^1 (1-x^2) \sin sx \, dx$$

$\therefore (1-x^2) \sin sx$  is an odd fn.  $\therefore \int_{-1}^1 (1-x^2) \sin sx \, dx = 0$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_0^1 (1-x^2) \cos sx \, dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}} \left[ (1-x^2) \frac{\sin sx}{s} - (-2x) \frac{-\cos sx}{s} + (-2) \frac{-\sin sx}{s} \right]_0^1$$

$$= -2\sqrt{\frac{2}{\pi}} \left[ \frac{x \cos sx}{s^2} - \frac{\sin sx}{s^3} \right]_0^1$$

$$= -2\sqrt{\frac{2}{\pi}} \left[ \frac{\cos s}{s^2} - \frac{\sin s}{s^3} - (0) \right]$$

$$= -2\sqrt{\frac{2}{\pi}} \left[ \frac{s \cos s - \sin s}{s^3} \right]$$

$$F(s) = 2\sqrt{\frac{2}{\pi}} \frac{\sin s - s \cos s}{s^3}$$

Deduction: 1

By inverse Fourier transform of  $F(s)$ ,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2\sqrt{\frac{2}{\pi}} \frac{\sin s - s \cos s}{s^3} e^{-isx} \, ds$$

$$= \frac{2}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{\sin s - s \cos s}{s^3} (\cos sx - i \sin sx) \, ds$$

$$= \pi \int_{-\infty}^{\infty} \frac{\sin s - s \cos s}{s^3} (\cos sx) \, ds - i \int_{-\infty}^{\infty} \frac{\sin s - s \cos s}{s^3} (\sin sx) \, ds$$

$$f(x) = \pi \int_0^{\infty} \frac{\sin s - s \cos s}{s^3} \cos sx \, ds \quad \because \frac{\sin s - s \cos s}{s^3} (\sin sx) \text{ is an odd function}$$

$$\int_0^{\infty} \frac{\sin s - s \cos s}{s^3} \cos sx \, ds = \frac{\pi}{4} f(x)$$

Put  $x = \frac{1}{2}$

$$\int_0^{\infty} \frac{\sin s - s \cos s}{s^3} \cos \frac{s}{2} \, ds = \frac{\pi}{4} f\left(\frac{1}{2}\right)$$

$$\int_0^{\infty} \frac{\sin s - s \cos s}{s^3} \cos \frac{s}{2} \, ds = \frac{\pi}{4} \times \frac{3}{4}$$

$$\therefore f(x) = 1-x^2 \Rightarrow f\left(\frac{1}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\int_0^{\infty} \frac{(\sin s - s \cos s)^2}{s^3} ds = \frac{3\pi}{16}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2 ds = \int_{-\infty}^{\infty} [f(x)]^2 dx$$

$$\int_{-\infty}^{\infty} \frac{2\sqrt{\frac{2}{\pi}} (\sin s - s \cos s)^2}{s^3} ds = \int_{-1}^1 (1-x^2)^2 dx$$

$$\frac{8}{\pi} \int_0^{\infty} \frac{(\sin sa - as \cos sa)^2}{s^3} ds = 2 \int_0^1 (1-2x^2+x^4) dx$$

$\therefore (1-x^2)^2$  and  $\frac{(\sin s - s \cos s)^2}{s^3}$  are an even functions

$$\frac{8}{\pi} \int_0^{\infty} \frac{(\sin s - s \cos s)^2}{s^3} ds = \left[ x - \frac{2x^3}{3} + \frac{x^5}{5} \right]_0^1$$

$$\frac{8}{\pi} \int_0^{\infty} \frac{(\sin s - s \cos s)^2}{s^3} ds = \left[ 1 - \frac{2}{3} + \frac{1}{5} \right]$$

$$\frac{8}{\pi} \int_0^{\infty} \frac{(\sin s - s \cos s)^2}{s^3} ds = \frac{15-10+3}{15}$$

$$\int_0^{\infty} \frac{(\sin s - s \cos s)^2}{s^3} ds = \frac{8}{15} \times \frac{\pi}{8}$$

Put  $s=t$  we get,

$$\int_0^{\infty} \frac{(\sin t - t \cos t)^2}{t^3} dx = \frac{\pi}{15}$$

7. Find the Fourier cosine and sine transform of  $f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2-x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$ .

**Solution:**

$$\text{Given } f(x) = \begin{cases} x, & \text{for } 0 < x < 1 \\ 2-x, & \text{for } 1 < x < 2 \\ 0, & \text{for } x > 2 \end{cases}$$

The Fourier Cosine transform of  $f(x)$  is

$$F_c[f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 x \cos sx dx + \int_1^2 (2-x) \cos sx dx + \int_2^{\infty} 0 \cos sx dx$$

$$\begin{aligned}
 &= \sqrt{\frac{2}{\pi}} \left[ (x) \frac{\sin sx}{s} - (1) \frac{\cos sx}{s^2} + (2-x) \frac{\sin sx}{s} - (-1) \frac{\cos sx}{s^2} \right] \\
 &= \sqrt{\frac{2}{\pi}} \left[ x \frac{\sin sx}{s} - \frac{\cos sx}{s^2} + (2-x) \frac{\sin sx}{s} - \frac{\cos sx}{s^2} \right] \\
 &= \sqrt{\frac{2}{\pi}} \left[ \frac{\sin s}{s} + \frac{\cos s}{s^2} + \frac{1}{s^2} - \frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2} \right] \\
 &= \sqrt{\frac{2}{\pi}} \left[ \frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2} - \frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2} \right]
 \end{aligned}$$

$$F_c(s) = \sqrt{\frac{2}{\pi}} \frac{2 \cos s - \cos 2s - 1}{s^2}$$

The Fourier sine transform of f(x) is

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx.$$

$$\begin{aligned}
 &= \sqrt{\frac{2}{\pi}} \int_0^1 x \sin sx \, dx + \int_1^2 (2-x) \sin sx \, dx + \int_2^{\infty} 0 \sin sx \, dx \\
 &= \sqrt{\frac{2}{\pi}} \left[ (x) \frac{-\cos sx}{s} - (1) \frac{-\sin sx}{s^2} + (2-x) \frac{-\cos sx}{s} - (-1) \frac{-\sin sx}{s^2} \right] \\
 &= \sqrt{\frac{2}{\pi}} \left[ -x \frac{\cos sx}{s} + \frac{\sin sx}{s^2} + (2-x) \frac{\cos sx}{s} - \frac{\sin sx}{s^2} \right] \\
 &= \sqrt{\frac{2}{\pi}} \left[ -\frac{\cos s}{s} + \frac{\sin s}{s^2} + (0) - \frac{\sin 2s}{s^2} + \frac{\cos s}{s} - \frac{\sin s}{s^2} \right] \\
 &= \sqrt{\frac{2}{\pi}} \left[ -\frac{\cos s}{s} + \frac{\sin s}{s^2} - \frac{\sin 2s}{s^2} + \frac{\cos s}{s} - \frac{\sin s}{s^2} \right]
 \end{aligned}$$

$$F_s(s) = \sqrt{\frac{2}{\pi}} \frac{2 \sin s - \sin 2s}{s^2}$$

8. Find Fourier transform of  $e^{-a|x|}$  and hence deduce that

$$(a) \int_{-\infty}^{\infty} \frac{\cos xt}{a^2 + t^2} dt = \frac{\pi}{2a} e^{-a|x|} \quad (b) F[e^{-a|x|}] = i \sqrt{\frac{2}{\pi}} \left( \frac{2as}{s^2 + a^2} \right).$$

The Fourier transform of f(x) is

$$\begin{aligned}
 F[f(x)] = F(s) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|x|} (\cos sx + i \sin sx) dx
 \end{aligned}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax} \cos sx \, dx + i \int_{-\infty}^{\infty} e^{-ax} \sin sx \, dx$$

$\because e^{-ax} \sin sx$  is an odd fn.  $\therefore \int_{-\infty}^{\infty} e^{-ax} \sin sx \, dx = 0$

$$= \frac{1}{\sqrt{2\pi}} 2 \int_0^{\infty} e^{-ax} \cos sx \, dx$$

$$= \frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}} \int_0^{\infty} e^{-ax} \cos sx \, dx$$

$$F(s) = \int_{-\infty}^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \quad \text{here } a = a; b = s$$

**Deduction (a):**

By inverse Fourier transform of  $F(s)$  is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2} e^{-isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{1}{a^2 + s^2} (\cos sx - i \sin sx) \, ds$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{a^2 + s^2} (\cos sx) \, ds - ia \int_{-\infty}^{\infty} \frac{1}{a^2 + s^2} (\sin sx) \, ds$$

$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{1}{a^2 + s^2} \cos sx \, ds$   $\because \frac{1}{a^2 + s^2} (\sin sx)$  is an odd function

$$\int_0^{\infty} \frac{1}{a^2 + s^2} \cos sx \, ds = \frac{1}{2a} f(x)$$

$$\int_0^{\infty} \frac{\cos sx}{a^2 + s^2} \, ds = \frac{\pi}{2a} e^{-a|x|}$$

Put  $s=t$

$$\int_0^{\infty} \frac{\cos tx}{t^2 + a^2} \, dt = \frac{\pi}{2a} e^{-a|x|}$$

**Deduction (b):**

By Property

$$F[xf(x)] = -i \frac{d}{ds} [F(s)]$$

$$F[xe^{-ax}] = -i \frac{d}{ds} F(e^{-ax})$$

$$= -i \frac{d}{ds} \left[ \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2} \right]$$



$$= -ia \sqrt{\frac{2}{\pi}} \frac{-1}{(a^2 + s^2)^2} (0 + 2s) = i \sqrt{\frac{2}{\pi}} \frac{2as}{(a^2 + s^2)^2}$$

$$F_s [x e^{-ax}] = i \sqrt{\frac{2}{\pi}} \frac{2as}{(s^2 + a^2)^2}$$

9. Find the Fourier sine and cosine transform of  $e^{-ax}$ ,  $a > 0$  and deduce that

i)  $\int_0^{\infty} \frac{s}{s^2 + a^2} \sin sx \, dx = \frac{\pi}{2} e^{-ax}$ .

ii)  $\int_0^{\infty} \frac{1}{s + a} \cos sx \, dx = \frac{\pi}{2a} e^{-ax}$

**Solution:**

The Fourier sine transform of  $f(x)$  is

$$F_s [f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx$$

$$F_s(s) = F_s [e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{s}{a^2 + s^2}$$

$$\therefore \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2} \text{ here } a = a; b = s$$

The Fourier cosine transform of  $f(x)$  is

$$F_c [f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx \, dx$$

$$F_c(s) = F_c [e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2}$$

$$\therefore \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \text{ here } a = a; b = s$$

The inverse Fourier sine transform of  $F_s(s)$  is

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(s) \sin sx \, ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{s}{a^2 + s^2} \sin sx \, ds$$

$$= \frac{2}{\pi} \int_0^{\infty} \frac{s}{a^2 + s^2} \sin sx \, ds$$

$$\int_0^{\infty} \frac{s}{a^2 + s^2} \sin sx \, ds = \frac{\pi}{2} f(x)$$

$$\int_0^{\infty} \frac{s}{a^2 + s^2} \sin sx \, ds = \frac{\pi}{2} e^{-ax}$$

The inverse Fourier Cosine transform of  $F_c(s)$  is

$$\begin{aligned}
 f(x) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx \\
 &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2} \cos sx \, dx \\
 &= \frac{2a}{\pi} \int_0^{\infty} \frac{1}{a^2 + s^2} \cos sx \, dx \\
 \int_0^{\infty} \frac{a}{a^2 + s^2} \cos sx \, dx &= \frac{\pi}{2} f(x)
 \end{aligned}$$

$$\int_0^{\infty} \frac{a}{a^2 + s^2} \cos sx \, dx = \frac{\pi}{2a} e^{-ax}$$

10. Find the Fourier sine and cosine transform of  $e^{-ax}$ ,  $a > 0$  and hence find  $F_s[xe^{-ax}]$  and  $F_c[xe^{-ax}]$ .

**Solution:**

The Fourier sine transform  $f(x)$  is

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx$$

$$F_s(s) = F_s[xe^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{s}{a^2 + s^2} \quad \because \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2} \text{ here } a = a, b = s$$

The Fourier cosine transform  $f(x)$  is

$$F_c[f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx.$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx \, dx$$

$$F_c(s) = F_c[xe^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2} \quad \because \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \text{ here } a = a, b = s$$

We know that

$$\text{i) } F_s[xf(x)] = - \frac{d}{ds} \{F_c[f(x)]\} = - \frac{d}{ds} [F_c(s)]$$

$$F_s[xe^{-ax}] = - \frac{d}{ds} \left\{ F_c[xe^{-ax}] \right\} = - \frac{d}{ds} \left[ \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2} \right]$$

$$= -a \sqrt{\frac{2}{\pi}} \frac{d}{ds} \left[ \frac{1}{a^2 + s^2} \right]$$

$$= -a \sqrt{\frac{2}{\pi}} \frac{-1}{(a^2 + s^2)^2} (0 + 2s)$$

$$F_s \{ x e^{-ax} \} = \frac{\sqrt{2}}{\pi} \frac{2as}{(a^2 + s^2)^2} F$$

$$\text{ii) } F_c \{ x f(x) \} = \frac{d}{ds} \{ F_s \{ f(x) \} \} = \frac{d}{ds} [F_s(s)]$$

$$\begin{aligned} F_s \{ x e^{-ax} \} &= \frac{d}{ds} \{ F_c \{ e^{-ax} \} \} = \frac{d}{ds} \left\{ \frac{\sqrt{2}}{\pi} \frac{s}{a^2 + s^2} \right\} \\ &= \frac{\sqrt{2}}{\pi} \frac{(a^2 + s^2)(1) - s(0 + 2s)}{(a^2 + s^2)^2} \\ &= \frac{\sqrt{2}}{\pi} \frac{a^2 + s^2 - 2s^2}{(a^2 + s^2)^2} \end{aligned}$$

$$F_s \{ x e^{-ax} \} = \frac{\sqrt{2}}{\pi} \frac{a^2 - s^2}{(a^2 + s^2)^2} F$$

11.

Find the Fourier sine transform of  $\frac{e^{-ax}}{x}$ ,  $a > 0$  and hence find  $F_s \left\{ \frac{e^{-ax} - e^{-bx}}{x} \right\}$ .

**Solution:**

The Fourier sine transform of  $f(x)$  is

$$F_s \{ f(x) \} = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$

$$F_s \left\{ \frac{e^{-ax}}{x} \right\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx \, dx$$

Taking diff. on both sides w.r.to  $s$

$$\frac{d}{ds} F_s \left\{ \frac{e^{-ax}}{x} \right\} = \frac{d}{ds} \left\{ \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \sin sx \, dx \right\}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \frac{\partial}{\partial s} (\sin sx) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} (\cos sx) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx \, dx$$

$$\frac{d}{ds} F_s \left\{ \frac{e^{-ax}}{x} \right\} = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2}$$

Integrating on both sides w.r.to  $s$

$$F_s \left\{ \frac{e^{-ax}}{x} \right\} = \sqrt{\frac{2}{\pi}} \int \frac{a}{a^2 + s^2} ds$$

$$F_s \frac{e^{-ax}}{x} = \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{s}{a}$$

$$\therefore \int \frac{a}{x^2 + a^2} dx = \tan^{-1} \frac{x}{a}$$

Similarly  $F_s \frac{e^{-bx}}{x} = \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{s}{b}$

Deduction:

$$\begin{aligned} F_s \frac{e^{-ax} - e^{-bx}}{x} &= F_s \frac{e^{-ax}}{x} - \frac{e^{-bx}}{x} \\ &= F_s \frac{e^{-ax}}{x} - F_s \frac{e^{-bx}}{x} \\ &= \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{s}{a} - \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{s}{b} \end{aligned}$$

$$F_s \frac{e^{-ax} - e^{-bx}}{x} = \sqrt{\frac{2}{\pi}} \left[ \tan^{-1} \frac{s}{a} - \tan^{-1} \frac{s}{b} \right]$$

12.

Find the Fourier cosine transform of  $\frac{e^{-ax}}{x}, a > 0$  and hence find  $F_c \frac{e^{-ax} - e^{-bx}}{x}$

**Solution:**

The Fourier cosine transform f(x) is

$$F_c [f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$$

$$F_c \frac{e^{-ax}}{x} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \cos sx \, dx$$

Taking diff. on both sides w.r.to s

$$\frac{d}{ds} F_c \frac{e^{-ax}}{x} = \frac{d}{ds} \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} \frac{\partial}{\partial s} (\cos sx) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-ax}}{x} (-\sin sx) \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx$$

$$\frac{d}{ds} F_c \frac{e^{-ax}}{x} = -\sqrt{\frac{2}{\pi}} \frac{s}{a^2 + s^2}$$

Integrating on on both sides w.r.to s

$$F_c \frac{e^{-ax}}{x} = -\sqrt{\frac{2}{\pi}} \int \frac{s}{a^2 + s^2} ds$$

$$\begin{aligned}
 &= -\sqrt{\frac{2}{\pi}} \int \frac{s}{a^2 + s^2} ds \\
 &= -\sqrt{\frac{2}{\pi}} \frac{1}{2} \int \frac{2s}{a^2 + s^2} ds \\
 &= -\sqrt{\frac{2}{\pi}} \frac{1}{2} \log(s^2 + a^2) \quad \because \int \frac{f'(x)}{f(x)} dx = \log[f(x)] \\
 &= -\frac{1}{\sqrt{2\pi}} \log(s^2 + a^2) \\
 &= \frac{1}{\sqrt{2\pi}} \log \frac{1}{s^2 + a^2}
 \end{aligned}$$

$$F_c \frac{e^{-ax}}{x} = \frac{1}{\sqrt{2\pi}} \log \frac{1}{s^2 + a^2}$$

Similarly  $F_c \frac{e^{-bx}}{x} = \frac{1}{\sqrt{2\pi}} \log \frac{1}{s^2 + b^2}$

**Deduction:**

$$F_c \frac{e^{-ax} - e^{-bx}}{x} = F_c \frac{e^{-ax}}{x} - \frac{e^{-bx}}{x}$$

$$\begin{aligned}
 &= F_c \frac{e^{-ax}}{x} - F_c \frac{e^{-bx}}{x} \\
 &= \frac{1}{\sqrt{2\pi}} \log \frac{1}{s^2 + a^2} - \frac{1}{\sqrt{2\pi}} \log \frac{1}{s^2 + b^2} \\
 &= \frac{1}{\sqrt{2\pi}} \log \frac{s^2 + b^2}{s^2 + a^2}
 \end{aligned}$$

$$F_s \frac{e^{-ax} - e^{-bx}}{x} = \frac{1}{\sqrt{2\pi}} \log \frac{s^2 + b^2}{s^2 + a^2}$$

13. Using Parseval's identity evaluate the following integrals.

1)  $\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2}$

2)  $\int_0^{\infty} \frac{x^2}{(x^2 + a^2)^2} dx$ , where  $a > 0$ .

**Solution:**

Assume  $f(x) = e^{-ax}$

The Fourier sine transform  $f(x)$  is

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx .$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx \, dx$$

$$F_s(s) = F_s e^{-ax} = \sqrt{\frac{2}{\pi}} \frac{s}{a^2 + s^2}$$

$$\therefore \int_0^{\infty} e^{-ax} \sin bx \, dx = \frac{b}{a^2 + b^2} \quad \text{here } a = a; b = s$$

The Fourier cosine transform f(x) is

$$F_c[f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos x \, dx$$

$$F_c(s) = F_c e^{-ax} = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2}$$

$$\therefore \int_0^{\infty} e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \quad \text{here } a = a; b = s$$

(i) The Parseval's identity for Fourier cosine transform is

$$\int_0^{\infty} |F_c(s)|^2 \, ds = \int_0^{\infty} |f(x)|^2 \, dx$$

$$\int_0^{\infty} \left( \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2} \right)^2 \, ds = \int_0^{\infty} (e^{-ax})^2 \, dx$$

$$\frac{2a^2}{\pi} \int_0^{\infty} \frac{1}{(a^2 + s^2)^2} \, ds = \int_0^{\infty} e^{-2ax} \, dx$$

$$\int_0^{\infty} \frac{1}{(a^2 + s^2)^2} \, ds = \frac{\pi}{2a^2} \frac{e^{-2ax}}{-2a}$$

$$\int_0^{\infty} \frac{1}{a^2 + s^2} \, ds = \frac{-\pi}{4a^3} (e^{-\infty} - e^{-0})$$

$$\int_0^{\infty} \frac{1}{(a^2 + s^2)^2} \, ds = \frac{-\pi}{4a^3} [0 - 1] \quad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\int_0^{\infty} \frac{1}{a^2 + s^2} \, ds = \frac{\pi}{4a^3}$$

Put s=x we get

$$\int_0^{\infty} \frac{1}{(a^2 + x^2)^2} \, dx = \frac{\pi}{4a^3}$$

(ii) The Parseval's identity for Fourier sine transform is

$$\int_0^{\infty} |F_s(s)|^2 \, ds = \int_0^{\infty} |f(x)|^2 \, dx$$

$$\int_0^{\infty} \left( \sqrt{\frac{2}{\pi}} \frac{s}{a^2 + s^2} \right)^2 \, ds = \int_0^{\infty} (e^{-ax})^2 \, dx$$

$$\frac{2}{\pi} \int_0^{\infty} \frac{s^2}{(a^2 + s^2)^2} ds = \int_0^{\infty} e^{-2ax} dx$$

$$\int_0^{\infty} \frac{s^2}{(a^2 + s^2)^2} ds = \frac{\pi}{2} \frac{e^{-2ax}}{-2a} \Big|_0^{\infty}$$

$$\int_0^{\infty} \frac{s^2}{(a^2 + s^2)^2} ds = \frac{-\pi}{4a} [e^{-\infty} - e^{-0}]$$

$$\int_0^{\infty} \frac{s^2}{(a^2 + s^2)^2} ds = \frac{-\pi}{4a} [0 - 1] \quad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\int_0^{\infty} \frac{s^2}{(a^2 + s^2)^2} ds = \frac{\pi}{4a}$$

Put s=x we get

$$\int_0^{\infty} \frac{x^2}{(a^2 + x^2)^2} dx = \frac{\pi}{4a}$$

14.

**Evaluate (a)**  $\int_0^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx$  **(b)**  $\int_0^{\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)} dx$  using Fourier transforms.

**Solution:**

**(a) Assume**  $f(x) = e^{-ax}$  ;  $g(x) = e^{-bx}$

The Fourier sine transform f(x) is

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx dx$$

$$F_s(s) = F_s \int_0^{\infty} e^{-ax} \sin sx dx = \sqrt{\frac{2}{\pi}} \frac{s}{a^2 + s^2}$$

$$\because \int_0^{\infty} e^{-ax} \sin bx dx = \frac{b}{a^2 + b^2} \quad \text{here } a = a; b = s$$

Similarly

$$G_s(s) = F_s \int_0^{\infty} e^{-bx} \sin sx dx = \sqrt{\frac{2}{\pi}} \frac{s}{b^2 + s^2}$$

We know that

$$\int_0^{\infty} F_s(s) G_s(s) ds = \int_0^{\infty} f(x) g(x) dx$$

$$\int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{s}{a^2 + s^2} \sqrt{\frac{2}{\pi}} \frac{s}{b^2 + s^2} ds = \int_0^{\infty} e^{-ax} e^{-bx} dx$$

$$\int_0^{\infty} \frac{s^2}{(a^2 + s^2)(b^2 + s^2)} ds = \int_0^{\infty} e^{-ax} \cos bx dx$$

$$\int_0^{\infty} \frac{s^2}{(s^2 + a^2)(s^2 + b^2)} ds = \frac{\pi}{2} \int_0^{\infty} e^{-(a+b)x} dx$$

$$= \frac{\pi}{2} \int_0^{\infty} e^{-(a+b)x} dx$$

$$= \frac{\pi}{2} \left[ \frac{e^{-(a+b)x}}{-(a+b)} \right]_0^{\infty}$$

$$= \frac{-\pi}{2(a+b)} [e^{-\infty} - e^{-0}]$$

$$= \frac{-\pi}{2(a+b)} [0 - 1] \quad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\int_0^{\infty} \frac{s^2 + a^2}{(s^2 + a^2)(s^2 + b^2)} ds = \frac{\pi}{2(a+b)}$$

Put s=x we get

$$\int_0^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{2(a+b)}$$

(b) Assume  $f(x) = e^{-ax}$ ;  $g(x) = e^{-bx}$

The Fourier cosine transform f(x) is

$$F_c[f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx dx$$

$$F_c(s) = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2}$$

$$\because \int_0^{\infty} e^{-ax} \cos bx dx = \frac{a}{a^2 + b^2} \text{ here } a = a; b = s$$

Similarly

$$G_c(s) = F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}} \frac{b}{b^2 + s^2}$$

We know that

$$\int_0^{\infty} F_c(s) G_c(s) ds = \int_0^{\infty} f(x) g(x) dx$$

$$\int_0^{\infty} \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2} \sqrt{\frac{2}{\pi}} \frac{b}{b^2 + s^2} ds = \int_0^{\infty} e^{-ax} e^{-bx} dx$$



$$\frac{2ab}{\pi} \int_0^{\infty} \frac{1}{(a_2 + s_2)(b_2 + s_2)} ds = \int_0^{\infty} e^{-ax - bx} dx$$

$$\int_0^{\infty} \frac{1}{(s_2 + a_2)(s_2 + b_2)} ds = \frac{\pi}{2ab} \int_0^{\infty} e^{-(a+b)x} dx$$

$$= \frac{\pi}{2ab} \int_0^{\infty} e^{-(a+b)x} dx$$

$$= \frac{\pi}{2ab} \frac{e^{-(a+b)x}}{-(a+b)} \Big|_0^{\infty}$$

$$= \frac{-\pi}{2ab(a+b)} [e^{-\infty} - e^{-0}]$$

$$= \frac{-\pi}{2ab(a+b)} [0 - 1] \quad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\int_0^{\infty} \frac{1}{(s^2 + a^2)(s^2 + b^2)} ds = \frac{\pi}{2ab(a+b)}$$

Put s=x we get

$$\int_0^{\infty} \frac{1}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{2ab(a+b)}$$

**Evaluate (a)**  $\int_0^{\infty} \frac{x^2}{(x^2 + 9)(x^2 + 16)} dx$ , **(b)**  $\int_0^{\infty} \frac{1}{(x^2 + 1)(x^2 + 4)} dx$  using Fourier transforms.

**Solution:**

**(a) Assume**  $f(x) = e^{-ax}$  ;  $g(x) = e^{-bx}$

The Fourier sine transform f(x) is

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \sin sx dx$$

$$F_s(s) = F_s \int_0^{\infty} e^{-ax} = \sqrt{\frac{2}{\pi}} \frac{s}{a^2 + s^2}$$

$$\therefore \int_0^{\infty} e^{-ax} \sin bx dx = \frac{b}{a^2 + b^2} \quad \text{here } a = a; b = s$$

Similarly

$$G_s(s) = G_s \int_0^{\infty} e^{-bx} = \sqrt{\frac{2}{\pi}} \frac{s}{b^2 + s^2}$$

We know that

$$\int_0^{\infty} F_s(s)G_s(s)ds = \int_0^{\infty} f(x)g(x)dx$$

$$\int_0^{\infty} \frac{s^2}{\sqrt{\pi(a^2 + s^2)} \sqrt{\pi(b^2 + s^2)}} ds = \int_0^{\infty} e^{-ax} e^{-bx} dx$$

$$\frac{2}{\pi} \int_0^{\infty} \frac{s^2}{(a^2 + s^2)(b^2 + s^2)} ds = \int_0^{\infty} e^{-ax} e^{-bx} dx$$

$$\int_0^{\infty} \frac{s^2}{(s^2 + a^2)(s^2 + b^2)} ds = \frac{1}{2} \int_0^{\infty} e^{-(a+b)x} dx$$

$$= \frac{1}{2} \int_0^{\infty} e^{-(a+b)x} dx$$

$$= \frac{1}{2} \left[ \frac{e^{-(a+b)x}}{-(a+b)} \right]_0^{\infty}$$

$$= \frac{-1}{2(a+b)} [e^{-\infty} - e^{-0}]$$

$$= \frac{-1}{2(a+b)} [0 - 1] \quad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\int_0^{\infty} \frac{s^2}{(s^2 + a^2)(s^2 + b^2)} ds = \frac{\pi}{2(a+b)} \quad \text{-----(1)}$$

Put  $a=3$  &  $b=4$  and  $s=x$  we get

$$(1) \Rightarrow \int_0^{\infty} \frac{x^2}{(x^2 + 9)(x^2 + 16)} dx = \frac{\pi}{2(3+4)}$$

$$\int_0^{\infty} \frac{x^2}{(x^2 + 9)(x^2 + 16)} dx = \frac{\pi}{14}$$

(b) Assume  $f(x) = e^{-ax}$  ;  $g(x) = e^{-bx}$

The Fourier cosine transform  $f(x)$  is

$$F_c[f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-ax} \cos sx dx$$

$$F_c(s) = F_c[e^{-ax}] = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + s^2}$$

$$\because \int_0^{\infty} e^{-ax} \cos bx dx = \frac{a}{a^2 + b^2} \quad \text{here } a = a; b = s$$

Similarly

$$G_c(s) = F_c[e^{-bx}] = \sqrt{\frac{2}{\pi}} \frac{b}{b^2 + s^2}$$

We know that

$$\int_{-\infty}^{\infty} F_c(s)G_c(s)ds = \int_{-\infty}^{\infty} f(x)g(x)dx$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi(a+s)}} \frac{1}{\sqrt{\pi(b+s)}} ds = \int_0^{\infty} e^{-ax} e^{-bx} dx$$

$$\frac{2ab}{\pi} \int_0^{\infty} \frac{1}{(a+s)(b+s)} ds = \int_0^{\infty} e^{-ax} e^{-bx} dx$$

$$\int_0^{\infty} \frac{1}{(s+a)(s+b)} ds = \frac{\pi}{2ab} \int_0^{\infty} e^{-(a+b)x} dx$$

$$= \frac{\pi}{2ab} \int_0^{\infty} e^{-(a+b)x} dx$$

$$= \frac{\pi}{2ab} \frac{e^{-(a+b)x}}{-(a+b)} \Big|_0^{\infty}$$

$$= \frac{-\pi}{2ab(a+b)} [e^{-\infty} - e^{-0}]$$

$$= \frac{-\pi}{2ab(a+b)} [0 - 1] \quad \because e^{-\infty} = 0; e^{-0} = 1$$

$$\int_0^{\infty} \frac{1}{(s^2+a^2)(s^2+b^2)} ds = \frac{\pi}{2ab(a+b)}$$

Put a=1 & b=2 s=x we get

$$(1) \Rightarrow \int_0^{\infty} \frac{1}{(x^2+1)(x^2+4)} dx = \frac{\pi}{2(1)(2)(1+2)}$$

$$\int_0^{\infty} \frac{1}{(x^2+1)(x^2+4)} dx = \frac{\pi}{12}$$

**Self reciprocal:**

If a transformation of a function  $f(x)$  is equal to  $f(s)$  then the function  $f(x)$  is called self reciprocal.

14.

**Find the Fourier transform of  $e^{-a^2 x^2}$ . Hence prove that  $e^{-\frac{x^2}{2}}$  is self reciprocal with respect to Fourier Transforms.**

**Solution:**

The Fourier transform  $f(x)$  is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx$$

$$F[e^{-a^2 x^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2 x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2 x^2 + isx} dx$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} dx \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{isx^2}{2a}\right)^2 - \frac{is^2x^2}{4a^2}} dx \\
 &= \frac{1}{\sqrt{2\pi}} e^{-\frac{is^2}{4a^2}} \int_{-\infty}^{\infty} e^{-\left(ax - \frac{isx^2}{2a}\right)^2} dx \\
 &= \frac{1}{\sqrt{2\pi}} e^{-\frac{is^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a}
 \end{aligned}$$

$$(A - B)^2 = A^2 - 2AB + B^2$$

$$2AB = isx$$

$$\text{Here } A = ax, B = \frac{is}{2a}$$

Let  $u = ax - \frac{isx^2}{2a} \Rightarrow du = adx \Rightarrow dx = \frac{du}{a}; u: -\infty \text{ to } \infty$

$$\begin{aligned}
 &= \frac{1}{\sqrt{2\pi}} e^{-\frac{is^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a} \\
 &= \frac{1}{a\sqrt{2\pi}} e^{-\frac{is^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} du \quad \because i^2 = -1 \\
 &= \frac{1}{a\sqrt{2\pi}} e^{-\frac{is^2}{4a^2}} 2 \int_0^{\infty} e^{-u^2} du \quad \because e^{-u^2} \text{ is an even function} \\
 &= \frac{1}{a\sqrt{2\pi}} e^{-\frac{is^2}{4a^2}} 2 \frac{\sqrt{\pi}}{2} \quad \because \int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}
 \end{aligned}$$

$$F \int_{-\infty}^{\infty} e^{-a^2x^2} dx = \frac{1}{a} \sqrt{\frac{\pi}{a^2}} \quad (1)$$

**Deduction:**

To prove  $e^{-\frac{x^2}{2}}$  is self reciprocal

It is enough to prove that  $F \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$  is  $e^{-\frac{s^2}{2}}$

Put  $a = \frac{1}{\sqrt{2}}$  in (1)

$$F \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx = \frac{1}{\frac{1}{\sqrt{2}}} \sqrt{\frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)^2}} = \sqrt{2} \sqrt{\pi} = \sqrt{2\pi}$$

$$F \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = e^{-\frac{s^2}{2}}$$

$$F \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = e^{-\frac{s^2}{2}}$$

$\therefore e^{-\frac{x^2}{2}}$  is self reciprocal.

15. Find the Fourier transform of  $e^{-\frac{x^2}{2}}$ .

(or) Show that  $e^{-\frac{x^2}{2}}$  is self reciprocal with respect to Fourier Transforms.

**Solution:**

Let  $f(x) = e^{-\frac{x^2}{2}}$

Assume  $f(x) = e^{-a^2x^2}$  where  $a = \frac{1}{\sqrt{2}}$

The Fourier transform  $f(x)$  is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx} dx$$

$$F[e^{-a^2x^2}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2 + isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(a^2x^2 - isx + \frac{(is)^2}{4a^2} - \frac{(is)^2}{4a^2}\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(a^2x^2 - isx + \frac{(is)^2}{4a^2}\right)} e^{\frac{(is)^2}{4a^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{(is)^2}{4a^2}} \int_{-\infty}^{\infty} e^{-\left(a^2x^2 - isx + \frac{(is)^2}{4a^2}\right)} dx$$

Let  $u = ax - \frac{is}{2a} \Rightarrow du = adx \Rightarrow dx = \frac{du}{a}$ ;  $u: -\infty$  to  $\infty$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{(is)^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a}$$

$$= \frac{1}{a\sqrt{2\pi}} e^{\frac{(is)^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} du \quad \because i^2 = -1$$

$$= \frac{1}{a\sqrt{2\pi}} e^{\frac{(is)^2}{4a^2}} 2 \int_0^{\infty} e^{-u^2} du \quad \because e^{-u^2} \text{ is an even function}$$

$$= \frac{1}{a\sqrt{2\pi}} e^{\frac{(is)^2}{4a^2}} 2 \frac{\sqrt{\pi}}{2} \quad \because \int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}$$

$$F[e^{-a^2x^2}] = \frac{1}{a} e^{-\frac{(is)^2}{4a^2}} \quad \text{-----(1)}$$

**Deduction:**

$$(A - B)^2 = A^2 - 2AB + B^2$$

$$2AB = isx$$

Here  $A = ax, B = \frac{is}{2a}$

To prove  $e^{-x^2/2}$  is self reciprocal

It is enough to prove that  $F\left[e^{-x^2/2}\right]$  is  $e^{-s^2/2}$

Put  $a = \frac{1}{2}$  in (1)

$$F\left[e^{-\frac{1}{2}x^2}\right] = \frac{1}{\sqrt{2}} e^{-\frac{s^2}{4}}$$

$$F\left[e^{-\frac{x^2}{2}}\right] = e^{-\frac{s^2}{4}}$$

$$F\left[e^{-\frac{x^2}{2}}\right] = e^{-\frac{s^2}{4}}$$

$\therefore e^{-x^2/2}$  is self reciprocal.

16.

Find the Fourier cosine transform of  $e^{-a^2x^2}$ . Hence find  $F_s[xe^{-a^2x^2}]$ .

**Solution:**

Let  $f(x) = e^{-a^2x^2}$

The Fourier cosine transform  $f(x)$  is

$$F_c[f(x)] = F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx \, dx \quad \because \int_0^{\infty} f(x) \, dx = \frac{1}{2} \int_{-\infty}^{\infty} f(x) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-a^2x^2} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{2} \int_{-\infty}^{\infty} e^{-a^2x^2} \cos sx \, dx$$

$$F_c[f(x)] = \text{R.P. of } \frac{1}{2} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} \, dx \quad \because \cos sx = \text{R.P. of } e^{isx}$$

$$F_c[f(x)] = \text{R.P. of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} \, dx$$

$$= \text{R.P. of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2} e^{isx} \, dx$$

$$= \text{R.P. of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a^2x^2 + isx} \, dx$$

$$= \text{R.P. of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(a^2x^2 - isx)} \, dx$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$-2ab = isx$$

Here  $a = ax$

$$= \text{R.P. of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(ax - is)^2}{2a^2} - \frac{is^2}{2a^2}} dx$$

$$= \text{R.P. of } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{2a^2}{2a^2} \frac{(ax - is)^2}{2a^2} - \frac{is^2}{2a^2}} dx$$

$$= \text{R.P. of } \frac{1}{\sqrt{2\pi}} e^{-\frac{is^2}{2a^2}} \int_{-\infty}^{\infty} e^{-\frac{(ax - is)^2}{2a^2}} dx$$

Let  $u = ax - \frac{is}{a} \Rightarrow du = adx \Rightarrow dx = \frac{du}{a}$ ;  $u: -\infty \text{ to } \infty$

$$= \text{R.P. of } \frac{1}{\sqrt{2\pi}} e^{-\frac{i^2 s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{a}$$

$$= \text{R.P. of } \frac{1}{a\sqrt{2\pi}} e^{-\frac{-s^2}{4a^2}} \int_{-\infty}^{\infty} e^{-u^2} du \quad \because i^2 = -1$$

$$= \text{R.P. of } \frac{1}{a\sqrt{2\pi}} e^{\frac{-s^2}{4a^2}} 2 \int_0^{\infty} e^{-u^2} du \quad \because e^{-u^2} \text{ is an even function}$$

$$= \text{R.P. of } \frac{1}{a\sqrt{2}} \frac{1}{\sqrt{\pi}} e^{\frac{-s^2}{4a^2}} 2 \frac{\sqrt{\pi}}{2} \quad \because \int_0^{\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2}$$

$$F_s [e^{-a^2 x^2}] = \frac{1}{a} e^{-\frac{s^2}{4a^2}} \quad \text{--- (1)}$$

Deduction:

$$F_s [xf(x)] = -\frac{d}{ds} \{F_c [f(x)]\} = -\frac{d}{ds} [F_c (s)]$$

$$F_s [xe^{-a^2 x^2}] = -\frac{d}{ds} \left\{ F_c [e^{-a^2 x^2}] \right\}$$

$$= -\frac{d}{ds} \left[ \frac{1}{a} e^{-\frac{s^2}{4a^2}} \right]$$

$$= -\frac{1}{a} e^{-\frac{s^2}{4a^2}} \left[ \frac{-2s}{4a^2} \right]$$

$$F_s [xe^{-a^2 x^2}] = \frac{s}{2a^3} e^{-\frac{s^2}{4a^2}}$$

17. Solve for  $f(x)$ , the integral equation  $\int_0^{\infty} f(x) \sin sxdx = \begin{cases} 1, & 0 \leq s < 1 \\ 2, & 1 \leq s < 2 \\ 0, & s \geq 2 \end{cases}$

**Solution:**

Given  $\int_0^{\infty} f(x) \sin sxdx = \begin{cases} 1, & 0 \leq s < 1 \\ 2, & 1 \leq s < 2 \\ 0, & s \geq 2 \end{cases}$  ----- (1)

**We know that**

$$F_s [f(x)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \begin{cases} F^{-1} \int_0^x F(s) \sin sx \, ds, & 0 \leq x < 1 \\ F^{-1} \int_0^1 F(s) \sin sx \, ds + \int_1^x 2 \sin sx \, ds, & 1 \leq x < 2 \\ F^{-1} \int_0^1 F(s) \sin sx \, ds, & x \geq 2 \end{cases}$$

$$F^{-1} [F(s)] = f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(s) \sin sx \, ds$$

$$f(x) = \sqrt{\frac{2}{\pi}} \left[ \int_0^1 1 \sin sx \, ds + \int_1^2 2 \sin sx \, ds + \int_2^{\infty} 0 \sin sx \, ds \right]$$

$$= \frac{2}{\pi} \left[ \int_0^1 1 \sin sx \, ds + \int_1^2 2 \sin sx \, ds \right]$$

$$= \frac{2}{\pi} \left[ \frac{-\cos sx}{s} \Big|_0^1 + 2 \frac{-\cos sx}{s} \Big|_1^2 \right]$$

$$= \frac{2}{\pi} \left[ \frac{-\cos x}{x} + \frac{\cos 0}{x} + 2 \frac{-\cos 2x}{x} + \frac{\cos x}{x} \right]$$

$$= \frac{2}{\pi} \left[ \frac{-\cos x}{x} + \frac{1}{x} - 2 \frac{\cos 2x}{x} + \frac{\cos x}{x} \right]$$

$$= \frac{2}{\pi x} [1 - \cos x - 2 \cos 2x + 2 \cos x]$$

$$f(x) = \frac{2}{\pi x} [1 + \cos x - 2 \cos 2x]$$

18. Find the Fourier cosine and sine transform of  $x^{n-1}$ . Hence show that  $\frac{1}{\sqrt{x}}$  is self reciprocal under

**Fourier cosine and sine transforms.**

**Solution:**

By definition of Gamma integral

$$\int_0^{\infty} e^{-ax} x^{n-1} \, dx = \frac{\Gamma n}{a^n}, \quad a > 0, n > 0$$

Put  $a = is$

$$\int_0^{\infty} e^{-isx} x^{n-1} \, dx = \frac{\Gamma n}{(is)^n}, \quad a > 0, n > 0$$

$$\int_0^{\infty} x^{n-1} e^{-isx} \, dx = \frac{\Gamma n}{i^n s^n}$$

$$= \frac{\Gamma n}{s^n} (-i)^n$$

$$= \frac{\Gamma n}{s^n} \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \quad \because e^{-i\frac{\pi}{2}} = \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} = -i$$

$$= \frac{\Gamma n}{s^n} \cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2} \quad \because \text{by Demorives theorem } (\cos \theta \pm i \sin \theta)^n = \cos n\theta \pm i \sin n\theta$$

$$\int_0^{\infty} x^{n-1} (\cos sx - i \sin sx) \, dx = \frac{\Gamma n}{s^n} \cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2}$$

$$\int_0^{\infty} x^{n-1} \cos sx \, dx - i \int_0^{\infty} x^{n-1} \sin sx \, dx = \frac{\Gamma n}{s^n} \cos \frac{n\pi}{2} - i \frac{\Gamma n}{s^n} \sin \frac{n\pi}{2}$$



Equating real and imaginary parts on both sides

$$\int_0^{\infty} x^{n-1} \cos sx \, dx = \frac{\Gamma n}{s^n} \cos \frac{n\pi}{2} \qquad \int_0^{\infty} x^{n-1} \sin sx \, dx = \frac{\Gamma n}{s^n} \sin \frac{n\pi}{2}$$

$$\sqrt{\frac{2}{\pi}} \int_0^{\infty} x^{n-1} \cos sx \, dx = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \cos \frac{n\pi}{2} \qquad \sqrt{\frac{2}{\pi}} \int_0^{\infty} x^{n-1} \sin sx \, dx = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \sin \frac{n\pi}{2}$$

$$F_c [x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \cos \frac{n\pi}{2}$$

$$F_s [x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \sin \frac{n\pi}{2}$$

Deduction:

To prove  $\frac{1}{\sqrt{x}}$  is self reciprocal under Fourier cosine and sine transforms.

It is enough to prove that  $F_c \left[ \frac{1}{\sqrt{x}} \right] = \frac{1}{\sqrt{s}}$  and  $F_s \left[ \frac{1}{\sqrt{x}} \right] = \frac{1}{\sqrt{s}}$

We know that

$$F_c [x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \cos \frac{n\pi}{2} \qquad F_s [x^{n-1}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma n}{s^n} \sin \frac{n\pi}{2}$$

Put  $n = \frac{1}{2}$

$$F_c [x^{-1/2}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma \frac{1}{2}}{s^{1/2}} \cos \frac{\pi}{4} \qquad F_s [x^{-1/2}] = \sqrt{\frac{2}{\pi}} \frac{\Gamma \frac{1}{2}}{s^{1/2}} \sin \frac{\pi}{4}$$

$$F_c [x^{-1/2}] = \frac{1}{\sqrt{s}} \cos \frac{\pi}{4} \qquad F_s [x^{-1/2}] = \frac{1}{\sqrt{s}} \sin \frac{\pi}{4} \qquad \because \cos \frac{\pi}{4} = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \Gamma \frac{1}{2} = \sqrt{\pi}$$

$$F_c \left[ \frac{1}{\sqrt{x}} \right] = \frac{1}{\sqrt{s}}$$

$$F_s \left[ \frac{1}{\sqrt{x}} \right] = \frac{1}{\sqrt{s}}$$

$\therefore \frac{1}{\sqrt{x}}$  is self reciprocal under Fourier cosine and sine transforms.

19.

Find the function  $f(x)$  if its sine transform is  $\frac{e^{-as}}{s}$

**Solution:**

Given  $F_s [f(x)] = F_s(s) = \frac{e^{-as}}{s}$

$$f(x) = F_s^{-1} [F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(s) \sin sx \, ds$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-as}}{s} \sin sx \, ds$$

Taking diff on both sides w.r.to x

$$\frac{d}{dx} [f(x)] = \frac{d}{dx} \left[ \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-as}}{s} \sin sx \, ds \right]$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-as}}{s} \frac{\partial}{\partial x} (\sin sx) ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{e^{-as}}{s} \cos sx \times s ds$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-as} \cos sx ds$$

$$\frac{d}{dx} [f(x)] = \sqrt{\frac{2}{\pi}} \frac{a}{a^2 + x^2} \quad \because \int_0^{\infty} e^{-ax} \cos bx dx = \frac{a}{a^2 + b^2} \text{ here } a = a, b = x$$

**Integrating on w.r.to x**

$$f(x) = \sqrt{\frac{2}{\pi}} a \int \frac{1}{a^2 + x^2} \quad \because \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$= \sqrt{\frac{2}{\pi}} a \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \tan^{-1} \frac{x}{a}$$

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