DEPARTMENT OF MATHEMATICS

NAME OF THE SUBJECT : TRANSFORMS & PARTIAL DIFFERENTIAL EQUATION

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UNIT – III : APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATION

UNIT III APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS
PART – A

One Dimensional Wave Equation: $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

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Vibrations of a string

↓

Zero Initial velocity

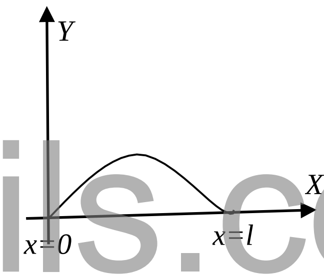
Hint:
1) Velocity will not be given in the question
2) The word released from the rest will be there in the question

The Boundary and Initial conditions are:
i) $y(0,t)=0$
ii) $y(l,t)=0$
iii) $\frac{\partial y}{\partial t}(x,0)=0$
iv) $y(x,0)=f(x),\ 0 \le x \le l$

Non Zero Initial velocity

Hint:
1) Velocity will be given in the question
2) The word initially at rest will be there in the question

The Boundary and Initial conditions are:
i) $y(0,t)=0$
ii) $y(l,t)=0$
iii) $y(x,0)=0$
iv) $\frac{\partial y}{\partial t}(x,0)=f(x),\ 0 \le x \le l$



The Correct (or) Suitable Solution is
$y(x,t)=(A\cos px + B\sin px)(C\cos pat + D\sin pat)$

$y(x,t) \rightarrow$ The displacement of the string at a distance *x* from one end at time *t*

1) **In the wave equation $\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2}$, what does C^2 stands for?**

Solution:

One dimensional heat equation is $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$

$C^2 = T/m$, where *T* is the tension and *m* is the mass of the string.

2) **Write all possible solutions of the transverse vibration of the string in one dimension.**

Solution:

(i) $y(x,t) = (Ae^{px} + Be^{-px})(Ce^{pat} + De^{-pat})$

(ii) $y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$.

(iii) $y(x,t) = (Ax + B)(Ct + D$

PART - B

1. **A uniform string is stretched and fastened to two points $'l'$ apart. Motion is started by displacing the string into the form of the curve $y = kx(l - x)$ and then releasing it from this position at time $t = 0$. Find the displacement of the point of the string at a distance x from one end at time t.**

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad ----- (1)$$

The Boundary and Initial conditions are

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $\frac{\partial y}{\partial t}(x,0) = 0$

iv) $y(x,0) = f(x) = kx(l - x), \quad 0 < x < l$

Applying condn (i) in (1)

$(1) \Rightarrow y(0,t) = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$

$0 = (A)(C\cos pat + D\sin pat)$

Here $(C\cos pat + D\sin pat) \neq 0 \quad \therefore \boxed{A = 0}$

Sub A=0 in (1)

$(1) \Rightarrow y(x,t) = (B\sin px)(C\cos pat + D\sin pat) \quad ----- (2)$

Applying condn (ii) in (2)

$y(l,t) = (B\sin pl)(C\cos pat + D\sin pat)$

$0 = (B\sin pl)(C\cos pat + D\sin pat)$

Here $B \neq 0, (C\cos pat + D\sin pat) \neq 0$

$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$

Sub the value of p in (2)

$$(2) \Rightarrow y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi a}{l}t + D\sin\frac{n\pi a}{l}t\right) \quad ----- (3)$$

Diff (3) partially w.r.to 't'

$$(2) \Rightarrow \frac{\partial y}{\partial t}(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left[-C\sin\frac{n\pi a}{l}t \times \left(\frac{n\pi a}{l}\right) + D\cos\frac{n\pi a}{l}t \times \left(\frac{n\pi a}{l}\right)\right]$$

Apply condn. (iii) in the above equation

$$\frac{\partial y}{\partial t}(x,0) = \left(B\sin\frac{n\pi}{l}x\right)\left[-C\sin 0\times\left(\frac{n\pi a}{l}\right) + D\cos 0\times\left(\frac{n\pi a}{l}\right)\right]$$

$$0 = \left(B\sin\frac{n\pi}{l}x\right)\left[D\times\left(\frac{n\pi a}{l}\right)\right]$$

Here $B \neq 0$, $\sin\frac{n\pi}{l}x \neq 0$, $\frac{n\pi a}{l} \neq 0$, $\therefore$ $\boxed{D = 0}$

Sub the value of D in (3)

$$(3) \Rightarrow \quad y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi a}{l}t + 0\right)$$

$$y(x,t) = BC\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t$$

$$y(x,t) = b_1\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t \qquad \text{let } BD = b_1$$

The most general solution is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t} \text{ - - - - - (4)}$$

Applying condn (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{l}x\cos 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{l}x \qquad \because \cos 0 = 1$$

Which is half range Fourier sine series in $(0,l)$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\int_0^l kx(l-x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2k}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[\frac{-2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{-4kl^3}{ln^3\pi^3}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-4kl^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$b_n = \begin{cases} 0 & if\ \ n\ \ is\ \ even \\ \dfrac{8kl^2}{n^3\pi^3} & if\ \ n\ \ is\ \ odd \end{cases}$$

Sub b_n in (4)

$$y(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin\frac{n\pi}{l}x \cos\frac{n\pi a}{l}t$$

(or) $$y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{(2n-1)\pi}{l}x\cos\frac{(2n-1)\pi a}{l}t$$

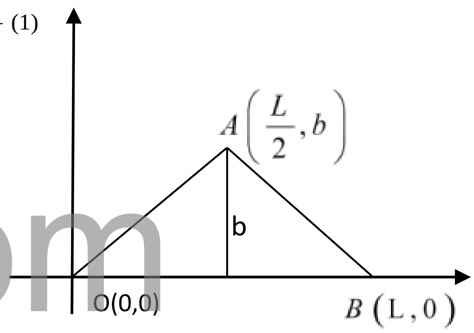
2. A string of length *2l* is fastened at both ends. The midpoint of the string is displaced transversely through a small distance '*b*' and the string is released from the rest in that position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.

Solution:

One dimensional wave equation is $\dfrac{\partial^2 y}{\partial t^2} = a^2\dfrac{\partial^2 y}{\partial x^2}$ where $a^2 = \dfrac{T}{m}$

The correct solution is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad -----(1)$$



The Boundary and Initial conditions are

Assume $2l=L$

i) $y(0,t) = 0$

ii) $y(L,t) = 0$

iii) $\dfrac{\partial y}{\partial t}(x,0) = 0$

iv) $y(x,0) = f(x) = ?$

To find *f(x)*:

The equation of line joining two points is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

Eqution of OA is *O(0,0)& A(l/2,b)*:

$$\frac{x-0}{\frac{L}{2}-0} = \frac{y-0}{b-0} \Rightarrow \frac{2x}{L} = \frac{y}{b} \Rightarrow y = \frac{2b}{L}x,\ 0 < x < \frac{L}{2}$$

Eqution of AB is *A(L/2,b)&B(L,0)*:

$$\frac{x-\frac{L}{2}}{L-\frac{L}{2}} = \frac{y-b}{0-b} \Rightarrow \frac{\frac{2x-L}{2}}{\frac{L}{2}} = \frac{y-b}{-b} \Rightarrow \frac{2x-L}{L} = \frac{y-b}{-b}$$

$$-2xb + lb = yl - lb$$

$$-2xb + Lb + Lb = yL \Rightarrow -2xb + 2Lb = yL$$

$$y = \frac{2b}{L}(L - x), \quad \frac{L}{2} < x < L$$

$$y = f(x) = \begin{cases} \frac{2b}{L}x, & 0 < x < \frac{L}{2} \\ \frac{2b}{L}(L - x), & \frac{L}{2} < x < L \end{cases}$$

Applying condn (i) in (1)

$$(1) \Rightarrow \; y(0,t) = (A\cos 0 + \cancel{B\sin 0})(C\cos pat + D\sin pat)$$

$$0 = (A)(C\cos pat + D\sin pat)$$

Here $(C\cos pat + D\sin pat) \neq 0 \;\therefore\; \boxed{A = 0}$

Sub A=0 in (1)

$$(1) \Rightarrow \; y(x,t) = (B\sin px)(C\cos pat + D\sin pat) \quad -----(2)$$

Applying condn (ii) in (2)

$$y(L,t) = (B\sin pL)(C\cos pat + D\sin pat)$$

$$0 = (B\sin pL)(C\cos pat + D\sin pat)$$

Here $B \neq 0, (C\cos pat + D\sin pat) \neq 0$

$$\therefore \sin pL = 0 \;\Rightarrow \sin pL = \sin n\pi \;\Rightarrow\; pL = n\pi \;\Rightarrow\; \boxed{p = \frac{n\pi}{L}}$$

Sub the value of p in (2)

$$(2) \Rightarrow \; y(x,t) = \left(B\sin\frac{n\pi}{L}x\right)\left(C\cos\frac{n\pi a}{L}t + D\sin\frac{n\pi a}{L}t\right) \quad -----(3)$$

Diff (3) partially w.r.to 't'

$$(2) \Rightarrow \; \frac{\partial y}{\partial t}(x,t) = \left(B\sin\frac{n\pi}{L}x\right)\left[-C\sin\frac{n\pi a}{L}t \times \left(\frac{n\pi a}{L}\right) + D\cos\frac{n\pi a}{L}t \times \left(\frac{n\pi a}{L}\right)\right]$$

Apply condn. (iii) in the above equation

$$\frac{\partial y}{\partial t}(x,0) = \left(B\sin\frac{n\pi}{L}x\right)\left[\cancel{-C\sin 0 \times \left(\frac{n\pi a}{L}\right)} + D\cos 0 \times \left(\frac{n\pi a}{L}\right)\right]$$

$$0 = \left(B\sin\frac{n\pi}{L}x\right)\left[D \times \left(\frac{n\pi a}{L}\right)\right]$$

Here $B \neq 0$, $\sin\frac{n\pi}{L}x \neq 0$, $\frac{n\pi a}{L} \neq 0$, $\therefore D = 0$

Sub the value of D in (3)

$$(3) \Rightarrow \; y(x,t) = \left(B\sin\frac{n\pi}{L}x\right)\left(C\cos\frac{n\pi a}{L}t + 0\right)$$

$$y(x,t) = BC \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t$$

$$y(x,t) = b_1 \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t \qquad \text{let } BD = b_1$$

The most general solution is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t} \quad ------(4)$$

Applying condn (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \cos 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \quad \because \cos 0 = 1$$

Which is half range Fourier sine series in $(0,l)$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$= \frac{2}{L}\left[\int_0^{\frac{L}{2}} \frac{2b}{L} x \sin \frac{n\pi x}{L} dx + \int_{\frac{L}{2}}^{L} \frac{2b}{L} (L-x) \sin \frac{n\pi x}{L} dx \right] \quad \because y = f(x) = \begin{cases} \frac{2b}{L} x, & 0 < x < \frac{L}{2} \\ \frac{2b}{L}(L-x), & \frac{L}{2} < x < L \end{cases}$$

$$= \frac{4b}{L^2}\left[\int_0^{\frac{L}{2}} x \sin \frac{n\pi x}{L} dx + \int_{\frac{L}{2}}^{L} (L-x) \sin \frac{n\pi x}{L} dx \right]$$

$$= \frac{4b}{L^2}\left\{ \left[(x)\left(\frac{-\cos \frac{n\pi x}{L}}{\frac{n\pi}{L}} \right) - (1)\left(\frac{-\sin \frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}} \right) \right]_0^{\frac{L}{2}} + \left[(L-x)\left(\frac{-\cos \frac{n\pi x}{L}}{\frac{n\pi}{L}} \right) - (-1)\left(\frac{-\sin \frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}} \right) \right]_{\frac{L}{2}}^{L} \right\}$$

$$= \frac{4b}{L^2}\left\{ \left[-\frac{L}{n\pi} x \cos \frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi x}{L} \right]_0^{\frac{L}{2}} + \left[-\frac{L}{n\pi}(L-x) \cos \frac{n\pi x}{L} - \frac{L^2}{n^2\pi^2} \sin \frac{n\pi x}{L} \right]_{\frac{L}{2}}^{L} \right\}$$

$$= \frac{4b}{L^2}\left\{ \left[\left(-\frac{L}{n\pi} \frac{L}{2} \cos \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right) - (0) \right]_0^{\frac{L}{2}} + \left[(0) - \left(-\frac{L}{n\pi}\left(\frac{L}{2}\right) \cos \frac{n\pi}{2} - \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right) \right]_{\frac{L}{2}}^{L} \right\}$$

$$= \frac{4b}{L^2}\left[-\frac{L^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{L^2}{2n\pi} \cos \frac{n\pi}{2} + \frac{L^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$$= \frac{4b}{L^2}\left[\frac{2L^2}{n^2\pi^2} \sin \frac{n\pi}{2} \right]$$

$$b_n = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Sub b_n in (4)

$$\therefore\ y(x,t) = \sum_{n=1}^{\infty}\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$$

$$\therefore\ y(x,t) = \frac{8b}{\pi^2}\sum_{n=1}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l}$$

This is the required displacement.

3. **A string of length *l* is fastened at both ends. The midpoint of the string is displaced transversely through a small distance '*b*' and the string is released from the rest in that position. Find an expression for the transeverse displacement of the string at any time during the subsequent motion.**

Solution:

Replace *L by l* in the above problem

4. **A tightly stretched string with fixed end points *x=0* and *x=l* is initially in a position given by $y(x,0) = y_0\sin^3\frac{\pi x}{l}$. If it is released from rest from this position, find the displacement *y* at any distance *x* from one end any time *t*.**

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad ------(1)$$

The Boundary and Initial conditions are

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $\frac{\partial y}{\partial t}(x,0) = 0$

iv) $y(x,0) = f(x) = y_0\sin^3\frac{\pi x}{l}, \quad 0 < x < l$

Applying condn (i) in (1)

$(1) \Rightarrow y(0,t) = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$

$0 = (A)(C\cos pat + D\sin pat)$

Here $(C\cos pat + D\sin pat) \neq 0 \ \therefore \boxed{A = 0}$

Sub A=0 in (1)

$(1) \Rightarrow y(x,t) = (B\sin px)(C\cos pat + D\sin pat) \quad ------(2)$

Applying condn (ii) in (2)

$y(l,t) = (B\sin pl)(C\cos pat + D\sin pat)$

$0 = (B\sin pl)(C\cos pat + D\sin pat)$

Here $B \neq 0, (C\cos pat + D\sin pat) \neq 0$

$$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2) \Rightarrow \quad y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi a}{l}t + D\sin\frac{n\pi a}{l}t\right) \quad ------(3)$$

Diff (3) partially w.r.to ‘t’

$$(2) \Rightarrow \quad \frac{\partial y}{\partial t}(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left[-C\sin\frac{n\pi a}{l}t \times \left(\frac{n\pi a}{l}\right) + D\cos\frac{n\pi a}{l}t \times \left(\frac{n\pi a}{l}\right)\right]$$

Apply condn. (iii) in the above equation

$$\frac{\partial y}{\partial t}(x,0) = \left(B\sin\frac{n\pi}{l}x\right)\left[-C\sin 0 \times \left(\frac{n\pi a}{l}\right) + D\cos 0 \times \left(\frac{n\pi a}{l}\right)\right]$$

$$0 = \left(B\sin\frac{n\pi}{l}x\right)\left[D \times \left(\frac{n\pi a}{l}\right)\right]$$

Here $B \neq 0$, $\sin\frac{n\pi}{l}x \neq 0$, $\frac{n\pi a}{l} \neq 0$, $\therefore$ $\boxed{D = 0}$

Sub the value of D in (3)

$$(3) \Rightarrow \quad y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi a}{l}t + 0\right)$$

$$y(x,t) = BC\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t$$

$$y(x,t) = b_1\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t \qquad \text{let } BD = b_1$$

The most general solution is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t} \quad ------(4)$$

Applying condn (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\cos 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \qquad \because \cos 0 = 1$$

$$y_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$$

$$y_0 \sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \qquad \boxed{\sin^3\theta =} \frac{1}{4}[3\sin\theta - \sin 3\theta]$$

$$y_0\left\{\frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]\right\} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x$$

$$\frac{3y_0}{4}\sin\frac{\pi x}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l} = b_1\sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + b_3\sin\frac{3\pi x}{l} + b_4\sin\frac{4\pi x}{l} + \dots$$

Equating co-efficients of likely terms on both sides

$$b_1 = \frac{3y_0}{4};\ b_2 = 0;\ b_3 - \frac{y_0}{4};\ b_4 = b_5 = b_6 \dots = 0.$$

Sub these values in (4)

$$(4) \Rightarrow y(x,t) = b_1\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} + b_2\sin\frac{2\pi x}{l}\cos\frac{2\pi at}{l} + b_3\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l} + \dots$$

$$\boxed{y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} - \frac{y}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}}$$

5. **A tightly stretched string end points *x=0* and *x=l* is initially at restin its equilibrium position. If it is set vibrating giving each point a velocity $\lambda(lx - x^2)$, then show that the displacement of given string is**

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{l}\sin\frac{(2n-1)\pi at}{l}.$$

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) - - - - - (1)$$

The Boundary and Initial conditions are

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $y(x,0) = 0$

iv) $\frac{\partial y}{\partial t}(x,0) = \lambda(lx - x^2),\ 0 < x < l$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0,t) = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$$

$$0 = (A)(C\cos pat + D\sin pat)$$

Here $(C\cos pat + D\sin pat) \neq 0\ \therefore\ \boxed{A = 0}$

Sub A=0 in (1)

$$(1) \Rightarrow y(x,t) = (B\sin px)(C\cos pat + D\sin pat) - - - - - (2)$$

Applying condn (ii) in (2)

$$y(l,t) = (B\sin pl)(C\cos pat + D\sin pat)$$

$$0 = (B\sin pl)(C\cos pat + D\sin pat)$$

Here $B \neq 0, (C\cos pat + D\sin pat) \neq 0$

$$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2) \Rightarrow y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi a}{l}t + D\sin\frac{n\pi a}{l}t\right) ------(3)$$

Apply condn. (iii) in the above equation

$$(3) \Rightarrow y(x,0) = \left(B\sin\frac{n\pi}{l}x\right)(C\cos 0 + \cancel{D\sin 0})$$

$$0 = \left(B\sin\frac{n\pi}{l}x\right)(C)$$

Here $B \neq 0$, $\sin\frac{n\pi}{l}x \neq 0$, $\boxed{\therefore C = 0}$

Sub the value of C in (3)

$$(3) \Rightarrow y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(D\sin\frac{n\pi a}{l}t\right)$$

$$y(x,t) = BD\sin\frac{n\pi}{l}x\sin\frac{n\pi a}{l}t$$

$$y(x,t) = b_1\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t \quad \text{let } BD = b_1$$

The most general solution is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{l}x\sin\frac{n\pi a}{l}t} ------(4)$$

Diff (4) partially w.r.to 't'

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t \times \left(\frac{n\pi a}{l}\right)$$

Applying condn (iv) in the above eqn.

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{l}x\cos 0 \times \left(\frac{n\pi a}{l}\right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l} \times \left(\frac{n\pi a}{l}\right) \quad \boxed{\cos 0 = 1}$$

$$y(x,0) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l} \quad \text{let } B_n = b_n\frac{n\pi a}{l}$$

$$f(x) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi}{l}x$$

Which is half range Fourier sine series in $(0,l)$

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l \lambda x(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2\lambda}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2\lambda}{l}\left[\frac{-2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{-4\lambda l^3}{ln^3\pi^3}\left[\cos n\pi - \cos 0\right]$$

$$b_n\frac{n\pi a}{l} = \frac{-4\lambda l^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$b_n = \frac{-4\lambda l^3}{n^4\pi^4 a}\left[(-1)^n - 1\right]$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{8\lambda l^3}{n^4\pi^4 a} & \text{if } n \text{ is odd} \end{cases}$$

Sub b_n in (4)

$$y(x,t) = \sum_{n=1,3,5,\ldots}^{\infty}\frac{8\lambda l^3}{n^4\pi^4 a}\sin\frac{n\pi}{l}x\,\sin\frac{n\pi a}{l}t$$

(or) $$y(x,t) = \frac{8\lambda l^3}{\pi^4 a}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi}{l}x\,\sin\frac{(2n-1)\pi a}{l}t$$

6. **If a string of length l is initially at rest in its equilibrium position whose ends are fixed and each of its points is given a velocity v such that** $v = \begin{cases} cx\,;\ 0 < x < \frac{l}{2} \\ c(l-x)\,;\ \frac{l}{2} < x < l \end{cases}$ **, find the displacement of the string at any time t .**

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad -----(1)$$

The Boundary and Initial conditions are

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $y(x,0) = 0$

iv) $\frac{\partial y}{\partial t}(x,0) = \begin{cases} cx \,;\; 0 < x < \frac{l}{2} \\ c(l-x) \,;\; \frac{l}{2} < x < l \end{cases}$

Applying condn (i) in (1)

$(1) \Rightarrow \; y(0,t) = (A\cos 0 + \cancel{B\sin 0})(C\cos pat + D\sin pat)$

$0 = (A)(C\cos pat + D\sin pat)$

Here $(C\cos pat + D\sin pat) \neq 0$ $\boxed{\therefore A = 0}$

Sub A=0 in (1)

$(1) \Rightarrow \; y(x,t) = (B\sin px)(C\cos pat + D\sin pat) \;\; ------(2)$

Applying condn (ii) in (2)

$y(l,t) = (B\sin pl)(C\cos pat + D\sin pat)$

$0 = (B\sin pl)(C\cos pat + D\sin pat)$

Here $B \neq 0, \; (C\cos pat + D\sin pat) \neq 0$

$\therefore \sin pl = 0 \;\Rightarrow\; \sin pl = \sin n\pi \;\Rightarrow\; pl = n\pi \;\Rightarrow\; \boxed{p = \frac{n\pi}{l}}$

Sub the value of *p* in (2)

$(2) \Rightarrow \; y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi a}{l}t + D\sin\frac{n\pi a}{l}t\right) \;\; ------(3)$

Apply condn. (iii) in the above equation

$(3) \Rightarrow \; y(x,0) = \left(B\sin\frac{n\pi}{l}x\right)(C\cos 0 + \cancel{D\sin 0})$

$0 = \left(B\sin\frac{n\pi}{l}x\right)(C)$

Here $B \neq 0 \,, \; \sin\frac{n\pi}{l}x \neq 0, \; \therefore C = 0$

Sub the value of C in (3)

$(3) \Rightarrow \; y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(D\sin\frac{n\pi a}{l}t\right)$

$y(x,t) = BD\sin\frac{n\pi}{l}x\sin\frac{n\pi a}{l}t$

$y(x,t) = b_1\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t \qquad \text{let } BD = b_1$

The most general solution is

$$\boxed{y(x,t)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \sin\frac{n\pi a}{l}t} \quad -----(4)$$

Diff (4) partially w.r.to 't'

$$\frac{\partial y}{\partial t}(x,t)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\ \cos\frac{n\pi a}{l}t\times\left(\frac{n\pi a}{l}\right)$$

Applying condn (iv) in the above eqn.

$$\frac{\partial y}{\partial t}(x,0)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\ \cos 0\times\left(\frac{n\pi a}{l}\right)$$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\times\left(\frac{n\pi a}{l}\right) \qquad \boxed{\cos 0=1}$$

$$y(x,0)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \qquad \text{let } B_n=b_n\frac{n\pi a}{l}$$

$$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x$$

Which is half range Fourier sine series in $(0,l)$

$$B_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\left[\int_0^{\frac{l}{2}} cx\sin\frac{n\pi x}{l}\,dx+\int_{\frac{l}{2}}^{l} c(l-x)\sin\frac{n\pi x}{l}\,dx\right] \qquad \because f(x)=\begin{cases} cx\ ;\ 0<x<\dfrac{l}{2}\\ c(l-x)\ ;\ \dfrac{l}{2}<x<l\end{cases}$$

$$=\frac{2c}{l}\left[\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}\,dx+\int_{\frac{l}{2}}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$=\frac{2c}{l}\left\{\left[(x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{\frac{l}{2}}+\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{\frac{l}{2}}^{l}\right\}$$

$$=\frac{2c}{l}\left\{\left[-\frac{l}{n\pi}x\cos\frac{n\pi x}{l}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^{\frac{L}{2}}+\left[-\frac{l}{n\pi}(l-x)\cos\frac{n\pi x}{l}-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_{\frac{L}{2}}^{L}\right\}$$

$$=\frac{2c}{l}\left\{\left[\left(-\frac{l}{n\pi}\frac{l}{2}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)-(0)\right]_0^{\frac{l}{2}}+\left[(0)-\left(-\frac{l}{n\pi}\left(\frac{l}{2}\right)\cos\frac{n\pi}{2}-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]_{\frac{l}{2}}^{l}\right\}$$

$$= \frac{2c}{l}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} \right]$$

$$\frac{2c}{l}\left[\frac{2l^2}{n\ \ \pi} \quad \frac{n\pi}{2} \right]$$

$$b\,\frac{n\pi a}{l} = \frac{4cl}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad B_n = b\,\frac{n\pi a}{l}$$

$$\boxed{b = \frac{4cl^2}{n^3\pi^3 a}\sin\frac{n\pi}{2}}$$

Sub b_n in (4)

$$\therefore\ y(x,t) = \sum_{n=1}^{\infty} \frac{4cl^2}{n^3\pi^3 a}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$\therefore\ y(x,t) = \frac{4cl^2}{\pi^3 a}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

7. **If a string of length of *l* is initially at rest in its equilibrium position and each of its point is given the velocity** $\frac{\partial y}{\partial t}(x,0) = V_0 \sin^3\frac{\pi x}{l}$**; *0<x<l*. Determine the displacement function *y(x,t)*.**

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad -----(1)$$

The Boundary and Initial conditions are

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $y(x,0) = 0$

iv) $\frac{\partial y}{\partial t}(x,0) = V_0\sin^3\frac{\pi x}{l},\ 0 < x < l$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0,t) = (A\cos 0 + B\sin 0)(C\cos pat + D\sin pat)$$

$$0 = (A)(C\cos pat + D\sin pat)$$

Here $(C\cos pat + D\sin pat) \neq 0\ \therefore\ \boxed{A = 0}$

Sub A=0 in (1)

$$(1) \Rightarrow y(x,t) = (B\sin px)(C\cos pat + D\sin pat) \quad -----(2)$$

Applying condn (ii) in (2)

$$y(l,t) = (B\sin pl)(C\cos pat + D\sin pat)$$

$0 = (B\sin pl)(C\cos pat + D\sin pat)$

Here $B \neq 0,\ (C\cos pat + D\sin pat) \neq 0$

$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$

Sub the value of p in (2)

$$(2) \Rightarrow\ y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi a}{l}t + D\sin\frac{n\pi a}{l}t\right) \text{ ------ } (3)$$

Apply condn. (iii) in the above equation

$$(3) \Rightarrow\ y(x,0) = \left(B\sin\frac{n\pi}{l}x\right)(C\cos 0 + \cancel{D\sin 0})$$

$$0 = \left(B\sin\frac{n\pi}{l}x\right)(C)$$

Here $B \neq 0$, $\sin\frac{n\pi}{l}x \neq 0$, $\therefore \boxed{C = 0}$

Sub the value of C in (3)

$$(3) \Rightarrow\ y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(D\sin\frac{n\pi a}{l}t\right)$$

$$y(x,t) = BD\sin\frac{n\pi}{l}x\sin\frac{n\pi a}{l}t$$

$$y(x,t) = b_1\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t \qquad \text{let } BD = b_1$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{l}x\sin\frac{n\pi a}{l}t \text{ ------ } (4)$$

Diff (4) partially w.r.to 't'

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t \times \left(\frac{n\pi a}{l}\right)$$

Applying condn (iv) in the above eqn.

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{l}x\cos 0 \times \left(\frac{n\pi a}{l}\right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l} \times \left(\frac{n\pi a}{l}\right) \qquad \boxed{\cos 0 = 1}$$

$$V_0\sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n\frac{n\pi a}{l}\sin\frac{n\pi x}{l}$$

$$V_0\left\{\frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]\right\} = \sum_{n=1}^{\infty} b_n\frac{n\pi a}{l}\sin\frac{n\pi x}{l} \qquad \boxed{\sin^3\theta = }\frac{1}{4}[3\sin\theta - \sin 3\theta]$$

$$\frac{3V_0}{4}\sin\frac{\pi x}{l} - \frac{V_0}{4}\sin\frac{3\pi x}{l} = b\frac{\pi a}{l}\sin\frac{\pi x}{l} + b\frac{2\pi a}{l}\sin\frac{2\pi x}{l} + b\frac{3\pi a}{l}\sin\frac{3\pi x}{l} + b\frac{4\pi a}{l}\sin\frac{4\pi x}{l} + \ldots$$

Equating co-efficients of likely terms on both sides

$_1$ $\frac{\pi a}{}$ $\frac{3V_0}{}$ $_2$ $\frac{2\pi a}{}$ $_3$ $\frac{3\pi a}{}$ $\frac{-V_0}{}$ $_4$ $_5$ $_6$

$_1$ 3V l $_2$ $_3$ $-V\ l$ $_4$ $_5$ $_6$

Sub these values in (4)

$_1$ $\frac{\pi x}{}$ $\frac{\pi a t}{}$ $_2$ $\frac{2\pi x}{}$ $\frac{2\pi a t}{}$ $_3$ $\frac{3\pi x}{}$ $\frac{3\pi a t}{}$

$$y(x,t) = \frac{3V_0 l}{4\pi a}\sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} - \frac{V_0 l}{12\pi a}\sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l}$$

One Dimensional Heat Equation

1. One dimensional heat equation is

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} \text{ where } \alpha^2 = \frac{k}{\rho c} = \frac{\text{Thermal conductivity}}{\text{density} \times \text{Specific heat capacity}}$$

2. $u(x,t) \to$ The temperature distribution at any point x from one end at time t.

3. The various Possible Solution of 1-D heat equation.

(i) $u(x,t) = (Ae^{px} + Be^{-px})Ce^{\alpha^2 p^2 t}$

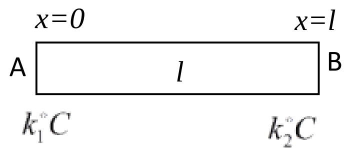
(ii) $u(x,t) = (A\cos px + B\sin px)Ce^{-\alpha^2 p^2 t}$

(iii) $u(x,t) = Ax + B$

4. The boundary and initial conditions.

i) $u(0,t) = k^\circ C$

iii) $u(x,0) = f(x)$



5. The correct solution is $u(x,t) = (A\cos px + B\sin px)Ce^{-\alpha^2 p^2 t}$

6. The steady state solution in 1-D heat equation:

Solution:

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} - - - - - - (1)$$

In steady state $t=0$ then $\frac{\partial u}{\partial t} = 0$

$$\alpha^2\frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{d^2 u}{dx^2} = 0$$

Integrating twice we get $\boxed{u(x) = Ax + B}$

PART - B

1. **A rod of length *l* has its ends *A* and *B* are kept at $0^\circ C$ and $100^\circ C$ until steady state condition prevail. If the temperature at *B* is reduced suddenly to $0^\circ C$ and kept so while that of *A* is maintained. Find the temperature $u(x,t)$ at a distance x from *A* and at time *t*.**

Solution:

The 1-D heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ where $\alpha^2 = \frac{k}{\rho c}$

<u>To find steady state solution</u> $u(x,0) = u(x)$

In steady state $t=0$ then $\frac{\partial u}{\partial t} = 0$

$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{d^2 u}{dx^2} = 0$

Integrating twice we get $\boxed{u(x) = Ax + B}$ ----(1)

The boundary conditions are i) $u(0) = 0$ °C ii) $u(l) = 100$ °C

Applying condn (i) in (1)

$(1) \Rightarrow u(0) = 0 + B \Rightarrow \boxed{B = 0}$

Sub B in (1)

$u(x) = Ax$ ----(2)

Applying condn (ii) in (2)

$u(l) = Al \Rightarrow 100 = Al \Rightarrow \boxed{A = \frac{100}{l}}$

Sub A in (2)

$\boxed{u(x) = \frac{100x}{l}}$

The boundary and initial conditions are

i) $u(0,t) = 0$ °C

ii) $u(l,t) = 100$ °C

iii) $u(x,0) = f(x) = \frac{100x}{l}, \ 0 \le x \le l$

The correct solution is

$u(x,t) = (A\cos px + B\sin px)Ce^{-\alpha^2 p^2 t}$ ------(1)

Apply condn. (i) in (1)

$u(0,t) = (A\cos 0 + \cancel{B\sin 0})Ce^{-\alpha^2 p^2 t}$

$0 = ACe^{-\alpha^2 p^2 t}$

Here $C \ne 0, \ e^{-\alpha^2 p^2 t} \ne 0 \ \therefore \boxed{A = 0}$

Sub A in (1)

$u(x,t) = (B\sin px)Ce^{-\alpha^2 p^2 t}$ ------(2)

Apply condn. (ii) in (2)

$u(l,t) = (B\sin pl)Ce^{-\alpha^2 p^2 t}$

$0 = (B\sin pl)Ce^{-\alpha^2 p^2 t}$

Here $B \ne 0, C \ne 0, e^{-\alpha^2 p^2 t} \ne 0 \ \therefore \boxed{\sin pl = 0}$

$\sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$

Sub p in (2)

$$u(x,t) = \left(B \sin \frac{n\pi x}{l} \right) C e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

$$u(x,t) = b_1 \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}} \qquad BC = b_1 (\text{say})$$

The most general solution is

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}} \qquad ------ (3)$$

Apply condn (iii) in (3)

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-0}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad \boxed{e^{-0} = 1}$$

This is Fourier sine series of *f(x)* in *(0,l)*

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l \frac{100x}{l} \sin \frac{n\pi x}{l} dx$$

$$= \frac{200}{l^2} \int_0^l x \sin \frac{n\pi x}{l} dx$$

$$= \frac{200}{l^2} \left[(x) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (1) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2 \pi^2}{l^2}} \right) \right]_0^l$$

$$= \frac{200}{l^2} \left(\frac{-l}{n\pi} \right) \left[x \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{-200}{l n\pi} [l \cos n\pi - 0]$$

$$= \frac{-200(-1)^n}{n\pi}$$

$$\boxed{b_n = \frac{200}{n\pi} (-1)^{n+1}}$$

Sub b_n in (3)

$$u(x,t) = \sum_{n=1}^{\infty} \frac{200}{n\pi} (-1)^{n+1} \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

$$\boxed{u(x,t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}}$$

This is the required temperature.

3. **The ends A and B of a rod *l* cm long have their temperatures kept at $30^\circ C$ and $80^\circ C$, until steady state conditions prevail. The temperature of the end B is suddenly reduced to $60^\circ C$ and that of A is increased to $40^\circ C$. Find the steady state temperature distribution in the rod after time t.**

Solution:

The 1-D heat equation is $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ where $\alpha^2 = \frac{k}{\rho c}$

To find steady state solution 1 $u(x,0) = u(x)$

In steady state $t=0$ then $\frac{\partial u}{\partial t} = 0$

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 0 \Rightarrow \frac{d^2 u}{dx^2} = 0$$

Integrating twice we get $\boxed{u(x) = Ax + B}$ $----(1)$

The boundary conditions are i) $u(0) = 30^\circ C$ ii) $u(l) = 80^\circ C$

Applying condn (i) in (1)

$(1) \Rightarrow u(0) = 0 + B \Rightarrow \boxed{B = 30}$

Sub B in (1)

$u(x) = Ax + 30 ----(2)$

Applying condn (ii) in (2)

$u(l) = Al + 30 \Rightarrow 80 = Al + 30 \Rightarrow \boxed{A = \frac{50}{l}}$

Sub A in (2)

$u(x) = \frac{50x}{l} + 30 = f(x)$

This $u(x)$ will be treated as the initial conditions $u(x,0) = f(x)$

To find steady state solution 2 $u(x,0) = u(x)$

Integrating twice we get $\boxed{u_t(x) = Ax + B}$ $----(3)$

The boundary conditions are i) $u_t(0) = 40^\circ C$ ii) $u_t(l) = 60^\circ C$

Applying condn (i) in (3)

$(3) \Rightarrow u_t(0) = 0 + B \Rightarrow \boxed{B = 40}$

Sub B in (1)

$u_t(x) = Ax + 40 ----(4)$

Applying condn (ii) in (4)

$u_t(l) = Al + 40 \Rightarrow 60 = Al + 40 \Rightarrow \boxed{A = \frac{20}{l}}$

Sub A in (2)

$u_t(x) = \frac{20x}{l} + 40$

This $u_t(x)$ will be treated as the transient state temperature.

The required temperature is

$$u(x,t) = u_t(x,0) + (A\cos px + B\sin px)Ce^{-\alpha^2 p^2 t}$$

$$u(x,t) = \frac{20x}{l} + 40 + (A\cos px + B\sin px)Ce^{-\alpha^2 p^2 t} -----(5)$$

The boundary and initial conditions are

i) $u(0,t) = 40\,^{\circ}C$

ii) $u(l,t) = 60\,^{\circ}C$

iii) $u(x,0) = f(x) = \dfrac{50x}{l} + 30,\ 0 \le x \le l$

Apply condn (i) in (5)

$$u(0,t) = 0 + 40 + (A\cos 0 + B\sin 0)Ce^{-\alpha^2 p^2 t}$$

$$40 = 0 + 40 + (A\cos 0 + \cancel{B\sin 0})Ce^{-\alpha^2 p^2 t}$$

$$0 = ACe^{-\alpha^2 p^2 t}$$

This $C \neq 0, e^{-\alpha^2 p^2 t} \neq 0 \therefore \boxed{A = 0}$

Sub A in (5)

$$u(x,t) = \frac{20x}{l} + 40 + (B\sin px)Ce^{-\alpha^2 p^2 t} ------(6)$$

Apply condn (ii) in (6)

$$u(l,t) = 20 + 40 + (B\sin pl)Ce^{-\alpha^2 p^2 t}$$

$$60 = 20 + 40 + (B\sin pl)Ce^{-\alpha^2 p^2 t}$$

$$0 = (B\sin pl)Ce^{-\alpha^2 p^2 t}$$

$$B \neq 0, C \neq 0, e^{-\alpha^2 p^2 t} \neq 0 \quad \therefore \sin pl = 0$$

$$\sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub p in (6)

$$u(x,t) = \frac{20x}{l} + 40 + \left(B\sin\frac{n\pi x}{l}\right)Ce^{-\frac{\alpha^2 n^2 \pi^2 t}{l}}$$

$$u(x,t) = \frac{20x}{l} + 40 + b_1 \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}} \qquad \boxed{BC = b_1}$$

The most general solution is

$$u(x,t) = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}} ------(7)$$

Apply condn (iii) in (7)

$$u(x,0) = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} e^{-0}$$

$$\frac{50x}{l} + 30 = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\frac{30x}{l} - 10 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

To find b_n:

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{\cancel{l}}\int_0^l \left(\frac{30x}{\cancel{l}} - 10\right)\sin\frac{n\pi x}{\cancel{l}}dx$$

$$= \frac{2}{l}\left[\left(\frac{30x}{l}-10\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(\frac{30}{l}\right)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left(\frac{-l}{n\pi}\right)\left(\frac{30x}{l}-10\right)\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{-2}{n\pi}\left[(20)\cos n\pi + 10\right]$$

$$\boxed{b_n = \frac{-20}{n\pi}\left[2(-1)^n + 1\right]}$$

Sub b_n in (7)

$$u(x,t) = \frac{20x}{l} + 40 + \sum_{n=1}^{\infty} \frac{-20}{n\pi}\left[2(-1)^n + 1\right]\sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}}$$

$$\boxed{u(x,t) = \frac{20x}{l} + 40 - \frac{20}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left[2(-1)^n + 1\right]\sin\frac{n\pi x}{l}\, e^{-\frac{\alpha^2 n^2 \pi^2 t}{l}}}$$

TWO DIMENSIONAL HEAT EQUATION (LAPLACE EQUATION)

1) The 2-D heat equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

2) The various possible solution of 2-D heat equation is

i) $u(x,y) = (A\cos px + B\sin px)(C e^{py} + e^{-py})$

ii) $u(x,y) = (Ae^{px} + Be^{-px})(C\cos py + \sin py)$

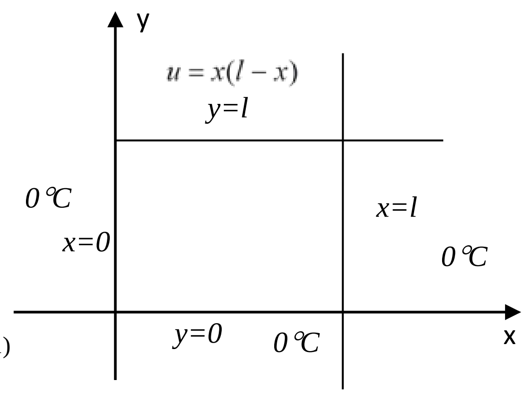
iii) $u(x,y) = (Ax+B)(Cy+D)$

1. **A square plate is bounded by the lines $x = 0, x = l, y = 0$ and $y = l$, its faces are insulated. The temperature along upper horizontal edge is given by $u = x(l-x)$ when $0 < x < l$. while the other three edges are kept at $0^\circ C$. Find steady state solution in the plate.**

Solution:

The 2-D heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



The Boundary conditions are

i) $u(0,y) = 0$

ii) $u(l,y) = 0$

iii) $u(x,0) = 0$

iv) $u(x,l) = f(x) = x(l-x), \quad 0 \le x \le l$

Here the non zero temperature is parallel to x axis then the Correct solution is

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \quad - - - - - (1)$$

Apply condn (i) in (1)

$$u(0,y) = (A\cos 0 + B\sin 0)(Ce^{py} + De^{-py})$$

$$0 = A(Ce^{py} + De^{-py})$$

Here $Ce^{py} + De^{-py} \neq 0 \quad \therefore A = 0$

Sub A in (1)

$$u(x,y) = (B\sin px)\left(Ce^{py} + De^{-py}\right) \quad -----(2)$$

Apply condn (ii) in (2)

$$u(l,y) = (B\sin pl)\left(Ce^{py} + De^{-py}\right)$$

$$0 = (B\sin pl)\left(Ce^{py} + De^{-py}\right)$$

Here $Ce^{py} + De^{-py} \neq 0,\ B \neq 0 \ \therefore \sin pl = 0$

$$\sin pl = \sin n\pi \ \Rightarrow\ pl = n\pi \ \Rightarrow\ p = \frac{n\pi}{l}$$

Sub p in (2)

$$u(x,y) = \left(B\sin\frac{n\pi x}{l}\right)\left(Ce^{\frac{n\pi y}{l}} + De^{-\frac{n\pi y}{l}}\right) \quad -----(3)$$

Apply condn (iii) in (3)

$$u(x,0) = \left(B\sin\frac{n\pi x}{l}\right)\left(Ce^{0} + De^{-0}\right)$$

$$0 = \left(B\sin\frac{n\pi x}{l}\right)(C + D)$$

Here $\sin\frac{n\pi x}{l} \neq 0, B \neq 0,\ \therefore C + D = 0 \ \Rightarrow\ D = -C$

Sub $D = -C$ in (3)

$$u(x,y) = \left(B\sin\frac{n\pi x}{l}\right)\left(Ce^{\frac{n\pi y}{l}} - Ce^{-\frac{n\pi y}{l}}\right)$$

$$u(x,y) = BC\sin\frac{n\pi x}{l}\left(e^{\frac{n\pi y}{l}} - e^{-\frac{n\pi y}{l}}\right)$$

$$u(x,y) = BC\sin\frac{n\pi x}{l}\left(2\sinh\frac{n\pi y}{l}\right) \qquad \frac{e^{\theta} - e^{-\theta}}{2} = \sinh\theta$$

$$u(x,y) = b_1\sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l} \qquad let\ 2BC = b_1$$

The most general Solution is

$$u(x,y) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}\sinh\frac{n\pi y}{l} \quad -----(4)$$

Apply condn (iv) in (4)

$$u(x,l) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l}\sinh n\pi$$

$$f(x) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l} \qquad let\ B_n = b_n\sinh n\pi$$

This is Fourier sine series in *(0,l)*

$$B_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l x(l - x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{-2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{-4l^3}{ln^3\pi^3}\left[\cos n\pi - \cos 0\right]$$

$$b_n \sinh n\pi = \frac{-4l^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

$$b_n = \frac{-4l^2}{n^3\pi^3 \sinh n\pi}\left[(-1)^n - 1\right]$$

$$b_n = \begin{cases} 0 & if\ n\ is\ even \\ \frac{8l^2}{n^3\pi^3 \sinh n\pi} & if\ n\ is\ odd \end{cases}$$

Sub b_n in (4)

$$u(x,y) = \sum_{n=1,3,5...}^{\infty} \frac{8l^2}{n^3\pi^3 \sinh n\pi} \sin\frac{n\pi x}{l} \sinh\frac{n\pi y}{l}$$

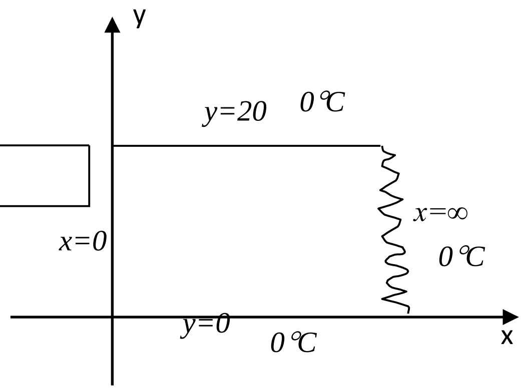
$$u(x,y) = \frac{8l^2}{\pi^3} \sum_{n=1,3,5...}^{\infty} \frac{1}{n^3 \sinh n\pi} \sin\frac{n\pi x}{l} \sinh\frac{n\pi y}{l}$$

2. **A rectangular plate with insulated surface is 20cm wide and so long compared to its width that it may be considered infinite in length without introducing appreciable error. The temperature at short edge $x = 0$ is given by** $u = \begin{cases} 10y, & 0 \le y \le 10 \\ 10(20-y), & 10 \le y \le 20 \end{cases}$ **and the two long edges as well as the other short edges are kept at** $0^\circ C$**. Find the steady state temperature distribution in the plate.**

Solution:

The 2-D heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



The Boundary conditions are

$i)\ u(x,0) = 0$

$ii)\ u(x,20) = 0$

$iii)\ u(\infty, y) = 0$

$$iv)\ u(0,y) = \begin{cases} 20y, & 0 \le y \le 10 \\ 20(20-y), & 10 < y \le 20 \end{cases}$$

The correct solution is

$$u(x,y) = \left(Ae^{px} + Be^{-px}\right)\left(C\cos py + D\sin py\right) -----(1)$$

Apply condn (i) in (1)

$$u(x,0) = \left(Ae^{px} + Be^{-px}\right)\left(C\cos 0 + D\sin 0\right)$$

$$0 = \left(Ae^{px} + Be^{-px}\right)C$$

Here $\left(Ae^{px} + Be^{-px}\right) \neq 0$, $\boxed{\therefore C = 0}$

Sub C in (1)

$$u(x,y) = \left(Ae^{px} + Be^{-px}\right)\left(D\sin py\right) - - - - - (2)$$

Apply condn (ii) in (2)

$$u(x,20) = \left(Ae^{px} + Be^{-px}\right)\left(D\sin 20p\right)$$

$$0 = \left(Ae^{px} + Be^{-px}\right)\left(D\sin 20p\right)$$

Here $\left(Ae^{px} + Be^{-px}\right) \neq 0$, $D \neq 0$

$$\therefore \sin 20p = 0 \Rightarrow \sin 20p = \sin n\pi \Rightarrow 20p = n\pi \Rightarrow \boxed{p = \frac{n\pi}{20}}$$

Sub p in (2)

$$(2) \Rightarrow u(x,y) = \left(Ae^{\frac{n\pi x}{20}} + Be^{-\frac{n\pi x}{20}}\right)\left(D\sin\frac{n\pi y}{20}\right) - - - - - (3)$$

Apply codn (iii) in (3)

$$u(\infty,y) = \left(Ae^{\infty} + Be^{-\infty}\right)\left(D\sin\frac{n\pi y}{20}\right)$$

$$0 = \left(Ae^{\infty}\right)\left(D\sin\frac{n\pi y}{20}\right)$$

temperature $e^{\infty} \neq 0, D \neq 0, \sin\frac{n\pi y}{20} \neq 0, \therefore$ $\boxed{A = 0}$

sub A in (3)

$$(3) \Rightarrow u(x,y) = \left(Be^{-\frac{n\pi x}{20}}\right)\left(D\sin\frac{n\pi y}{20}\right)$$

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{20} e^{-\frac{n\pi x}{10}} - - - - - - - - -(4)$$

Apply condn (iv) in (4)

$$u(0,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{20} e^{-0}$$

$$f(y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{20}$$

This is half range sine series in (0,20)

$$b_n = \frac{2}{l}\int_0^l f(y)s\sin\frac{n\pi y}{l}dy$$

$$= \frac{2}{20}\int_0^{20} f(y)\sin\frac{n\pi y}{20}dy$$

$$= \frac{1}{10}\left[\int_0^{10} 10y\sin\frac{n\pi y}{20}dy + \int_{10}^{20} 10(20-y)\sin\frac{n\pi y}{20}dy\right]$$

$$= \frac{10}{10}\left[\int_0^{10} y\sin\frac{n\pi y}{20}dy + \int_{10}^{20} (20-y)\sin\frac{n\pi y}{20}dy\right]$$

$$= \left[(y)\left(\frac{-\cos\frac{n\pi y}{20}}{\frac{n\pi}{20}} \right) - (1)\left(\frac{-\sin\frac{n\pi y}{20}}{\frac{n^2\pi^2}{400}} \right) \right]_0^{10} + \left[(20-y)\left(\frac{-\cos\frac{n\pi y}{20}}{\frac{n\pi}{20}} \right) - (-1)\left(\frac{-\sin\frac{n\pi y}{20}}{\frac{n^2\pi^2}{400}} \right) \right]_{10}^{20}$$

$$= \left[-\frac{20}{n\pi}(y)\cos\frac{n\pi y}{20} + \frac{400}{n^2\pi^2}\sin\frac{n\pi y}{20} \right]_0^{10} + \left[-\frac{20}{n\pi}(20-y)\cos\frac{n\pi y}{20} - \frac{400}{n^2\pi^2}\sin\frac{n\pi y}{20} \right]_{10}^{20}$$

$$= \left[\left(-\frac{20}{n\pi}(10)\cos\frac{n\pi}{2} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2} \right) - (0) \right] + \left[(0) - \left(-\frac{20}{n\pi}(10)\cos\frac{n\pi}{2} - \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2} \right) \right]$$

$$= -\frac{200}{n\pi}\cos\frac{n\pi}{2} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{200}{n\pi}\cos\frac{n\pi}{2} + \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\boxed{b_n = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}}$$

Sub b_n in (4)

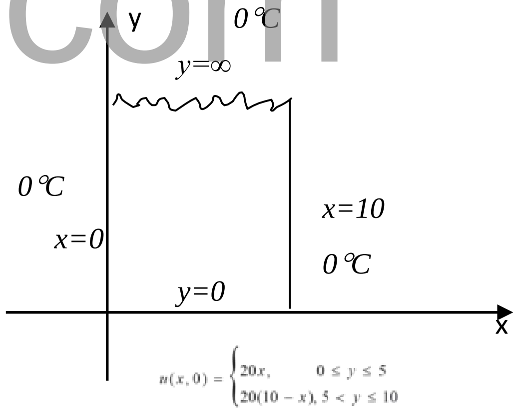
$$\sum^{\infty} \frac{800}{n\pi}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{20} e^{\frac{n\pi x}{10}} \qquad \boxed{u(x,y) = \frac{800}{n^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi y}{20} e^{-\frac{n\pi x}{10}}}$$

3. **An infinitely long rectangular plate is of width 10cm. The temperature along the short edge *y=0* is given by** $u = \begin{cases} 20x, & 0 \le x \le 5 \\ 20(10-x), & 5 \le x \le 10 \end{cases}$ **. If all the other edges are kept at zero temperature. Find the steady state temperature at any point on it.**

Solution:

The 2-D heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$



The Boundary conditions are

$i)\ u(0,y) = 0$

$ii)\ u(10,y) = 0$

$iii)\ u(x,\infty) = 0$

$iv)\ u(x,0) = \begin{cases} 20x, & 0 \le x \le 5 \\ 20(10-x), & 5 < x \le 10 \end{cases}$

Here the non zero boundary condition is parallel to x axis then
The correct solution is

$$u(x,y) = (A\cos px + B\sin px)(Ce^{py} + De^{-py}) \quad ------(1)$$

Apply condn (i) in (1)

$$u(0,y) = (A\cos 0 + B\sin 0)(Ce^{py} + De^{-py})$$

$$0 = A(Ce^{py} + De^{-py})$$

Here $(Ce^{py} + De^{-py}) \neq 0, \ \therefore \boxed{A = 0}$

Sub C in (1)

$$u(x,y) = (B\sin px)(Ce^{py} + De^{-py}) \quad -----(2)$$

Apply condn (ii) in (2)

$$u(10,y) = (B\sin 10p)(Ce^{py} + De^{-py})$$

$$0 = (B\sin 10p)(Ce^{py} + De^{-py})$$

Here $B \neq 0, (Ce^{py} + De^{-py}) \neq 0 \ \therefore \sin 10p = 0$

$$\therefore \sin 10p = 0 \Rightarrow \sin 10p = \sin n\pi \ \Rightarrow 10p = n\pi \ \Rightarrow \boxed{p = \frac{n\pi}{10}}$$

Sub p in (2)

$$u(x,y) = \left(B\sin\frac{n\pi x}{10}\right)\left(Ce^{\frac{n\pi y}{10}} + De^{-\frac{n\pi y}{10}}\right) - - - - - (3)$$

Apply codn (iii) in (3)

$$u(x,\infty) = \left(B\sin\frac{n\pi x}{10}\right)(Ce^{\infty} + De^{-\infty})$$

$$0 = \left(B\sin\frac{n\pi x}{10}\right)(Ce^{\infty} + De^{-\infty})$$

Here $B \neq 0, e^{\infty} \neq 0, \sin\frac{n\pi x}{10} \neq 0 \ \therefore C = 0$

sub C in (3)

$$u(x,y) = \left(B\sin\frac{n\pi x}{10}\right)\left(De^{-\frac{n\pi y}{10}}\right)$$

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} e^{-\frac{n\pi y}{10}} - - - - - (4)$$

Apply condn (iv) in (4)

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} e^{-0}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10}$$

This is half range sine series in (0,10)

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi x}{10}dx$$

$$= \frac{1}{10}\left[\int_0^5 20x\sin\frac{n\pi x}{20}dx + \int_5^{10} 20(10-x)\sin\frac{n\pi x}{20}dx\right]$$

$$= \frac{20}{10}\left[\int_0^5 x\sin\frac{n\pi x}{10}dx + \int_5^{10}(10-x)\sin\frac{n\pi x}{10}dx\right]$$

$$= 2\left\{\left[(x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right]_0^5 + \left[(10-x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}}\right)\right]_5^{10}\right\}$$

$$= 2\left\{\left[-\frac{10}{n\pi}(x)\cos\frac{n\pi x}{10} + \frac{100}{n^2\pi^2}\sin\frac{n\pi x}{10}\right]_0^5 + \left[-\frac{10}{n\pi}(10-x)\cos\frac{n\pi x}{10} - \frac{100}{n^2\pi^2}\sin\frac{n\pi x}{10}\right]_5^{10}\right\}$$

	$= 2\left\{\left[\left(-\frac{10}{n\pi}(5)\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right) - (0)\right] + \left[(0) - \left(-\frac{10}{n\pi}(5)\cos\frac{n\pi}{2} - \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]\right\}$ $= 2\left[-\frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$ $b_n = \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}$ Sub b_n in (4) $u(x,y) = \sum_{n=1}^{\infty}\frac{400}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}e^{-\frac{n\pi y}{10}} \Rightarrow$ $\boxed{u(x,y) = \frac{400}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{10}e^{-\frac{n\pi y}{10}}}$
4.	**An infinite long rectangular plate with insulated surfaces is 10 cm wide. The two long edges and one short edge are kept at 0°C, while the other short edge x = 0 is kept at temperature** $u = \begin{cases} 20y, & 0 \le y \le 5 \\ 20(10-y), & 5 < y \le 10 \end{cases}$. **Find the steady state temperature distribution in the plate.** Solution: The 2-D heat equation is $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ The Boundary conditions are $i)\ u(x,0) = 0$ $ii)\ u(x,10) = 0$ $iii)\ u(\infty, y) = 0$ $iv)\ u(0,y) = \begin{cases} 20y, & 0 \le y \le 5 \\ 20(10-y), & 5 < y \le 10 \end{cases}$ The correct solution is $u(x,y) = \left(Ae^{px} + Be^{-px}\right)\left(C\cos py + D\sin py\right) - - - - - (1)$ Apply condn (i) in (1) $u(x,0) = \left(Ae^{px} + Be^{-px}\right)\left(C\cos 0 + D\sin 0\right)$ $0 = \left(Ae^{px} + Be^{-px}\right)C$ Here $\left(Ae^{px} + Be^{-px}\right) \ne 0, \ \therefore C = 0$ Sub C in (1) $u(x,y) = \left(Ae^{px} + Be^{-px}\right)\left(D\sin py\right) - - - - - (2)$ Apply condn (ii) in (2) $u(x,10) = \left(Ae^{px} + Be^{-px}\right)\left(D\sin 10p\right)$ $0 = \left(Ae^{px} + Be^{-px}\right)\left(D\sin 10p\right)$ Here $\left(Ae^{px} + Be^{-px}\right) \ne 0,\ D \ne 0$ $\therefore \sin 10p = 0 \Rightarrow \sin 10p = 0\sin n\pi \Rightarrow 10p = n\pi \Rightarrow p = \frac{n\pi}{10}$ Sub p in (2) $(2) \Rightarrow u(x,y) = \left(Ae^{\frac{n\pi x}{10}} + Be^{-\frac{n\pi x}{10}}\right)\left(D\sin\frac{n\pi y}{10}\right) - - - - - (3)$ Apply codn (iii) in (3)

$$u(\infty, y) = \left(Ae^{\infty} + Be^{-\infty}\right)\left(D\sin\frac{n\pi y}{10}\right)$$

$$0 = \left(Ae^{\infty}\right)\left(D\sin\frac{n\pi y}{10}\right)$$

temperature $e^{\infty} \neq 0,\ D \neq 0,\ \sin\frac{n\pi y}{10} \neq 0,\ \therefore A = 0$

sub A in (3)

$$(3) \Rightarrow u(x,y) = \left(Be^{-\frac{n\pi x}{10}}\right)\left(D\sin\frac{n\pi y}{10}\right)$$

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{10}\, e^{-\frac{n\pi x}{10}} \quad --------(4)$$

Apply condn (iv) in (4)

$$u(0,y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{10}\, e^{-0}$$

$$f(y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi y}{10}$$

Which is half range sine series in (0,10)

$$b_n = \frac{2}{l}\int_0^l f(y)s\sin\frac{n\pi y}{10}\,dy$$

$$b_n = \frac{2}{10}\int_0^{10} f(y)s\sin\frac{n\pi y}{10}\,dy$$

$$b_n = \frac{2}{10}\left[\int_0^5 20y\sin\frac{n\pi y}{10}\,dy + \int_5^{10} 20(10-y)\sin\frac{n\pi y}{10}\,dy\right]$$

$$b_n = \frac{40}{10}\left[\int_0^5 y\sin\frac{n\pi y}{10}\,dy + \int_5^{10}(10-y)\sin\frac{n\pi y}{10}\,dy\right]$$

$$= \frac{20}{5}\left[\left(y\left(\frac{-\cos\frac{n\pi y}{10}}{\frac{n\pi}{10}}\right) - (1)\left(\frac{-\sin\frac{n\pi y}{10}}{\frac{n^2\pi^2}{100}}\right)\right)_0^5 + \left((10-x)\left(\frac{-\cos\frac{n\pi y}{10}}{\frac{n\pi}{10}}\right) - (-1)\left(\frac{-\sin\frac{n\pi y}{10}}{\frac{n^2\pi^2}{100}}\right)\right)_5^{10}\right]$$

$$b_n = 4\left[\frac{-50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{50}{n\pi}\cos\frac{n\pi}{2} + \frac{100}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$b_n = \frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Sub b_n in (4)

$$u(x,y) = \sum_{n=1}^{\infty}\frac{800}{n^2\pi^2}\sin\frac{n\pi}{2}\,\sin\frac{n\pi y}{10}\,e^{-\frac{n\pi x}{10}}$$

$$u(x,y) = \frac{800}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\,\sin\frac{n\pi y}{10}\,e^{-\frac{n\pi x}{10}}$$