DEPARTMENT OF MATHEMATICS

NAME OF THE SUBJECT : TRANSFORMS & PARTIAL DIFFERENTIAL EQUATION

SUBJECT CODE : MA8353

REGULATION : 2017

UNIT – II : FOURIER SERIES

SOME IMPORTANT DIFFERENTIATION AND INTEGRATION FORMULAE:

1. $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$
2. $\frac{d}{dx}\left(e^{ax}\right) = ae^{ax}$
3. $\frac{d}{dx}(\cos nx) = -n\sin nx$
4. $\frac{d}{dx}(\sin nx) = n\cos nx$
5. $\int x^n dx = \frac{x^{n+1}}{n+1} + c \ , n \neq -1$
6. $\int e^{ax} dx = \frac{e^{ax}}{a} + c$
7. $\int \cos nx\, dx = \frac{\sin nx}{n} + c$
8. $\int \sin nx\, dx = \frac{-\cos nx}{n} + c$
9. **Bernoulli's formula:**
   $\int u\, v\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + ...$
   Here $u', u'', u''', ...$ are successive Differentiation.
   And $v_1, v_2, v_3, ...$ are successive Integration.
10. $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$
11. $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$
12. $\int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2+b^2}$
13. $\int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2+b^2}$
14. $\int_{-a}^{a} f(x)dx = \begin{cases} 0 & , \text{ if } f(x) \text{ is odd} \\ 2\int_0^a f(x)\, dx, & \text{if } f(x) \text{ is even} \end{cases}$
15. $\cos n\pi = (-1)^n \Rightarrow \cos n\pi = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$
16. $\sin n\pi = 0 \text{ if } n = 1, 2, 3, 4, ...$
17. $\sin 0 = 0 \ \& \cos 0 = 1$
18. $\cos(-\theta) = \cos\theta \ \& \sin(-\theta) = -\sin\theta$
19. $\sin(A+B) = \sin A\cos B + \cos A\sin B$
20. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
21. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
22. $\cos(A-B) = \cos A\cos B + \sin A\sin B$
23. $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
24. $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
25. 25. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
26. 26. $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

EVEN & ODD FUNCTIONS

1) If f(–x)= f(x), then f(x) is said to be even function.

2) If f(–x)= – f(x), then f(x) is said to be odd function.

3) Let $f(x) = \begin{cases} f_1(x) & -\pi < x < 0 \\ f_2(x) & 0 < x < \pi \end{cases}$

If $f_1(-x) = f_2(x)$ and $f_2(-x) = f_1(x)$ then *f(x)* is said to be an even function

If $f_1(-x) = -f_2(x)$ and $f_2(-x) = -f_1(x)$ then *f(x)* is said to be an odd function

GENERAL FOURIER SERIES

Problem Identification

Full Range

Full Range $l = \dfrac{\text{Upper Limit-Lower Limit}}{2}$

For eg: If the Question starts with Find the Fourier series (or) Determine the Fourier Series (or) Obtain the Fourier series, it is full range

Half Range

Half Range $l = \text{Upper Limit-Lower Limit}$

For eg: If the Question starts with Find the Half range Fourier cosine series (or) Half Range Fourier sine Series (or) Find the cosine series (or) Find the cosine series, it is Half range.

Cosine Series B2

For eg. $(0,l),(0,\pi),(0,1),(0,2)$....

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}$$

$$a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)\,dx$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

Sine Series B3

For eg. $(0,l),(0,\pi),(0,1),(0,2)$....

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

Full Range cases

If Interval Starts with Zero then we use Formula **A**

For eg. $(0,2l),(0,2\pi),(0,1),(0,2)$....

Formula A

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

$$a_0 = \frac{1}{\ell}\int_0^{2\ell} f(x)\,dx$$

$$a_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

$$b_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

If Interval Starts with (-) then we should check whether the given function is odd or Even function.

For eg. $(-l,l),(-\pi,\pi),(-1,1),(-2,2)$....

Formula B1

If the given function is **Neither Even Nor Odd** then we use **Formula B1**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

Formula B2

If the given function is **Even** then we use **Formula B2**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}$$

$$a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)\,dx$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

$$b_n = 0$$

Formula B3

If the given function is **Odd** then we use **Formula B3**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

$$a_0 = a_n = 0$$

1. Convergence of Fourier series : (For deduction purpose)

Full range:

i) Let $x = a$ be a point of continuity then the Fourier series converges to $f(a)$

ii) Let $x = a$ be a point of discontinuity then the Fourier series converges to averages of the end points *i.e.*, $f(a) = \dfrac{f(a-) + f(a+)}{2}$.

iii) At end point c or c+2*l* in (c, c+2*l*), Fourier series converges to $\dfrac{f(c) + f(c+2\ell)}{2}$

Half Range:

<u>Convergence of Fourier Cosine Series</u> :

- If x = a be end point or mid point of the given interval then Fourier Cosine series converges to **f(a).**

<u>Convergence of Fourier Sine Series</u> :

- At **inner point** x = a Fourier Sine series converges to **f(a).**
- At both **end points** Fourier Sine series converges to **0**.

2. Parseval's Identity: (For deduction purpose)

Full Range:

i) Parseval's Identity in the interval $(0,2l)$: $\dfrac{1}{\ell}\displaystyle\int_0^{2\ell} [f(x)]^2\,dx = \dfrac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2$

ii) Parseval's Identity in the interval $(-l,l)$: $\dfrac{1}{\ell}\displaystyle\int_{-\ell}^{\ell} [f(x)]^2\,dx = \dfrac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2$

Half range:

i) Parseval's Identity Cosine series: $\dfrac{2}{\ell}\displaystyle\int_0^{\ell} [f(x)]^2\,dx = \dfrac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$

ii) Parseval's Identity Cosine series: $\dfrac{2}{\ell}\displaystyle\int_0^{\ell} [f(x)]^2\,dx = \sum_{n=1}^{\infty} b_n^2$

NOTE: For the interval $(0, 2\pi)$ or $(-\pi, \pi)$ replace $\ell = \pi$ in above formula

3.

Complex form of the Fourier Series

Full range $l = \dfrac{\text{Upper Limit-Lower Limit}}{2}$

In the interval $(0,2l),(0,2\pi),(0,1),(0,2)....$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\left(\frac{in\pi x}{\ell}\right)}$$

where $c_n = \dfrac{1}{2\ell}\displaystyle\int_0^{2\ell} f(x)\, e^{-\left(\frac{in\pi x}{\ell}\right)} dx$

In the interval $(-l,l),(-\pi,\pi),(-1,1),(-2,2)....$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\left(\frac{in\pi x}{\ell}\right)}$$

where $c_n = \dfrac{1}{2\ell}\displaystyle\int_{-\ell}^{\ell} f(x)\, e^{-\left(\frac{in\pi x}{\ell}\right)} dx$

4. Root Mean Square Value: (RMS Value):

The Root Mean square (**RMS**) value (or) Effective value of the function *y = f(x)* in the interval $a \le x \le b$ is

$$\bar{y} = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2\,dx}$$

5. **Harmonic Analysis:** $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right)$

where $a_0 = \frac{2}{K}\sum y$, $a_n = \frac{2}{K}\sum y\cos\left(\frac{n\pi x}{\ell}\right)$, $b_n = \frac{2}{K}\sum y\sin\left(\frac{n\pi x}{\ell}\right)$

PART A

1. **State Dirichlet's conditions for the existence of Fourier series of *f(x)* in the interval $(0,2\pi)$.**

A function f(x) can be expanded as a Fourier series in the interval $(0,2\pi)$ if the following conditions are satisfied.

(i) f(x) is periodic, single valued and finite in $(0,2\pi)$

(ii) f(x) has only finite number of finite discontinuities and no infinite discontinuities in $(0,2\pi)$.

(iii) f(x) has only finite number of maxima and minima in $(0,2\pi)$

2. **Does $f(x) = \tan x$ possess a Fourier expansion?**

Solution:

$\tan x$ does not possess a Fourier expansion because the function $f(x) = \tan x = \frac{\sin x}{\cos x}$ has the infinite discontinuity at the point $x = \frac{\pi}{2}$.

3. **Determine the value of a_n & a_0 in the Fourier series expansion of $f(x) = x^3$ in $-\pi < x < \pi$**

Solution:

$f(x) = x^3 \Rightarrow f(-x) = (-x)^3 = -x^3 = -f(x) \Rightarrow f(x)$ is an odd function $\therefore a_n = a_0 = 0$

4. **Find the Fourier constant b_n for *x sin x* in $-\pi < x < \pi$, when expressed as a Fourier series.**

Solution:

$f(x) = x\sin x,\ -\pi < x < \pi$

$f(-x) = (-x)\sin(-x) = x\sin x = f(x)$

$\therefore f(x)$ is an even function $\therefore b_n = 0$

5. **Find the constant term of the Fourier series for the function *f(x)* = x^2, $-\pi < x < \pi$**

Solution:

$f(x) = x^2, -\pi < x < \pi$

$f(-x) = (-x)^2 = x^2 = f(x)$

∴*f(x)* is an even function

$l = \frac{U.L - L.L}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} = \pi$

$a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right] = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$

$\therefore$ Constant term $= \frac{a_0}{2} = \frac{\pi^2}{3}$

6. **Find the root mean square value of the function *f(x)* = *x* in *(0,l)***

RMS value = $\bar{y} = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2 dx} = \sqrt{\frac{1}{l-0}\int_0^l x^2 dx} = \sqrt{\frac{1}{l}\left[\frac{x^3}{3}\right]_0^l} = \sqrt{\frac{l^3}{3l}} = \frac{l}{\sqrt{3}}$

7. **What do you mean by Harmonic analysis ?**

The process of finding the harmonics in the Fourier series expansion of a function numerically is known as harmonic analysis.

8. **Find the constant term in the expression of $\cos^2 x$ as a Fourier series in the interval $(-\pi,\pi)$.**

Solution:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 x\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{2\pi}\left[x + \frac{\sin 2x}{2}\right]_{-\pi}^{\pi}\quad a_0 = \frac{1}{2\pi}\left[(\pi)-(-\pi)\right] = \frac{2\pi}{2\pi} = 1$$

$\therefore$ **Constant term =** $\boxed{\frac{a_0}{2} = \frac{1}{2}}$

PART B

1. **Find the Fourier series $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $0 < x < 2\pi$. Hence show that**

(i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots = \frac{\pi^2}{6}$. **(ii)** $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots = \frac{\pi^2}{12}$.

Solution:

Given $f(x) = \left(\frac{\pi - x}{2}\right)^2 = \frac{1}{4}(\pi - x)^2 = \frac{1}{4}\left(\pi^2 - 2\pi x + x^2\right), 0 < x < 2\pi$

General Fourier is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \frac{2\pi}{2} = \pi \quad \therefore \ell = \pi$

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx} \quad ------ (1)$$

To Find a_0:

$$a_0 = \frac{1}{\ell}\int_0^{2\ell} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}\left(\pi^2 - 2\pi x + x^2\right)dx = \frac{1}{4\pi}\int_0^{2\pi}\left(\pi^2 - 2\pi x + x^2\right)dx$$

$$= \frac{1}{4\pi}\left[\pi^2 x - \frac{2\pi x^2}{2} + \frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{4\pi}\left[\left(2\pi^3 - 4\pi^3 + \frac{8\pi^3}{3}\right) - (0)\right]$$

$$= \frac{\pi^3}{4\pi}\left[-2 + \frac{8}{3}\right] = \frac{\pi^2}{4}\left[\frac{-6+8}{3}\right] = \frac{\pi^2}{4}\left[\frac{2}{3}\right]$$

$$\boxed{a_0 = \frac{\pi^2}{6}}$$

To Find a_n

$$a_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}\left(\pi^2 - 2\pi x + x^2\right)\cos nx\,dx = \frac{1}{4\pi}\int_0^{2\pi}\left(\pi^2 - 2\pi x + x^2\right)\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[\left(\pi^2 - 2\pi x + x^2\right)\left(\frac{\sin nx}{n}\right) - (-2\pi + 2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{1}{n^2}(-2\pi + 2x)\cos nx\right]_0^{2\pi}$$

$$= \frac{1}{4\pi n^2}\left[(2\pi\cos 2n\pi) - (-2\pi\cos 0)\right]$$

$$a_n = \frac{1}{4\pi n^2}\left[2\pi + 2\pi\right] = \frac{4\pi}{4\pi n^2} \quad \because \cos 2n\pi = 1 \ \& \ \cos 0 = 1$$

$$\boxed{a_n = \frac{1}{n^2}}$$

To find b_n:

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}\left(\pi^2 - 2\pi x + x^2\right)\sin nx\,dx = \frac{1}{4\pi}\int_0^{2\pi}\left(\pi^2 - 2\pi x + x^2\right)\sin nx\,dx$$

$$= \frac{1}{4\pi}\left[\left(\pi^2 - 2\pi x + x^2\right)\left(\frac{-\cos nx}{n}\right) - (-2\pi + 2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{-1}{n}\left(\pi^2 - 2\pi x + x^2\right)\cos nx + \frac{2}{n^3}\cos nx\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(\frac{-1}{n}\left(\pi^2 - 4\pi^2 + 4\pi^2\right)\cos 2n\pi + \frac{2}{n^3}\cos 2n\pi\right) - \left(\frac{-1}{n}\pi^2\cos 0 + \frac{2}{n^3}\cos 0\right)\right]$$

$$= \frac{1}{4\pi}\left(\frac{-\pi^2}{n} + \frac{-\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right)$$

$$\boxed{b_n = 0}$$

Substitute a_0, a_n, b_n in (1)

$$f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx + \sum_{n=1}^{\infty} 0\sin nx$$

$$\therefore \boxed{f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx} \text{-------}(2)$$

Deduction:

(i) Let $x = 0$ be a point of discontinuity

$$(2) \Rightarrow f(0) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos 0 \qquad f(0) = \frac{f(0) + f(2\pi)}{2} = \frac{\frac{\pi^2}{4} + \frac{\pi^2}{4}}{2} = \frac{2\frac{\pi^2}{4}}{2} = \frac{\pi^2}{4}$$

$$\because f(x) = \left(\frac{\pi - x}{2}\right)^2$$

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{4\pi^2}{48} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \Rightarrow \boxed{\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{6}}$$

ii) Let $x = \pi$ be a point of continuity

$$f(\pi) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi \qquad \because f(x) = \left(\frac{\pi - x}{2}\right)^2 \Rightarrow f(0) = \left(\frac{\pi - \pi}{2}\right)^2 = 0$$

$$0=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12}=\frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\ldots$$

$$\boxed{\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots=\frac{\pi^2}{12}}$$

2. **Obtain the Fourier series for *f(x)* of period *2l* and defined as follows** $f(x)=\begin{cases} l-x & , 0<x\le l \\ 0 & , l\le x<2l\end{cases}$. **Hence deduce that (i)** $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4}$ **(ii)** $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}$

Solution: General Fourier is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$

Here $\ell=\frac{\text{Upper Limit}-\text{Lower Limit}}{2}=\frac{l-(-l)}{2}=\frac{2l}{2}=l \quad \therefore \ell=l$

To find a_0

$$a_0=\frac{1}{\ell}\int_0^{2\ell}f(x)\,dx=\frac{1}{\ell}\left[\int_0^{\ell}(\ell-x)\,dx+\int_{\ell}^{2\ell}0\,dx\right]$$

$$=\frac{1}{\ell}\left[lx-\frac{x^2}{2}\right]_0^{\ell}=\frac{1}{\ell}\left[l^2-\frac{l^2}{2}\right]=\frac{1}{\ell}\left[\frac{l^2}{2}\right]$$

$$\boxed{a_0=\frac{\ell}{2}}$$

$$a_n=\frac{1}{\ell}\int_0^{2\ell}f(x)\cos\frac{n\pi x}{\ell}\,dx=\frac{1}{\ell}\left[\int_0^{\ell}(\ell-x)\cos\frac{n\pi x}{\ell}\,dx+\int_{\ell}^{2\ell}0\,dx\right]$$

$$=\frac{1}{\ell}\left[(\ell-x)\left(\frac{\sin\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right)-(-1)\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right)\right]_0^{\ell}$$

$$=\frac{1}{\ell}\left[\frac{-l^2}{n^2\pi^2}\cos\frac{n\pi x}{\ell}\right]_0^{l}$$

$$=\frac{-\ell^2}{l\,n^2\pi^2}\left[\cos n\pi-\cos 0\right]=\frac{\ell}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$\boxed{a_n=\begin{cases}\frac{2\ell}{n^2\pi^2}, & n=1,3,5,\ldots\\ 0 & , n=2,4,6,\ldots\end{cases}}$$

$$b_n=\frac{1}{\ell}\int_0^{2\ell}f(x)\sin\frac{n\pi x}{\ell}\,dx=\frac{1}{\ell}\left[\int_0^{\ell}(\ell-x)\sin\frac{n\pi x}{\ell}\,dx+\int_{\ell}^{2\ell}0\,dx\right]$$

$$b_n = \frac{1}{l}\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^l$$

$$= \frac{1}{l}\left[\frac{-l}{n\pi}(l-x)\cos\frac{n\pi x}{l}\right]_0^l = \frac{-l}{n\pi}\left[0 - l\cos 0\right] = \frac{l^2}{l n\pi}$$

$$\boxed{b_n = \frac{l}{n\pi}}$$

Substitute a_0, a_n, b_n in (1)

$$f(x) = \frac{\ell}{4} + \sum_{n=1,3,5}^{\infty} \frac{2\ell}{n^2\pi^2}\cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty}\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}$$

$$\therefore\ f(x) = \frac{l}{4} + \frac{2l}{\pi^2}\sum_{n=1,3,5...}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{l} + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l} \quad ------(2)$$

Deduction:

i) Let $x = l/2$ be a point of continuity

$$(2) \Rightarrow f(l/2) = \frac{l}{4} + \frac{2l}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{2} + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2} \qquad f(x) = \begin{cases} l-x, & 0 < x \le l \\ 0, & l \le x < 2l \end{cases}$$

$$\Rightarrow \frac{l}{2} = \frac{l}{4} + \frac{2l}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{2} + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2} \qquad \because f\left(\frac{l}{2}\right) = l - \frac{l}{2} = \frac{l}{2}$$

$$\Rightarrow \frac{l}{2} - \frac{l}{4} = 0 + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2} \qquad \cos\frac{n\pi}{2} = 0 \text{ if n is odd}$$

$$\Rightarrow \frac{l}{4} = \frac{l}{\pi}\left[\frac{1}{1}\sin\frac{\pi}{2} + \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} + ..\right]$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{1}(1) + \frac{1}{2}(0) + \frac{1}{3}(-1) + ... \qquad \sin\frac{\pi}{2} = 1; \sin\frac{3\pi}{2} = -1; \sin\frac{5\pi}{2} = 1; \sin\frac{7\pi}{2} = -1 \text{ etc.}$$

$$\boxed{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}}$$

i) Let $x = \ell$ be a point of continuity

$$(2) \Rightarrow f(l) = \frac{l}{4} + \frac{2l}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\cos n\pi + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi \qquad \because f(x) = \begin{cases} l-x, & 0 < x \le l \\ 0, & l \le x < 2l \end{cases}$$

$$0 = \frac{l}{4} + \frac{2l}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}(-1)^n + 0 \qquad \Rightarrow f(l) = l - l = 0$$

$$-\frac{l}{4} = \frac{2l}{\pi^2}\left[-\frac{1}{1^2} + \frac{1}{3^2} - \frac{1}{5^2} + ...\right]$$

$$-\frac{\pi^2}{8} = -\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} ...\right]$$

$$\boxed{\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - ... = \frac{\pi^2}{8}}$$

3. **Find the Fourier series expansion of** $f(x) = \begin{cases} x, & 0 \le x \le 3 \\ 6-x, & 3 \le x \le 6 \end{cases}$ **and hence find the value**

of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots.$

Solution:

Given $f(x) = \begin{cases} x, & 0 \le x \le 3 \\ 6 - x, & 3 \le x \le 6 \end{cases}$

General Fourier is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{}$

Here $= \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{6-0}{2} = 3 \quad \therefore \quad = 3$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3} \quad ------- (1)$$

$$a_0 = \frac{1}{}\int^{2} f(x)\, dx = \frac{1}{3}\int_0^6 f(x)\, dx = \frac{1}{3}\left[\int_0^3 x\,dx + \int_3^6 (6-x)\,dx\right]$$

$$= \frac{1}{3}\left\{\left[\frac{x^2}{2}\right]_0^3 + \left[\frac{(6-x)^2}{-2}\right]_3^6\right\}$$

$$= \frac{1}{3}\left\{\left[\left\{\frac{3^2}{2}\right\} - \{0\}\right] + \left[\{0\} - \left\{\frac{(3)^2}{-2}\right\}\right]\right\} = \frac{1}{3}\left[\frac{9}{2} + \frac{9}{2}\right] = \frac{9}{3}$$

$$\boxed{a_0 = 3}$$

$$a_n = \frac{1}{}\int^{2} f(x) \cos \frac{n\pi x}{}\, dx$$

$$= \frac{1}{3}\int_0^6 f(x) \cos \frac{n\pi x}{3}\, dx$$

$$= \frac{1}{3}\left[\int_0^3 x \cos \frac{n\pi x}{3}\, dx + \int_3^6 (6-x) \cos \frac{n\pi x}{3}\, dx\right]$$

$$= \frac{1}{3}\left\{\left[(x)\left(\frac{\sin \frac{n\pi x}{3}}{\frac{n\pi}{3}}\right) - (1)\left(\frac{-\cos \frac{n\pi x}{3}}{\frac{n^2\pi^2}{9}}\right)\right]_0^3 + \left[(6-x)\left(\frac{\sin \frac{n\pi x}{3}}{\frac{n\pi}{3}}\right) - (-1)\left(\frac{-\cos \frac{n\pi x}{3}}{\frac{n^2\pi^2}{9}}\right)\right]_3^6\right\}$$

$$= \frac{1}{3}\left\{\left[\frac{9}{n^2\pi^2} \cos \frac{n\pi x}{3}\right]_0^3 - \left[\frac{9}{n^2\pi^2} \cos \frac{n\pi x}{3}\right]_3^6\right\}$$

$$= \frac{9}{3n^2\pi^2}\left\{\left[\cos \frac{n\pi x}{3}\right]_0^3 - \left[\cos \frac{n\pi x}{3}\right]_3^6\right\}$$

$$= \frac{3}{n^2\pi^2}\left\{[\cos n\pi - \cos 0] - [\cos 2n\pi - \cos n\pi]\right\}$$

$$= \frac{3}{n^2\pi^2}[\cos n\pi - 1 - 1 + \cos n\pi] = \frac{3}{n^2\pi^2}\left[2(-1)^n - 2\right] = \frac{6}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$= \frac{6}{n^2\pi^2}\begin{cases} -2, & n = 1,3,5,\ldots \\ 0, & n = 2,4,6,\ldots \end{cases}$$

$$a_n = \begin{cases} \frac{-12}{n^2\pi^2}, & n=1,3,5,\dots \\ 0, & n=2,4,6,\dots \end{cases}$$

$$b_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\sin\frac{n\pi x}{\ell}dx = \frac{1}{3}\int_0^6 f(x)\sin\frac{n\pi x}{3}dx = \frac{1}{3}\left[\int_0^3 x\sin\frac{n\pi x}{3}dx + \int_3^6 (6-x)\sin\frac{n\pi x}{3}dx\right]$$

$$= \frac{1}{3}\left\{\left[(x)\left(\frac{-\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{3}}{\frac{n^2\pi^2}{9}}\right)\right]_0^3 + \left[(6-x)\left(\frac{-\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{3}}{\frac{n^2\pi^2}{9}}\right)\right]_3^6\right\}$$

$$= \frac{1}{3}\left\{\left[\frac{-3}{n\pi}x\cos\frac{n\pi x}{3}\right]_0^3 + \left[\frac{-3}{n\pi}(6-x)\cos\frac{n\pi x}{3}\right]_3^6\right\}$$

$$= \frac{-3}{3n\pi}\{[3\cos n\pi - 0] + [0 - 3\cos n\pi]\} = \frac{-3}{3n\pi}[3\cos n\pi - 3\cos n\pi]$$

$$\therefore \boxed{b_n = 0}$$

Substitute a_0, a_n, b_n in (1)

$$f(x) = \frac{3}{2} + \sum_{n=1,3,5}^{\infty} \frac{-12}{n^2\pi^2}\cos\frac{n\pi x}{3} + \sum_{n=1}^{\infty} 0 \sin\frac{n\pi x}{3}$$

$$\therefore \boxed{f(x) = \frac{3}{2} - \frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{3}} \quad ------(2)$$

Deduction:

Let $x = 0$ be a point of continuity

$$f(0) = \frac{3}{2} - \frac{12}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\cos 0 \qquad f(x) = \begin{cases} x, & 0 \le x \le 3 \\ 6-x, & 3 \le x \le 6 \end{cases}$$

$$\Rightarrow 0 = \frac{3}{2} - \frac{12}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \frac{12}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2} = \frac{3}{2}$$

$$\therefore \sum_{n=1,3,5}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8} \quad \Rightarrow \boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots = \frac{\pi^2}{8}}$$

4. **Find the Fourier series for** $f(x) = |x|$ **in** $-\pi < x < \pi$ **and deduce that** $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\dots = \frac{\pi^2}{8}$.

Solution:

Given $f(x) = |x|, \quad -\pi < x < \pi$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} \quad \therefore \ell = \pi$

Now, $f(-x) = |-x| = |x| = f(x)$

$\therefore f(x)$ is an even function.

$\therefore \boxed{b_n = 0}$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$\therefore f(x)=\frac{a_0}{2}+\sum_{1}^{\infty} a_n \cos nx \quad -----(1) \qquad \boxed{l=\pi}$$

To Find a_0:

$$a_0=\frac{2}{l}\int_0^l f(x)\,dx=\frac{2}{\pi}\int_0^{\pi}|x|\,dx=\frac{2}{\pi}\int_0^{\pi}x\,dx$$

$$=\frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}=\frac{2}{2\pi}\left(\pi^2-0\right)=\pi$$

$$\therefore \boxed{a_0=\pi}$$

To Find a_n:

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{\pi}\int_0^{\pi}|x|\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx$$

$$=\frac{2}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right)-(1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\left(\frac{1}{n^2}\right)\cos nx\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left(\frac{1}{n^2}\right)\left[\cos n\pi-\cos 0\right]$$

$$=\frac{2}{\pi n^2}\left[(-1)^n-1\right]$$

$$\boxed{a_n=\begin{cases}\dfrac{-4}{\pi n^2}, & if\ n=1,3,5,\ldots\\ 0, & if\ n=2,4,6,\ldots\end{cases}}$$

$$\therefore (1)\Rightarrow f(x)=\frac{\pi}{2}+\sum_{n=1,3,5\ldots}^{\infty}\frac{-4}{n^2\pi}\cos nx$$

$$\therefore \boxed{f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\cos nx} \quad --------(2)$$

Deduction:

Let $x=0$ be a point of continuity

$$(2)\Rightarrow f(0)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\cos 0$$

$$\Rightarrow 0=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2} \qquad f(x)=|x| \Rightarrow f(0)=0$$

$$\Rightarrow \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}=\frac{\pi}{2}$$

$$\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{8}$$

$$\therefore \boxed{1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}}$$

5. nd the Fourier series for $f(x) = x^2$ in $-\pi \le x \le \pi$ and deduce that

(a) $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$.

(b) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}=\frac{\pi^2}{12}$.

(c) $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots=\frac{\pi^4}{90}$

lution:

Given $f(x)=x^2, -\pi \le x \le \pi$

Here $=\frac{\text{Upper Limit}-\text{Lower Limit}}{2}=\frac{\pi-(-\pi)}{2}=\frac{2\pi}{2}=\pi \quad \therefore =\pi$

Now, $f(-x)=x^2=(-x)^2=x^2=f(x)$

$\therefore f(x)$ is an even function.

$\therefore b_n = 0$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$\therefore f(x)=\frac{a_0}{2}+\sum_{1}^{\infty} a_n \cos nx \quad -----(1) \qquad \because l=\pi$$

To Find a_0:

$$a_0=\frac{2}{l}\int_0 f(x)\,dx=\frac{2}{\pi}\int_0 x^2\,dx=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}=\frac{2}{3\pi}\left[\pi^3-0\right]=\frac{2\pi^2}{3}$$

$$\boxed{a_0=\frac{2\pi^2}{3}}$$

To find a_n:

$$a_n=\frac{2}{l}\int_0 f(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$=\frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right)-(2x)\left(\frac{-\cos nx}{n^2}\right)+(2)\left(\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\left(\frac{2}{n^2}\right)x\cos nx\right]_0^{\pi}$$

$$=\frac{4}{\pi n^2}(\pi\cos n\pi-0)$$

$$\boxed{a_n=\frac{4(-1)^n}{n^2}}$$

$$\therefore (1) \Rightarrow f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

$$\therefore \boxed{f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx} \quad ----(2)$$

Deduction: a

Let $x = \pi$ be a point of continuity

$$(2) \Rightarrow f(\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)}{n^2}\cos\pi$$

$$\Rightarrow \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2} \qquad f(x) = x^2 \Rightarrow f(\pi) = \pi^2$$

$$\Rightarrow \pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{12} = \sum_{n=1}^{\infty} \frac{1}{n^2} \qquad (-1)^{2n} = 1$$

$$\therefore \boxed{1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}}$$

Deduction: b

Let $x = 0$ be a point of continuity

$$(2) \Rightarrow f(0) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)}{n^2}\cos 0$$

$$\Rightarrow 0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \qquad f(x) = x^2 \Rightarrow f(0) = 0$$

$$\Rightarrow -\frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\Rightarrow -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} = -\frac{\pi^2}{12}$$

$$\Rightarrow \boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} = \frac{\pi^2}{12}}$$

Deduction: c

By Parsevals identity for Fourier series,

$$\frac{2}{l}\int_0^l \left[f(x)\right]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^\pi \left[x^2\right]^2 dx = \frac{\frac{4\pi^4}{9}}{2} + \sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\right)^2$$

$$\frac{2}{\pi}\int_0^\pi x^4 dx = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^4}$$

$$\frac{2}{\pi}\left[\left(\frac{\pi^5}{5}\right) - (0)\right] = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4} \qquad (-1)^{2n} = 1$$

$$\frac{2\pi^5}{5\pi} = \frac{2\pi^4}{9} + 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{2\pi^4}{5}-\frac{2\pi^4}{9}=16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$2\pi^4\left(\frac{1}{5}-\frac{1}{9}\right)=16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$2\pi^4\times\frac{4}{45}=16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{8\pi^4}{45\times 16}=\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{90}=\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots=\frac{\pi^4}{90}$$

6. Obtain the Fourier series of $f(x) = x + x^2$ in $-\pi < x < \pi$. Hence show that

$i)\ \sum_{1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ $ii)\ \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\dots=\frac{\pi^2}{12}$

Solution:

Given $f(x)=x+x^2$ **in** $-\pi\le x\le\pi$

Here $l=\frac{\text{Upper Limit}-\text{Lower Limit}}{2}=\frac{\pi-(-\pi)}{2}=\frac{2\pi}{2}=\pi \quad \therefore l=\pi$

Now, $f(-x)=-x+(-x)^2$

$=-x+x^2\neq f(x)$

$=-(x-x^2)\neq -f(x)$

$\therefore f(-x)\neq f(x)\ \&\ f(-x)\neq -f(x)$

$\therefore f(x)$ is Neither even Nor odd function.

General Fourier is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}$

$\therefore f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \quad ------(1) \quad l=\pi$

To Find a_0:

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)dx=\frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left\{\left(\frac{\pi^2}{2}+\frac{\pi^3}{3}\right)-\left(\frac{\pi^2}{2}-\frac{\pi^3}{3}\right)\right\}=\frac{1}{\pi}\left\{\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right\}=\frac{1}{\pi}\left\{\frac{2\pi^3}{3}\right\}=\frac{2\pi^2}{3}$$

$$\boxed{a_0=\frac{2\pi^2}{3}}$$

To Find a_n:

$$a_n=\frac{1}{\ell}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left[\left(x+x^2\right)\left(\frac{\sin nx}{n}\right) - (1+2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{n^2}\right)(1+2x)\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\{[(1+2\pi)\cos n\pi -] - [(1-2\pi)\cos(-n\pi)]\}$$

$$= \frac{1}{\pi^2}\left[(-1)^n\left(1+2\pi-1+2\pi\right)\right] \qquad \because \cos(-n\pi) = \cos n\pi = (-1)^n$$

$$= \frac{(-1)^n}{n^2\pi}[4\pi] = \frac{4(-1)^n}{n^2}$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

To Find b_n:

$$b_n = \frac{1}{l}\int^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\left(x+x^2\right)\left(\frac{-\cos nx}{n}\right) - (1+2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-1}{n}\right)\left(x+x^2\right)\cos nx + \left(\frac{2}{n^3}\right)\cos nx\right]_{-\pi}^{\pi} \qquad \because \cos(-n\pi) = \cos n\pi = (-1)^n$$

$$= \frac{1}{\pi}\left\{\left[\left(\frac{-1}{n}\right)\left(\pi+\pi^2\right)\cos n\pi + \left(\frac{2}{n^3}\right)\cos n\pi\right] - \left[\left(\frac{-1}{n}\right)\left(-\pi+\pi^2\right)\cos n\pi + \left(\frac{2}{n^3}\right)\cos(-n\pi)\right]\right\}$$

$$= \frac{(-1)^n}{\pi}\left(-\frac{\pi}{n} - \frac{\pi^2}{n} + \frac{2}{n^3} - \frac{\pi}{n} + \frac{\pi^2}{n} - \frac{2}{n^3}\right) = -\frac{2\pi(-1)^n}{n\pi}$$

$$\therefore \boxed{b_n = \frac{-2}{n}(-1)^n}$$

$$(1) \Rightarrow f(x) = \frac{\pi^2}{3} + \sum_{1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx + \sum_{1}^{\infty}\frac{-2(-1)^n}{n}\sin nx$$

$$\boxed{f(x) = \frac{\pi^2}{3} + 4\sum_{1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{1}^{\infty}\frac{(-1)^n}{n}\sin nx} \qquad -----(2)$$

Deduction: 1

Let $x = \pi$ be the point of discontinuity

$(2) \Rightarrow f(\pi) = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos n\pi - 2\sum_1^\infty \frac{(-1)^n}{n}\sin n\pi$

$\pi^2 = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{1}{n^2}$ $\qquad f(x) = x + x^2 \Rightarrow f(\pi) = \frac{f(-\pi)+f(\pi)}{2} = \frac{-\pi+\pi^2+\pi+\pi^2}{2} = \frac{2\pi^2}{2} = \pi^2$

$\pi^2 - \frac{\pi^2}{3} = 4\sum_1^\infty \frac{1}{n^2}$

$\frac{2\pi^2}{3\times 4} = \sum_1^\infty \frac{1}{n^2}$

$$\boxed{\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}}$$

Deduction: 2

Let $x = 0$ be the point of continuity

$(2) \Rightarrow f(0) = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos 0 - 2\sum_1^\infty \frac{(-1)^n}{n}\sin 0$

$0 = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}$ $\qquad f(x) = x + x^2 \Rightarrow f(0) = 0$

$-\frac{\pi^2}{3} = 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^3}\dots\right]$

$$\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^3} - \dots = \frac{\pi^2}{12}}$$

7. Find the Fourier series expansion of f (x) where $f(x) = \begin{cases} \pi + x, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$ and hence deduce that $\sum_1^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$.

Solution:

Given $f(x) = \begin{cases} \pi + x, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$

Here $l = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi-(-\pi)}{2} = \frac{2\pi}{2} = \pi \quad \therefore l = \pi$

Let $f(x) = \begin{cases} f_1(x), & -\pi \le x \le 0 \\ f_2(x), & 0 \le x \le \pi \end{cases}$

Where

$f_1(x) = \pi + x \qquad f_2(x) = \pi - x$

$f_1(-x) = \pi - x = f_2(x) \qquad f_2(-x) = \pi + x = f_1(x)$

$\therefore f(x)$ is an even function.

$\therefore \boxed{b_n = 0}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos\frac{n\pi x}{l}$$

$$\therefore \boxed{f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos nx} \text{ ------(1)} \quad \because l = \pi$$

To Find a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left(\pi x - \frac{x^2}{2}\right)_0^{\pi} = \frac{2}{\pi}\left(\pi^2 - \frac{\pi^2}{2}\right) = \frac{2}{\pi}\times\frac{\pi^2}{2} = \pi$$

$$\therefore \boxed{a_0 = \pi}$$

To Find a_n:

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{-1}{n^2}\right)\cos nx\right]_0^{\pi}$$

$$= \frac{-2}{n^2\pi}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$\boxed{a_n = \begin{cases}\dfrac{4}{\pi n^2}, & if\ n = 1,3,5,\ldots \\ 0, & if\ n = 2,4,6,\ldots\end{cases}}$$

$$\therefore (1) \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1,3,5\ldots}^{\infty}\frac{4}{n^2\pi}\cos nx$$

$$\therefore \boxed{f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\cos nx} \text{ --------(2)}$$

Deduction:

Let $x = 0$ be a point of continuity

$$(2) \Rightarrow f(0) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\cos 0$$

$$\Rightarrow \pi = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}$$

$$\because f(x) = \begin{cases}\pi + x, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi\end{cases} \Rightarrow f(0) = \pi$$

$$\Rightarrow \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2} = \pi - \frac{\pi}{2}$$

$$\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2} = \frac{\pi}{2}\times\frac{\pi}{4} = \frac{\pi^2}{8}$$

$$\therefore \boxed{\sum_{1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.}$$

$$\because \sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n} = \sum_{n=1}^{\infty}\frac{1}{2n-1}$$

8. Determine the Fourier series expansion of f(x) where $f(x)=\begin{cases}-1+x, & -\pi \le x \le 0\\ 1+x, & 0 \le x \le \pi\end{cases}$ with $f(x+2\pi)=f(x)$.

Solution:

Given $f(x)=\begin{cases}-1+x, & -\pi \le x \le 0\\ 1+x, & 0 \le x \le \pi\end{cases}$

Here $l=\frac{\text{Upper Limit}-\text{Lower Limit}}{2}=\frac{\pi-(-\pi)}{2}=\frac{2\pi}{2}=\pi \quad \therefore l=\pi$

Let $f(x)=\begin{cases}f_1(x), & -\pi \le x \le 0\\ f_2(x), & 0 \le x \le \pi\end{cases}$

Where

$f_1(x)=-1+x$ $\qquad$ $f_2(x)=1+x$

$f_1(-x)=-1-x=-(1+x)=-f_2(x)$ $\qquad$ $f_2(-x)=1-x=-(-1+x)=-f_1(x)$

$\therefore f(x)$ is an odd function.

$\therefore \boxed{a_0=a_n=0}$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$\therefore f(x)=\sum_{1}^{\infty} b_n \sin nx \quad -----(1) \quad \boxed{l=\pi}$$

To Find b_n:

$$b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx=\frac{2}{\pi}\int_0^{\pi}(1+x)\sin nx\,dx$$

$$=\frac{2}{\pi}\left[(1+x)\left(\frac{-\cos nx}{n}\right)-(1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\left(\frac{-1}{n}\right)(1+x)\cos nx\right]_0^{\pi}$$

$$=\frac{-2}{n\pi}\left[(1+\pi)\cos n\pi-(1+0)\cos 0\right]$$

$$=\frac{-2}{n\pi}\left[(1+\pi)(-1)^n-1\right]$$

$$\boxed{b_n=\frac{2}{n\pi}\left[1-(1+\pi)(-1)^n\right]}$$

$$\therefore (1)\Rightarrow f(x)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[1-(1+\pi)(-1)^n\right]\sin nx$$

$$\therefore \boxed{f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1-(1+\pi)(-1)^n\right]\sin nx}$$

9. Find the Fourier series for f (x) where $\phi(\xi)=\begin{cases}0, & -1<\xi<0\\ 1, & 0<\xi<1\end{cases}$.

Solution:

Given $f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$

Here $l = \dfrac{\text{Upper Limit} - \text{Lower Limit}}{2} = \dfrac{1-(-1)}{2} = \dfrac{2}{2} = 1 \quad \therefore l = 1$

Let $f(x) = \begin{cases} f_1(x), & -1 \le x \le 0 \\ f_2(x), & 0 \le x \le 1 \end{cases}$

Where

$f_1(x) = 0 \qquad\qquad f_2(x) = 1$

$f_1(-x) = 0 \neq f_2(x) \qquad\qquad f_2(-x) = 1 \neq f_1(x)$

$f_1(-x) = 0 \neq -f_2(x) \qquad\qquad f_2(-x) = 1 \neq -f_1(x)$

$\therefore f(x)$ is Neither even Nor odd function.

General Fourier is $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\dfrac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{l}$

$$\therefore \boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x} \quad \text{-------} (1) \quad \boxed{l = 1}$$

To Find a_0:

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 0\,dx + \int_{0}^{1} 1\,dx = [x]_0^1 = 1 - 0 = 1$$

$$\boxed{\therefore a_0 = 1}$$

To Find a_n:

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{1}\,dx$$

$$= \int_{-1}^{0} 0\,dx + \int_{0}^{1} 1\cos n\pi x\,dx$$

$$= \left[\frac{\sin n\pi x}{n\pi}\right]_0^1$$

$$= [\sin n\pi - \sin 0] = 0$$

$$\boxed{a_n = 0}$$

To Find b_n:

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{1}\,dx$$

$$= \int_{-1}^{0} 0\,dx + \int_{0}^{1} 1\sin n\pi x\,dx$$

$$= \left[\frac{-\cos n\pi x}{n\pi}\right]_0^1$$

$= \frac{-1}{n\pi}[\cos n\pi x]_0^1$

$= \frac{-1}{n\pi}[\cos n\pi - \cos 0]_0^1$

$= \frac{1}{n\pi}\left[1-(-1)^n\right]$

$$\boxed{b_n = \begin{cases} \frac{2}{n\pi}, & if \;\; n = 1,3,5,... \\ 0, & if \;\; n = 2,4,6.... \end{cases}}$$

$$(1) \Rightarrow f(x) = \frac{1}{2} + \sum_{1,3}^{\infty} \frac{2}{n\pi} \sin n\pi x.$$

$$\boxed{f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{1,3}^{\infty} \frac{1}{n} \sin n\pi x}$$

10. Find the Fourier series for $f(x) = |\cos x|$ in the interval (-π,π).

Solution:

Given $f(x) = |\cos x|, \quad -\pi < x < \pi$

Here $= \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi-(-\pi)}{2} = \frac{2\pi}{2} \quad \therefore \; = \pi$

Now, $f(-x) = |\cos(-x)| = |\cos x| = f(x)$

$\therefore f(x)$ is an even function.

$\therefore \boxed{b_n = 0}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx \;\; -----(1) \qquad \because l = \pi$$

To find a_0:

$a_0 = \frac{2}{l}\int_0^l f(x)dx$

$= \frac{2}{\pi}\int_0^{\pi} |\cos x| dx$

$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} -\cos x\, dx\right] \quad \because |\cos x| = \begin{cases} \cos x, & 0 < x < \frac{\pi}{2} \\ -\cos x, & \frac{\pi}{2} < x < \pi \end{cases}$

$= \frac{2}{\pi}\left[(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\right]$

$= \frac{2}{\pi}\left[\left(\sin\frac{\pi}{2} - \sin 0\right) - \left(\sin\pi - \sin\frac{\pi}{2}\right)\right] = \frac{2}{\pi}(1-(-1))$

$\therefore \boxed{a_0 = \frac{4}{\pi}}$

To find a_n:

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx$$

$$a_n = \frac{2}{\pi}\int_0^\pi |\cos x|\cos nx dx$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\cos nx dx + \int_{\pi/2}^{\pi} -\cos x\cos nx dx\right]$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2}\cos nx\cos x dx - \int_{\pi/2}^{\pi}\cos nx\cos x dx\right]$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right] \quad Here \quad A = nx \quad B = x$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2}\frac{1}{2}\left[\cos(n+1)x+\cos(n-1)x\right]dx - \int_{\pi/2}^{\pi}\frac{1}{2}\left[\cos(n+1)x+\cos(n-1)x\right]dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2}\left[\cos(n+1)x+\cos(n-1)x\right]dx - \int_{\pi/2}^{\pi}\left[\cos(n+1)x+\cos(n-1)x\right]dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} - \left[\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left(\frac{\sin(n+1)\frac{\pi}{2}}{n+1}+\frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right) - (0)\right] - \frac{1}{\pi}\left[(0) - \left(\frac{\sin(n+1)\frac{\pi}{2}}{n+1}+\frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{\sin\left(\frac{n\pi}{2}+\frac{\pi}{2}\right)}{n+1}+\frac{\sin\left(\frac{n\pi}{2}-\frac{\pi}{2}\right)}{n-1}+\frac{\sin\left(\frac{n\pi}{2}+\frac{\pi}{2}\right)}{n+1}+\frac{\sin\left(\frac{n\pi}{2}-\frac{\pi}{2}\right)}{n-1}\right)$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin\left(\frac{n\pi}{2}+\frac{\pi}{2}\right) = \sin\frac{n\pi}{2}\cos\frac{\pi}{2}+\cos\frac{n\pi}{2}\sin\frac{\pi}{2} = \cos\frac{n\pi}{2} \quad \because \cos\frac{\pi}{2} = 0 \,\&\, \sin\frac{\pi}{2} = 1$$

$$\sin\left(\frac{n\pi}{2}-\frac{\pi}{2}\right) = \sin\frac{n\pi}{2}\cos\frac{\pi}{2}-\cos\frac{n\pi}{2}\sin\frac{\pi}{2} = -\cos\frac{n\pi}{2} \quad \because \cos\frac{\pi}{2} = 0 \,\&\, \sin\frac{\pi}{2} = 1$$

$$a_n = \frac{1}{\pi}\left(\frac{\cos\frac{n\pi}{2}}{n+1}-\frac{\cos\frac{n\pi}{2}}{n-1}+\frac{\cos\frac{n\pi}{2}}{n+1}-\frac{\cos\frac{n\pi}{2}}{n-1}\right)$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}\left(\frac{1}{n+1}-\frac{1}{n-1}\right)$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}\left(\frac{n-1-n-1}{(n+1)(n-1)}\right)$$

$$a_n = \frac{2}{\pi}\cos\frac{n\pi}{2}\left(\frac{-2}{(n^2-1)}\right)$$

$$a_n = \frac{-4}{\pi(n^2-1)}\cos\frac{n\pi}{2}$$, Provided $n \neq 1$

When $n=1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos x dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos x dx + \int_{\pi/2}^{\pi} -\cos x \cos x dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x dx - \int_{\pi/2}^{\pi} \cos^2 x dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx - \int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{2}{2\pi}\left[\int_0^{\pi/2}(1+\cos 2x)\,dx - \int_{\pi/2}^{\pi}(1+\cos 2x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(x+\frac{\sin 2x}{2}\right)_0^{\pi/2} - \left(x+\frac{\sin 2x}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left\{\left[\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)-(0)\right]-\left[\left(\pi+\frac{\sin 2\pi}{2}\right)-\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)\right]\right\}$$

$$a_1 = \frac{1}{\pi}\left[\frac{\pi}{2}-\pi+\frac{\pi}{2}\right]$$

$$a_1 = 0$$

$$\therefore f(x) = \frac{\frac{4}{\pi}}{2} + \sum_1^{\infty}\frac{-4}{\pi(n^2-1)}\cos\frac{n\pi}{2}\cos nx$$

$$\therefore f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_1^{\infty}\frac{1}{(n^2-1)}\cos\frac{n\pi}{2}\cos nx$$

11. Find the half range Fourier sine series for $f(x)=x(\pi-x)$ **in the interval** $(0,\pi)$ **and deduce that**

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\ldots\infty$$

Solution:

Given $f(x)=x(\pi-x)=\pi x - x^2, \quad (0,\pi)$

: General Fourier is $f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$

Here ℓ = Upper Limit – Lower Limit = π – 0 = π ∴ ℓ = π

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx \qquad ------(1)$$

To Find b_n**:**

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}dx$$

$$= \frac{2}{\ell}\int_0^{\ell} x(\pi-x)\sin nx dx$$

$$= \frac{2}{\ell}\int_0^{\ell}(\pi x - x^2)\sin nx\,dx$$

$$b_n = \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^3}\cos nx\right]_0^{\pi}$$

$$= \frac{-4}{\pi n^3}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-4}{\pi n^3}\left[(-1)^n - 1\right]$$

$$\boxed{b_n = \begin{cases} \frac{8}{\pi n^3}, & if\ n = 1,3,5,\ldots \\ 0, & if\ n = 2,4,6,\ldots \end{cases}}$$

The required Fourier sine series be

$$(1) \Rightarrow f(x) = \sum_{n=1,3,5\ldots}^{\infty} \frac{8}{\pi n^3}\sin nx$$

$$f(x) = \frac{8}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^3}\sin nx \quad ------(2)$$

Deduction:

Let $x = \frac{\pi}{2}$ be a point of continuity.

$$(2) \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{8}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi}{2}\right)$$

$$\because f(x) = x(\pi - x) \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2} - \frac{\pi^2}{4} = \frac{2\pi^2 - \pi^2}{4}$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{8}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi}{2}\right) = \frac{\pi^3}{32}$$

$$\Rightarrow \frac{1}{1^3}\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\frac{\pi}{2} + \frac{1}{5^3}\sin\frac{\pi}{2} + \ldots = \frac{\pi^3}{32}$$

$$\Rightarrow \boxed{\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots = \frac{\pi^3}{32}}$$

12. Find the half – range cosine series for f (x) = (x – 1)^2 in (0, 1). Hence show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$.

Solution:

Given:

Here Upper Limit – Lower Limit=1-0=1 ∴ =1

Let the cosine series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\pi x \quad l=1 \text{ -------(1)}$$

To Find a_0:

$$a_0=\frac{2}{l}\int_0^l f(x)\,dx=\frac{2}{1}\int_0^1\left(x^2-2x+1\right)dx=2\left[\frac{x^3}{3}-2\frac{x^2}{2}+x\right]_0^1=2\left(\frac{1}{3}-1+1\right)=\frac{2}{3}$$

$$\boxed{a_0=\frac{2}{3}}$$

To Find a_n:

$$a_n=2\int_0^1(x^2-2x+1)\cos n\pi x\,dx$$

$$=2\left[(x^2-2x+1)\left(\frac{\sin n\pi x}{n\pi}\right)-(2x-2)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)+(2)\left(\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$=2\left[\left(\frac{1}{n^2\pi^2}\right)(2x-2)\cos n\pi x\right]_0^1$$

$$=\frac{2}{n^2\pi^2}\left[(0)-(-2\cos 0)\right]$$

$$a_n=\frac{4}{n^2\pi^2}$$

$$(1)\Rightarrow f(x)=\frac{2/3}{2}+\sum_1^{\infty}\frac{4}{n^2\pi^2}\cos n\pi x$$

$$\therefore \boxed{f(x)=\frac{1}{3}+\frac{4}{\pi^2}\sum_1^{\infty}\frac{1}{n^2}\cos n\pi x} \text{ -----(2)}$$

Deduction:

Let $x=0$ be a point of discontinuity

$$(2)\Rightarrow f(0)=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 0$$

$$\Rightarrow \frac{1}{2}=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \qquad \because f(x)=(x-1)^2\Rightarrow f(0)=\frac{f(0)+f(1)}{2}=\frac{1+0}{2}=\frac{1}{2}$$

$$\Rightarrow \frac{1}{2}-\frac{1}{3}=\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \frac{1}{6}\times\frac{\pi^2}{4}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+....=\frac{\pi^2}{24}}$$

13. Find the Fourier cosine series for $x(\pi-x)$ in $0<x<\pi$. Hence show that $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+....=\frac{\pi^4}{90}$.

Solution:

Given:

Let $f(x)=x(\pi-x),\quad 0<x<\pi$

$$f(x)=\pi x-x^2$$

Here ℓ = Upper Limit – Lower Limit = π-0 = π

Let the cosine series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \because l = \pi \text{ ------ } (1)$$

To Find a_0 :

$$a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\,dx = \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right) - (0)\right]$$

$$= \frac{2}{\pi}\left[\frac{3\pi^3 - 2\pi^3}{6}\right] = \frac{2}{\pi}\left[\frac{\pi^3}{6}\right] = \frac{\pi^2}{3}$$

$$\boxed{a_0 = \frac{\pi^2}{3}}$$

To Find a_n:

$$a_n = 2\int_0^1 (x^2 - 2x + 1)\cos n\pi x\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{1}{n^2}\right)(\pi - 2x)\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left[-\pi\cos n\pi - \pi\cos 0\right] = \frac{-2\pi}{\pi n^2}\left[(-1)^n + 1\right]$$

$$= \frac{-2}{n^2}\left[(-1)^n + 1\right]$$

$$\boxed{a_n = \begin{cases} -\frac{4}{n^2}, & \text{if } n = 2,4,6\ldots \\ 0, & \text{if } n = 1,3,5\ldots \end{cases}}$$

$$\therefore \boxed{f(x) = \frac{\pi^2}{6} + \sum_{n=2,4,\ldots}^{\infty} -\frac{4}{n^2}\cos nx.}$$

Deduction:

Let the Parseval's identity for Fourier cosine series be

$$\frac{2}{l}\int_0^l [f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi}[\pi x - x^2]^2\,dx = \frac{\pi^4}{2} + \sum_{n=2,4,6,\ldots}^{\infty}\left(\frac{-4}{n^2}\right)^2$$

$$\frac{2}{\pi}\int_0^{\pi}[\pi^2 x^2 - 2\pi x^3 + x^4]\,dx = \frac{\pi^4}{18} + 16\sum_{n=2,4,6,\ldots}^{\infty}\frac{1}{n^4}$$

$$\frac{2}{\pi}\left[\left(\frac{\pi^5}{3}-\frac{\pi^5}{2}+\frac{\pi^5}{5}\right)-(0)\right]=\frac{\pi^4}{18}+16\sum_{n=1}^{\infty}\frac{1}{(2n)^4} \quad \therefore \sum_{n=2,4,6,..}^{\infty}\frac{1}{n}=\sum_{n=1}^{\infty}\frac{1}{2n}$$

$$\frac{2\pi^5}{\pi}\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right)=\frac{\pi^4}{18}+\frac{16}{16}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$2\pi^4\left(\frac{10-15+6}{30}\right)-\frac{\pi^4}{18}=\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{15}-\frac{\pi^4}{18}=\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{6\pi^4-5\pi^4}{90}$$

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots=\frac{\pi^4}{90}}$$

14. Find the cosine series for f (x) = x in (0, π) and then using Parseval's theorem, show that

$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots=\frac{\pi^4}{96}.$$

Solution:

Given:

Let $f(x)=x, \quad 0<x<\pi$

Here Upper Limit – Lower Limit = π-0 = π $\quad\therefore \; = \pi$

Let the cosine series be $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\pi \qquad \boxed{l=\pi}$$

To Find a_0:

$$a_0=\frac{2}{\ell}\int_0^{\ell} f(x)\,dx=\frac{2}{\pi}\int_0^{\pi} x\,dx=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=\frac{2}{2\pi}\left[\pi^2-0\right]=\pi$$

$$\boxed{a_0=\pi}$$

To Find a_n:

$$a_n=\frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$$

$$=\frac{2}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right)-(1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\left(\frac{1}{n^2}\right)\cos nx\right]_0^{\pi}$$

$$=\frac{2}{\pi n^2}\left[\cos n\pi-\cos 0\right]$$

$$=\frac{2}{n^2\pi}\left[(-1)^n-1\right]$$

$$a_n = \begin{cases} \dfrac{-4}{n^2\pi}, & if \;\; n = 1,3,5,\ldots \\ 0, & if \;\; n = 2,4,6,\ldots \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5\ldots}^{\infty} \frac{-4}{\pi n^2} \cos nx$$

Deduction:

Let the Parseval's identity for Fourier cosine series be

$$\frac{2}{l}\int_0^l [f(x)]^2 \, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{\pi^2}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \left(\frac{-4}{n^2\pi}\right)^2$$

$$\frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{\pi^2}{2} + \sum_{n=1,3,5\ldots}^{\infty} \frac{16}{\pi^2 n^4}$$

$$\frac{2}{3\pi}\left(\pi^3 - 0\right) = \frac{\pi^2}{2} + \frac{16}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^4}$$

$$\frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^2}{6} \times \frac{\pi^2}{16} = \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^4}$$

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\infty = \frac{\pi^4}{96}$$

15. Find the complex form of the Fourier series of $f(x) = \cos ax$ in $(-\pi, \pi)$, where a is not an integer.

Solution:

Given $f(x) = \cos ax$ **in** $(-\pi, \pi)$

Here $l = \dfrac{\text{Upper Limit} - \text{Lower Limit}}{2} = \dfrac{\pi - (-\pi)}{2} = \dfrac{2\pi}{2} \quad \therefore l = \pi$

Let the complex form of the Fourier series be

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \, e^{\frac{in\pi x}{l}}$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \, e^{inx} \qquad l = \pi$$

$$C_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax \, e^{-inx} \, dx$$

$$\because \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2}\left[a \cos bx + b \sin bx\right] \qquad \text{Here } a = -in \;\&\; b = a$$

$$= \frac{1}{2\pi}\left\{ \frac{e^{-inx}}{\left((in)^2 - a^2\right)} \left(-in \cos ax + a \sin ax\right) \right\}_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi\left(a^2-n^2\right)}\left[e^{-in\pi}\left(-in\cos a\pi - a\sin a\pi\right) - e^{in\pi}\left(-in\cos a\pi + a\sin a\pi\right)\right]$$

$$\boxed{e^{in\pi} = \cos n\pi + i\sin n\pi = \cos n\pi = (-1)^n \quad \text{and } e^{-in\pi} = \cos n\pi - i\sin n\pi = \cos n\pi = (-1)^n \quad \because \sin n\pi = 0}$$

$$= \frac{1}{2\pi\left(a^2-n^2\right)}\left[(-1)^n\left[-in\cos a\pi + a\sin a\pi + in\cos a\pi + a\sin a\pi\right]\right]$$

$$= \frac{1}{2\pi\left(a^2-n^2\right)}\left[(-1)^n(2a\sin a\pi)\right]$$

$$\boxed{C_n = \frac{(-1)^n a\sin a\pi}{\pi\left(a^2-n^2\right)}}$$

$$\therefore \boxed{f(x) = \frac{a\sin a\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n}{\left(a^2-n^2\right)}e^{inx}}$$

16. Find complex form of the Fourier series of the function $f(x) = e^{-x}, -1 < x < 1$

Solution:

Given $f(x) = e^{-x}, -1 < x < 1$

Here $l = \dfrac{\text{Upper Limit} - \text{Lower Limit}}{2} = \dfrac{1-(-1)}{2} = \dfrac{2}{2} = 1 \quad \therefore l = 1$

Let the complex form of the Fourier series be

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}}$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x} \quad \boxed{l = 1}$$

$$C_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} dx = \frac{1}{2}\int_{-1}^{1} e^{-x} e^{-in\pi x} dx$$

$$C_n = \frac{1}{2}\int_{-1}^{1} e^{-x-in\pi x} dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x} dx$$

$$= \frac{1}{2}\left\{\frac{e^{-(1+in\pi x)}}{-(1+in\pi)}\right\}_{-1}^{1}$$

$$= -\frac{-1}{2(1+in\pi)}\left\{e^{-(1+in\pi)} - e^{(1+in\pi x)}\right\}$$

$$= -\frac{-1}{2(1+in\pi)}\left\{e^{-1}e^{-in\pi} - e^{1}e^{in\pi}\right\}$$

$$\boxed{e^{in\pi} = \cos n\pi + i\sin n\pi = \cos n\pi = (-1)^n \quad \text{and } e^{-in\pi} = \cos n\pi - i\sin n\pi = \cos n\pi = (-1)^n \quad \because \sin n\pi = 0}$$

$$= \frac{-(1-in\pi)}{2\left(1+n^2\pi^2\right)}\left\{e^{-1}(-1)^n - e^{1}(-1)^n\right\}$$

$$= \frac{(1-in\pi)}{2(1+n^2\pi^2)}(-1)^n\left(e^1 - e^{-1}\right)$$

$$\boxed{C_n = \frac{(1-in\pi)}{2(1+n^2\pi^2)}(-1)^n\, 2\sinh(1)} \qquad \sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

$$\therefore \boxed{f(x) = \sum_{-\infty}^{\infty}\frac{(1-in\pi)}{(1+n^2\pi^2)}(-1)^n \sinh(1)\, e^{in\pi x}}$$

17. Calculate the first two harmonic of the Fourier series of $f(x)$ from the following data

x	0	30	60	90	120	150	180	210	240	270	300	330
f(x)	1.8	1.1	0.3	0.16	0.5	1.3	2.16	1.25	1.3	1.52	1.76	2.0

Solution:

x	0	30	60	90	120	150	180	210	240	270	300	330	~~360~~
f(x)	1.8	1.1	0.3	0.16	0.5	1.3	2.16	1.25	1.3	1.52	1.76	2.0	~~1.8~~

We know that $360 = 2\pi$

Here $\frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \pi \quad \therefore \ = \pi$

K=12

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad = \pi$$

$$\boxed{f(x) = \frac{a_0}{2} + (a_1\cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots}$$

Where

$a_0 = \frac{2}{K}\sum y \qquad b_1 = \frac{2}{K}\sum y\sin x$

$a_1 = \frac{2}{K}\sum y\cos x \qquad b_2 = \frac{2}{K}\sum y\sin 2x$

$a_2 = \frac{2}{K}\sum y\cos 2x$

x	y	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1.8	1.8	1.8	0	0
30	1.1	0.95	0.55	0.55	0.95
60	0.3	0.15	-0.15	0.26	0.26
90	0.16	0.00	-0.16	0.16	0.00
120	0.5	-0.25	-0.25	0.43	-0.43
150	1.3	-1.13	0.65	0.65	-1.12
180	2.16	-2.16	2.16	0.00	0.01
210	1.25	-1.08	0.62	-0.63	1.08
240	1.3	-0.65	-0.65	-1.13	1.12

270	1.52	0.00	-1.52	-1.52	-0.01
300	1.76	0.88	-0.87	-1.52	-1.53
330	2.0	1.73	1.01	-1.00	-1.73
Total	15.15	0.26	3.18	-3.74	-1.39

$a_0 = \frac{2}{12}(15.15) = 2.52 \quad b_1 = \frac{2}{12}(-3.74) = -0.62$

$a_1 = \frac{2}{12}(0.26) = 0.043 \quad b_2 = \frac{2}{12}(-1.39) = -0.23$

$a_2 = \frac{2}{12}(3.18) = 0.53$

$$\boxed{f(x) = 1.26 + (0.043\cos x - 0.62\sin x) + (0.53\cos 2x - 0.23\sin 2x) + \ldots}$$

18. Find the first two harmonic of the Fourier series of f (x) given by

x	**0**	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
f (x)	**1**	**1.4**	**1.9**	**1.7**	**1.5**	**1.2**	**1.0**

Solution:

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
f (x)=y	1	1.4	1.9	1.7	1.5	1.2	1.0

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \pi \quad \therefore \ell = \pi$

K=6

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ell = \pi$$

$$\boxed{f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots}$$

Where $a_0 = \frac{2}{K}\sum y \quad b_1 = \frac{2}{K}\sum y\sin x$

$a_1 = \frac{2}{K}\sum y\cos x \quad b_2 = \frac{2}{K}\sum y\sin 2x$

$a_2 = \frac{2}{K}\sum y\cos 2x$

x	y	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1	1	1	0	0
$\frac{\pi}{3} = 60$	1.4	0.7	-0.7	1.212	1.212
$\frac{2\pi}{3} = 120$	1.9	-0.95	-0.95	1.65	-1.645
$\pi = 180$	1.7	-1.7	1.7	0	0
$\frac{4\pi}{3} = 240$	1.5	-0.75	-0.75	-1.299	1.299

$\frac{5\pi}{3} = 300$	1.2	0.6	-0.6	-1.039	-1.039
Total	8.7	-1.1	-0.3	0.5196	-0.1732

$a_0 = \frac{2}{6}(8.7) = 2.9$ $b_1 = \frac{2}{6}(0.5196) = 0.17$

$a_1 = \frac{2}{6}(-1.1) = -0.37$ $b_2 = \frac{2}{6}(-0.1732) = -0.06$

$a_2 = \frac{2}{6}(-0.3) = -0.1$

$$\boxed{f(x) = 1.45 + (-0.37\cos x + 0.17\sin x) + (-0.1\cos 2x - 0.06\sin 2x) + \ldots}$$

19. Find the first three harmonic of the Fourier series of f (x) given by

X	0	1	2	3	4	5
f (x)	9	18	24	28	26	20

Solution:

X	0	1	2	3	4	5	6
f (x)	9	18	24	28	26	20	~~9~~

Here $\frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{6-0}{2} = 3 \quad \therefore \; = 3$

K=6

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{3} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{3}$$

$$\boxed{f(x) = \frac{a_0}{2} + \left(a_1 \cos \frac{\pi x}{3} + b_1 \sin \frac{\pi x}{3}\right) + \left(a_2 \cos \frac{2\pi x}{3} + b_2 \sin \frac{2\pi x}{3}\right) + \left(a_3 \cos \frac{3\pi x}{3} + b_3 \sin \frac{3\pi x}{3}\right) + \ldots}$$

Where

$a_0 = \frac{2}{K}\sum y$	$b_1 = \frac{2}{K}\sum y \sin\left(\frac{\pi x}{3}\right)$
$a_1 = \frac{2}{K}\sum y \cos\left(\frac{\pi x}{3}\right)$	$b_2 = \frac{2}{K}\sum y \sin\left(\frac{2\pi x}{3}\right)$
$a_2 = \frac{2}{K}\sum y \cos\left(\frac{2\pi x}{3}\right)$	$b_3 = \frac{2}{K}\sum y \sin\left(\frac{3\pi x}{3}\right)$
$a_3 = \frac{2}{K}\sum y \cos\left(\frac{3\pi x}{3}\right)$	

x	y	$y\cos\frac{\pi x}{3}$ (or) $y\cos 60x$	$y\cos\frac{2\pi x}{3}$ $y\cos 120x$	$y\cos\frac{3\pi x}{3}$ $y\cos 180x$	$y\sin\frac{\pi x}{3}$ $y\sin 60x$	$y\sin\frac{2\pi x}{3}$ $y\sin 120x$	$y\sin\frac{3\pi x}{3}$ $y\cos 180x$
0	9	9	9	9	0	0	0
1	18	9	-9	-18	15.7	15.6	0
2	24	-12	-24	24	20.9	-20.784	0
3	28	-28	28	-28	0	0	0
4	26	-13	-13	26	-22.6	22.6	0

5	20	10	-10	-20	-17.4	-17.4	0
Total	125	-25	-19	-7	-3.4	20.8	0

$a_0 = \frac{2}{6}(125) = 41.66$ $b_1 = \frac{2}{6}(-3.4) = -1.13$

$a_1 = \frac{2}{6}(-25) = -8.33$ $b_2 = \frac{2}{6}(20.8) = 6.9$

$a_2 = \frac{2}{6}(-19) = -6.33$ $b_3 = \frac{2}{6}(0) = 0$

$a_3 = \frac{2}{6}(-7) = -2.3$

The required Harmonic of the Fourier series be

$$f(x) = \frac{41.66}{2} + \left(-8.33\cos\frac{\pi x}{3} + -1.13\sin\frac{\pi x}{3}\right) + \left(-6.33\cos\frac{2\pi x}{3} + 6.9\sin\frac{2\pi x}{3}\right) + \left(-2.3\cos\frac{3\pi x}{3} + 0\right) + \ldots$$

$$\boxed{f(x) = 20.83 + \left(-8.33\cos\frac{\pi x}{3} + -1.13\sin\frac{\pi x}{3}\right) + \left(-6.33\cos\frac{2\pi x}{3} + 6.9\sin\frac{2\pi x}{3}\right) + \left(-2.3\cos\frac{3\pi x}{3}\right) + \ldots}$$

20. The following table gives the variations of a periodic function over a period T

x	*0*	*T/6*	*T/3*	*T/2*	*2T/3*	*5T/6*	*T*
f(x)	**1.98**	**1.3**	**1.05**	**1.3**	**- 0.88**	**- 0.25**	**1.98**

find *f(x)* upto first harmonic.

Solution:

Assume $X = \frac{2\pi x}{T}$

X	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y=f(θ)	**1.98**	**1.3**	**1.05**	**1.3**	**- 0.88**	**- 0.25**	**1.98**

Here $\frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \pi \quad \therefore \quad = \pi$

K=6

$$f(X) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi X}{} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi X}{}$$

$$f(X) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nX + \sum_{n=1}^{\infty} b_n \sin nX$$

$$\boxed{f(x) = \frac{a_0}{2} + (a_1 \cos nX + b_1 \sin nX) + (a_2 \cos 2X + b_2 \sin 2X) + (a_3 \cos 3X + b_3 \sin 3X) + \ldots}$$

Where

$a_0 = \frac{2}{K}\sum y$ $a_1 = \frac{2}{K}\sum y\cos X$				$b_1 = \frac{2}{K}\sum y\sin X$
x	y	$y\cos X$	$y\sin X$	
0	1.98	1.98	0	
$\frac{\pi}{3}$	1.30	0.65	1.1258	

$\frac{2\pi}{3}$	1.05	-0.525	0.9093
π	1.30	-1.3	0
$\frac{4\pi}{3}$	-0.88	0.44	0.762
$\frac{5\pi}{3}$	-0.25	-0.125	0.2165
Total	4.6	1.12	3.013

$$a_0 = \frac{2}{6}\sum y = \frac{4.6}{3} = 1.5 \quad a_1 = \frac{2}{6}(1.12) = 0.37 \quad b_1 = \frac{2}{6}(3.013) = 1.005$$

$$\therefore \boxed{f(x) = 0.75 + 0.37\cos X + 1.005\sin X}$$