DEPARTMENT OF MATHEMATICS

NAME OF THE SUBJECT : TRANSFORMS & PARTIAL DIFFERENTIAL EQUATION

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UNIT – I : PARTIAL DIFFERENTIAL EQUATION

UNIT I - PARTIAL DIFFERENTIAL EQUATIONS

Notations: If $z = f(x, y)$ then

$$p = \frac{\partial z}{\partial x};\ q = \frac{\partial z}{\partial y};\ r = \frac{\partial^2 z}{\partial x^2};\ s = \frac{\partial^2 z}{\partial x \partial y};\ t = \frac{\partial^2 z}{\partial y^2}$$

Formation of PDE by eliminating arbitrary constants:

Let the given equation be $z = f(x, y, a, b)$ ------ (1)

Step 1: Differentiating (1) partially with respect to x

$$\frac{\partial z}{\partial x} = p = f'(x, y, a, b) \text{ ------ } (2)$$

Step 2: Differentiating (1) partially with respect to y

$$\frac{\partial z}{\partial y} = q = f'(x, y, a, b) \text{ ------ } (3)$$

Step 3: Eliminate a & b from (1) using (2) & (3)

1. **Obtain partial differential equation by eliminating arbitrary constant 'a' and 'b' from** $z = (x-a)^2 + (y-b)^2$

Solution:

Given $z = (x-a)^2 + (y-b)^2$ ---- (1)

Diff Partially w.r.t x

$$\frac{\partial z}{\partial x} = 2(x-a) + 0$$

$p = 2(x-a)$ ---- (2)

Diff Partially w.r.t y

$$\frac{\partial z}{\partial y} = 0 + 2(y-b)$$

$q = 2(y-b)$ ---- (3)

Eliminate a & b from (1) using (2) & (3)

$(2) \Rightarrow (x-a) = \frac{p}{2}$ ---- (4)

$(3) \Rightarrow y - b = \frac{q}{2}$ ---- (5)

Sub (4) & (5) in (1)

$$(1) \Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

The required the PDE is

$$\boxed{p^2 + q^2 = 4z}$$

2. **Form the partial differential equation by eliminating the arbitrary constants 'a' & 'b' from** $z = (x^2 + a)(y^2 + b)$.

Solution:

Given $z = (x^2 + a)(y^2 + b)$ ---- (1)

Diff Partially w.r.t x

$$\frac{\partial z}{\partial x} = p = 2x(y^2 + b) \text{ ---- } (2)$$

Diff Partially w.r.t y

$$\frac{\partial z}{\partial y} = q = 2y(x^2 + a) \text{ ---- } (3)$$

Eliminate a & b from (1) using (2) & (3)

	$(2) \Rightarrow (y^2+b) = \frac{p}{2x} ----(4)$ $(3) \Rightarrow x^2+b = \frac{q}{2y} ----(5)$ Sub (4) & (5) in (1) $(1) \Rightarrow z = \left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right)$ The required the PDE is $\boxed{4xyz = pq}$
3.	**Find the PDE of all planes having equal intercepts on the x and y axis.** **Solution:** The intercept form of the plane equation is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ Given that equal intercepts on the x & y axis $\Rightarrow a=b$ $\therefore \frac{x}{a}+\frac{y}{a}+\frac{z}{c}=1 -----(1)$ Diff Partially w.r.t x $\frac{1}{a}+0+\frac{1}{c}\frac{\partial z}{\partial x}=0 \Rightarrow \frac{1}{a}=\frac{-1}{c}p ----(2)$ Diff Partially w.r.t y $0+\frac{1}{a}(1)+\frac{1}{c}\frac{\partial z}{\partial y}=0 \Rightarrow \frac{1}{a}=\frac{-1}{c}q ----(3)$ From (2) & (3) $\frac{-1}{c}p=\frac{-1}{c}q$ The required the PDE is $\boxed{p=q}$
4.	**Obtain the partial differential equation by eliminating arbitrary constants 'a' and ' b' from $(x-a)^2+(y-b)^2+z^2=r^2$** **Solution:** $(x-a)^2+(y-b)^2+z^2=1 ----(1)$ - (1) Diff Partially w.r.t x $2(x-a)(1-0)+0+2z\frac{\partial z}{\partial x}=0$ $\Rightarrow 2(x-a)+2zp=0 ------ (2)$ Diff Partially w.r.t y $0+2(y-b)(1-0)+2z\frac{\partial z}{\partial y}=0$ $\Rightarrow 2(y-b)+2zq=0 ------ (3)$ Eliminate a & b from (1) using (2) & (3) $(2) \Rightarrow x-a=-zp ------ (4)$ $(3) \Rightarrow y-b=-zq ------ (5)$ Sub (4) & (5) in (1) $(-zp)^2+(-zq)^2+z^2=1$ The required PDE is $\boxed{z^2\left(p^2+q^2+1\right)=1}$
Formation of PDE by eliminating arbitrary functions:	
1.	**Eliminate the arbitrary function f from $z=f\left(\frac{y}{x}\right)$ and form the PDE.** **Solution:**

$z = f\left(\frac{y}{x}\right)$ ----(1)

Diff Partially w.r.t x

$$\frac{\partial z}{\partial x} = p = f'\left(\frac{y}{x}\right)\times\left(\frac{-y}{x^2}\right) \Rightarrow f'\left(\frac{y}{x}\right) = \frac{-px^2}{y} \text{ ---- (2)}$$

Diff Partially w.r.t x

$$\frac{\partial z}{\partial y} = q = f'\left(\frac{y}{x}\right)\times\left(\frac{1}{x}\right) \text{ ----(3)}$$

From (1) & (2) $\frac{p}{q} = \frac{-y}{x} \Rightarrow \boxed{px + qy = 0}$

2. **Form the partial differential equation by eliminating *f* from** $z = x^2 + 2f\left(\frac{1}{y} + \log x\right)$.

Solution:

Given $z = x^2 + 2f\left(\frac{1}{y} + \log x\right)$ -----(1)

Differentiate (1) partially w.r. t x

$$\frac{\partial z}{\partial x} = 2x + 2f'\left(\frac{1}{y} + \log x\right)\left(0 + \frac{1}{x}\right)$$

$$p = 2x + 2f'\left(\frac{1}{y} + \log x\right)\left(\frac{1}{x}\right) \Rightarrow f'\left(\frac{1}{y} + \log x\right) = (p - 2x)\frac{x}{2} \text{ ----(2)}$$

$$\frac{\partial z}{\partial y} = 2f'\left(\frac{1}{y} + \log x\right)\left(\frac{-1}{y^2} + 0\right)$$

$$q = \frac{-2}{y^2} f'\left(\frac{1}{y} + \log x\right) \Rightarrow f'\left(\frac{1}{y} + \log x\right) = \frac{-qy^2}{2} \text{ -----(3)}$$

Eliminating f' from (2) & (3)

$$(p - 2x)\frac{x}{2} = \frac{-qy^2}{2} \Rightarrow \left(px - 2x^2\right) = -qy^2$$

$$\Rightarrow \boxed{px + qy^2 = 2x^2}$$

Formation of PDE by eliminating *f* from $f(u,v) = 0$ ------ (1)

Method 1:

The required PDE of (1) is $\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$

Method 2:

The required PDE is $Pp + Qq = R$

Where

$P = \begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix}$; $Q = \begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix}$; $R = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}$

1. **Form the PDE from** $\phi\left(ax + by + cz, x^2 + y^2 + z^2\right) = 0$

Solution:

Given $\phi\left(ax + by + cz, x^2 + y^2 + z^2\right) = 0$

This is of the form $f(u,v) = 0$ where $u = ax + by + cz$ & $v = x^2 + y^2 + z^2$

The required PDE of (1) is $\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$

$$\begin{vmatrix} p & q & -1 \\ a & b & c \\ 2x & 2y & 2z \end{vmatrix} = 0$$

$\Rightarrow \; p(2bz - 2cy) - q(2az - 2cx) + 1(2az - 2cx) = 0$

$\div 2 \Rightarrow$ $\boxed{(bz - cy)p + (cx - az)q + (az - cx) = 0}$

2. **Form the PDE from $\phi\left(x^2 + y^2 + z^2, xyz\right) = 0$**

Solution:

Given $\phi\left(x^2 + y^2 + z^2, xyz\right) = 0$

This is of the form $f(u,v) = 0$ where $u = x^2 + y^2 + z^2$ & $v = xyz$

The required PDE of (1) is $\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$

$$\begin{vmatrix} p & q & -1 \\ 2x & 2y & 2z \\ yz & xz & xy \end{vmatrix} = 0$$

$\Rightarrow \; p(2xy^2 - 2xz^2) - q(2x^2y - 2yz^2) + 1(2x^2z - 2y^2z) = 0$

$\div 2 \Rightarrow$ $\boxed{x(y^2 - z^2)p + y(z^2 - x^2) + z(x^2 - y^2) = 0}$

3. **Form the PDE from $\phi\left(\frac{y}{x}, x^2 + y^2 + z^2\right) = 0$**

Solution:

Given $\phi\left(\frac{y}{x}, x^2 + y^2 + z^2\right) = 0$

This is of the form $f(u,v) = 0$ where $u = \frac{y}{x}$ & $v = x^2 + y^2 + z^2$

The required PDE of (1) is $\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$

$$\begin{vmatrix} p & q & -1 \\ \frac{-y}{x^2} & \frac{1}{x} & 0 \\ 2x & 2y & 2z \end{vmatrix} = 0$$

$\Rightarrow \; p\left(\frac{2z}{x} - 0\right) - q\left(\frac{-2yz}{x^2} - 0\right) + 1\left(\frac{-2y^2}{x^2} - \frac{2x}{x}\right) = 0$

$\Rightarrow \; \frac{2zp}{x} + \frac{2yzq}{x^2} - 2\left(\frac{y^2}{x^2} + 1\right) = 0$

$\div 2 \Rightarrow \; \frac{zp}{x} + \frac{yzq}{x^2} - \left(\frac{y^2 + x^2}{x^2}\right) = 0$

$\Rightarrow \dfrac{xzp + yzq - \left(y^2 + x^2\right)}{x^2} = 0$

$\boxed{xzp + yzq - \left(y^2 + x^2\right) = 0}$

Solutions of standard types of First order PDE's:

Different solutions of PDE:

Complete Integral (or) Complete Solution:

If the number of arbitrary constants is equal to number of independent variables, then the solution is called Complete integral.

Singular Integral (or) Singular Solution:

Consider a PDE of first order as $f(x, y, z, p, q) = 0 ------(1)$

It's complete integral may be, $f(x, y, z, a, b) = 0 ------(2)$

Diff (2) partially with respect to $a \,\&\, b$ respectively,

$\dfrac{\partial g}{\partial a} = 0 \;------(3)$

$\dfrac{\partial g}{\partial b} = 0 \;------(4)$

Eliminating $a \,\&\, b$ from (3) & (4) will get the Singular integral.

General Integral (or) Complete solution:

A Solution which contains number of arbitrary functions is equal to the order of the given PDE.
(or) A solution which contains the maximum possible number of arbitrary functions.

Type I:

Equations of the form $f(p, q) = 0 ------(1)$

To find Complete Integral:

Let the complete solution of (1) is $z = ax + by + c \;----(2)$

Let $p = a$ & $q = b$ in (1)

$f(a, b) = 0$ and represent $b = \phi(a)$

$\therefore (1) \Rightarrow z = ax + \phi(a)y + c \;----(3)$

To Find Singular Integral:

Diff (3) partially with respect to c

$0 = 1$ which is impossible

There is no singular integral for this type.

To find General integral:

Put $c = g(a)$ in (3)

$(3) \Rightarrow z = ax + \phi(a)y + g(a) \;----(4)$

Diff (4) partially with respect to a

$0 = x(1) + \phi'(a)y + g'(a) \;----(5)$

Eliminating a from (4) & (5) we get general integral.

1. **Find the complete integral of $p + q = pq$**

Solution:

Given $p + q = pq \;----(1)$

This of the form $f(p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = ax + by + c \;----(2)$

Let $p = a$ & $q = b$ in (1)

$(1) \Rightarrow a + b = ab \Rightarrow a + b - ab = 0 \Rightarrow b = \dfrac{a}{a-1}$

Sub b in (2)

$\boxed{z = ax + \left(\dfrac{a}{a-1}\right)y + c}$

This is the required complete integral.

2. **Find the complete integral of $p+q=1$**

Solution:

Given $p+q=1 \quad ----(1)$

This of the form $f(p,q)=0$

To find Complete Integral:

Let the complete solution of (1) is $z=ax+by+c \quad ----(2)$

Let $p=a$ & $q=b$ in (1)

$(1) \Rightarrow a+b=1 \Rightarrow b=1-a$

Sub b in (2)

$$\boxed{z=ax+(1-a)y+c}$$

This is the required complete integral.

3. **Solve $\sqrt{p}+\sqrt{q}=1$**

Solution:

Given $\sqrt{p}+\sqrt{q}=1 \quad ----(1)$

This of the form $f(p,q)=0$

To find Complete Integral:

Let the complete solution of (1) is $z=ax+by+c \quad ----(2)$

Let $p=a$ & $q=b$ in (1)

$(1) \Rightarrow \sqrt{a}+\sqrt{b}=1 \Rightarrow \sqrt{b}=1-\sqrt{a} \Rightarrow b=\left(1-\sqrt{a}\right)^2$

Sub b in (2)

$$\boxed{z=ax+\left(1-\sqrt{a}\right)^2 y+c} \quad ----(3)$$

This is the required complete integral.

To Find Singular Integral:

Diff (3) partially with respect to c

$0=1$ which is impossible

There is no singular integral for this type.

To find General integral:

Put $c=f(a)$ in (3)

$(3) \Rightarrow \quad z=ax+\left(1-\sqrt{a}\right)^2 y+f(a) \quad ----(4)$

Diff. (4) partially with respect to a

$0=x(1)+2\left(1-\sqrt{a}\right)\frac{1}{2\sqrt{a}}y+f'(a) \quad ----(5)$

Eliminate a from (4) & (5) we get the general integral.

Type II:

Equations of the form $z=px+qy+f(p,q) \quad ------(1)$

To find Complete Integral:

Put $p=a$ & $q=b$ in (1)

$\therefore (1) \Rightarrow z=ax+by+f(a,b) \quad ------(2)$

To Find Singular Integral:

Diff (2) partially with respect to a

$0=x(1)+0+f'(a,b) \quad ----(3)$

Diff (2) partially with respect to b

$0=0+y(1)+f'(a,b) \quad ----(4)$

Eliminating a & b from (2) using (3) &(4), we get the singular integral.

To find General integral:

Put $b=\phi(a)$ in (2)

$(2) \Rightarrow z=ax+\phi(a)y+g(a) \quad ------(5)$

Diff (4) partially with respect to a

$0 = x(1) + \phi'(a) y + g'(a) \quad ----(6)$

Eliminating a from (5) & (6) we get General integral.

1. **Solve** $z = px + qy + p^2q^2$

Solution:

Given $z = px + qy + p^2q^2 \quad -----(1)$

Equations of the form $z = px + qy + f(p,q)$

To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$\therefore (1) \Rightarrow \boxed{z = ax + by + a^2b^2} \quad ------(2)$

This is the required complete integral

To Find Singular Integral:

Diff (2) partially with respect to a

$0 = x(1) + 0 + 2ab^2 \Rightarrow x + 2ab^2 = 0 \Rightarrow x = -2ab^2 \quad ----(3)$

Diff (2) partially with respect to b

$0 = 0 + y(1) + 0 + 2a^2b \Rightarrow y + 2a^2b = 0 \Rightarrow y = -2a^2b \quad ----(4)$

Eliminating a & b from (2) using (3) &(4)

$(3) \Rightarrow \frac{x}{b} = -2ab \quad ----(5)$

$(4) \Rightarrow \frac{y}{a} = -2ab \quad ----(6)$

From (5) & (6)

$\frac{x}{b} = \frac{y}{a} = k$ (say)

$\Rightarrow \frac{x}{b} = k$ & $\frac{y}{a} = k$

$\Rightarrow b = \frac{x}{k}$ & $a = \frac{y}{k} \quad ----(7)$

Sub a & b in (2)

$(2) \Rightarrow z = \frac{y}{k}x + \frac{x}{k}y + \frac{y^2}{k^2}\frac{x^2}{k^2}$

$z = \frac{xy}{k} + \frac{xy}{k} + \frac{x^2y^2}{k^4}$

$z = \frac{2xy}{k} + \frac{x^2y^2}{k^4} \quad ----(8)$

To find k

Sub (7) in (3) (or) (4)

$(3) \Rightarrow x = -2\frac{y}{k}\frac{x^2}{k^2} \Rightarrow x = \frac{-2x^2y}{k^3} \Rightarrow k^3 = -2xy$

$(8) \Rightarrow z = \frac{2xy}{k} + \frac{x^2y^2}{k(-2xy)} \Rightarrow z = \frac{2xy}{k} - \frac{xy}{2k}$

$z = \frac{4xy - xy}{2k} \Rightarrow z = \frac{3xy}{2k} \Rightarrow z^3 = \frac{27x^3y^3}{8k^3} \Rightarrow z^3 = \frac{27x^3y^3}{8(-2xy)}$

$\boxed{16z^3 = -27x^2y^2}$

This is the required singular integral.

To find General integral:

Put $b = \phi(a)$ in (2)

$(2) \Rightarrow z = ax + f(a)y + a^2[f(a)]^2 \quad ------(9)$

Diff (9) partially with respect to a

$$0 = x(1) + f'(a)y + 2f(a)f'(a) \quad ----(10)$$

Eliminating a from (9) & (10) we get General integral.

2. **Find the singular integral of** $z = px + qy + p^2 + pq + q^2$

Solution:

Given $z = px + qy + p^2 + pq + q^2 \quad -----(1)$

Equations of the form $z = px + qy + f(p,q)$

To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$$\therefore (1) \Rightarrow \boxed{z = ax + by + a^2 + ab + b^2} \quad ------(2)$$

This is the required complete integral

To Find Singular Integral:

Diff (2) partially with respect to a

$$0 = x(1) + 0 + 2a + b + 0 \Rightarrow 2a + b = -x \quad ----(3)$$

Diff (2) partially with respect to b

$$0 = 0 + y(1) + 0 + a + 2b \Rightarrow a + 2b = -y \quad ----(4)$$

Eliminating a & b from (2) using (3) &(4)

$$(4) \times 2 \Rightarrow 2a + 4b = -2y \quad -----(5)$$

$$(3) - (5) \Rightarrow -3b = -x + 2y \Rightarrow \boxed{b = \frac{x - 2y}{3}}$$

Sub the value of b in (3)

$$2a + b = -x \Rightarrow 2a = -x - b \Rightarrow 2a = -x - \left(\frac{x - 2y}{3}\right)$$

$$2a = \frac{-3x - x + 2y}{3} \Rightarrow 6a = -4x + 2y \Rightarrow \boxed{a = \frac{y - 2x}{3}}$$

Sub the value of a & b in (2)

$$z = \left(\frac{y-2x}{3}\right)x + \left(\frac{x-2y}{3}\right)y + \left(\frac{y-2x}{3}\right)^2 + \left(\frac{y-2x}{3}\right)\left(\frac{x-2y}{3}\right) + \left(\frac{x-2y}{3}\right)^2$$

$$z = \frac{xy - 2x^2}{3} + \frac{xy - 2y^2}{3} + \frac{y^2 - 4xy + 4y^2}{9} + \frac{xy - 2y^2 - 2x^2 + 4xy}{9} + \frac{x^2 - 4xy + 4y^2}{9}$$

$$z = \frac{3xy - 3x^2 + 3xy - 6y^2 + y^2 - 4xy + 4y^2 + xy - 2y^2 - 2x^2 + 4xy + x^2 - 4xy + 4y^2}{9}$$

$$\boxed{9z = -4x^2 + y^2 + xy}$$

3. **Solve** $z = px + qy + \sqrt{p^2 + q^2 + 1}$

Solution:

Given

$$z = px + qy + \sqrt{p^2 + q^2 + 1} \quad -----(1)$$

To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$$(1) \Rightarrow \boxed{z = ax + by + \sqrt{a^2 + b^2 + 1}} \quad ----(2)$$

This is required complete integral.

To Find Singular Integral:

Diff (2) partially with respect to a

$$0 = x(1) + 0 + \frac{1}{2\sqrt{a^2 + b^2 + 1}}(2a) \Rightarrow \boxed{x = \frac{-a}{\sqrt{a^2 + b^2 + 1}}} \quad ----(3)$$

Diff (2) partially with respect to b

	$0 = 0 + y(1) + \dfrac{1}{2\sqrt{a^2+b^2+1}}(2b) \Rightarrow \boxed{y = \dfrac{-b}{\sqrt{a^2+b^2+1}}}$ ----(4) Eliminating a & b from (2) using (3) &(4) $(3)^2 + (4)^2 \Rightarrow x^2 + y^2 = \left(\dfrac{-a}{\sqrt{a^2+b^2+1}}\right)^2 + \left(\dfrac{-b}{\sqrt{a^2+b^2+1}}\right)^2$ $x^2 + y^2 = \dfrac{a^2}{a^2+b^2+1} + \dfrac{b^2}{a^2+b^2+1}$ $x^2 + y^2 = \dfrac{a^2+b^2}{a^2+b^2+1}$ $1-(x^2+y^2) = 1 - \dfrac{a^2+b^2}{a^2+b^2+1}$ $1-x^2-y^2 = \dfrac{a^2+b^2+1-a^2-b^2}{a^2+b^2+1}$ $1-x^2-y^2 = \dfrac{1}{a^2+b^2+1}$ $\Rightarrow 1+a^2+b^2 = \dfrac{1}{1-x^2-y^2}$ Taking square root on both sides $\Rightarrow \boxed{\sqrt{1+a^2+b^2} = \dfrac{1}{\sqrt{1-x^2-y^2}}}$ ----(5) Sub (5) in (2) and (3) $(3) \Rightarrow x = \dfrac{-a}{\dfrac{1}{\sqrt{1-x^2-y^2}}} \Rightarrow x = -a\sqrt{1-x^2-y^2} \Rightarrow \boxed{a = \dfrac{-x}{\sqrt{1-x^2-y^2}}}$ $(4) \Rightarrow y = \dfrac{-b}{\dfrac{1}{\sqrt{1-x^2-y^2}}} \Rightarrow y = -b\sqrt{1-x^2-y^2} \Rightarrow \boxed{b = \dfrac{-y}{\sqrt{1-x^2-y^2}}}$ Sub (5), a & b in (2) $(2) \Rightarrow z = \left(\dfrac{-x}{\sqrt{1-x^2-y^2}}\right)x + \left(\dfrac{-y}{\sqrt{1-x^2-y^2}}\right)y + \dfrac{1}{\sqrt{1-x^2-y^2}}$ $\Rightarrow z = \dfrac{-x^2}{\sqrt{1-x^2-y^2}} + \dfrac{-y^2}{\sqrt{1-x^2-y^2}} + \dfrac{1}{\sqrt{1-x^2-y^2}}$ $\Rightarrow z = \dfrac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$ $\Rightarrow z = \sqrt{1-x^2-y^2}$ Squaring on both sides $z^2 = 1-x^2-y^2 \Rightarrow \boxed{x^2+y^2+z^2 = 1}$
4.	**Find the singular integral of** $z = px + qy + p^2 - q^2$ **Solution:** Given $z = px + qy + p^2 - q^2$ -----(1) Equations of the form $z = px + qy + f(p,q)$ To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$\therefore (1) \Rightarrow \boxed{z = ax + by + a^2 - b^2} ------(2)$

This is the required complete integral

To Find Singular Integral:

Diff (2) partially with respect to a

$0 = x(1) + 0 + 2a + 0 \Rightarrow a = \frac{-x}{2} ----(3)$

Diff (2) partially with respect to b

$0 = 0 + y(1) + 0 - 2b \Rightarrow b = \frac{y}{2} ----(4)$

Sub a & b in (2)

$\therefore (2) \Rightarrow z = \left(\frac{-x}{2}\right)x + \left(\frac{y}{2}\right)y + \left(\frac{-x}{2}\right)^2 - \left(\frac{y}{2}\right)^2$

$\therefore (2) \Rightarrow z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$

$\therefore (2) \Rightarrow z = \frac{-2x^2 + 2y^2 + x^2 - y^2}{4} \Rightarrow \boxed{4z = -x^2 + y^2}$

This is the required singular integral.

Type III:

Equations of the form $f(z, p, q) = 0 -----(1)$

In this type x & y do not appear explicitly.

To find Complete Integral:

Let the complete solution of (1) is $z = f(x + ay) ------(2)$

Let $x + ay = u$

$(2) \Rightarrow z = f(u) ------(3)$

By total derivative,

$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \Rightarrow p = \frac{dz}{dx}(1) \qquad \because u = x + ay \Rightarrow \frac{\partial u}{\partial x} = 1$

$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} \Rightarrow q = a\frac{dz}{du} \qquad \because u = x + ay \Rightarrow \frac{\partial u}{\partial y} = a$

Substitute the value of p & q in (1)

$(1) \Rightarrow f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$

This may be solve by method of separation of variables

Other solutions can obtain as usual.

1. **Solve $p(1 + q) = qz$.**

Solution:

Given $p(1 + q) = qz ----(1)$

This is of the form $f(z, p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = f(x + ay) ------(2)$

Let $x + ay = u \Rightarrow z = f(u)$

Then $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$

Substitute the value of p & q in (1)

$(1) \Rightarrow \frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du}z$

	$1+a\frac{dz}{du}=az$ $\frac{dz}{du}=az-1$ $\frac{dz}{az-1}=du$ Integrating on both sides $\int\frac{dz}{az-1}=\int du$ $u=\log(az-1)+c \qquad \because \int\frac{f'(x)}{f(x)}dx=\log f(x)$ $\boxed{x+ay=\log(az-1)+c}$ This is the required complete integral. Other solutions can be obtained as usual.
2.	**Solve** $z^2=1+p^2+q^2$ Solution: Given $z^2=1+p^2+q^2 ----(1)$ This is of the form $f(z,p,q)=0$ **To find Complete Integral:** Let the complete solution of (1) is $z=f(x+ay)$ ------(2) Let $x+ay=u \Rightarrow z=f(u)$ Then $p=\frac{dz}{du}$ & $q=a\frac{dz}{du}$ Substitute the value of p & q in (1) $(1)\Rightarrow z^2=\left(\frac{dz}{du}\right)^2+\left(a\frac{dz}{du}\right)^2+1$ $\Rightarrow\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2=z^2-1$ $\Rightarrow(1+a^2)\left(\frac{dz}{du}\right)^2=z^2-1$ $\Rightarrow\left(\frac{dz}{du}\right)^2=\frac{z^2-1}{1+a^2}$ Taking square root on both sides $\frac{dz}{du}=\frac{\sqrt{z^2-1}}{\sqrt{1+a^2}}$ $\Rightarrow\frac{dz}{\sqrt{z^2-1}}=\frac{du}{\sqrt{1+a^2}}$ Integrating on both sides $\cosh^{-1}z=\frac{1}{\sqrt{a^2-1}}u+c \qquad \because\int\frac{dz}{\sqrt{x^2-1}}=\cosh^{-1}x$ $\boxed{\cosh^{-1}z=\frac{1}{\sqrt{a^2-1}}(x+ay)+c} \qquad \because u=x+ay$ This is the required complete integral. Other solutions can be obtained as usual.

3. **Solve** $p\left(1-q^2\right) = q(1-z)$

Solution:

Given $p\left(1-q^2\right) = q(1-z) \quad ----(1)$

This is of the form $f(z, p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = f(x+ay) \quad ------(2)$

Let $x + ay = u \Rightarrow z = f(u)$

Then $p = \dfrac{dz}{du}$ & $q = a\dfrac{dz}{du}$

Substitute the value of p & q in (1)

$$(1) \Rightarrow \frac{dz}{du}\left(1-\left(a\frac{dz}{du}\right)^2\right) = a\frac{dz}{du}(1-z)$$

$$1 - a + az = a^2\left(\frac{dz}{du}\right)^2$$

Taking square root on both sides

$$a\frac{dz}{du} = \sqrt{1-a+az}$$

$$\frac{a\,dz}{\sqrt{1-a+az}} = du$$

Integrating on both sides

$$a\left(\frac{2\sqrt{1-a+az}}{a}\right) = u + c \qquad \because \int \frac{1}{\sqrt{ax}}dx = \frac{1}{a}\left(2\sqrt{x}\right)$$

$$\boxed{2\sqrt{1-a+az} = x + ay + c} \qquad \because u = x + ay$$

This is the required complete integral.
Other solutions can be obtained as usual.

Type IV:

Equations of the form $f_1(x, p) = f_2(y, q) \quad ------(1)$

To find Complete Integral:

Let $f_1(x, p) = f_2(y, q) = a$ (say)

$\therefore f_1(x, p) = a \quad ; \quad f_2(y, q) = a$

From the above we get $p = f_1(x, a)$; $q = f_2(y, b)$

Substitute the value of p & q in $z = \int p\,dx + \int q\,dy$

Integrating we get complete integral
Other solutions can obtain as usual.

1. **Solve** $p^2 + q^2 = x^2 + y^2$

Solution:

Given $p^2 + q^2 = x^2 + y^2$

$p^2 - x^2 = y^2 - q^2 \quad -----(1)$

This is of the form $f_1(x, p) = f_2(y, q)$

To find Complete Integral:

Let $p^2 - x^2 = y^2 - q^2 = a^2$ (say)

$\therefore p^2 - x^2 = a^2 \quad ; \quad y^2 - q^2 = a^2$

$\therefore p^2 = a^2 + x^2 \quad ; \quad q^2 = y^2 - a^2$

$\boxed{p = \sqrt{a^2 + x^2}} \quad ; \quad \boxed{q = \sqrt{y^2 - a^2}}$

Substitute the value of p & q in

$$z = \int \sqrt{x^2 + a^2}\, dx + \int \sqrt{y^2 - a^2}\, dy$$

$$\boxed{z = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2 - a^2} + \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + c}$$

$$\because \int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right); \quad \int \sqrt{y^2 - a^2}\, dy = \frac{y}{2}\sqrt{y^2 - a^2} + \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right)$$

Integrating we get complete integral

Other solutions can obtain as usual.

2. **Find the complete integral of** $p^2 y(1 + x^2) = qx^2$

Solution:

Given $p^2 y(1 + x^2) = qx^2$

$$\frac{p^2(1 + x^2)}{x^2} = \frac{q}{y} \text{ - - - - - - } (1)$$

This is of the form $f_1(x, p) = f_2(y, q)$

To find Complete Integral:

Let $\frac{p^2(1 + x^2)}{x^2} = \frac{q}{y} = a$ (say)

$$\frac{p^2(1 + x^2)}{x^2} = a \;;\; \frac{q}{y} = a$$

$$p^2 = a\frac{x^2}{1 + x^2} \;;\; q = ay$$

$$\boxed{p = \frac{\sqrt{a}\, x}{\sqrt{1 + x^2}}} \;;\; \boxed{q = ay}$$

Substitute the value of p & q in

$$z = \int \frac{\sqrt{a}\, x}{\sqrt{1 + x^2}} dx + \int ay\, dy$$

let $1 + x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{dt}{2}$

$$z = \sqrt{a}\int \frac{1}{\sqrt{t}}\frac{dt}{2} + a\frac{y^2}{2}$$

$$z = \frac{\sqrt{a}}{2} 2\sqrt{t} + \frac{ay^2}{2} \qquad \because \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x}$$

$$\boxed{z = \sqrt{a}\sqrt{1 + x^2} + \frac{ay^2}{2} + c}$$

3. **Find the complete integral of** $p + q = \sin x + \sin y$

Solution:

Given $p + q = \sin x + \sin y$

$p - \sin x = \sin y - q$ - - - - - - (1)

This is of the form $f_1(x, p) = f_2(y, q)$

To find Complete Integral:

Let $p - \sin x = \sin y - q = a$ (say)

$\therefore p - \sin x = a \;;\; \sin y - q = a$

$\therefore \boxed{p = \sin x + a} \;;\; \boxed{q = \sin y - a}$

Substitute the value of p & q in

	$z = \int (\sin x + a)\,dx + \int (\sin y - a)\,dy$ $z = \cos x + ax + \cos y - ay + c$ $\boxed{z = \cos x + \cos y - a(x-y) + c}$ This is the required complete integral
Type V: Equations reducible to standard form (Identification:	
1.	**Solve** $z^2(p^2+q^2) = x^2+y^2$ **Solution:** Given $z^2(p^2+q^2) = x^2+y^2$ This is of the form equations reducible to standard form $z^2p^2 + z^2q^2 = x^2+y^2$ $(zp)^2 + (zq)^2 = x^2+y^2$ $\left(\frac{z\,\partial z}{\partial x}\right)^2 + \left(\frac{z\,\partial z}{\partial y}\right)^2 = x^2+y^2 -----(1) \qquad \because p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}$ Let $z\partial z = \partial Z$; $\partial x = \partial X$; $\partial y = \partial Y$ Integrating $\boxed{\frac{z^2}{2} = Z \; ; \; x = X \; ; \; y = Y}$ $(1) \Rightarrow \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = X^2+Y^2$ $\boxed{P^2+Q^2 = X^2+Y^2} \qquad \because P = \frac{\partial Z}{\partial X}; Q = \frac{\partial Z}{\partial Y}$ This is of the form $F_1(X,P) = F_2(Y,Q)$ **To find Complete Integral:** Let $P^2 - X^2 = Y^2 - Q^2 = a^2$ (say) $\therefore P^2 - X^2 = a^2$; $Y^2 - Q^2 = a^2$ $\therefore P^2 = a^2 + X^2$; $Q^2 = Y^2 - a^2$ $\boxed{P = \sqrt{a^2+X^2} \; ; Q = \sqrt{Y^2-a^2}}$ Substitute the value of p & q in $z = \int \sqrt{X^2+a^2}\,dX + \int \sqrt{Y^2-a^2}\,dY$ $Z = \frac{X}{2}\sqrt{X^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{X}{a}\right) + \frac{Y}{2}\sqrt{Y^2-a^2} + \frac{a^2}{2}\cosh^{-1}\left(\frac{Y}{a}\right) + c$ $\frac{z^2}{2} = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + \frac{y}{2}\sqrt{y^2-a^2} + \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right) + c$ $\boxed{z^2 = x\sqrt{x^2+a^2} + a^2\sinh^{-1}\left(\frac{x}{a}\right) + y\sqrt{y^2-a^2} + a^2\cosh^{-1}\left(\frac{y}{a}\right) + 2c}$ $\because \int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right); \int \sqrt{y^2-a^2}\,dy = \frac{y}{2}\sqrt{y^2-a^2} + \frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right)$ This is the required complete integral Other solutions can obtain as usual.
2.	**Solve** $p^2 + x^2y^2q^2 = x^2z^2$ **Solution:** Given $p^2 + x^2y^2q^2 = x^2z^2$ This is of the form equations reducible to standard form

$\div x$ on bothsides $\frac{p^2}{x^2} + y^2q^2 = z^2$

$\left(\frac{p}{x}\right)^2 + (yq)^2 = z^2$

$\left(\frac{1}{x}\frac{\partial z}{\partial x}\right)^2 + \left(y\frac{\partial z}{\partial y}\right)^2 = z^2 \qquad \because p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}$

$\left(\frac{\partial z}{x\partial x}\right)^2 + \left(\frac{\partial z}{\frac{\partial y}{y}}\right)^2 = z^2 \quad -----(1) \qquad \because p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}$

Let $\partial z = \partial Z \;;\; x\partial x = \partial X \;;\; \frac{\partial y}{y} = \partial Y$

Integrating $\boxed{z = Z \;;\; \frac{x^2}{2} = X \;;\; \log y = Y}$

$(1) \Rightarrow \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = Z^2$

$\boxed{P^2 + Q^2 = Z^2} \; ----(2) \qquad \because P = \frac{\partial Z}{\partial X}; Q = \frac{\partial Z}{\partial Y}$

This is of the form $F(Z, P, Q) = 0$

To find Complete Integral:

Let the complete solution of (2) is $Z = F(X + aY)$

Let $X + aY = U \Rightarrow Z = F(U)$

Then $P = \frac{dZ}{dU}$ & $Q = a\frac{dZ}{dU}$

Substitute the value of P & Q in (2)

$(2) \Rightarrow \left(\frac{dZ}{dU}\right)^2 + \left(a\frac{dZ}{dU}\right)^2 = Z^2$

$\left(\frac{dZ}{dU}\right)^2 = \frac{Z^2}{(1+a^2)}$

Taking square root on both sides

$\frac{dZ}{dU} = \frac{Z}{\sqrt{1+a^2}}$

$\frac{dZ}{Z} = \frac{dU}{\sqrt{1+a^2}}$

Integrating on both sides

$\log Z = \frac{1}{\sqrt{1+a^2}}U$

$\log Z = \frac{1}{\sqrt{1+a^2}}(X + aY)$

This is complete integral of (2)

$\boxed{\log z = \frac{1}{\sqrt{1+a^2}}\left(\frac{x^2}{2} + a\log y\right) + c}$

This is the required complete integral .

Other solutions can be obtained as usual.

3. **Find the complete integral of** $z^2\left(x^2p^2+q^2\right)=1$

Solution:

Given $z^2\left(x^2p^2+q^2\right)=1$

This is of the form equations reducible to standard form

$$x^2p^2+q^2=\frac{1}{z^2}$$

$$(xp)^2+q^2=\frac{1}{z^2}$$

$$\left(x\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\frac{1}{z^2}\qquad \because p=\frac{\partial z}{\partial x};q=\frac{\partial z}{\partial y}$$

$$\left(\frac{\partial z}{\frac{\partial x}{x}}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2=\frac{1}{z^2}\;-----(1)$$

Let $\partial z=\partial Z$; $\frac{\partial x}{x}=\partial X$; $\partial y=\partial Y$

Integrating $\boxed{z=Z\ ;\ \log x=X\ ;\ y=Y}$

$$(1)\Rightarrow\left(\frac{\partial Z}{\partial X}\right)^2+\left(\frac{\partial Z}{\partial Y}\right)^2=\frac{1}{Z^2}$$

$$\boxed{P^2+Q^2=\frac{1}{Z^2}}\;----(2)\qquad \because P=\frac{\partial Z}{\partial X};Q=\frac{\partial Z}{\partial Y}$$

This is of the form $F(Z,P,Q)=0$

To find Complete Integral:

Let the complete solution of (2) is $Z=F(X+aY)$

Let $X+aY=U \Rightarrow Z=F(U)$

This is of the form $F(Z,P,Q)=0$

Then $P=\frac{dZ}{dU}$ & $Q=a\frac{dZ}{dU}$

Substitute the value of $P\,\&\,Q$ in (2)

$$(2)\Rightarrow\left(\frac{dZ}{dU}\right)^2+\left(a\frac{dZ}{dU}\right)^2=\frac{1}{Z^2}$$

$$\left(\frac{dZ}{dU}\right)^2=\frac{1}{Z^2(1+a^2)}$$

Taking square root on both sides

$$\frac{dZ}{dU}=\frac{1}{Z\sqrt{1+a^2}}$$

$$ZdZ=\frac{dU}{\sqrt{1+a^2}}$$

Integrating on both sides

$$\frac{Z^2}{2}=\frac{1}{\sqrt{1+a^2}}U$$

$$\frac{Z^2}{2}=\frac{1}{\sqrt{1+a^2}}(X+aY)$$

	This is the complete integral of (2) $\frac{z^2}{2} = \frac{1}{\sqrt{1+a^2}}\left(\log x + ay\right) + c$ This is the required complete integral . Other solutions can be obtained as usual.
4.	Solve $x^2p^2 + y^2q^2 = z^2$
5.	Solve $x^4p^2 + y^2zq = 2z^2$

Lagrange's Linear Differential Equations:

Equations of the form $Pp + Qq = R$ (or) $P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R$

Where P, Q, R are functions of x, y, z or constants.

Procedure :

1. Write the auxiliary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

2. Solve the auxiliary equation by using
   a) Method of grouping
   b) Method of multipliers

a) Method of grouping: In the auxiliary equation, if the variables can be separated in any pair of equations, then we get a solution of the form $u(x, y, z) = c_1$ & $v(x, y, z) = c_2$

∴ The general solution is $\phi(u, v) = 0$

b) Method of Multipliers:

i) Choose any three multipliers l, m, n which may be constants or functions of x, y, z we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR}$$

If it is possible to choose l, m, n such that $lP + mQ + nR = 0$ then $ldx + mdy + ndz = 0$

Integrating this we get $u(x, y, z) = c_1$

ii) Choose another any three multipliers l', m', n' which may be constants or functions of x, y, z we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}$$

If it is possible to choose l', m', n' such that $l'P + m'Q + n'R = 0$ then $l'dx + m'dy + n'dz = 0$

Integrating this we get $v(x, y, z) = c_2$

∴ The general solution is $\phi(u, v) = 0$

1. **Solve** $x(y - z)p + y(z - x)q = z(x - y)$

Solution:

Given $x(y - z)p + y(z - x)q = z(x - y)$

This is of the form $Pp + Qq = R$

Where $P = x(y - z);\ Q = y(z - x);\ R = z(x - y)$

The auxiliary equation be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y - z)} = \frac{dy}{y(z - x)} = \frac{dz}{z(x - y)} \quad ----(1)$$

i) Choose the multipliers as (1,1,1)

$$(1) \Rightarrow \frac{dx}{x(y - z)} = \frac{dy}{y(z - x)} = \frac{dz}{z(x - y)}$$

	$= \frac{dx+dy+dz}{xy-xz+yz-xy+xz-yz} = \frac{dx+dy+dz}{0}$ $\therefore dx + dy + dz = 0$ Integrating $\int dx + \int dy + \int dz = 0$ $\boxed{x+y+z=c_1}$ ii) Choose the multipliers as $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ $(1) \Rightarrow \frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$ $= \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y-z+z-x+x-y} = \frac{dx+dy+dz}{0}$ $\therefore \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z} = 0$ Integrating $\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$ $\log x + \log y + \log z = \log c_2$ $\log xyz = \log c_2 \quad \Rightarrow \boxed{xyz = c_2}$ $\therefore$The general solution is $\boxed{\phi\left(x+y+z, xyz\right) = 0}$
2.	**Solve** $x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$ **Solution:** Given $x(z^2-y^2)p + y(x^2-z^2)q = z(y^2-x^2)$ This is of the form $Pp+Qq=R$ Where $P = x(z^2-y^2);\ Q = y(x^2-z^2);\ R = z(y^2-x^2)$ The auxiliary equation be $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ $\frac{dx}{x(z^2-y^2)} = \frac{dy}{y(x^2-z^2)} = \frac{dz}{z(y^2-x^2)} \quad ----(1)$ i) Choose the multipliers as (x,y,z) $(1) \Rightarrow \frac{xdx}{x^2(z^2-y^2)} = \frac{ydy}{y^2(x^2-z^2)} = \frac{zdz}{z^2(y^2-x^2)}$ $= \frac{xdx+ydy+zdz}{x^2z^2-x^2y^2+x^2y^2-y^2z^2+y^2z^2-x^2z^2} = \frac{xdx+ydy+zdz}{0}$ $\therefore xdx + ydy + zdz = 0$ Integrating $\int xdx + \int ydy + \int zdz = 0$ $\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = \frac{c_1^2}{2}$ $\boxed{x^2+y^2+z^2 = c_1^2}$ ii) Choose the multipliers as $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

$$(1) \Rightarrow \frac{\frac{dx}{x}}{\frac{x}{x}(z^2 - y^2)} = \frac{\frac{dy}{y}}{\frac{y}{y}(x^2 - z^2)} = \frac{\frac{dz}{z}}{\frac{z}{z}(y^2 - x^2)}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{z^2 - y^2 + x^2 - z^2 + y^2 - x^2} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating $\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$

$\log x + \log y + \log z = \log c_2$

$\log xyz = \log c_2 \quad \Rightarrow \boxed{xyz = c_2}$

$\therefore$The general solution is $\boxed{\phi\left(x^2 + y^2 + z^2, xyz\right) = 0}$

3. **Solve** $(x^2 - y^2 - z^2)\,p + 2xyq = 2zx$

Solution:

Given $(x^2 - y^2 - z^2)\,p + 2xyq = 2zx$

This is of the form $Pp + Qq = R$

Where $P = (x^2 - y^2 - z^2)$; $Q = 2xy$; $R = 2zx$

The auxiliary equation be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2zx} ----(1)$$

i) by method of grouping, from last two ratios

$$(1) \Rightarrow \frac{dy}{2xy} = \frac{dz}{2zx}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating $\int \frac{dy}{y} = \int \frac{dz}{z}$

$\log y = \log z + \log c_1$

$\log y - \log z = \log c_1$

$\log \frac{y}{z} = \log c_1$

$$\boxed{\frac{y}{z} = c_1}$$

ii) Choose the multipliers as (x,y,z)

$$(1) \Rightarrow \frac{xdx}{x\left(x^2 - y^2 - z^2\right)} = \frac{ydy}{y\left(2xy\right)} = \frac{zdz}{z\left(2zx\right)}$$

$$= \frac{xdx + ydy + zdz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{xdx + ydy + zdz}{x^3 + xy^2 + xz^2}$$

	$= \dfrac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$ From 3[rd] and last ratio $\dfrac{dz}{(2zx)} = \dfrac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$ $\dfrac{dz}{z} = \dfrac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)}$ Integrating $\int \dfrac{dz}{z} = \int \dfrac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)}$ $\log z = \log\left(x^2 + y^2 + z^2\right) \qquad \because \int \dfrac{f'(x)}{f(x)} dx = \log\left[f(x)\right]$ $\log z = \log\left(x^2 + y^2 + z^2\right) + \log c_2$ $\log z - \log\left(x^2 + y^2 + z^2\right) = \log c_2$ $\log \dfrac{z}{x^2 + y^2 + z^2} = \log c_2$ $\boxed{\dfrac{z}{x^2 + y^2 + z^2} = c_2}$ The general solution is $\boxed{\phi\left(\dfrac{y}{z}, \dfrac{z}{x^2 + y^2 + z^2}\right) = 0}$
4.	**Solve** $(3z - 4y)p + (4x - 2z)q = 2y - 3x$ **Hint:** The multipliers are (x, y, z) & $(2, 3, 4)$ The general solution is $\phi\left(x^2 + y^2 + z^2, 2x + 3y + 4z\right) = 0$
5.	**Solve** $(y - xz)p + (yz - x)q = (x + y)(x - y)$ **Hint:** The multipliers are (x, y, z) & $(y, x, 1)$ The general solution is $\phi\left(x^2 + y^2 + z^2, xy + z\right) = 0$
6.	**Solve** $x(y^2 + z)p + y(x^2 + z)q = z(x^2 - y^2)$ **Hint:** The multipliers are $\left(\dfrac{1}{x}, \dfrac{-1}{y}, \dfrac{1}{z}\right)$ & $(x, -y, -1)$ The general solution is $\phi\left(\dfrac{xz}{y}, x^2 - y^2 - 2z\right) = 0$

Homogeneous Linear PDE of second and higher order with constant co-efficient:

Consider the second order homogeneous linear PDE

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = f(x, y) \quad ----- \quad (1)$$

Let the differential operator $D = \dfrac{\partial}{\partial x}$ & $D' = \dfrac{\partial}{\partial y}$

$$(1) \Rightarrow \left(D^2 + a_1 DD' + a_2 D'^2\right) z = f(x, y) \quad ----- \quad (2)$$

The general solution of equation (2) is

z = complementary function+ Particular Integral=C.F+P.I

To find complementary Function:

1. Write the Auxiliary equation by putting $D = m, D' = 1, z = 1,$ & $RHS = 0$

   $(2) \Rightarrow m^2 + a_1 m + a_2 = 0$

2. Solve the auxiliary equation, we get the roots of m. Say the roots are m_1, m_2
3. Comparing the roots of m and write the complementary function.

   Case 1: The Roots are real and distinct : say $m_1 \neq m_2$

   $$C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$$

   Case 2: The Roots are real and equal : say $m_1 = m_2 = m$

   $$C.F = f_1(y + mx) + x f_2(y + mx)$$

Note: If the roots are $m = \alpha \pm i\beta$

then $C.F = f_1[y + (\alpha + i\beta)] + f_2[y + (\alpha - i\beta)]$

To find Particular Integral :

Type : I

If $RHS = e^{ax+by}$ then

$$P.I = \frac{1}{f(D, D')} e^{ax+by}$$

Rule: $D = a$ & $D' = b$

$$P.I = \frac{1}{f(a,b)} e^{ax+by}, \text{Provided Denominator} \neq 0$$

If Denominator $= 0$, then 1) multiply the numerator by x 2) differentiating denominator partially w.r.to D

$$P.I = \frac{x}{f'(D, D')} e^{ax+by}$$

$$P.I = \frac{1}{f'(a,b)} e^{ax+by}, \text{ Provided Denominator} \neq 0 \because \text{Replace } D = a \text{ \& } D' = b$$

Continuing this process until we get $Dr \neq 0$.

Type : II

If $RHS = \sin(ax + by)$ (or) $RHS = \cos(ax + by)$ then

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax + by)$$

Rule: $D^2 = -(a^2); DD' = (-ab); D'^2 = -(b^2)$

$$P.I = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax + by) \text{ ,Provided Dr} \neq 0$$

Note 1: After substitutions the denominator will be in terms of D & D'. Multiply and divide by D so that the denominator will have D^2 & DD' terms.

Note 2: After substitutions the denominator will be in terms of D & constant terms,

For eg. $P.I = \frac{1}{D-5} \sin(x - 2y)$

Take conjugate of denominator with constant term and multiplied with both numerator and denominator.

$$P.I = \frac{1}{D-5} \times \frac{D+5}{D+5} \sin(x - 2y) = \frac{1}{D^2 - 25} \sin(x - 2y)$$

Then apply the rule as usual.

Type : III

If $RHS = x^m y^n$ (polynomial type) **then**

$P.I = \dfrac{1}{f(D,D)} x^m y^n$, we bring this into a standard binomial format, by taking out highest power term of D.

(i.e) $P.I = [1 \pm f(D,D')]^{-1} x^m y^n$

This will be expanded by using the formulae

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$

Note:

$\dfrac{1}{D} = \int dx \; , D' = \dfrac{\partial}{\partial y}$

Type : IV (Exponential shifted rule)

If $\boxed{RHS = e^{ax+by}\cos(ax+by)}$ (or) $\boxed{RHS = e^{ax+by}\sin(ax+by)}$ (or) $\boxed{RHS = e^{ax+by}x^m y^n}$ then

$$P.I = \frac{1}{f(D,D')} e^{ax+by}\sin(ax+by)$$

$$P.I = e^{ax+by}\frac{1}{f(D+a, D'+b)}\sin(ax+by)$$

Here after apply the rule as we discussed in Type II&III

Type : V

If $\boxed{RHS = y\cos x}$ (or) $\boxed{RHS = y\sin x}$ then

Case 1:

$$P.I = \frac{1}{D - mD'} y\cos x$$

$$P.I = \frac{1}{D - mD'}\int (c - mx)\cos x\,dx \qquad \because Rule: \boxed{y = c - mx}$$

Case 2:

$$P.I = \frac{1}{D + mD'} y\cos x$$

$$P.I = \frac{1}{D + mD'}\int (c + mx)\cos x\,dx \qquad \because Rule: y = c - mx$$

Note: After integration we have to replace $c - mx = y$

1. **Solve** $\left(D^2 - DD' - 20D'^2\right) z = e^{5x+y} + \sin(4x - y)$

Solution:

Given $\left(D^2 - DD' - 20D'^2\right) z = e^{5x+y} + \sin(4x - y)$

To find C.F

The auxiliary equation is

$m^2 - m - 20 = 0 \qquad \because$ replace $D = m, D' = 1, z = 0$

$(m-5)(m+4) = 0$

$\boxed{m = 5, -4}$

$\therefore \boxed{C.F = f_1(y - 4x) + f_2(y + 5x)}$ $\because$The roots are real and distinct $\Rightarrow C.F = f_1(y + mx) + xf_2(y + mx)$

To find P.I

$$P.I = \frac{1}{D^2 - DD' - 20D'^2}\left[e^{5x+y} + \sin(4x - y)\right]$$

$P.I = P.I_1 + P.I_2$

	$P.I_1 = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y}$ Rule: replace $D = 5$ & $D' = 1$ Type:1 $= \frac{1}{25-5-20} e^{5x+y} = \frac{1}{0} e^{5x+y}$ $\because$ Introducing x in Nr.Diff Dr.partially w.r.to D $= \frac{x}{2D - D'} e^{5x+y}$ Rule: replace $D = 5$ & $D' = 1$ $= \frac{x}{10-1} e^{5x+y}$ $\boxed{P.I_1 = \frac{x}{9} e^{5x+y}}$ $P.I_2 = \frac{1}{D^2 - DD' - 20D'^2} \sin(4x - y)$ here $a = 4$ & $b = -1$ (Type:2) Rule: replace $D^2 = -(a^2) = -16; D'^2 = -(b^2) = -1$ & $DD' = -(ab) = 4$ $P.I_2 = \frac{1}{-16-4-20(-1)} \sin(4x - y) = \frac{1}{0} \sin(4x - y)$ $= \frac{x}{2D - D'} \sin(4x - y)$ $\because$ Introducing x in Nr.Diff Dr.partially w.r.to D $= \frac{x}{2D - D'} \times \frac{D}{D} \sin(4x - y)$ $= \frac{xD}{2D^2 - DD'} \sin(4x - y)$ $= \frac{xD}{2(-16) - 4} \sin(4x - y)$ $= \frac{xD\left(\sin(4x - y)\right)}{-32 - 4}$ $P.I_2 = \frac{4x\cos(4x+3)}{-36} = \frac{-1}{9} x \cos(4x + 3)$ $\because \frac{d}{dx}(\sin nx) = n\cos nx$ $\boxed{P.I_2 \frac{-1}{9} x \cos(4x+3)}$ $\boxed{P.I = \frac{x}{9} e^{5x+y} - \frac{1}{9} x \cos(4x+3) = \frac{x}{9}\left[e^{5x+y} - \cos(4x+3)\right]}$ The general solution is $z = C.F + P.I$ $\boxed{z = f_1(y - 4x) + f_2(y + 5x) + \frac{x}{9}\left[e^{5x+y} - \cos(4x+3)\right]}$
2.	Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial x} = e^{x+2y} + 4\sin(x + y)$ **Solution: same as previous problem** **Hint:** Given $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial x} = e^{x+2y} + 4\sin(x + y)$ $\left(D^3 - 2D^2D'\right) z = e^{x+2y} + 4\sin(x + y)$ $\because D = \frac{\partial}{\partial x}$ & $D' = \frac{\partial}{\partial y}$ $\boxed{m = 0, 0, 2}$ $C.F = f_1(y) + xf_2(y) + f_2(y + 2x)$

$$P.I = \frac{-1}{3} e^{x+2y} - 4\cos(x+2y)$$

3. **Solve** $\left(D^2 + DD' - 6D'^2\right) z = x^2 y + e^{3x+y}$

Solution:

Given $\left(D^2 + DD' - 6D'^2\right) z = x^2 y + e^{3x+y}$

To find C.F

The auxiliary equation is

$m^2 + m - 6 = 0$ $\quad\because$ replace $D = m, D' = 1, z = 0$

$(m-2)(m+3) = 0$

$\boxed{m = 2, -3}$

$\therefore \boxed{C.F = f_1(y-3x) + f_2(y+2x)}$ $\quad\because$The roots are real and distinct $\Rightarrow C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$

To find P.I

$$P.I = \frac{1}{D^2 + DD' - 6D'^2}\left[x^2 y + e^{3x+y}\right]$$

$$P.I = P.I_1 + P.I_2$$

$$P.I_1 = \frac{1}{D^2 + DD' - 6D'^2} x^2 y \qquad (\text{Type:3})$$

$$= \frac{1}{D^2\left[1 + \left(\frac{DD' - 6D'^2}{D^2}\right)\right]} x^2 y$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right)\right]^{-1} x^2 y$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{D'}{D} - \frac{6D'^2}{D^2}\right) + \dots\right] x^2 y \qquad \because (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$= \frac{1}{D^2}\left[1 - \frac{D'}{D} + \frac{6D'^2}{D^2}\right] x^2 y \qquad \because D' = \frac{\partial}{\partial y}$$

$$= \frac{1}{D^2}\left[x^2 y - \frac{D'(x^2 y)}{D} + \frac{6D'^2(x^2 y)}{D^2}\right]$$

$$= \frac{1}{D^2}\left[x^2 y - \frac{x^2}{D}\right]$$

$$= \frac{1}{D^2}\left[x^2 y - \int x^2 dx\right] \qquad \because \frac{1}{D} = \int dx$$

$$= \frac{1}{D^2}\left[x^2 y - \frac{x^3}{3}\right]$$

$$= \frac{1}{D}\int\left[x^2 y + \frac{x^3}{3}\right] dx$$

$$= \int\left[\frac{x^3 y}{3} - \frac{x^4}{12}\right] dx$$

$$\boxed{P.I_1 = \frac{x^4 y}{12} - \frac{x^5}{60}}$$

$$P.I_2 = \frac{1}{D^2 + DD' - 6D'^2} e^{3x+y}$$ Rule: replace $D = a = 3$ & $D' = b = 1$ $\quad$ Type:1

$$= \frac{1}{9+3-6(1)} e^{3x+y} = \frac{1}{6} e^{3x+y}$$

$$\boxed{P.I_2 = \frac{1}{6} e^{3x+y}}$$

$$\boxed{P.I = \frac{x^4 y}{12} - \frac{x^5}{60} + \frac{1}{6} e^{3x+y}}$$

The general solution is

$z = C.F + P.I$

$$\boxed{z = f_1(y-3x) + f_2(y+2x) + \frac{x^4 y}{12} - \frac{x^5}{60} + \frac{1}{6} e^{3x+y}}$$

4.

Solve $\left(D^2 + 2DD' + D'^2\right) z = x^2 y + e^{x-y}$

Solution: same as previous problem

Hint:

$m = -1, -1$

$C.F = f_1(y-x) + x f_2(y-x)$

$$P.I = \frac{x^4 y}{12} - \frac{x^5}{30} + \frac{x^2}{2} e^{x-y}$$

5.

Solve $\left(D^2 - 6DD' + 5D'^2\right) z = xy + e^x \sinh y$

Solution:

Given $\left(D^2 - 6DD' + 5D'^2\right) z = xy + e^x \sinh y$

$$= xy + e^x \left[\frac{e^y - e^{-y}}{2}\right] \qquad \because \boxed{\sinh\theta = \frac{e^\theta - e^{-\theta}}{2}}, \text{ here } \theta = y$$

$$= xy + \left[\frac{e^x e^y - e^x e^{-y}}{2}\right]$$

$$\left(D^2 - 6DD' + 5D'^2\right) z = xy + \frac{e^{x+y}}{2} - \frac{e^{x-y}}{2} \qquad \because e^a e^b = e^{a+b}$$

To find C.F

The auxiliary equation is

$m^2 - 6m + 5 = 0$ $\qquad \because$ replace $D = m, D' = 1, z = 0$

$(m-1)(m-5) = 0$

$\boxed{m = 1, 2}$

$\therefore \boxed{C.F = f_1(y+x) + f_2(y+5x)}$ $\qquad \because$ The roots are real and distinct $\Rightarrow C.F = f_1(y + m_1 x) + x f_2(y + m_2 x)$

To find P.I

$$P.I = \frac{1}{D^2 - 6DD' + 5D'^2}\left[xy + \frac{e^{x+y}}{2} - \frac{e^{x-y}}{2}\right]$$

$P.I = P.I_1 + P.I_2 - P.I_3 \quad -----(1)$

To find $P.I_1$

$$P.I_1 = \frac{1}{D^2 - 6DD' + 5D'^2} xy \qquad (\text{Type:3})$$

$$= \frac{1}{D^2\left[1 + \left(\frac{-6DD' + 5D'^2}{D^2}\right)\right]} xy$$

$$= \frac{1}{D^2}\left[1 + \left(\frac{-6D'}{D} + \frac{5D'^2}{D^2}\right)\right]^{-1} xy$$

$$= \frac{1}{D^2}\left[1 - \left(\frac{-6D'}{D} + \frac{5D'^2}{D^2}\right) + ...\right] xy \quad \because (1+x)^{-1} = 1 - x + x^2 - x^3 + ...$$

$$= \frac{1}{D^2}\left[1 + \frac{6D'}{D} - \frac{5D'^2}{D^2} + ...\right] xy$$

$$= \frac{1}{D^2}\left[xy + \frac{6x}{D} - \frac{5(0)}{D^2} + ...\right] xy \quad \because D'(y) = \frac{\partial}{\partial y}(y) = 1 \ \& \ D'^2(y) = \frac{\partial^2}{\partial y^2}(y) = 0$$

$$= \frac{1}{D^2}\left[xy + 6\int x dx\right] = \frac{1}{D^2}\left[xy + \frac{6x^2}{2}\right] \quad \because \frac{1}{D} = \int dx$$

$$= \frac{1}{D}\int \left(xy + 3x^2 \right) dx = \frac{1}{D}\left[\frac{x^2 y}{2} + \frac{3x^3}{3}\right] = \int \left[\frac{x^2 y}{2} + x^3\right] dx$$

$$\boxed{P.I_1 = \frac{x^3 y}{6} + \frac{x^4}{4}}$$

To find P.I₂

$$P.I_2 = \frac{1}{D^2 - 6DD' + 5D'^2} \frac{e^{x+y}}{2} \qquad \text{Type:1}$$

$$= \left(\frac{1}{2}\right)\frac{1}{1-6+5} e^{x+y} = \left(\frac{1}{2}\right)\frac{1}{0} e^{x+y} \qquad \text{Rule: replace } D = a = 1 \ \& \ D' = b = 1$$

$$= \left(\frac{1}{2}\right)\frac{x}{2D - 6D' + 0} e^{x+y}$$

$$= \left(\frac{1}{2}\right)\frac{x}{2-6} e^{x+y}$$

$$= \left(\frac{1}{2}\right)\frac{-x}{4} e^{x+y}$$

$$\boxed{P.I_2 = \frac{-x}{8} e^{x+y}}$$

To find P.I₃

$$P.I_3 = \frac{1}{D^2 - 6DD' + 5D'^2} \frac{e^{x-y}}{2} \qquad \text{Type:1}$$

$$= \left(\frac{1}{2}\right)\frac{1}{1+6+5} e^{x+y} \qquad \text{Rule: replace } D = a = 1 \ \& \ D' = b = -1$$

$$\boxed{P.I_3 = \frac{1}{24} e^{x+y}}$$

$$(1) \Rightarrow P.I = P.I_1 + P.I_2 - P.I_3 = \boxed{\frac{x^3 y}{6} + \frac{x^4}{4} - \frac{x}{8} e^{x+y} - \frac{1}{24} e^{x+y}}$$

The general solution is

$z = C.F + P.I$

$$\boxed{z = f_1(y+x) + f_2(y+5x) + \frac{x^3 y}{6} + \frac{x^4}{4} - \frac{x}{8} e^{x+y} - \frac{1}{24} e^{x+y}}$$

6. **Solve** $\left(D^2 + 2DD' + D'^2\right) z = \sinh(x+y) + e^{x+2y}$

Solution: same as previous problem

Hint:

$$\left(D^2 + 2DD' + D'^2\right) z = \sinh(x+y) + e^{x+2y}$$

$$\left(D^2 + 2DD' + D'^2\right) z = \frac{e^{x+y} - e^{(x-y)}}{2} + e^{x+2y} = \frac{e^{x+y}}{2} - \frac{e^{-x-y}}{2} + e^{x+2y}$$

$$\left(D^2 + 2DD' + D'^2\right) z = \frac{e^{x+y}}{2} - \frac{e^{-x-y}}{2} + e^{x+2y}$$

$m = -1, -1$

$C.F = f_1(y-x) + x f_2(y-x)$

$$P.I = \frac{e^{x+y}}{8} - \frac{e^{-x-y}}{8} - \frac{e^{x+2y}}{9}$$

7. Solve $\left(D^2 - 4DD' + 4D'^2\right) z = e^{x+2y}$

Solution:

Hint: $m = 2, 2$

$C.F = f_1(y+x) + f_2(y+5x)$

$$P.I = \frac{x^2}{2} e^{2x+y}$$

8. Solve $\left(2D^2 - 5DD' + 2D'^2\right) z = 5\sin(2x+y)$

Solution:

Given $\left(2D^2 - 5DD' + 2D'^2\right) z = 5\sin(2x+y)$

To find C.F

The auxiliary equation is

$2m^2 - 5m + 2 = 0$ $\quad \because$ replace $D = m, D' = 1, z = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{here } a = 2, b = -5, c = 2$$

$$= \frac{-(-5) \pm \sqrt{25 - 4(2)(2)}}{2(2)}$$

$$= \frac{5 \pm \sqrt{25-16}}{4} = \frac{5 \pm \sqrt{9}}{4}$$

$$= \frac{5 \pm \sqrt{25-16}}{4} = \frac{5 \pm \sqrt{9}}{4} = \frac{5 \pm 3}{4}$$

$$m = \frac{5+3}{4}, \; m = \frac{5-3}{4} \Rightarrow \boxed{m = 2, \; m = \frac{1}{2}}$$

$\therefore \boxed{C.F = f_1(y+2x) + f_2\left(y + \frac{x}{2}\right)}$ $\quad \because$ The roots are real and distinct $\Rightarrow C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$

To find P.I

$$P.I = \frac{1}{2D^2 - 5DD' + 2D'^2} 5\sin(2x+y) \qquad \text{Type:2}$$

$$P.I = \frac{1}{2D^2 - 5DD' + 2D'^2} 5\sin(2x+y) \qquad \text{here } a = 2 \;\&\; b = 1$$

Rule: replace $D^2 = -(a^2) = -4; D'^2 = -(b^2) = -1 \;\&\; DD' = -(ab) = -2$

$$P.I = \frac{1}{2(-4) - 5(-2) + 2(-1)} 5\sin(2x+y)$$

$$P.I = \frac{1}{-8 + 10 - 2} 5\sin(2x+y) = \frac{1}{0} 5\sin(2x+y)$$

$$= \frac{x}{4D - 5D' + 0} 5\sin(2x + y) \quad \{\text{Introducing } x \text{ in Nr.\& Diff Dr.partially w.r.to } D\}$$

$$= \frac{x}{4D - 5D'} \times \frac{D}{D} 5\sin(2x + y)$$

$$= 5\frac{xD}{4D^2 - 5DD'} \sin(2x + y)$$

$$= 5\frac{xD}{4(-4) - 5(-2)} \sin(2x + y) = 5\frac{xD[\sin(2x + y)]}{-16 + 10}$$

$$= \frac{5}{-6} x[2\cos(2x + y)]$$

$$\boxed{P.I = \frac{-5}{3} x\cos(2x + y)}$$

The general solution is

$z = C.F + P.I$

$$\boxed{z = f_1(y + 2x) + f_2\left(y + \frac{x}{2}\right) - \frac{5}{3} x\cos(2x + y)}$$

9. Solve the equation $\left(D^3 + D^2D' - 4DD'^2 - 4D'^3\right) z = \cos(2x + y)$

Solution:

Given $\left(D^3 + D^2D' - 4DD'^2 - 4D'^3\right) z = \cos(2x + y)$

To find C.F

The auxiliary equation is

$m^3 + m^2 - 4m - 4 = 0 \qquad \because$ replace $D = m, D' = 1, z = 0$

$m^2(m + 1) - 4(m + 1) = 0$

$(m + 1)\left(m^2 - 4\right) = 0$

$$\boxed{m = -1, -2, 2}$$

$$\therefore \boxed{C.F = f_1(y - x) + f_2(y - 2x) + f_3(y + 2x)}$$

To find P.I

$$P.I = \frac{1}{D^3 + D^2D' - 4DD'^2 - 4D'^3} \cos(2x + y) \quad here\ a = 2, b = 1 \quad (\text{Type:2})$$

Rule: replace $D^2 = -(a^2) = -4; D'^2 = -(b^2) = -1 \;\&\; DD' = -(ab) = -2$

$$P.I = \frac{1}{-4D - 4D' - 4D(-1) - 4(-1)D'} \cos(2x + y)$$

$$= \frac{1}{-4D - 4D' + 4D + 4D'} \cos(2x + y) == \frac{1}{0} \cos(2x + y)$$

$$P.I = \frac{x}{3D^2 + 2DD' - 4D'^2 - 0} \cos(2x + y) \quad \{\text{Introducing } x \text{ in Nr.\& Diff Dr.partially w.r.to } D\}$$

$$= \frac{x}{3(-4) + 2(-2) - 4(-1)} \cos(2x + y) = \frac{x}{-12 - 4 + 4} \cos(2x + y)$$

$$\boxed{P.I = \frac{-x}{12} \cos(2x + y)}$$

The general solution is

$z = C.F + P.I$

$$\boxed{z = f_1(y - x) + f_2(y - 2x) + f_3(y + 2x) - \frac{x}{12} \cos(2x + y)}$$

10. **Solve** $\left(D^3 - 7DD'^2 - 6D'^3\right) z = \cos(x+2y) + 4$

Solution:

Given $\left(D^3 - 7DD'^2 - 6D'^3\right) z = \cos(x+2y) + 4$

To find C.F

The auxiliary equation is

$m^3 - 7m - 6 = 0 \qquad \because$ replace $D = m, D' = 1, z = 0$

m=-1	1	0	-7	6
	0	-1	1	6
	1	-1	-6	0

$m = -1, m^2 - m - 6 = 0$

$m = -1, (m-3)(m+2) = 0$

$\boxed{m = -1, -2, 3}$

$\therefore \boxed{C.F = f_1(y-x) + f_2(y-2x) + f_3(y+3x)}$

To find P.I

$$P.I = \frac{1}{D^3 - 7DD'^2 - 6D'^3}\left[\cos(x+2y) + 4\right]$$

$P.I = P.I_1 + P.I_2$

$$P.I_1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3}\cos(x+2y) \quad here\ a = 1, b = 2 \ \ (\text{Type:}2)$$

Rule: replace $D^2 = -(a^2) = -1; D'^2 = -(b^2) = -4 \ \& \ DD' = -(ab) = -2$

$$P.I_1 = \frac{1}{(-1)D - 7D(-2) - 6(-2)D'}\cos(x+2y) = \frac{1}{-D + 14D + 12D'}\cos(x+2y)$$

$$= \frac{1}{13D + 12D'} \times \frac{D}{D}\cos(x+2y)$$

$$= \frac{D}{13D^2 + 12D'D}\cos(x+2y)$$

$$= \frac{D}{13(-1) + 12(-2)}\cos(x+2y) = \frac{D\cos(x+2y)}{-13-24}$$

$$\boxed{P.I_1 = \frac{-\sin(x+2y)}{37}}$$

To find P.I₂

$$P.I_2 = \frac{1}{D^3 - 7DD'^2 - 6D'^3}4e^{0x+0y} \qquad \because e^0 = 1 \qquad \text{Type:}1$$

$$= \frac{1}{D^3 - 7DD'^2 - 6D'^3}4e^{0x+0y} = \frac{1}{0}4e^{0x+0y} \qquad \text{Rule: Replace } D = 0, D' = 0$$

$$= \frac{x}{3D^2 - 7D'^2 - 0}4e^{0x+0y} = \frac{x}{0}4e^{0x+0y}$$

$$= \frac{x}{6D - 0}4e^{0x+0y} = \frac{x}{0}4e^{0x+0y}$$

$$= \frac{x^2}{6}4$$

$$P.I_2 = \frac{2x^2}{3}$$

$$P.I = \frac{-1}{37}\sin(x+2y) + \frac{2x^2}{3}$$

The general solution is

$z = C.F + P.I$

$$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{37}\sin(x+2y) + \frac{2x^2}{3}$$

11. **Solve** $\left(D^2 + 3DD' - 4D'^2\right) z = xy + \cos(2x+y)$

Solution: same as previous problem

Hint:

$m = -4, 1$

$C.F = f_1(y-4x) + xf_2(y+x)$

$$P.I = \frac{-1}{6}\cos(2x+y) + \frac{x^3 y}{6} - \frac{x^4}{8}$$

12. **Solve** $\left(D^2 + 3DD' - 4D'^2\right) z = x + \sin y$

Hint:

$m = -4, 1$

$C.F = f_1(y-4x) + xf_2(y+x)$

$$P.I = \frac{1}{D^2 + 3DD' - 4D'^2}\left[x + \sin(0x+y)\right] = \ldots = \frac{x^3}{6} + \frac{\sin y}{4}$$

13. **Solve** $\left(D^2 - DD' - 2D'^2\right) z = (2x+3y) + e^{3x+4y}$

Hint:

$m = -1, 2$

$C.F = f_1(y-x) + f_2(y+2x)$

$$P.I = \frac{5x^3}{6} + \frac{3x^2 y}{2} + \frac{1}{35}e^{3x+4y}$$

14. Solve $\left(D^2 - 2DD' + D'^2\right) z = (2+4x)e^{x+2y}$

Solution:

Given $\left(D^2 - 2DD' + D'^2\right) z = x^2 y^2 e^{x+2y}$

To find C.F

The auxiliary equation is

$m^2 - 2m + 1 = 0$ $\quad\because$ replace $D = m, D' = 1, z = 0$

$(m-1)(m-1) = 0$

$m = 1, 1$

$\therefore$ $C.F = f_1(y+x) + xf_2(y+x)$

To find P.I

$$P.I = \frac{1}{D^2 - 2DD' + D'^2}(2+4x)e^{x+2y} \quad \text{here } a = 2, b = 1 \ (\text{Type:4})$$

$$P.I = \frac{1}{(D-D')^2}(2+4x)\,e^{x+2y}$$

Rule:replace $D = D + a = D + 1; D' = D' + b = D' + 2$

$$P.I = e^{x+2y}\frac{1}{(D+1-D'-2)^2}(2+4x)$$

	$= e^{x+2y} \dfrac{1}{(D-D'-1)^2}(2+4x)$ $= e^{x+2y} \dfrac{1}{[-(1-D+D')]^2}(2+4x) = e^{x+2y} \dfrac{1}{[1-(D-D')]^2}(2+4x)$ $= e^{x+2y} [1-(D-D')]^{-2}(2+4x)$ $= e^{x+2y} [1+2(D-D')+3(D-D')^2+...](2+4x)$ $\because (1-x)^{-2} = 1+2x+3x^2+4x^3+...$ $= e^{x+2y} [1+2D-2D'+3(D^2-2DD'+D'^2)+...](2+4x)$ $= e^{x+2y} [1+2D-2D'+3D^2-6DD'+3D'^2](2+4x)$ $= e^{x+2y} [1+2D+3D^2](2+4x)$ $\because$ there is no y term in RHS, neglect the term D' $= e^{x+2y} [(2+4x)+2D(2+4x)+3D^2(2+4x)]$ $= e^{x+2y} [2+4x+2(4)+0]$ $P.I = e^{x+2y}[4x+10]$ **The general solution is** $z = C.F + P.I$ $z = f_1(y+x) + xf_2(y+x) + e^{x+2y}[4x+10]$
15.	**Solve** $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = e^{x-y}\sin(2x+3y)$ **Solution:** Given $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = e^{x-y}\sin(2x+3y)$ $(D^2 - D'^2)z = e^{x-y}\sin(2x+3y)$ **To find C.F** The auxiliary equation is $m^2 - 1 = 0$ $\because$ replace $D = m, D' = 1, z = 0$ $m^2 = 1 \Rightarrow m = -1, 1$ $\therefore C.F = f_1(y-x) + f_2(y+x)$ **To find P.I** $P.I = \dfrac{1}{D^2 - D'^2} e^{x-y}\sin(2x+3y)$ $here\ a = 1, b = -1$ (Type:4) Rule:replace $D = D + a = D + 1; D' = D' + b = D' - 1$ $P.I = e^{x-y} \dfrac{1}{(D+1)^2 - (D'-1)^2}\sin(2x+3y)$ $= e^{x-y} \dfrac{1}{D^2 + 2D + 1 - D'^2 + 2D' - 1}\sin(2x+3y)$ $= e^{x-y} \dfrac{1}{D^2 + 2D - D'^2 + 2D'}\sin(2x+3y)$ Here $a = 2, b = 3$ Rule: replace $D^2 = -(a^2) = -4; D'^2 = -(b^2) = -9$ & $DD' = -(ab) = -6$ $= e^{x-y} \dfrac{1}{-4 + 2D - (-9) + 2D'}\sin(2x+3y) = e^{x-y} \dfrac{1}{2D + 2D' + 5}\sin(2x+3y)$ $= e^{x-y} \dfrac{1}{2D + 2D' + 5} \times \dfrac{D}{D}\sin(2x+3y)$

$$= e^{x-y} \frac{D}{2D^2 + 2DD' + 5D} \sin(2x + 3y)$$

$$= e^{x-y} \frac{D}{-8 - 12 + 5D} \sin(2x + 3y) = e^{x-y} \frac{D}{5D - 20} \sin(2x + 3y)$$

If we multiply and divide by D, we can not get the term D^2, D'^2 term, so we take conjugate for constent term and multiplied with both Nr. & Dr.

$$= e^{x-y} \frac{D}{5D - 20} \times \frac{5D + 20}{5D + 20} \sin(2x + 3y)$$

$$= e^{x-y} \frac{5D^2 + 20D}{25D^2 - 400} \sin(2x + 3y)$$

$$= e^{x-y} \frac{5D^2 \sin(2x + 3y) + 20D \sin(2x + 3y)}{25(-4) - 400}$$

$$= e^{x-y} \frac{5D \cos(2x + 3y) \times 2 + 20 \cos(2x + 3y) \times 2}{-100 - 400}$$

$$= e^{x-y} \frac{-20 \sin(2x + 3y) + 40 \cos(2x + 3y)}{-500}$$

$$\boxed{P.I = \frac{e^{x-y}}{25} \left[\sin(2x + 3y) - 2\cos(2x + 3y)\right]}$$

The general solution is

$z = C.F + P.I$

$$\boxed{z = f_1(y - x) + f_2(y + x) + \frac{e^{x-y}}{25} \left[\sin(2x + 3y) - 2\cos(2x + 3y)\right]}$$

16. **Solve** $\left(D^2 + DD' - 6D'^2\right) z = y \cos x$

Solution:

Given $\left(D^2 + DD' - 6D'^2\right) z = y \cos x$

To find C.F

The auxiliary equation is

$m^2 + m - 6 = 0$ $\quad\because$ replace $D = m, D' = 1, z = 0$

$(m - 2)(m + 3) = 0$

$\boxed{m = 2, -3}$

$\therefore \boxed{C.F = f_1(y - 3x) + f_2(y + 2x)}$ $\quad\because$ The roots are real and distinct $\Rightarrow C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$

To find P.I

$$P.I = \frac{1}{D^2 + DD' - 6D'^2} y \cos x \qquad \text{Type: 5}$$

$$= \frac{1}{(D + 3D')(D - 2D')} y \cos x$$

$$= \frac{1}{(D + 3D')} \int (c - 2x) \cos x \, dx \qquad \because Rule: y = c - mx \text{ here } m = 2$$

$$= \frac{1}{(D + 3D')} \left[(c - 2x)(\sin x) - (-2)(-\cos x)\right] \qquad \because \int uv dx = uv_1 - u'v_2 + \ldots$$

$$= \frac{1}{(D + 3D')} \left[y \sin x - 2\cos x\right]$$

$$= \frac{1}{(D + 3D')} \int \left[(c + 3x) \sin x - 2\cos x\right] dx \qquad \because Rule: y = c + mx \text{ here } m = 3$$

$$= (c + 3x)(-\cos x) - (3)(-\sin x) - 2\sin x \qquad \because \ y = c + 3x$$

$= -y\cos x + 3\sin x - 2\sin x$

$\boxed{P.I = \sin x - y\cos x}$

The general solution is

$z = C.F + P.I$

$\boxed{z = f_1(y-3x) + f_2(y+2x) + \sin x - y\cos x}$

17. **Solve** $\left(D^2 - 5DD' + 6D'^2\right) z = y\sin x$

Solution:

Given $\left(D^2 - 5DD' + 6D'^2\right) z = y\sin x$

To find C.F

The auxiliary equation is

$m^2 - 5m + 6 = 0 \qquad \because$ replace $D = m, D' = 1, z = 0$

$(m-2)(m-3) = 0$

$\boxed{m = 2, 3}$

$\therefore \boxed{C.F = f_1(y+3x) + f_2(y+2x)} \quad \because$ The roots are real and distinct $\Rightarrow C.F = f_1(y+m_1x) + f_2(y+m_2x)$

To find P.I

$$P.I = \frac{1}{D^2 - 5DD' + 6D'^2}\, y\sin x \qquad \text{Type: 5}$$

$$= \frac{1}{(D-3D')(D-2D')}\, y\sin x$$

$$= \frac{1}{(D-3D')}\int (c-2x)\sin x\,dx \qquad \because Rule: y = c - mx \text{ here } m = 2$$

$$= \frac{1}{(D-3D')}\left[(c-2x)(-\cos x) - (-2)(-\sin x)\right] \qquad \because \int uv\,dx = uv_1 - u'v_2 + \dots$$

$$= \frac{1}{(D-3D')}\left[-y\cos x - 2\sin x\right] = \frac{-1}{(D-3D')}\left[y\cos x + 2\sin x\right]$$

$$= -\int \left[(c-3x)\cos x + 2\sin x\right]dx \qquad \because Rule: y = c - mx \text{ here } m = 3$$

$$= -\left[(c-3x)(\sin x) - (-3)(-\cos x) + 2(-\cos x)\right] \qquad \because y = c - 3x$$

$$= -\left[y\sin x - 3\cos x - 2\cos x\right]$$

$\boxed{P.I = 5\cos x - y\sin x}$

The general solution is

$z = C.F + P.I$

$\boxed{z = f_1(y+3x) + f_2(y+2x) + 5\cos x - y\sin x}$

Non-Homogeneous Linear PDE of second and higher order with constant co-efficient:

Consider the second order non-homogeneous linear PDE

$$\frac{\partial^2 z}{\partial x^2} + a_1\frac{\partial^2 z}{\partial x\partial y} + a_2\frac{\partial^2 z}{\partial y^2} + a_3\frac{\partial z}{\partial x} + a_4\frac{\partial z}{\partial y} = f(x,y) \quad ----- (1)$$

Let the differential operator $D = \dfrac{\partial}{\partial x}$ & $D' = \dfrac{\partial}{\partial y}$

$$(1) \Rightarrow \left(D^2 + a_1DD' + a_2D'^2 + a_3D + a_4D'\right) z = f(x,y) \quad ----- (2)$$

The general solution of equation (2) is

$\boxed{z = \text{complementary function+ Particular Integral=C.F+P.I}}$

To find complementary Function:

Case : I

The given PDE will bring into the form of $(D - m_1 D' - C_1)(D - m_2 D' - C_2)z = 0$

$$C.F = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x)$$

Case : II

The given PDE will bring into the form of $(D - mD' - C)^2 z = 0$

$$C.F = e^{cx} f_1(y + mx) + xe^{cx} f_2(y + mx)$$

Note: Particular Integral can be obtained, similar like in Homogeneous types.

1. **Solve** $\left(D^2 + 2DD' + D'^2 - 2D - 2D'\right) z = e^{3x+y} + \sin(x + 2y)$

Solution:

Given $\left(D^2 + 2DD' + D'^2 - 2D - 2D'\right) z = e^{3x+y} + \sin(x + 2y)$

To find C.F

$$\left((D + D')^2 - 2(D + D')\right) z = 0$$

$$(D + D')(D + D' - 2)z = 0 \quad ------(1)$$

This is of the form

$$(D - m_1 D' - C_1)(D - m_2 D' - C_2)z = 0 ------(2)$$

Comparing (1) & (2)

$$m_1 = -1, C_1 = 0, m_2 = -1, C_2 = 2$$

$$C.F = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x)$$

$$C.F = f_1(y - x) + e^{2x} f_2(y - x) \qquad \because e^0 = 1$$

To find P.I

$$P.I = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'}\left(e^{3x+y} + \sin(x + 2y)\right)$$

$$P.I = P.I_1 + P.I_2$$

To find P.I₁

$$P.I_1 = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} e^{3x+y} \qquad \text{Rule:replace } D = 3 \ \& \ D' = 1 \ \text{ Type:1}$$

$$= \frac{1}{9 + 6 + 1 - 6 - 2} e^{3x+y}$$

$$P.I_1 = \frac{1}{8} e^{3x+y}$$

To find P.I₂

$$P.I_2 = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin(x + 2y) \qquad \text{here } a = 1, b = 2 \quad \text{type:2}$$

Rule: replace $D^2 = -(a^2) = -1; D'^2 = -(b^2) = -4 \,\&\, DD' = -(ab) = -2$

$$P.I_2 = \frac{1}{-1 + 2(-2) - 4 - 2D - 2D'} \sin(x + 2y) = \frac{1}{-1 - 4 - 4 - 2D - 2D'} \sin(x + 2y)$$

$$= \frac{-1}{2D + 2D' + 9} \times \frac{D}{D} \sin(x + 2y)$$

$$= \frac{-D}{2D^2 + 2D'^2 + 9D} \sin(x + 2y)$$

$$= \frac{-D}{-2 - 8 + 9D} \sin(x + 2y) = \frac{-D}{9D - 10} \sin(x + 2y)$$

If we multiply and divide by D, we can not get the term D^2, D'^2 term, so we take conjugate for constent term and multiplied with both Nr. & Dr.

$$= \frac{-D}{9D - 10} \times \frac{9D + 10}{9D + 10} \sin(x + 2y)$$

$$= \frac{-9D^2 - 10}{81D^2 - 100} \sin(x+2y)$$

$$= \frac{-9D^2 \sin(x+2y) - 10D\sin(x+2y)}{-81-100}$$

$$= \frac{1}{-181}\left[-9D\cos(x+2y) - 10\cos(x+2y)\right]$$

$$P.I_2 = \frac{1}{181}\left[9\sin(x+2y) - 10\cos(x+2y)\right]$$

$$P.I = \frac{1}{8}e^{3x+y} + \frac{1}{181}\left[9\sin(x+2y) - 10\cos(x+2y)\right]$$

The general solution is

$z = C.F + P.I$

$$z = f_1(y-x) + e^{2x} f_2(y-x) + \frac{1}{8}e^{3x+y} + \frac{1}{181}\left[9\sin(x+2y) - 10\cos(x+2y)\right]$$

2. Solve $\left(D^2 - D'^2 - 3D + 3D'\right) z = e^{3x+y} + 4$

Solution:

Given $\left(D^2 - D'^2 - 3D + 3D'\right) z = e^{3x+y} + 4$

To find C.F

$\left((D+D')(D-D') - 3(D-D')\right) z = 0$

$(D-D')(D+D'-3)z = 0 \quad - - - - - (1)$

This is of the form

$(D - m_1D' - C_1)(D - m_2D' - C_2)z = 0 - - - - - (2)$

Comparing (1) & (2)

$m_1 = 1, C_1 = 0, m_2 = -1, C_2 = 3.$

$C.F = e^{c_1x} f_1(y + m_1x) + e^{c_2x} f_2(y + m_2x)$

$C.F = f_1(y+x) + e^{3x} f_2(y-x) \qquad \because e_0 = 1$

To find P.I

$$P.I = \frac{1}{D^2 - D'^2 - 3D + 3D'}\left(e^{3x+y} + 4\right)$$

$P.I = P.I_1 + P.I_2$

$$P.I_1 = \frac{1}{D^2 - D'^2 - 3D + 3D'} e^{3x+y} \qquad \text{here } a = 3, b = 1 \qquad \text{type : 1}$$

$$= \frac{1}{9-1-9+3} e^{3x+y} \qquad \text{Rule:Replace } D = 3, D' = 1$$

$$P.I_1 = \frac{1}{2} e^{3x+y}$$

$$P.I_2 = \frac{1}{D^2 - D'^2 - 3D + 3D'} 4e^{0x+0y} \qquad \text{here } a = 0, b = 0 \qquad \text{type : 1}$$

$$= \frac{1}{0} 4e^{0x+0y} \qquad \text{Rule:Replace } D = 0, D' = 0$$

Introduce x in Nr. and Diff. Dr. Partially w.r.to.D in the previous step

$$= \frac{x}{2D - 0 - 3 + 0} 4e^{0x+0y}$$

$$= \frac{4x}{-3}$$

$$P.I_2 = \frac{-4x}{3}$$

$$P.I = \frac{1}{2}e^{3x+y} - \frac{4x}{3}$$

The general solution is

$z = C.F + P.I$

$$z = f_1(y+x) + e^{3x} f_2(y-x) + \frac{1}{2}e^{3x+y} - \frac{4x}{3}$$

3. **Solve** $\left(2D^2 - DD' - D'^2 + 6D + 3D'\right) z = xe^y$

Solution:

Given $\left(2D^2 - DD' - D'^2 + 6D + 3D'\right) z = xe^y$

To find C.F

$\left(2D^2 - DD' - D'^2 + 6D + 3D'\right) z = 0$

$(2D + D')(D - D') + 3(2D + D')z = 0$

$(2D + D')(D - D' + 3)z = 0$

$\left(D + \frac{D'}{2}\right)(D - D' + 3)z = 0$

This is of the form

$(D - m_1D' - C_1)(D - m_2D' - C_2)z = 0 - - - - - (2)$

Comparing (1) & (2)

$$m_1 = \frac{-1}{2}, C_1 = 0, m_2 = 1, C_2 = -3.$$

$C.F = e^{c_1x} f_1(y + m_1x) + e^{c_2x} f_2(y + m_2x)$

$$C.F = f_1\left(y - \frac{x}{2}\right) + e^{-3x} f_2(y + x) \qquad \because e^0 = 1$$

To find P.I

$$P.I = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} xe^y \qquad \text{Type : 4}$$

$$P.I = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} xe^{0x+y} \qquad \text{here } a = 0, b = 1$$

Rule:replace $D = D + a = D + 0 = D;\ D' = D' + b = D' + 1$

$$P.I = e^{0x+y} \frac{1}{2D^2 - D(D'+1) - (D'+1)^2 + 6D + 3(D'+1)} x$$

$$= e^y \frac{1}{2D^2 - DD' - D - D'^2 - 2D' - 1 + 6D + 3D' + 3} x \qquad \text{Type : 3}$$

$$= e^y \frac{1}{2D^2 - DD' + 5D - D'^2 + D' + 2} x$$

$$= e^y \frac{1}{2\left[1 + \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2}\right)\right]} x$$

[normally we take out highest power term of D in the homogeneous type, but it is not necessary in the non-homogeneous type]

$$= \frac{e^y}{2}\left[1 + \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2}\right)\right]^{-1} x \qquad \because D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$$

$$= \frac{e^y}{2}\left[1 - \left(\frac{2D^2 - DD' + 5D - D'^2 + D'}{2}\right) + \cdots\right] x \quad \text{[neglect the terms D', since } D'(x) = 0]$$

$$= \frac{e^y}{2}\left[1 - \left(\frac{5D}{2}\right) + \cdots\right] x \qquad \because D^2(x) = 0$$

$$= \frac{e^y}{2}\left[x - \frac{5(1)}{2}\right] \qquad \because D^2(x) = 0$$

$$= \frac{e^y}{2}\left[\frac{2x-5}{2}\right]$$

$$\boxed{P.I = \frac{e^y}{4}[2x-5]}$$

The general solution is

$z = C.F + P.I$

$$\boxed{z = f_1(y+x) + e^{3x} f_2(y-x) + \frac{e^y}{4}[2x-5]}$$