

**DEPARTMENT OF MATHEMATICS**

**NAME OF THE SUBJECT : TRANSFORMS & PARTIAL  
DIFFERENTIAL  
EQUATION**

**SUBJECT CODE : MA8353**

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**UNIT - I : PARTIAL DIFFERENTIAL EQUATION**

**UNIT I - PARTIAL DIFFERENTIAL EQUATIONS**

**Notations:** If  $z = f(x, y)$  then

$$p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}; r = \frac{\partial^2 z}{\partial x^2}; s = \frac{\partial^2 z}{\partial x \partial y}; t = \frac{\partial^2 z}{\partial y^2}$$

**Formation of PDE by eliminating arbitrary constants:**

Let the given equation be  $z = f(x, y, a, b)$  ----- (1)

**Step 1:** Differentiating (1) partially with respect to  $x$

$$\frac{\partial z}{\partial x} = p = f'(x, y, a, b) \text{ ----- (2)}$$

**Step 2:** Differentiating (1) partially with respect to  $y$

$$\frac{\partial z}{\partial y} = q = f'(x, y, a, b) \text{ ----- (3)}$$

**Step 3:** Eliminate  $a$  &  $b$  from (1) using (2) & (3)

1. **Obtain partial differential equation by eliminating arbitrary constant 'a' and 'b' from**

$$z = (x - a)^2 + (y - b)^2$$

**Solution:**

**Given**  $z = (x - a)^2 + (y - b)^2$  ----- (1)

Diff Partially w.r.t  $x$

$$\frac{\partial z}{\partial x} = 2(x - a) + 0$$

$$p = 2(x - a) \text{ ----- (2)}$$

Diff Partially w.r.t  $y$

$$\frac{\partial z}{\partial y} = 0 + 2(y - b)$$

$$q = 2(y - b) \text{ ----- (3)}$$

Eliminate  $a$  &  $b$  from (1) using (2) & (3)

$$(2) \Rightarrow (x - a) = \frac{p}{2} \text{ ----- (4)}$$

$$(3) \Rightarrow y - b = \frac{q}{2} \text{ ----- (5)}$$

Sub (4) & (5) in (1)

$$(1) \Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

The required the PDE is

$$\boxed{p^2 + q^2 = 4z}$$

2. **Form the partial differential equation by eliminating the arbitrary constants 'a' & 'b' from**

$$z = (x^2 + a)(y^2 + b).$$

**Solution:**

**Given**  $z = (x^2 + a)(y^2 + b)$  ----- (1)

Diff Partially w.r.t  $x$

$$\frac{\partial z}{\partial x} = p = 2x(y^2 + b) \text{ ----- (2)}$$

Diff Partially w.r.t  $y$

$$\frac{\partial z}{\partial y} = q = 2y(x^2 + a) \text{ ----- (3)}$$

Eliminate  $a$  &  $b$  from (1) using (2) & (3)

	$(2) \Rightarrow (y^2 + b) = \frac{p}{2x} \text{-----(4)}$ $(3) \Rightarrow x^2 + b = \frac{q}{2y} \text{-----(5)}$ <p>Sub (4) &amp; (5) in (1)</p> $(1) \Rightarrow z = \frac{p}{2x} - \frac{q}{2y}$ <p>The required the PDE is</p> <div style="border: 1px solid black; padding: 2px; display: inline-block;"><math>4xyz = pq</math></div>
3.	<p><b>Find the PDE of all planes having equal intercepts on the x and y axis.</b></p> <p><b>Solution:</b></p> <p>The intercept form of the plane equation is <math>\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1</math></p> <p>Given that equal intercepts on the x &amp; y axis <math>\Rightarrow a = b</math></p> $\therefore \frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1 \text{-----(1)}$ <p>Diff Partially w.r.t x</p> $\frac{1}{a} + 0 + \frac{1}{c} \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{1}{a} = -\frac{1}{c}p \text{-----(2)}$ <p>Diff Partially w.r.t y</p> $0 + \frac{1}{a} + \frac{1}{c} \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{1}{a} = -\frac{1}{c}q \text{-----(3)}$ <p>From (2) &amp; (3) <math>\frac{-1}{c}p = \frac{-1}{c}q</math> The required the PDE is <div style="border: 1px solid black; padding: 2px; display: inline-block;"><math>p = q</math></div></p>
4.	<p><b>Obtain the partial differential equation by eliminating arbitrary constants 'a' and 'b' from</b></p> $(x - a)^2 + (y - b)^2 + z^2 = r^2$ <p><b>Solution:</b></p> $(x - a)^2 + (y - b)^2 + z^2 = 1 \text{-----(1) - (1)}$ <p>Diff Partially w.r.t x</p> $2(x - a)(1 - 0) + 0 + 2z \frac{\partial z}{\partial x} = 0$ $\Rightarrow 2(x - a) + 2zp = 0 \text{----- (2)}$ <p>Diff Partially w.r.t y</p> $0 + 2(y - b)(1 - 0) + 2z \frac{\partial z}{\partial y} = 0$ $\Rightarrow 2(y - b) + 2zq = 0 \text{----- (3)}$ <p>Eliminate a &amp; b from (1) using (2) &amp; (3)</p> $(2) \Rightarrow x - a = -zp \text{----- (4)}$ $(3) \Rightarrow y - b = -zq \text{----- (5)}$ <p>Sub (4) &amp; (5) in (1)</p> $(-zp)^2 + (-zq)^2 + z^2 = 1$ <p>The required PDE is <div style="border: 1px solid black; padding: 2px; display: inline-block;"><math>z^2 (p^2 + q^2 + 1) = 1</math></div></p>
<b>Formation of PDE by eliminating arbitrary functions:</b>	
1.	<p><b>Eliminate the arbitrary function f from <math>z = f\left(\frac{y}{x}\right)</math> and form the PDE.</b></p> <p><b>Solution:</b></p>

	$z = f\left(\frac{y}{x}\right) \text{ -----(1)}$ <p>Diff Partially w.r.t x</p> $\frac{\partial z}{\partial x} = p = f'\left(\frac{y}{x}\right) \times \frac{-y}{x^2} \Rightarrow f'\left(\frac{y}{x}\right) = \frac{-px^2}{y} \text{ -----(2)}$ <p>Diff Partially w.r.t y</p> $\frac{\partial z}{\partial y} = q = f'\left(\frac{y}{x}\right) \times \frac{1}{x} \text{ -----(3)}$ <p>From (1) &amp; (2) <math>\frac{p}{q} = \frac{-y}{x} \Rightarrow \boxed{px + qy = 0}</math></p>
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2.	<p><b>Form the partial differential equation by eliminating <math>f</math> from <math>z = x^2 + 2f\left(\frac{1}{y}\right) + \log x</math>.</b></p> <p><b>Solution:</b></p> <p><b>Given</b> <math>z = x^2 + 2f\left(\frac{1}{y}\right) + \log x \text{ -----(1)}</math></p> <p>Differentiate (1) partially w.r.t x</p> $\frac{\partial z}{\partial x} = 2x + 2f'\left(\frac{1}{y}\right) \times 0 + \frac{1}{x}$ $p = 2x + 2f'\left(\frac{1}{y}\right) \Rightarrow f'\left(\frac{1}{y}\right) = \frac{(p - 2x)x}{2} \text{ -----(2)}$ <p><math>\frac{\partial z}{\partial y} = 2f'\left(\frac{1}{y}\right) \times (-1) + 0</math></p> $q = \frac{-2}{y^2} f'\left(\frac{1}{y}\right) + \log x \Rightarrow f'\left(\frac{1}{y}\right) = \frac{-qy^2}{2} \text{ -----(3)}$ <p>Eliminating <math>f'</math> from (2) &amp; (3)</p> $\left(\frac{p - 2x}{2}\right) \frac{x}{2} = \frac{-qy^2}{2} \Rightarrow (px - 2x^2) = -qy^2$ $\Rightarrow \boxed{px + qy^2 = 2x^2}$
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<p><b>Formation of PDE by eliminating <math>f</math> from <math>f(u, v) = 0</math> -----(1)</b></p> <p><b>Method 1:</b></p> <p>The required PDE of (1) is <math display="block">\begin{vmatrix} p &amp; q &amp; -1 \\ u_x &amp; u_y &amp; u_z \\ v_x &amp; v_y &amp; v_z \end{vmatrix} = 0</math></p> <p><b>Method 2:</b></p> <p>The required PDE is <math>Pp + Qq = R</math></p> <p>Where</p> $P = \begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix}; Q = \begin{vmatrix} u_x & v_x \\ u_z & v_z \end{vmatrix}; R = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}$	
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1.	<p><b>Form the PDE from <math>\phi(ax + by + cz, x^2 + y^2 + z^2) = 0</math></b></p> <p><b>Solution:</b></p> <p>Given <math>\phi(ax + by + cz, x^2 + y^2 + z^2) = 0</math></p> <p>This is of the form <math>f(u, v) = 0</math> where <math>u = ax + by + cz</math> &amp; <math>v = x^2 + y^2 + z^2</math></p>
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	<p>The required PDE of (1) is <math display="block">\begin{vmatrix} p &amp; q &amp; -1 \\ u_x &amp; u_y &amp; u_z \\ v_x &amp; v_y &amp; v_z \end{vmatrix} = 0</math></p> $\begin{vmatrix} p & q & -1 \\ a & b & c \\ 2x & 2y & 2z \end{vmatrix} = 0$ $\Rightarrow p(2bz - 2cy) - q(2az - 2cx) + 1(2az - 2cx) = 0$ $\div 2 \Rightarrow (bz - cy)p + (cx - az)q + (az - cx) = 0$
2.	<p><b>Form the PDE from <math>\phi(x^2 + y^2 + z^2, xyz) = 0</math></b></p> <p><b>Solution:</b></p> <p>Given <math>\phi(x^2 + y^2 + z^2, xyz) = 0</math></p> <p>This is of the form <math>f(u, v) = 0</math> where <math>u = x^2 + y^2 + z^2</math> &amp; <math>v = xyz</math></p> <p>The required PDE of (1) is <math display="block">\begin{vmatrix} p &amp; q &amp; -1 \\ u_x &amp; u_y &amp; u_z \\ v_x &amp; v_y &amp; v_z \end{vmatrix} = 0</math></p> $\begin{vmatrix} p & q & -1 \\ 2x & 2y & 2z \\ yz & xz & xy \end{vmatrix} = 0$ $\Rightarrow p(2xy^2 - 2xz^2) - q(2x^2y - 2yz^2) + 1(2x^2z - 2y^2z) = 0$ $\div 2 \Rightarrow x(y^2 - z^2)p + y(z^2 - x^2)q + z(x^2 - y^2) = 0$
3.	<p><b>Form the PDE from <math>\phi\left(\frac{y}{x}, x^2 + y^2 + z^2\right) = 0</math></b></p> <p><b>Solution:</b></p> <p>Given <math>\phi\left(\frac{y}{x}, x^2 + y^2 + z^2\right) = 0</math></p> <p>This is of the form <math>f(u, v) = 0</math> where <math>u = \frac{y}{x}</math> &amp; <math>v = x^2 + y^2 + z^2</math></p> <p>The required PDE of (1) is <math display="block">\begin{vmatrix} p &amp; q &amp; -1 \\ u_x &amp; u_y &amp; u_z \\ v_x &amp; v_y &amp; v_z \end{vmatrix} = 0</math></p> $\begin{vmatrix} p & q & -1 \\ -\frac{y}{x^2} & \frac{1}{x} & 0 \\ 2x & 2y & 2z \end{vmatrix} = 0$ $\Rightarrow p\left(\frac{2z}{x} - 0\right) - q\left(\frac{-2yz}{x^2} - 0\right) + 1\left(\frac{-2y^2}{x^2} - \frac{2x}{x}\right) = 0$ $\Rightarrow \frac{2zp}{x} + \frac{2yzq}{x^2} - \frac{2y^2}{x^2} - \frac{2x}{x} = 0$ $\div 2 \Rightarrow \frac{zp}{x} + \frac{yzq}{x^2} - \frac{y^2 + x^2}{x^2} = 0$

$$\Rightarrow \frac{xzp + yzq - (y^2 + x^2)}{x^2} = 0$$

$$xzp + yzq - (y^2 + x^2) = 0$$

**Solutions of standard types of First order PDE's:**

**Different solutions of PDE:**

**Complete Integral (or) Complete Solution:**

If the number of arbitrary constants is equal to number of independent variables, then the solution is called Complete integral.

**Singular Integral (or) Singular Solution:**

Consider a PDE of first order as  $f(x, y, z, p, q) = 0$  ----- (1)

It's complete integral may be,  $f(x, y, z, a, b) = 0$  ----- (2)

Diff (2) partially with respect to  $a$  &  $b$  respectively,

$$\frac{\partial g}{\partial a} = 0$$
 ----- (3)

$$\frac{\partial g}{\partial b} = 0$$
 ----- (4)

Eliminating  $a$  &  $b$  from (3) & (4) will get the Singular integral.

**General Integral (or) Complete solution:**

A Solution which contains number of arbitrary functions is equal to the order of the given PDE.

(or) A solution which contains the maximum possible number of arbitrary functions.

**Type I:**

Equations of the form  $f(p, q) = 0$  ----- (1)

**To find Complete Integral:**

Let the complete solution of (1) is  $z = ax + by + c$  ----- (2)

Let  $p = a$  &  $q = b$  in (1)

$f(a, b) = 0$  and represent  $b = \Phi(a)$

$$\therefore (1) \Rightarrow z = ax + \Phi(a)y + c$$
 ----- (3)

**To Find Singular Integral:**

Diff (3) partially with respect to  $c$

$$0 = 1$$
 which is impossible

There is no singular integral for this type.

**To find General integral:**

Put  $c = g(a)$  in (3)

$$(3) \Rightarrow z = ax + \Phi(a)y + g(a)$$
 ----- (4)

Diff (4) partially with respect to  $a$

$$0 = x(1) + \Phi'(a)y + g'(a)$$
 ----- (5)

Eliminating  $a$  from (4) & (5) we get general integral.

1. **Find the complete integral of  $p + q = pq$**

**Solution:**

Given  $p + q = pq$  ----- (1)

This of the form  $f(p, q) = 0$

**To find Complete Integral:**

Let the complete solution of (1) is  $z = ax + by + c$  ----- (2)

Let  $p = a$  &  $q = b$  in (1)

$$(1) \Rightarrow a + b = ab \Rightarrow a + b - ab = 0 \Rightarrow b = \frac{a}{a-1}$$

Sub  $b$  in (2)

$$z = ax + \frac{a}{a-1}y + c$$

This is the required complete integral.

2.	<p><b>Find the complete integral of <math>p + q = 1</math></b></p> <p><b>Solution:</b> Given <math>p + q = 1</math> -----(1) This of the form <math>f(p, q) = 0</math></p> <p><b>To find Complete Integral:</b> Let the complete solution of (1) is <math>z = ax + by + c</math> -----(2) Let <math>p = a</math> &amp; <math>q = b</math> in (1) (1) <math>\Rightarrow a + b = 1 \Rightarrow b = 1 - a</math> Sub <math>b</math> in (2)</p> <div style="border: 1px solid black; padding: 2px; display: inline-block;"><math>z = ax + (1 - a)y + c</math></div> <p>This is the required complete integral.</p>
3.	<p><b>Solve <math>\sqrt{p} + \sqrt{q} = 1</math></b></p> <p><b>Solution:</b> Given <math>\sqrt{p} + \sqrt{q} = 1</math> -----(1) This of the form <math>f(p, q) = 0</math></p> <p><b>To find Complete Integral:</b> Let the complete solution of (1) is <math>z = ax + by + c</math> -----(2) Let <math>p = a</math> &amp; <math>q = b</math> in (1) (1) <math>\Rightarrow \sqrt{a} + \sqrt{b} = 1 \Rightarrow \sqrt{b} = 1 - \sqrt{a} \Rightarrow b = (1 - \sqrt{a})^2</math> Sub <math>b</math> in (2)</p> <div style="border: 1px solid black; padding: 2px; display: inline-block;"><math>z = ax + (1 - \sqrt{a})^2 y + c</math></div> -----(3) <p>This is the required complete integral.</p> <p><b>To Find Singular Integral:</b> Diff (3) partially with respect to <math>c</math> <math>0 = 1</math> which is impossible There is no singular integral for this type.</p> <p><b>To find General integral:</b> Put <math>c = f(a)</math> in (3) (3) <math>\Rightarrow z = ax + (1 - \sqrt{a})^2 y + f(a)</math> -----(4) Diff. (4) partially with respect to <math>a</math> <math>0 = x(1) + 2(1 - \sqrt{a}) \frac{1}{2\sqrt{a}} y + f'(a)</math> -----(5) Eliminate <math>a</math> from (4) &amp; (5) we get the general integral.</p>
<p><b>Type II:</b> Equations of the form <math>z = px + qy + f(p, q)</math> -----(1) To find Complete Integral: Put <math>p = a</math> &amp; <math>q = b</math> in (1) <math>\therefore (1) \Rightarrow z = ax + by + f(a, b)</math> -----(2)</p> <p><b>To Find Singular Integral:</b> Diff (2) partially with respect to <math>a</math> <math>0 = x(1) + 0 + f'(a, b)</math> -----(3) Diff (2) partially with respect to <math>b</math> <math>0 = 0 + y(1) + f'(a, b)</math> -----(4) Eliminating <math>a</math> &amp; <math>b</math> from (2) using (3) &amp; (4), we get the singular integral.</p> <p><b>To find General integral:</b> Put <math>b = \phi(a)</math> in (2) (2) <math>\Rightarrow z = ax + \phi(a)y + g(a)</math> -----(5) Diff (4) partially with respect to <math>a</math></p>	

$$0 = x(1) + \phi'(a)y + g'(a) \text{ --- (6)}$$

Eliminating  $a$  from (5) & (6) we get General integral.

1. **Solve**  $z = px + qy + p^2 q^2$

**Solution:**

Given  $z = px + qy + p^2 q^2$  --- (1)

Equations of the form  $z = px + qy + f(p, q)$

To find Complete Integral:

Put  $p = a$  &  $q = b$  in (1)

$$\therefore (1) \Rightarrow z = ax + by + a^2 b^2 \text{ --- (2)}$$

This is the required complete integral

**To Find Singular Integral:**

Diff (2) partially with respect to  $a$

$$0 = x(1) + 0 + 2ab^2 \Rightarrow x + 2ab^2 = 0 \Rightarrow x = -2ab^2 \text{ --- (3)}$$

Diff (2) partially with respect to  $b$

$$0 = 0 + y(1) + 0 + 2a^2 b \Rightarrow y + 2a^2 b = 0 \Rightarrow y = -2a^2 b \text{ --- (4)}$$

Eliminating  $a$  &  $b$  from (2) using (3) & (4)

$$(3) \Rightarrow \frac{x}{b} = -2ab \text{ --- (5)}$$

$$(4) \Rightarrow \frac{y}{a} = -2ab \text{ --- (6)}$$

From (5) & (6)

$$\frac{x}{b} = \frac{y}{a} = k \text{ (say)}$$

$$\Rightarrow \frac{x}{b} = k \text{ \& \ } \frac{y}{a} = k$$

$$\Rightarrow b = \frac{x}{k} \text{ \& \ } a = \frac{y}{k} \text{ --- (7)}$$

Sub  $a$  &  $b$  in (2)

$$(2) \Rightarrow z = \frac{y}{k} x + \frac{x}{k} y + \frac{y^2}{k^2} \frac{x^2}{k^2}$$

$$z = \frac{xy}{k} + \frac{xy}{k} + \frac{x^2 y^2}{k^4}$$

$$z = \frac{2xy}{k} + \frac{x^2 y^2}{k^4} \text{ --- (8)}$$

**To find k**

Sub (7) in (3) (or) (4)

$$(3) \Rightarrow x = -2 \frac{y x^2}{k k^2} \Rightarrow x = \frac{-2x^2 y}{k^3} \Rightarrow k^3 = -2xy$$

$$(8) \Rightarrow z = \frac{2xy}{k} + \frac{x^2 y^2}{k(-2xy)} \Rightarrow z = \frac{2xy}{k} - \frac{xy}{2k}$$

$$z = \frac{4xy - xy}{2k} \Rightarrow z = \frac{3xy}{2k} \Rightarrow z^3 = \frac{27x^3 y^3}{8k^3} \Rightarrow z^3 = \frac{27x^3 y^3}{8(-2xy)}$$

$$16z^3 = -27x^2 y^2$$

This is the required singular integral.

**To find General integral:**

Put  $b = \phi(a)$  in (2)

$$(2) \Rightarrow z = ax + f(a)y + a^2 [f(a)]^2 \text{ --- (9)}$$



	<p>Diff (9) partially with respect to <math>a</math>  <math>0 = x(1) + f'(a)y + 2f(a)f'(a) \dots\dots(10)</math>                      Eliminating <math>a</math> from (9) &amp; (10) we get General integral.</p>
2.	<p><b>Find the singular integral of <math>z = px + qy + p^2 + pq + q^2</math></b>  <b>Solution:</b>                      Given <math>z = px + qy + p^2 + pq + q^2 \dots\dots(1)</math>                      Equations of the form <math>z = px + qy + f(p, q)</math>                      To find Complete Integral:                      Put <math>p = a</math> &amp; <math>q = b</math> in (1)  <math>\therefore (1) \Rightarrow \boxed{z = ax + by + a^2 + ab + b^2} \dots\dots(2)</math>                      This is the required complete integral  <b>To Find Singular Integral:</b>                      Diff (2) partially with respect to <math>a</math>  <math>0 = x(1) + 0 + 2a + b + 0 \Rightarrow 2a + b = -x \dots\dots(3)</math>                      Diff (2) partially with respect to <math>b</math>  <math>0 = 0 + y(1) + 0 + a + 2b \Rightarrow a + 2b = -y \dots\dots(4)</math>                      Eliminating <math>a</math> &amp; <math>b</math> from (2) using (3) &amp; (4)  <math>(4) \times 2 \Rightarrow 2a + 4b = -2y \dots\dots(5)</math>  <math>(3) - (5) \Rightarrow -3b = -x + 2y \Rightarrow \boxed{b = \frac{x - 2y}{3}}</math>                      Sub the value of <math>b</math> in (3)  <math>2a + b = -x \Rightarrow 2a = -x - b \Rightarrow 2a = -x - \frac{x - 2y}{3}</math>  <math>2a = \frac{-3x - x + 2y}{3} \Rightarrow 6a = -4x + 2y \Rightarrow \boxed{a = \frac{y - 2x}{3}}</math>                      Sub the value of <math>a</math> &amp; <math>b</math> in (2)  <math>z = \frac{y - 2x}{3}x + \frac{x - 2y}{3}y + \left(\frac{y - 2x}{3}\right)^2 + \left(\frac{y - 2x}{3}\right)\left(\frac{x - 2y}{3}\right) + \left(\frac{x - 2y}{3}\right)^2</math>  <math>z = \frac{xy - 2x^2}{3} + \frac{xy - 2y^2}{3} + \frac{y^2 - 4xy + 4y^2}{9} + \frac{xy - 2y^2 - 2x^2 + 4xy}{9} + \frac{x^2 - 4xy + 4y^2}{9}</math>  <math>z = \frac{3xy - 3x^2 + 3xy - 6y^2 + y^2 - 4xy + 4y^2 + xy - 2y^2 - 2x^2 + 4xy + x^2 - 4xy + 4y^2}{9}</math>  <math>\boxed{9z = -4x^2 + y^2 + xy}</math></p>
3.	<p><b>Solve <math>z = px + qy + \sqrt{p^2 + q^2 + 1}</math></b>  <b>Solution:</b>                      Given  <math>z = px + qy + \sqrt{p^2 + q^2 + 1} \dots\dots(1)</math>                      To find Complete Integral:                      Put <math>p = a</math> &amp; <math>q = b</math> in (1)  <math>(1) \Rightarrow \boxed{z = ax + by + \sqrt{a^2 + b^2 + 1}} \dots\dots(2)</math>                      This is required complete integral.  <b>To Find Singular Integral:</b>                      Diff (2) partially with respect to <math>a</math>  <math>0 = x(1) + 0 + \frac{1}{2\sqrt{a^2 + b^2 + 1}}(2a) \Rightarrow \boxed{x = \frac{-a}{\sqrt{a^2 + b^2 + 1}}} \dots\dots(3)</math>                      Diff (2) partially with respect to <math>b</math></p>

$$0 = 0 + y(1) + \frac{1}{2\sqrt{a^2 + b^2 + 1}} (2b) \Rightarrow \boxed{y = \frac{-b}{\sqrt{a^2 + b^2 + 1}}} \text{ -----(4)}$$

Eliminating  $a$  &  $b$  from (2) using (3) & (4)

$$(3)^2 + (4)^2 \Rightarrow x^2 + y^2 = \frac{a^2}{\sqrt{a^2 + b^2 + 1}} + \frac{b^2}{\sqrt{a^2 + b^2 + 1}}$$

$$x^2 + y^2 = \frac{a^2}{a^2 + b^2 + 1} + \frac{b^2}{a^2 + b^2 + 1}$$

$$x^2 + y^2 = \frac{a^2 + b^2}{a^2 + b^2 + 1}$$

$$1 - (x^2 + y^2) = 1 - \frac{a^2 + b^2}{a^2 + b^2 + 1}$$

$$1 - x^2 - y^2 = \frac{a^2 + b^2 + 1 - a^2 - b^2}{a^2 + b^2 + 1}$$

$$1 - x^2 - y^2 = \frac{1}{a^2 + b^2 + 1}$$

$$\Rightarrow 1 + a^2 + b^2 = \frac{1}{1 - x^2 - y^2}$$

Taking square root on both sides

$$\Rightarrow \boxed{\sqrt{1 + a^2 + b^2} = \frac{1}{\sqrt{1 - x^2 - y^2}}} \text{ -----(5)}$$

Sub (5) in (2) and (3)

$$(3) \Rightarrow x = \frac{a}{\sqrt{1 - x^2 - y^2}} \Rightarrow x = -a \sqrt{1 - x^2 - y^2} \Rightarrow \boxed{a = \frac{-x}{\sqrt{1 - x^2 - y^2}}}$$

$$(4) \Rightarrow y = \frac{-b}{\sqrt{1 - x^2 - y^2}} \Rightarrow y = -b \sqrt{1 - x^2 - y^2} \Rightarrow \boxed{b = \frac{-y}{\sqrt{1 - x^2 - y^2}}}$$

Sub (5),  $a$  &  $b$  in (2)

$$(2) \Rightarrow z = \frac{-x}{\sqrt{1 - x^2 - y^2}} x + \frac{-y}{\sqrt{1 - x^2 - y^2}} y + \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$$\Rightarrow z = \frac{-x^2}{\sqrt{1 - x^2 - y^2}} + \frac{-y^2}{\sqrt{1 - x^2 - y^2}} + \frac{1}{\sqrt{1 - x^2 - y^2}}$$

$$\Rightarrow z = \frac{1 - x^2 - y^2}{\sqrt{1 - x^2 - y^2}}$$

$$\Rightarrow z = \sqrt{1 - x^2 - y^2}$$

Squaring on both sides

$$z^2 = 1 - x^2 - y^2 \Rightarrow \boxed{x^2 + y^2 + z^2 = 1}$$

4. Find the singular integral of  $z = px + qy + p^2 - q^2$

**Solution:**

$$\text{Given } z = px + qy + p^2 - q^2 \text{ -----(1)}$$

Equations of the form  $z = px + qy + f(p, q)$

To find Complete Integral:

Put  $p = a$  &  $q = b$  in (1)

$$\therefore (1) \Rightarrow z = ax + by + a^2 - b^2 \quad \text{---(2)}$$

This is the required complete integral

**To Find Singular Integral:**

Diff (2) partially with respect to  $a$

$$0 = x(1) + 0 + 2a + 0 \Rightarrow a = \frac{-x}{2} \quad \text{---(3)}$$

Diff (2) partially with respect to  $b$

$$0 = 0 + y(1) + 0 - 2b \Rightarrow b = \frac{y}{2} \quad \text{---(4)}$$

Sub  $a$  &  $b$  in (2)

$$\therefore (2) \Rightarrow z = \frac{-x}{2} + \frac{y}{2} + \left(\frac{-x}{2}\right)^2 - \left(\frac{y}{2}\right)^2$$

$$\therefore (2) \Rightarrow z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$

$$\therefore (2) \Rightarrow z = \frac{-2x^2 + 2y^2 + x^2 - y^2}{4} \Rightarrow \boxed{4z = -x^2 + y^2}$$

This is the required singular integral.

**Type III:**

Equations of the form  $f(z, p, q) = 0$  --- (1)

In this type  $x$  &  $y$  do not appear explicitly.

**To find Complete Integral:**

Let the complete solution of (1) is  $z = f(x + ay)$  --- (2)

Let  $x + ay = u$

$$(2) \Rightarrow z = f(u) \quad \text{---(3)}$$

By total derivative,

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \Rightarrow p = \frac{dz}{du} \quad \because u = x + ay \Rightarrow \frac{\partial u}{\partial x} = 1$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} \Rightarrow q = a \frac{dz}{du} \quad \because u = x + ay \Rightarrow \frac{\partial u}{\partial y} = a$$

Substitute the value of  $p$  &  $q$  in (1)

$$(1) \Rightarrow f\left(\frac{dz}{du}, a \frac{dz}{du}\right) = 0$$

This may be solve by method of separation of variables

Other solutions can obtain as usual.

**1. Solve  $p(1 + q) = qz$ .**

Solution:

$$\text{Given } p(1 + q) = qz \quad \text{---(1)}$$

This is of the form  $f(z, p, q) = 0$

**To find Complete Integral:**

Let the complete solution of (1) is  $z = f(x + ay)$  --- (2)

Let  $x + ay = u \Rightarrow z = f(u)$

$$\text{Then } p = \frac{dz}{du} \quad \& \quad q = a \frac{dz}{du}$$

Substitute the value of  $p$  &  $q$  in (1)

$$(1) \Rightarrow \frac{dz}{du} \left(1 + a \frac{dz}{du}\right) = a \frac{dz}{du} z$$

$$1 + a \frac{dz}{du} = a z$$

$$\frac{dz}{du} = a z - 1$$

$$\frac{dz}{a z - 1} = du$$

Integrating on both sides

$$\int \frac{dz}{a z - 1} = \int du$$

$$u = \log(a z - 1) + c \quad \because \int \frac{f'(x)}{f(x)} dx = \log f(x)$$

$$\boxed{x + ay = \log(a z - 1) + c}$$

This is the required complete integral.

Other solutions can be obtained as usual.

2.

**Solve**  $z^2 = 1 + p^2 + q^2$

Solution:

Given  $z^2 = 1 + p^2 + q^2$  -----(1)

This is of the form  $f(z, p, q) = 0$

**To find Complete Integral:**

Let the complete solution of (1) is  $z = f(x + ay)$  -----(2)

Let  $x + ay = u \Rightarrow z = f(u)$

Then  $p = \frac{dz}{du}$  &  $q = a \frac{dz}{du}$

Substitute the value of  $p$  &  $q$  in (1)

$$(1) \Rightarrow z^2 = \left(\frac{dz}{du}\right)^2 + a^2 \left(\frac{dz}{du}\right)^2 + 1$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 + a^2 \left(\frac{dz}{du}\right)^2 = z^2 - 1$$

$$\Rightarrow (1 + a^2) \left(\frac{dz}{du}\right)^2 = z^2 - 1$$

$$\Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{z^2 - 1}{1 + a^2}$$

Taking square root on both sides

$$\frac{dz}{du} = \frac{\sqrt{z^2 - 1}}{\sqrt{1 + a^2}}$$

$$\Rightarrow \frac{dz}{\sqrt{z^2 - 1}} = \frac{du}{\sqrt{1 + a^2}}$$

Integrating on both sides

$$\cosh^{-1} z = \frac{1}{\sqrt{a^2 - 1}} u + c \quad \because \int \frac{dz}{\sqrt{z^2 - 1}} = \cosh^{-1} z$$

$$\boxed{\cosh^{-1} z = \frac{1}{\sqrt{a^2 - 1}} (x + ay) + c}$$

$$\because u = x + ay$$

This is the required complete integral.

Other solutions can be obtained as usual.

**3. Solve**  $p(1 - q^2) = q(1 - z)$   
**Solution:**  
 Given  $p(1 - q^2) = q(1 - z)$  -----(1)  
 This is of the form  $f(z, p, q) = 0$   
**To find Complete Integral:**  
 Let the complete solution of (1) is  $z = f(x + ay)$  -----(2)  
 Let  $x + ay = u \Rightarrow z = f(u)$   
 Then  $p = \frac{dz}{du}$  &  $q = a \frac{dz}{du}$   
 Substitute the value of  $p$  &  $q$  in (1)  

$$(1) \Rightarrow \frac{dz}{du}(1 - a^2 \frac{dz^2}{du^2}) = a \frac{dz}{du}(1 - z)$$
  

$$1 - a + az = a^2 \frac{dz}{du}$$
  
 Taking square root on both sides  

$$a \frac{dz}{du} = \sqrt{-a + az}$$
  

$$\frac{a dz}{\sqrt{1 - a + az}} = du$$
  
 Integrating on both sides  

$$a \int \frac{1}{\sqrt{1 - a + az}} dz = u + c \quad \because \int \frac{1}{\sqrt{ax}} dx = \frac{1}{a} (2\sqrt{x})$$
  

$$2\sqrt{1 - a + az} = x + ay + c \quad \because u = x + ay$$
  
 This is the required complete integral.  
 Other solutions can be obtained as usual.

**Type IV:**  
 Equations of the form  $f_1(x, p) = f_2(y, q)$  -----(1)  
**To find Complete Integral:**  
 Let  $f_1(x, p) = f_2(y, q) = a$  (say)  
 $\therefore f_1(x, p) = a$  ;  $f_2(y, q) = a$   
 From the above we get  $p = f_1(x, a)$  ;  $q = f_2(y, b)$   
 Substitute the value of  $p$  &  $q$  in  $z = \int pdx + \int qdy$   
 Integrating we get complete integral  
 Other solutions can obtain as usual.

**1. Solve**  $p^2 + q^2 = x^2 + y^2$   
**Solution:**  
 Given  $p^2 + q^2 = x^2 + y^2$   
 $p^2 - x^2 = y^2 - q^2$  -----(1)  
 This is of the form  $f_1(x, p) = f_2(y, q)$   
**To find Complete Integral:**  
 Let  $p^2 - x^2 = y^2 - q^2 = a^2$  (say)  
 $\therefore p^2 - x^2 = a^2$  ;  $y^2 - q^2 = a^2$   
 $\therefore p^2 = a^2 + x^2$  ;  $q^2 = y^2 - a^2$   

$$p = \sqrt{a^2 + x^2} ; q = \sqrt{y^2 - a^2}$$

	<p>Substitute the value of <math>p</math> &amp; <math>q</math> in</p> $z = \int \sqrt{x^2 + a^2} dx + \int \sqrt{y^2 - a^2} dy$ <div style="border: 1px solid black; padding: 5px; margin: 5px 0;"> <math display="block">z = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a} + \frac{y}{2} \sqrt{y^2 - a^2} + \frac{a^2}{2} \cosh^{-1} \frac{y}{a} + c</math> </div> $\therefore \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a} \quad \& \quad \int \sqrt{y^2 - a^2} dy = \frac{y}{2} \sqrt{y^2 - a^2} + \frac{a^2}{2} \cosh^{-1} \frac{y}{a}$ <p>Integrating we get complete integral Other solutions can obtain as usual.</p>
2.	<p><b>Find the complete integral of <math>p^2 y(1 + x^2) = qx^2</math></b></p> <p><b>Solution:</b> Given <math>p^2 y(1 + x^2) = qx^2</math></p> $\frac{p^2(1 + x^2)}{x^2} = \frac{q}{y} \quad \text{--- (1)}$ <p>This is of the form <math>f_1(x, p) = f_2(y, q)</math></p> <p><b>To find Complete Integral:</b> Let <math>\frac{p^2(1 + x^2)}{x^2} = \frac{q}{y} = a</math> (say)</p> $\frac{p^2(1 + x^2)}{x^2} = a ; \frac{q}{y} = a$ $p^2 = a \frac{x^2}{1 + x^2} ; q = ay$ <div style="border: 1px solid black; padding: 5px; margin: 5px 0;"> <math display="block">p = \frac{\sqrt{ax}}{\sqrt{1 + x^2}} ; q = ay</math> </div> <p>Substitute the value of <math>p</math> &amp; <math>q</math> in</p> $z = \int \frac{\sqrt{ax}}{\sqrt{1 + x^2}} dx + \int ay dy$ <p>let <math>1 + x^2 = t \Rightarrow 2x dx = dt \Rightarrow x dx = \frac{dt}{2}</math></p> $z = \sqrt{a} \int \frac{1}{2} \frac{dt}{\sqrt{t}} + a \frac{y^2}{2}$ $z = \frac{\sqrt{a}}{2} 2\sqrt{t} + \frac{ay^2}{2} \quad \because \int \frac{1}{\sqrt{x}} dx = 2\sqrt{x}$ <div style="border: 1px solid black; padding: 5px; margin: 5px 0;"> <math display="block">z = \sqrt{a} \sqrt{1 + x^2} + \frac{ay^2}{2} + c</math> </div>
3.	<p><b>Find the complete integral of <math>p + q = \sin x + \sin y</math></b></p> <p><b>Solution:</b> Given <math>p + q = \sin x + \sin y</math></p> $p - \sin x = \sin y - q \quad \text{--- (1)}$ <p>This is of the form <math>f_1(x, p) = f_2(y, q)</math></p> <p><b>To find Complete Integral:</b> Let <math>p - \sin x = \sin y - q = a</math> (say)</p> $\therefore p - \sin x = a ; \sin y - q = a$ $\therefore p = \sin x + a ; q = \sin y - a$ <p>Substitute the value of <math>p</math> &amp; <math>q</math> in</p>

$$z = \int (\sin x + a) dx + \int (\sin y - a) dy$$

$$z = \cos x + ax + \cos y - ay + c$$

$$z = \cos x + \cos y - a(x - y) + c$$

This is the required complete integral

**Type V:**

Equations reducible to standard form (Identification:

1.

**Solve**  $z^2(p^2 + q^2) = x^2 + y^2$

**Solution:**

Given  $z^2(p^2 + q^2) = x^2 + y^2$

This is of the form equations reducible to standard form

$$z^2 p^2 + z^2 q^2 = x^2 + y^2$$

$$(zp)^2 + (zq)^2 = x^2 + y^2$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x^2 + y^2 \quad \dots \dots \dots (1) \quad \because p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x^2 + y^2 \quad \because p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}$$

Let  $z\partial z = \partial Z$  ;  $\partial x = \partial X$  ;  $\partial y = \partial Y$

Integrating  $\frac{z^2}{2} = Z$  ;  $x = X$  ;  $y = Y$

$$(1) \Rightarrow \frac{\partial Z}{\partial X} + \frac{\partial Z}{\partial Y} = X^2 + Y^2$$

$$P^2 + Q^2 = X^2 + Y^2 \quad \because P = \frac{\partial Z}{\partial X}; Q = \frac{\partial Z}{\partial Y}$$

This is of the form  $F_1(X, P) = F_2(Y, Q)$

**To find Complete Integral:**

Let  $P^2 - X^2 = Y^2 - Q^2 = a^2$  (say)

$$\therefore P^2 - X^2 = a^2 ; Y^2 - Q^2 = a^2$$

$$\therefore P^2 = a^2 + X^2 ; Q^2 = Y^2 - a^2$$

$$P = \sqrt{a^2 + X^2} ; Q = \sqrt{Y^2 - a^2}$$

Substitute the value of  $p$  &  $q$  in

$$z = \int \sqrt{X^2 + a^2} dX + \int \sqrt{Y^2 - a^2} dY$$

$$Z = \frac{X}{2} \sqrt{X^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \frac{X}{a} + \frac{Y}{2} \sqrt{Y^2 - a^2} + \frac{a^2}{2} \cosh^{-1} \frac{Y}{a} + c$$

$$\frac{z^2}{2} = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a} + \frac{y}{2} \sqrt{y^2 - a^2} + \frac{a^2}{2} \cosh^{-1} \frac{y}{a} + c$$

$$z^2 = x \sqrt{x^2 + a^2} + a^2 \sinh^{-1} \frac{x}{a} + y \sqrt{y^2 - a^2} + a^2 \cosh^{-1} \frac{y}{a} + 2c$$

$$\therefore \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \sinh^{-1} \frac{x}{a}; \int \sqrt{y^2 - a^2} dy = \frac{y}{2} \sqrt{y^2 - a^2} + \frac{a^2}{2} \cosh^{-1} \frac{y}{a}$$

This is the required complete integral

Other solutions can obtain as usual.

2.

**Solve**  $p^2 + x^2 y^2 q^2 = x^2 z^2$

**Solution:**

Given  $p^2 + x^2 y^2 q^2 = x^2 z^2$

This is of the form equations reducible to standard form

$$\div x \text{ on bothsides } \frac{P^2}{x^2} + y^2q^2 = z^2$$

$$\frac{p}{x} + (yq)^2 = z^2$$

$$\frac{1}{x} \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z^2 \quad \therefore p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{x \partial x} + \frac{\partial z}{\partial y} = z^2 \text{ -----(1)} \quad \therefore p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}$$

Let  $\partial z = \partial Z$  ;  $x \partial x = \partial X$  ;  $\frac{\partial y}{y} = \partial Y$

Integrating  $z = Z$  ;  $\frac{x^2}{2} = X$  ;  $\log y = Y$

$$(1) \Rightarrow \frac{\partial Z}{\partial X} + \frac{\partial Z}{\partial Y} = Z^2$$

$$P^2 + Q^2 = Z^2 \text{ -----(2)} \quad \therefore P = \frac{\partial Z}{\partial X}; Q = \frac{\partial Z}{\partial Y}$$

This is of the form  $F(Z, P, Q) = 0$

**To find Complete Integral:**

Let the complete solution of (2) is  $Z = F(X + aY)$

Let  $X + aY = U \Rightarrow Z = F(U)$

Then  $P = \frac{dZ}{dU}$  &  $Q = a \frac{dZ}{dU}$

Substitute the value of  $P$  &  $Q$  in (2)

$$(2) \Rightarrow \frac{dZ}{dU} + a \frac{dZ}{dU} = Z^2$$

$$\frac{dZ}{dU} = \frac{Z^2}{(1+a^2)}$$

Taking square root on both sides

$$\frac{dZ}{dU} = \frac{Z}{\sqrt{1+a^2}}$$

$$\frac{dZ}{Z} = \frac{dU}{\sqrt{1+a^2}}$$

Integrating on both sides

$$\log Z = \frac{1}{\sqrt{1+a^2}} U$$

$$\log Z = \frac{1}{\sqrt{1+a^2}} (X + aY)$$

This is complete integral of (2)

$$\log z = \frac{1}{\sqrt{1+a^2}} \frac{x^2}{2} + a \log y + c$$

This is the required complete integral .



Other solutions can be obtained as usual.

3. Find the complete integral of  $z^2 (x^2 p^2 + q^2) = 1$

**Solution:**

Given  $z^2 (x^2 p^2 + q^2) = 1$

This is of the form equations reducible to standard form

$$x^2 p^2 + q^2 = \frac{1}{z^2}$$

$$(xp)^2 + q^2 = \frac{1}{z^2}$$

$$\frac{\partial z}{\partial x} x^2 + \frac{\partial z}{\partial y} = \frac{1}{z^2} \quad \therefore p = \frac{\partial z}{\partial x}; q = \frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial x} x^2 + \frac{\partial z}{\partial y} = \frac{1}{z^2} \quad \text{----- (1)}$$

Let  $\partial z = \partial Z$  ;  $\frac{\partial x}{x} = \partial X$  ;  $\partial y = \partial Y$

Integrating  $\boxed{z = Z ; \log x = X ; y = Y}$

$$\frac{\partial Z}{\partial X} x^2 + \frac{\partial Z}{\partial Y} = \frac{1}{Z^2}$$

(1)  $\Rightarrow \frac{\partial Z}{\partial X} x^2 + \frac{\partial Z}{\partial Y} = \frac{1}{Z^2}$

$$\boxed{P^2 + Q^2 = \frac{1}{Z^2}} \quad \text{----- (2)} \quad \therefore P = \frac{\partial Z}{\partial X}; Q = \frac{\partial Z}{\partial Y}$$

This is of the form  $F(Z, P, Q) = 0$

**To find Complete Integral:**

Let the complete solution of (2) is  $Z = F(X + aY)$

Let  $X + aY = U \Rightarrow Z = F(U)$

This is of the form  $F(Z, P, Q) = 0$

Then  $P = \frac{dZ}{dU}$  &  $Q = a \frac{dZ}{dU}$

Substitute the value of  $P$  &  $Q$  in (2)

$$(2) \Rightarrow \frac{dZ}{dU}^2 + \frac{a^2}{dU}^2 = \frac{1}{Z^2}$$

$$\frac{dZ}{dU} = \frac{1}{Z \sqrt{1+a^2}}$$

Taking square root on both sides

$$\frac{dZ}{dU} = \frac{1}{Z \sqrt{1+a^2}}$$

$$Z dZ = \frac{dU}{\sqrt{1+a^2}}$$

Integrating on both sides

$$\frac{Z^2}{2} = \frac{1}{\sqrt{1+a^2}} U$$

$$\frac{Z^2}{2} = \frac{1}{\sqrt{1+a^2}} (X + aY)$$

	<p>This is the complete integral of (2)</p> $\frac{z}{2} = \frac{1}{\sqrt{1+a^2}} (\log x + ay)^+ + c$ <p>This is the required complete integral . Other solutions can be obtained as usual.</p>
4.	Solve $x^2 p^2 + y^2 q^2 = z^2$
5.	Solve $x^4 p^2 + y^2 zq = 2z^2$
<p><b>Lagrange's Linear Differential Equations:</b> Equations of the form <math>Pp + Qq = R</math> (or) <math>P \frac{\partial z}{\partial x} + Q \frac{\partial z}{\partial y} = R</math></p> <p>Where <math>P, Q, R</math> are functions of <math>x, y, z</math> or constants.</p> <p><b>Procedure :</b></p> <p>1. Write the auxiliary equation <math>\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}</math></p> <p>2. Solve the auxiliary equation by using a) Method of grouping b) Method of multipliers</p> <p><b>a) Method of grouping:</b> In the auxiliary equation, if the variables can be separated in any pair of equations, then we get a solution of the form <math>u(x, y, z) = c_1</math> &amp; <math>v(x, y, z) = c_2</math></p> <p>∴ The general solution is <math>\Phi(u, v) = 0</math></p> <p><b>b) Method of Multipliers:</b></p> <p>i) Choose any three multipliers <math>l, m, n</math> which may be constants or functions of <math>x, y, z</math> we have</p> $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{ldx + mdy + ndz}{lP + mQ + nR}$ <p>If it is possible to choose <math>l, m, n</math> such that <math>lP + mQ + nR = 0</math> then <math>ldx + mdy + ndz = 0</math></p> <p>Integrating this we get <math>u(x, y, z) = c_1</math></p> <p>ii) Choose another any three multipliers <math>l', m', n'</math> which may be constants or functions of <math>x, y, z</math> we have</p> $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}$ <p>If it is possible to choose <math>l', m', n'</math> such that <math>l'P + m'Q + n'R = 0</math> then <math>l'dx + m'dy + n'dz = 0</math></p> <p>Integrating this we get <math>v(x, y, z) = c_2</math></p> <p>∴ The general solution is <math>\Phi(u, v) = 0</math></p>	
1.	<p><b>Solve</b> <math>x(y - z)p + y(z - x)q = z(x - y)</math></p> <p><b>Solution:</b> Given <math>x(y - z)p + y(z - x)q = z(x - y)</math> This is of the form <math>Pp + Qq = R</math> Where <math>P = x(y - z)</math>; <math>Q = y(z - x)</math>; <math>R = z(x - y)</math> The auxiliary equation be</p> $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ $\frac{dx}{x(y - z)} = \frac{dy}{y(z - x)} = \frac{dz}{z(x - y)} \text{ -----(1)}$ <p>i) Choose the multipliers as (1,1,1)</p> $(1) \Rightarrow \frac{dx}{x(y - z)} = \frac{dy}{y(z - x)} = \frac{dz}{z(x - y)}$

$$= \frac{dx + dy + dz}{xy - xz + yz - xy + xz - yz} = \frac{dx + dy + dz}{0}$$

$$\therefore dx + dy + dz = 0$$

$$\text{Integrating } \int dx + \int dy + \int dz = 0$$

$$\boxed{x + y + z = c_1}$$

ii) Choose the multipliers as  $\begin{matrix} 1 & 1 & 1 \\ x & y & z \end{matrix}$

$$(1) \Rightarrow \frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y-z+z-x+x-y} = \frac{dx + dy + dz}{0}$$

$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

$$\text{Integrating } \int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\log xyz = \log c_2 \Rightarrow \boxed{xyz = c_2}$$

$\therefore$  The general solution is  $\boxed{\phi(x + y + z, xyz) = 0}$

2.

**Solve**  $x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)r$

**Solution:**

$$\text{Given } x(z^2 - y^2)p + y(x^2 - z^2)q = z(y^2 - x^2)r$$

This is of the form  $Pp + Qq = Rr$

$$\text{Where } P = x(z^2 - y^2); Q = y(x^2 - z^2); R = z(y^2 - x^2)$$

The auxiliary equation be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)} \text{ -----(1)}$$

i) Choose the multipliers as  $(x, y, z)$

$$(1) \Rightarrow \frac{xdx}{x^2(z^2 - y^2)} = \frac{ydy}{y^2(x^2 - z^2)} = \frac{zdz}{z^2(y^2 - x^2)}$$

$$= \frac{xdx + ydy + zdz}{x^2z^2 - x^2y^2 + x^2y^2 - y^2z^2 + y^2z^2 - x^2z^2} = \frac{xdx + ydy + zdz}{0}$$

$$\therefore xdx + ydy + zdz = 0$$

$$\text{Integrating } \int xdx + \int ydy + \int zdz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_1^2}{2}$$

$$\boxed{x^2 + y^2 + z^2 = c_1^2}$$

ii) Choose the multipliers as  $\begin{matrix} 1 & 1 & 1 \\ x & y & z \end{matrix}$

$$(1) \Rightarrow \frac{\frac{dx}{x}}{\frac{x}{x(z^2 - y^2)}} = \frac{\frac{dy}{y}}{\frac{y}{y(x^2 - z^2)}} = \frac{\frac{dz}{z}}{\frac{z}{z(y^2 - x^2)}}$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{z^2 - y^2 + x^2 - z^2 + y^2 - x^2} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0}$$

$$\therefore \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating  $\int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$

$$\log x + \log y + \log z = \log c_2$$

$$\log xyz = \log c_2 \Rightarrow \boxed{xyz = c_2}$$

$\therefore$  The general solution is  $\Phi(x^2 + y^2 + z^2, xyz) = 0$

3. **Solve**  $(x^2 - y^2 - z^2)p + 2xyq = 2zx$   
**Solution:**  
 Given  $(x^2 - y^2 - z^2)p + 2xyq = 2zx$   
 This is of the form  $Pp + Qq = R$   
 Where  $P = (x^2 - y^2 - z^2)$ ;  $Q = 2xy$ ;  $R = 2zx$

The auxiliary equation be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2zx} \quad \text{---(1)}$$

i) by method of grouping, from last two ratios

$$(1) \Rightarrow \frac{dy}{2xy} = \frac{dz}{2zx}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating  $\int \frac{dy}{y} = \int \frac{dz}{z}$

$$\log y = \log z + \log c_1$$

$$\log y - \log z = \log c_1$$

$$\log \frac{y}{z} = \log c_1$$

$$\boxed{\frac{y}{z} = c_1}$$

ii) Choose the multipliers as  $(x, y, z)$

$$(1) \Rightarrow \frac{xdx}{x(x^2 - y^2 - z^2)} = \frac{ydy}{y(2xy)} = \frac{zdz}{z(2zx)}$$

$$= \frac{xdx + ydy + zdz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{xdx + ydy + zdz}{x^3 + xy^2 + xz^2}$$

$$= \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$$

From 3<sup>rd</sup> and last ratio

$$\frac{dz}{(2zx)} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$$

$$\frac{dz}{z} = \frac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)}$$

Integrating  $\int \frac{dz}{z} = \int \frac{2xdx + 2ydy + 2zdz}{(x^2 + y^2 + z^2)}$

$$\log z = \log(x^2 + y^2 + z^2) \quad \because \int \frac{f'(x)}{f(x)} dx = \log[f(x)]$$

$$\log z = \log(x^2 + y^2 + z^2) + \log c_2$$

$$\log z - \log(x^2 + y^2 + z^2) = \log c_2$$

$$\log \frac{z}{x^2 + y^2 + z^2} = \log c$$

$$\frac{z}{x^2 + y^2 + z^2} = c$$

The general solution is  $\phi \left( \frac{y}{z}, \frac{z}{x^2 + y^2 + z^2} \right) = 0$

**4. Solve**  $(3z - 4y)p + (4x - 2z)q = 2y - 3x$   
**Hint:**  
 The multipliers are  $(x, y, z)$  &  $(2, 3, 4)$   
 The general solution is  $\phi(x^2 + y^2 + z^2, 2x + 3y + 4z) = 0$

**5. Solve**  $(y - xz)p + (yz - x)q = (x + y)(x - y)$   
**Hint:**  
 The multipliers are  $(x, y, z)$  &  $(y, x, 1)$   
 The general solution is  $\phi(x^2 + y^2 + z^2, xy + z) = 0$

**6. Solve**  $x(y^2 + z)p + y(x^2 + z)q = z(x^2 - y^2)$   
**Hint:**  
 The multipliers are  $\frac{1}{x}, -\frac{1}{y}, \frac{1}{z}$  &  $(x, -y, -1)$   
 The general solution is  $\phi\left(\frac{xz}{y}, x^2 - y^2 - 2z\right) = 0$

**Homogeneous Linear PDE of second and higher order with constant co-efficient:**  
 Consider the second order homogeneous linear PDE

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = f(x, y) \quad (1)$$

Let the differential operator  $D = \frac{\partial}{\partial x}$  &  $D' = \frac{\partial}{\partial y}$

$$(1) \Rightarrow (D^2 + a_1 DD' + a_2 D'^2)z = f(x, y) \quad (2)$$

**The general solution of equation (2) is**

$$z = \text{complementary function} + \text{Particular Integral} = \text{C.F.} + \text{P.I.}$$

**To find complementary Function:**

1. Write the Auxiliary equation by putting  $D = m, D' = 1, z = 1, \& RHS = 0$

$$(2) \Rightarrow m^2 + a_1 m + a_2 = 0$$

2. Solve the auxiliary equation, we get the roots of  $m$ . Say the roots are  $m_1, m_2$

3. Comparing the roots of  $m$  and write the complementary function.

**Case 1:** The Roots are real and distinct : say  $m_1 \neq m_2$

$$C.F = f_1(y + m_1 x) + f_2(y + m_2 x)$$

**Case 2:** The Roots are real and equal : say  $m_1 = m_2 = m$

$$C.F = f_1(y + mx) + x f_2(y + mx)$$

**Note:** If the roots are  $m = \alpha \pm i\beta$

$$\text{then } C.F = f_1[y + (\alpha + i\beta)] + f_2[y + (\alpha - i\beta)]$$

**To find Particular Integral :**

**Type : I**

If  $RHS = e^{ax+by}$  then

$$P.I = \frac{1}{f(D, D')} e^{ax+by}$$

$$\text{Rule: } D = a \& D' = b$$

$$P.I = \frac{1}{f(a, b)} e^{ax+by}, \text{ Provided Denominator } \neq 0$$

If Denominator = 0, then 1) multiply the numerator by  $x$  2) differentiating denominator partially w.r.to  $D$

$$P.I = \frac{x}{f'(D, D')} e^{ax+by}$$

$$P.I = \frac{1}{f'(a, b)} e^{ax+by}, \text{ Provided Denominator } \neq 0 \therefore \text{ Replace } D = a \& D' = b$$

Continuing this process until we get  $Dr \neq 0$ .

**Type : II**

If  $RHS = \sin(ax + by)$  (or)  $RHS = \cos(ax + by)$  then

$$P.I = \frac{1}{f(D^2, DD', D'^2)} \sin(ax + by)$$

$$\text{Rule: } D^2 = -(a^2); DD' = (-ab); D'^2 = -(b^2)$$

$$P.I = \frac{1}{f(-a^2, -ab, -b^2)} \sin(ax + by), \text{ Provided } Dr \neq 0$$

**Note 1:** After substitutions the denominator will be in terms of  $D$  &  $D'$ . Multiply and divide by  $D$  so that the denominator will have  $D^2$  &  $DD'$  terms.

**Note 2:** After substitutions the denominator will be in terms of  $D$  & constant terms,

For eg.  $P.I = \frac{1}{D - 5} \sin(x - 2y)$

Take conjugate of denominator with constant term and multiplied with both numerator and denominator.

$$P.I = \frac{1}{D - 5} \times \frac{D + 5}{D + 5} \sin(x - 2y) = \frac{1}{D^2 - 25} \sin(x - 2y)$$

Then apply the rule as usual.

**Type : III**

If  $RHS = x^m y^n$  (polynomial type) then

$P.I = \frac{1}{f(D, D')} x^m y^n$ , we bring this into a standard binomial format, by taking out highest power term of D.

(i.e)  $P.I = [1 \pm f(D, D')]^{-1} x^m y^n$

This will be expanded by using the formulae

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

Note:

$\frac{1}{D} = \int dx, D' = \frac{\partial}{\partial y}$

**Type : IV (Exponential shifted rule)**

If  $RHS = e^{ax+by} \cos(ax+by)$  (or)  $RHS = e^{ax+by} \sin(ax+by)$  (or)  $RHS = e^{ax+by} x^m y^n$  then

$P.I = \frac{1}{f(D, D')} e^{ax+by} \sin(ax+by)$

$P.I = e^{ax+by} \frac{1}{f(D+a, D'+b)} \sin(ax+by)$

Here after apply the rule as we discussed in Type II&III

**Type : V**

If  $RHS = y \cos x$  (or)  $RHS = y \sin x$  then

Case 1:

$P.I = \frac{1}{D - mD'} y \cos x$

$P.I = \frac{1}{D - mD'} \int (c - mx) \cos x dx \quad \therefore \text{Rule: } y = c - mx$

Case 2:

$P.I = \frac{1}{D + mD'} y \cos x$

$P.I = \frac{1}{D + mD'} \int (c + mx) \cos x dx \quad \therefore \text{Rule: } y = c - mx$

Note: After integration we have to replace  $c - mx = y$

1. Solve  $(D^2 - DD' - 20D'^2) z = e^{5x+y} + \sin(4x - y)$

**Solution:**

Given  $(D^2 - DD' - 20D'^2) z = e^{5x+y} + \sin(4x - y)$

To find C.F

The auxiliary equation is

$m^2 - m - 20 = 0 \quad \therefore \text{replace } D = m, D' = 1, z = 0$

$(m - 5)(m + 4) = 0$

$m = 5, -4$

$\therefore C.F = f_1(y - 4x) + f_2(y + 5x) \quad \therefore \text{The roots are real and distinct } \Rightarrow C.F = f_1(y + mx) + xf_2(y + mx)$

**To find P.I**

$P.I = \frac{1}{D^2 - DD' - 20D'^2} [e^{5x+y} + \sin(4x - y)]$

$P.I = P.I_1 + P.I_2$

$$P.I_1 = \frac{1}{D^2 - DD' - 20D'^2} e^{5x+y} \quad \text{Rule: replace } D = 5 \text{ \& } D' = 1 \quad \text{Type:1}$$

$$= \frac{1}{25 - 5 - 20} e^{5x+y} = \frac{1}{0} e^{5x+y} \quad \because \text{Introducing } x \text{ in Nr.Diff Dr. partially w.r.to } D$$

$$= \frac{x}{2D - D'} e^{5x+y} \quad \text{Rule: replace } D = 5 \text{ \& } D' = 1$$

$$= \frac{x}{10 - 1} e^{5x+y}$$

$$P.I_1 = \frac{x}{9} e^{5x+y}$$

$$P.I_2 = \frac{1}{D^2 - DD' - 20D'^2} \sin(4x - y) \quad \text{here } a = 4 \text{ \& } b = -1 \quad (\text{Type:2})$$

Rule: replace  $D^2 = -(a^2) = -16$ ;  $D'^2 = -(b^2) = -1$  &  $DD' = -(ab) = 4$

$$P.I_2 = \frac{1}{-16 - 4 - 20(-1)} \sin(4x - y) = \frac{1}{0} \sin(4x - y)$$

$$= \frac{x}{2D - D'} \sin(4x - y) \quad \because \text{Introducing } x \text{ in Nr.Diff Dr. partially w.r.to } D$$

$$= \frac{x}{2D - D'} \times \frac{D}{D} \sin(4x - y)$$

$$= \frac{x D}{2D^2 - DD'} \sin(4x - y)$$

$$= \frac{x D}{2(-16) - 4} \sin(4x - y)$$

$$= \frac{x D (\sin(4x - y))}{-32 - 4}$$

$$P.I_2 = \frac{4x \cos(4x + 3)}{-36} = \frac{-1}{9} x \cos(4x + 3) \quad \because \frac{d}{dx} (\sin nx) = n \cos nx$$

$$P.I_2 = \frac{-1}{9} x \cos(4x + 3)$$

$$P.I = \frac{x}{9} e^{5x+y} - \frac{1}{9} x \cos(4x + 3) = \frac{x}{9} e^{5x+y} - \cos(4x + 3)$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y - 4x) + f_2(y + 5x) + \frac{x}{9} e^{5x+y} - \cos(4x + 3)$$

2. Solve  $\frac{\partial^3 z}{\partial x^3} - 2 \frac{\partial^3 z}{\partial x^2 \partial x} = e^{x+2y} + 4 \sin(x + y)$

**Solution: same as previous problem**

**Hint:**

Given  $\frac{\partial^3 z}{\partial x^3} - 2 \frac{\partial^3 z}{\partial x^2 \partial x} = e^{x+2y} + 4 \sin(x + y)$

$$(D^3 - 2D^2 D') z = e^{x+2y} + 4 \sin(x + y) \quad \because D = \frac{\partial}{\partial x} \quad \& \quad D' = \frac{\partial}{\partial y}$$

$$m = 0, 0, 2$$

$$C.F = f_1(y) + x f_2(y) + f_2(y + 2x)$$



$$P.I = \frac{-1}{3} e^{x+2y} - 4 \cos(x+2y)$$

3. Solve  $(D^2 + DD' - 6D'^2)z = x^2y + e^{3x+y}$

**Solution:**

Given  $(D^2 + DD' - 6D'^2)z = x^2y + e^{3x+y}$

**To find C.F**

The auxiliary equation is

$$m^2 + m - 6 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m - 2)(m + 3) = 0$$

$$m = 2, -3$$

$$\therefore C.F = f_1(y - 3x) + f_2(y + 2x) \quad \because \text{The roots are real and distinct } \Rightarrow C.F = f_1(y + m_1x) + f_2(y + m_2x)$$

**To find P.I**

$$P.I = \frac{1}{D^2 + DD' - 6D'^2} (x^2y + e^{3x+y})$$

$$P.I = P.I_1 + P.I_2$$

$$P.I = \frac{1}{D^2 + DD' - 6D'^2} x^2y \quad (\text{Type:3})$$

$$= \frac{1}{D^2 + DD' - 6D'^2} x^2y$$

$$= \frac{1}{D^2 + DD' - 6D'^2} x^2y$$

$$= \frac{1}{D^2 + DD' - 6D'^2} x^2y$$

$$= \frac{1}{D^2 + DD' - 6D'^2} x^2y \quad \because (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$= \frac{1}{D^2 + DD' - 6D'^2} x^2y \quad \because D' = \frac{\partial}{\partial y}$$

$$= \frac{1}{D^2 + DD' - 6D'^2} x^2y$$

$$= \frac{1}{D^2 + DD' - 6D'^2} x^2y$$

$$= \frac{1}{D^2 + DD' - 6D'^2} x^2y$$

$$= \frac{1}{D^2 + DD' - 6D'^2} x^2y \quad \because \frac{1}{D} = \int dx$$

$$= \frac{1}{D^2 + DD' - 6D'^2} x^2y$$

$$= \frac{1}{D^2 + DD' - 6D'^2} x^2y$$

$$= \frac{1}{D^2 + DD' - 6D'^2} x^2y$$

$$= \frac{1}{D^2 + DD' - 6D'^2} x^2y$$

$$P.I_1 = \frac{x^4y}{12} - \frac{x^5}{60}$$

$$P.I = \frac{1}{D^2 + DD' - 6D'^2} e^{3x+y} \quad \text{Rule: replace } D = a = 3 \text{ \& } D' = b = 1 \quad \text{Type:1}$$

$$= \frac{1}{9+3-6(1)} e^{3x+y} = \frac{1}{6} e^{3x+y}$$

$$P.I = \frac{1}{6} e^{3x+y}$$

$$P.I = \frac{x^4 y}{12} - \frac{x^5}{60} + \frac{1}{6} e^{3x+y}$$

**The general solution is**  
 $z = C.F + P.I$

$$z = f_1(y-3x) + f_2(y+2x) + \frac{x^4 y}{12} - \frac{x^5}{60} + \frac{1}{6} e^{3x+y}$$

**4. Solve**  $(D^2 + 2DD' + D'^2) z = x^2 y + e^{x-y}$   
**Solution: same as previous problem**  
**Hint:**  
 $m = -1, -1$   
 $C.F = f_1(y-x) + x f_2(y-x)$   
 $P.I = \frac{x^4 y}{12} - \frac{x^5}{30} + \frac{x^2}{2} e^{x-y}$

**5. Solve**  $(D^2 - 6DD' + 5D'^2) z = xy + e^x \sinh y$   
**Solution:**  
 Given  $(D^2 - 6DD' + 5D'^2) z = xy + e^x \sinh y$

$$= xy + e^x \frac{e^y - e^{-y}}{2} \quad \therefore \sinh \theta = \frac{e^\theta - e^{-\theta}}{2}, \text{ here } \theta = y$$

$$= xy + \frac{e^{x+y} - e^{x-y}}{2}$$

$(D^2 - 6DD' + 5D'^2) z = xy + \frac{e^{x+y}}{2} - \frac{e^{x-y}}{2} \quad \therefore e^a e^b = e^{a+b}$

**To find C.F**  
 The auxiliary equation is  
 $m^2 - 6m + 5 = 0 \quad \therefore \text{replace } D = m, D' = 1, z = 0$   
 $(m-1)(m-5) = 0$   
 $m = 1, 2$

$\therefore C.F = f_1(y+x) + f_2(y+5x) \quad \therefore \text{The roots are real and distinct } \Rightarrow C.F = f_1(y+m_1x) + x f_2(y+m_2x)$

**To find P.I**

$$P.I = \frac{1}{D^2 - 6DD' + 5D'^2} \left[ xy + \frac{e^{x+y}}{2} - \frac{e^{x-y}}{2} \right]$$

$P.I = P.I_1 + P.I_2 - P.I_3 \quad \text{----- (1)}$

To find P.I<sub>1</sub>

$$P.I_1 = \frac{1}{D^2 - 6DD' + 5D'^2} xy \quad \text{(Type:3)}$$

$$= \frac{1}{D^2 - 6DD' + 5D'^2} xy$$

$$= \frac{1}{D^2} xy$$

$$\begin{aligned}
 &= \frac{1}{D^2} \frac{-6D'}{D} + \frac{5D'^2}{D^2} xy \\
 &= \frac{1}{D^2} \frac{-6D'}{D} + \frac{5D'^2}{D^2} xy \quad \because (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots \\
 &= \frac{1}{D^2} \frac{6D'}{D} - \frac{5D'^2}{D^2} xy + \dots xy \\
 &= \frac{1}{D^2} xy + \frac{6x}{D} - \frac{5(0)}{D^2} + \dots xy \quad \because D'(y) = \frac{\partial}{\partial y}(y) = 1 \text{ \& } D'^2(y) = \frac{\partial^2}{\partial y^2}(y) = 0 \\
 &= \frac{1}{D^2} xy + 6 \int x dx + \frac{1}{D^2} xy + \frac{6x^2}{2} \quad \because \frac{1}{D} = \int dx \\
 &= \frac{1}{D} \int (xy + 3x^2) dx = \frac{1}{D} \frac{x^2 y}{2} + \frac{3x^3}{3} = \int \frac{x^2 y}{2} + x^3 dx
 \end{aligned}$$

$$P.I_1 = \frac{x^3 y}{6} + \frac{x^4}{4}$$

To find P.I<sub>2</sub>

$$\begin{aligned}
 P.I_2 &= \frac{1}{D^2 - 6DD' + 5D'^2} e^{x+y} \quad \text{Type:1} \\
 &= \frac{1}{2-2} \frac{1}{1-6+5} e^{x+y} = \frac{1}{2-2} \frac{1}{2-2} e^{x+y} \quad \text{Rule: replace } D = a = 1 \text{ \& } D' = b = 1 \\
 &= \frac{1}{2-2} \frac{x}{2D - 6D' + 0} e^{x+y} \\
 &= \frac{1}{2-2} \frac{x}{2-6} e^{x+y} \\
 &= \frac{1}{2-2} \frac{-x}{4} e^{x+y}
 \end{aligned}$$

$$P.I = \frac{-x}{2} e^{x+y}$$

To find P.I<sub>3</sub>

$$\begin{aligned}
 P.I_3 &= \frac{1}{D^2 - 6DD' + 5D'^2} e^{x-y} \quad \text{Type:1} \\
 &= \frac{1}{2-2} \frac{1}{1+6+5} e^{x-y} \quad \text{Rule: replace } D = a = 1 \text{ \& } D' = b = -1
 \end{aligned}$$

$$P.I = \frac{1}{3} e^{x+y}$$

$$(1) \Rightarrow P.I = P.I_1 + P.I_2 - P.I_3 = \frac{x^3 y}{6} + \frac{x^4}{4} - \frac{x}{8} e^{x+y} - \frac{1}{24} e^{x+y}$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y+x) + f_2(y+5x) + \frac{x^3 y}{6} + \frac{x^4}{4} - \frac{x}{8} e^{x+y} - \frac{1}{24} e^{x+y}$$

6. Solve  $(D^2 + 2DD' + D'^2)z = \sinh(x+y) + e^{x+2y}$   
Solution: same as previous problem

	<p><b>Hint:</b></p> $(D^2 + 2DD' + D'^2)z = \sinh(x+y) + e^{x+2y}$ $(D^2 + 2DD' + D'^2)z = \frac{e^{x+y} - e^{-(x-y)}}{2} + e^{x+2y} = \frac{e^{x+y}}{2} - \frac{e^{-x-y}}{2} + e^{x+2y}$ $(D^2 + 2DD' + D'^2)z = \frac{e^{x+y}}{2} - \frac{e^{-x-y}}{2} + e^{x+2y}$ <p><math>m = -1, -1</math></p> <p><math>C.F = f_1(y-x) + xf_2(y-x)</math></p> $P.I = \frac{e^{x+y}}{8} - \frac{e^{-x-y}}{8} - \frac{e^{x+2y}}{9}$
7.	<p>Solve <math>(D^2 - 4DD' + 4D'^2)z = e^{x+2y}</math></p> <p><b>Solution:</b></p> <p>Hint: <math>m = 2, 2</math></p> <p><math>C.F = f_1(y+x) + f_2(y+5x)</math></p> $P.I = \frac{x^2}{2} e^{2x+y}$
8.	<p>Solve <math>(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y)</math></p> <p><b>Solution:</b></p> <p>Given <math>(2D^2 - 5DD' + 2D'^2)z = 5\sin(2x+y)</math></p> <p><b>To find C.F</b></p> <p>The auxiliary equation is</p> $2m^2 - 5m + 2 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$ $m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{here } a = 2, b = -5, c = 2$ $= \frac{-(-5) \pm \sqrt{25 - 4(2)(2)}}{2(2)}$ $= \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5 \pm 3}{4}$ $= \frac{5 \pm \sqrt{25 - 16}}{4} = \frac{5 \pm 3}{4} = \frac{5+3}{4}$ $m = \frac{5+3}{4}, m = \frac{5-3}{4} \Rightarrow m = 2, m = \frac{1}{2}$ <p><math>\therefore C.F = f_1(y+2x) + f_2\left(\frac{y+x}{2}\right)</math> <span style="margin-left: 20px;"><math>\because</math> The roots are real and distinct <math>\Rightarrow C.F = f_1(y+m_1x) + f_2(y+m_2x)</math></span></p> <p><b>To find P.I</b></p> $P.I = \frac{1}{2D^2 - 5DD' + 2D'^2} 5\sin(2x+y) \quad \text{Type:2}$ $P.I = \frac{1}{2D^2 - 5DD' + 2D'^2} 5\sin(2x+y) \quad \text{here } a = 2 \& b = 1$ <p>Rule: replace <math>D^2 = -(a^2) = -4; D'^2 = -(b^2) = -1 \&amp; DD' = -(ab) = -2</math></p> $P.I = \frac{1}{2(-4) - 5(-2) + 2(-1)} 5\sin(2x+y)$ $P.I = \frac{1}{-8+10-2} 5\sin(2x+y) = \frac{1}{0} 5\sin(2x+y)$

$$\begin{aligned}
 &= \frac{x}{4D - 5D' + 0} 5\sin(2x + y) \quad \{\text{Introducing } x \text{ in Nr. \& Diff Dr. partially w.r.to } D\} \\
 &= \frac{x}{4D - 5D'} \times \frac{D}{D} 5\sin(2x + y) \\
 &= 5 \frac{x D}{4D^2 - 5DD'} \sin(2x + y) \\
 &= 5 \frac{x D}{4(-4) - 5(-2)} \sin(2x + y) = 5 \frac{x D [\sin(2x + y)]}{-16 + 10} \\
 &= \frac{5}{-6} x [2 \cos(2x + y)]
 \end{aligned}$$

$$P.I = \frac{-5}{3} x \cos(2x + y)$$

The general solution is  
 $z = C.F + P.I$

$$z = f_1(y + 2x) + f_2\left(\frac{y}{2}\right) + \frac{x}{2} - \frac{5}{3} x \cos(2x + y)$$

9. Solve the equation  $(D^3 + D^2 D' - 4DD'^2 - 4D'^3) z = \cos(2x + y)$

**Solution:**

Given  $(D^3 + D^2 D' - 4DD'^2 - 4D'^3) z = \cos(2x + y)$

**To find C.F**

The auxiliary equation is

$$m^3 + m^2 - 4m - 4 = 0 \quad \therefore \text{replace } D = m, D' = 1, z = 0$$

$$m^2(m+1) - 4(m+1) = 0$$

$$(m+1)(m^2 - 4) = 0$$

$$m = -1, -2, 2$$

$$\therefore C.F = f_1(y - x) + f_2(y - 2x) + f_3(y + 2x)$$

**To find P.I**

$$P.I = \frac{1}{D^3 + D^2 D' - 4DD'^2 - 4D'^3} \cos(2x + y) \quad \text{here } a = 2, b = 1 \text{ (Type:2)}$$

Rule: replace  $D^2 = -(a^2) = -4$ ;  $D'^2 = -(b^2) = -1$  &  $DD' = -(ab) = -2$

$$\begin{aligned}
 P.I &= \frac{1}{-4D - 4D' - 4D(-1) - 4(-1)D'} \cos(2x + y) \\
 &= \frac{1}{-4D - 4D' + 4D + 4D'} \cos(2x + y) = \frac{1}{0} \cos(2x + y)
 \end{aligned}$$

$$P.I = \frac{x}{3D^2 + 2DD' - 4D'^2 - 0} \cos(2x + y) \quad \{\text{Introducing } x \text{ in Nr. \& Diff Dr. partially w.r.to } D\}$$

$$= \frac{x}{3(-4) + 2(-2) - 4(-1)} \cos(2x + y) = \frac{x}{-12 - 4 + 4} \cos(2x + y)$$

$$P.I = \frac{-x}{12} \cos(2x + y)$$

The general solution is  
 $z = C.F + P.I$

$$z = f_1(y - x) + f_2(y - 2x) + f_3(y + 2x) - \frac{x}{12} \cos(2x + y)$$

10. Solve  $(D^3 - 7DD'^2 - 6D'^3)z = \cos(x + 2y) + 4$

**Solution:**

Given  $(D^3 - 7DD'^2 - 6D'^3)z = \cos(x + 2y) + 4$

**To find C.F**

The auxiliary equation is

$m^3 - 7m - 6 = 0$        $\therefore$  replace  $D = m, D' = 1, z = 0$

m=-1	1	0	-7	6
	0	-1	1	6
	1	-1	-6	0

$m = -1, m^2 - m - 6 = 0$

$m = -1, (m - 3)(m + 2) = 0$

$m = -1, -2, 3$

$\therefore$  C.F =  $f_1(y - x) + f_2(y - 2x) + f_3(y + 3x)$

**To find P.I**

$P.I = \frac{1}{D^3 - 7DD'^2 - 6D'^3} [\cos(x + 2y) + 4]$

$P.I = P.I_1 + P.I_2$

$P.I_1 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} \cos(x + 2y)$  here  $a = 1, b = 2$  (Type:2)

Rule: replace  $D^2 = -(a^2) = -1; D'^2 = -(b^2) = -4$  &  $DD' = -(ab) = -2$

$$\begin{aligned}
 P.I_1 &= \frac{1}{(-1)D - 7D(-2) - 6(-2)D'} \cos(x + 2y) = \frac{1}{-D + 14D + 12D'} \cos(x + 2y) \\
 &= \frac{1}{13D + 12D'} \times \frac{D}{D} \cos(x + 2y) \\
 &= \frac{D}{13D^2 + 12D'D} \cos(x + 2y) \\
 &= \frac{D}{13(-1) + 12(-2)} \cos(x + 2y) = \frac{D \cos(x + 2y)}{-13 - 24}
 \end{aligned}$$

$P.I_1 = \frac{-\sin(x + 2y)}{37}$

**To find P.I<sub>2</sub>**

$P.I_2 = \frac{1}{D^3 - 7DD'^2 - 6D'^3} 4e^{0x+0y}$        $\therefore e^0 = 1$       Type:1

$= \frac{1}{D^3 - 7DD'^2 - 6D'^3} 4e^{0x+0y} = \frac{1}{0} 4e^{0x+0y}$       Rule: Replace  $D = 0, D' = 0$

$= \frac{x}{3D^2 - 7D'^2 - 0} 4e^{0x+0y} = \frac{x}{0} 4e^{0x+0y}$

$= \frac{x}{6D - 0} 4e^{0x+0y} = \frac{x}{0} 4e^{0x+0y}$

$= \frac{x^2}{6} 4$

	$P.I_2 = \frac{2x^2}{3}$ $P.I = \frac{-1}{37} \sin(x+2y) + \frac{2x^2}{3}$ <p><b>The general solution is</b>  <math>z = C.F + P.I</math></p> $z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{37} \sin(x+2y) + \frac{2x^2}{3}$
<b>11.</b>	<p><b>Solve</b> <math>(D^2 + 3DD' - 4D'^2) z = xy + \cos(2x + y)</math></p> <p><b>Solution:</b> same as previous problem</p> <p><b>Hint:</b>  <math>m = -4, 1</math></p> <p><math>C.F = f_1(y - 4x) + xf_2(y + x)</math></p> <p><math>P.I = \frac{-1}{6} \cos(2x + y) + \frac{x^3 y}{6} - \frac{x^4}{8}</math></p>
<b>12.</b>	<p><b>Solve</b> <math>(D^2 + 3DD' - 4D'^2) z = x + \sin y</math></p> <p><b>Hint:</b>  <math>m = -4, 1</math></p> <p><math>C.F = f_1(y - 4x) + xf_2(y + x)</math></p> <p><math>P.I = \frac{1}{D^2 + 3DD' - 4D'^2} [x + \sin(0x + y)] = \dots = \frac{x^3}{6} + \frac{\sin y}{4}</math></p>
<b>13.</b>	<p><b>Solve</b> <math>(D^2 - DD' - 2D'^2) z = (2x + 3y) + e^{3x+4y}</math></p> <p><b>Hint:</b>  <math>m = -1, 2</math></p> <p><math>C.F = f_1(y - x) + f_2(y + 2x)</math></p> <p><math>P.I = \frac{5x^3}{6} + \frac{3x^2 y}{2} + \frac{1}{35} e^{3x+4y}</math></p>
<b>14.</b>	<p><b>Solve</b> <math>(D^2 - 2DD' + D'^2) z = (2 + 4x)e^{x+2y}</math></p> <p><b>Solution:</b></p> <p>Given <math>(D^2 - 2DD' + D'^2) z = x^2 y^2 e^{x+2y}</math></p> <p><b>To find C.F</b>          The auxiliary equation is  <math>m^2 - 2m + 1 = 0 \quad \therefore \text{replace } D = m, D' = 1, z = 0</math>  <math>(m-1)(m-1) = 0</math>  <math>m = 1, 1</math></p> <p><math>\therefore C.F = f_1(y + x) + xf_2(y + x)</math></p> <p><b>To find P.I</b></p> <p><math>P.I = \frac{1}{D^2 - 2DD' + D'^2} (2 + 4x)e^{x+2y} \quad \text{here } a = 2, b = 1 \text{ (Type:4)}</math></p> <p><math>P.I = \frac{1}{(D - D')^2} (2 + 4x)e^{x+2y}</math></p> <p>Rule: replace <math>D = D + a = D + 1; D' = D' + b = D' + 2</math></p> <p><math>P.I = e^{x+2y} \frac{1}{(D+1 - D' - 2)^2} (2 + 4x)</math></p>

$$\begin{aligned}
 &= e^{x+2y} \frac{1}{(D - D' - 1)^2} (2 + 4x) \\
 &= e^{x+2y} \frac{1}{[-(1 - D + D')]^2} (2 + 4x) = e^{x+2y} \frac{1}{[1 - (D - D')]^2} (2 + 4x) \\
 &= e^{x+2y} [1 - (D - D')]^{-2} (2 + 4x) \\
 &= e^{x+2y} [1 + 2(D - D') + 3(D - D')^2 + \dots] (2 + 4x) \quad \because (1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots \\
 &= e^{x+2y} [1 + 2D - 2D' + 3(D^2 - 2DD' + D'^2) + \dots] (2 + 4x) \\
 &= e^{x+2y} [1 + 2D - 2D' + 3D^2 - 6DD' + 3D'^2] (2 + 4x) \\
 &= e^{x+2y} [1 + 2D + 3D^2] (2 + 4x) \quad \because \text{there is no } y \text{ term in RHS, neglect the term } D' \\
 &= e^{x+2y} [2 + 4x + 2D(2 + 4x) + 3D^2(2 + 4x)] \\
 &= e^{x+2y} [2 + 4x + 2(4) + 0]
 \end{aligned}$$

$$P.I = e^{x+2y} [4x + 10]$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y-x) + f_2(y+x) + e^{x+2y} [4x + 10]$$

15.

Solve  $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x-y} \sin(2x + 3y)$

**Solution:**

Given  $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x-y} \sin(2x + 3y)$   
 $(D^2 - D'^2) z = e^{x-y} \sin(2x + 3y)$

**To find C.F**

The auxiliary equation is

$$m^2 - 1 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$m^2 = 1 \Rightarrow m = -1, 1$$

$$\therefore C.F = f_1(y - x) + f_2(y + x)$$

**To find P.I**

$$P.I = \frac{1}{D^2 - D'^2} e^{x-y} \sin(2x + 3y) \quad \text{here } a = 1, b = -1 \text{ (Type:4)}$$

$$\text{Rule: replace } D = D + a = D + 1; D' = D' + b = D' - 1$$

$$P.I = e^{x-y} \frac{1}{(D+1)^2 - (D'-1)^2} \sin(2x + 3y)$$

$$= e^{x-y} \frac{1}{D^2 + 2D + 1 - D'^2 + 2D' - 1} \sin(2x + 3y)$$

$$= e^{x-y} \frac{1}{D^2 + 2D - D'^2 + 2D'} \sin(2x + 3y) \quad \text{Here } a = 2, b = 3$$

$$\text{Rule: replace } D^2 = -(a^2) = -4; D'^2 = -(b^2) = -9 \text{ \& } DD' = -(ab) = -6$$

$$= e^{x-y} \frac{1}{-4 + 2D - (-9) + 2D'} \sin(2x + 3y) = e^{x-y} \frac{1}{2D + 2D' + 5} \sin(2x + 3y)$$

$$= e^{x-y} \frac{1}{2D + 2D' + 5} \times \frac{D}{D} \sin(2x + 3y)$$



$$= e^{x-y} \frac{D}{2D^2 + 2DD' + 5D} \sin(2x + 3y)$$

$$= e^{x-y} \frac{D}{-8 - 12 + 5D} \sin(2x + 3y) = e^{x-y} \frac{D}{5D - 20} \sin(2x + 3y)$$

If we multiply and divide by D, we can not get the term  $D^2, D'^2$  term, so we take conjugate for constant term and multiplied with both Nr. & Dr.

$$= e^{x-y} \frac{D}{5D - 20} \times \frac{5D + 20}{5D + 20} \sin(2x + 3y)$$

$$= e^{x-y} \frac{D}{5D^2 + 20D} \sin(2x + 3y)$$

$$= e^{x-y} \frac{25D^2 - 400}{5D^2 \sin(2x + 3y) + 20D \sin(2x + 3y)}$$

$$= \frac{e^{x-y}}{e} \frac{25(-4) - 400}{25(-4) - 400}$$

$$= e^{x-y} \frac{5D \cos(2x + 3y) \times 2 + 20 \cos(2x + 3y) \times 2}{-100 - 400}$$

$$= e^{x-y} \frac{-20 \sin(2x + 3y) + 40 \cos(2x + 3y)}{-500}$$

$$P.I = \frac{e^{x-y}}{25} [\sin(2x + 3y) - 2 \cos(2x + 3y)]$$

**The general solution is**

$$z = C.F + P.I$$

$$z = f_1(y - x) + f_2(y + x) + \frac{e^{x-y}}{25} [\sin(2x + 3y) - 2 \cos(2x + 3y)]$$

16.

**Solve**  $(D^2 + DD' - 6D'^2)z = y \cos x$

**Solution:**

Given  $(D^2 + DD' - 6D'^2)z = y \cos x$

**To find C.F**

The auxiliary equation is

$$m^2 + m - 6 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m - 2)(m + 3) = 0$$

$$m = 2, -3$$

$\therefore C.F = f_1(y - 3x) + f_2(y + 2x)$   $\because$  The roots are real and distinct  $\Rightarrow C.F = f_1(y + m_1x) + f_2(y + m_2x)$

**To find P.I**

$$P.I = \frac{1}{D^2 + DD' - 6D'^2} y \cos x \quad \text{Type: 5}$$

$$= \frac{1}{(D + 3D')(D - 2D')} y \cos x$$

$$= \frac{1}{(D + 3D')} \int (c - 2x) \cos x \, dx \quad \because \text{Rule: } y = c - mx \text{ here } m = 2$$

$$= \frac{1}{(D + 3D')} [(c - 2x)(\sin x) - (-2)(-\cos x)] \quad \because \int uv \, dx = uv_1 - u'v_2 + \dots$$

$$= \frac{1}{(D + 3D')} [y \sin x - 2 \cos x]$$

$$= \frac{1}{(D + 3D')} \int [(c + 3x) \sin x - 2 \cos x] \, dx \quad \because \text{Rule: } y = c + mx \text{ here } m = 3$$

$$= (c + 3x)(-\cos x) - (3)(-\sin x) - 2 \sin x \quad \because y = c + 3x$$

$$= -y \cos x + 3 \sin x - 2 \sin x$$

$$P.I = \sin x - y \cos x$$

**The general solution is**

$$z = C.F + P.I$$

$$z = f_1(y - 3x) + f_2(y + 2x) + \sin x - y \cos x$$

17.

**Solve**  $(D^2 - 5DD' + 6D'^2)z = y \sin x$

**Solution:**

Given  $(D^2 - 5DD' + 6D'^2)z = y \sin x$

**To find C.F**

The auxiliary equation is

$$m^2 - 5m + 6 = 0 \quad \because \text{replace } D = m, D' = 1, z = 0$$

$$(m - 2)(m - 3) = 0$$

$$m = 2, 3$$

$$\therefore C.F = f_1(y + 3x) + f_2(y + 2x) \quad \because \text{The roots are real and distinct} \Rightarrow C.F = f_1(y + m_1x) + f_2(y + m_2x)$$

**To find P.I**

$$P.I = \frac{1}{D^2 - 5DD' + 6D'^2} y \sin x \quad \text{Type: 5}$$

$$= \frac{1}{(D - 3D')(D - 2D')} y \sin x$$

$$= \frac{1}{(D - 3D')} \int (c - 2x) \sin x dx \quad \because \text{Rule: } y = c - mx \text{ here } m = 2$$

$$= \frac{1}{(D - 3D')} [(c - 2x)(-\cos x) - (-2)(-\sin x)] \quad \because \int uv dx = uv - u'v + \dots$$

$$= \frac{1}{(D - 3D')} [-y \cos x - 2 \sin x] = \frac{-1}{(D - 3D')} [y \cos x + 2 \sin x]$$

$$= - \int [(c - 3x) \cos x + 2 \sin x] dx \quad \because \text{Rule: } y = c - mx \text{ here } m = 3$$

$$= - [(c - 3x)(\sin x) - (-3)(-\cos x) + 2(-\cos x)] \quad \because y = c - 3x$$

$$= - [y \sin x - 3 \cos x - 2 \cos x]$$

$$P.I = 5 \cos x - y \sin x$$

**The general solution is**

$$z = C.F + P.I$$

$$z = f_1(y + 3x) + f_2(y + 2x) + 5 \cos x - y \sin x$$

**Non-Homogeneous Linear PDE of second and higher order with constant co-efficient:**

Consider the second order non-homogeneous linear PDE

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} + a_3 \frac{\partial z}{\partial x} + a_4 \frac{\partial z}{\partial y} = f(x, y) \quad \text{----- (1)}$$

Let the differential operator  $D = \frac{\partial}{\partial x}$  &  $D' = \frac{\partial}{\partial y}$

$$(1) \Rightarrow (D^2 + a_1 DD' + a_2 D'^2 + a_3 D + a_4 D')z = f(x, y) \quad \text{----- (2)}$$

**The general solution of equation (2) is**

$$z = \text{complementary function} + \text{Particular Integral} = C.F + P.I$$

**To find complementary Function:**

**Case : I**

The given PDE will bring into the form of  $(D - m_1D' - C_1)(D - m_2D' - C_2)z = 0$

$$C.F = e^{c_1x} f_1(y + m_1x) + e^{c_2x} f_2(y + m_2x)$$

**Case : II**

The given PDE will bring into the form of  $(D - mD' - C)^2 z = 0$

$$C.F = e^{cx} f_1(y + mx) + xe^{cx} f_2(y + mx)$$

**Note:** Particular Integral can be obtained, similar like in Homogeneous types.

1. **Solve**  $(D^2 + 2DD' + D'^2 - 2D - 2D')z = e^{3x+y} + \sin(x + 2y)$

**Solution:**

Given  $(D^2 + 2DD' + D'^2 - 2D - 2D')z = e^{3x+y} + \sin(x + 2y)$

**To find C.F**

$$((D + D')^2 - 2(D + D'))z = 0$$

$$(D + D')(D + D' - 2)z = 0 \text{ ----- (1)}$$

This is of the form

$$(D - m_1D' - C_1)(D - m_2D' - C_2)z = 0 \text{ ----- (2)}$$

Comparing (1) & (2)

$$m_1 = -1, C_1 = 0, m_2 = -1, C_2 = 2$$

$$C.F = e^{c_1x} f_1(y + m_1x) + e^{c_2x} f_2(y + m_2x)$$

$$C.F = f_1(y - x) + e_{2x} f_2(y - x) \quad \because e_0 = 1$$

**To find P.I**

$$P.I = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} (e^{3x+y} + \sin(x + 2y))$$

$$P.I = P.I_1 + P.I_2$$

**To find P.I<sub>1</sub>**

$$P.I_1 = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} e^{3x+y} \quad \text{Rule: replace } D = 3 \text{ \& } D' = 1 \text{ Type:1}$$

$$= \frac{1}{9 + 6 + 1 - 6 - 2} e^{3x+y}$$

$$P.I_1 = \frac{1}{8} e^{3x+y}$$

**To find P.I<sub>2</sub>**

$$P.I_2 = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin(x + 2y) \quad \text{here } a = 1, b = 2 \text{ type:2}$$

Rule: replace  $D^2 = -(a^2) = -1$ ;  $D'^2 = -(b^2) = -4$  &  $DD' = -(ab) = -2$

$$P.I_2 = \frac{1}{-1 + 2(-2) - 4 - 2D - 2D'} \sin(x + 2y) = \frac{1}{-1 - 4 - 4 - 2D - 2D'} \sin(x + 2y)$$

$$= \frac{-1}{2D + 2D' + 9} \times \frac{D}{D} \sin(x + 2y)$$

$$= \frac{-D}{2D^2 + 2D'^2 + 9D} \sin(x + 2y)$$

$$= \frac{-D}{-2 - 8 + 9D} \sin(x + 2y) = \frac{-D}{9D - 10} \sin(x + 2y)$$

If we multiply and divide by D, we can not get the term  $D^3, D'^2$  term, so we take conjugate for constant term and multiplied with both Nr. & Dr.

$$= \frac{-D}{9D - 10} \times \frac{9D + 10}{9D + 10} \sin(x + 2y)$$

$$= \frac{-9D^2 - 10}{81D^2 - 100} \sin(x + 2y)$$

$$= \frac{-9D^2 \sin(x + 2y) - 10D \sin(x + 2y)}{-81 - 100}$$

$$= \frac{1}{-181} [-9D \cos(x + 2y) - 10 \cos(x + 2y)]$$

$$P.I = \frac{1}{181} [9 \sin(x + 2y) - 10 \cos(x + 2y)]$$

$$P.I = \frac{1}{8} e^{3x+y} + \frac{1}{181} [9 \sin(x + 2y) - 10 \cos(x + 2y)]$$

**The general solution is**

$$z = C.F + P.I$$

$$z = f_1(y - x) + e^{2x} f_2(y - x) + \frac{1}{8} e^{3x+y} + \frac{1}{181} [9 \sin(x + 2y) - 10 \cos(x + 2y)]$$

2.

Solve  $(D^2 - D'^2 - 3D + 3D') z = e^{3x+y} + 4$

**Solution:**

Given  $(D^2 - D'^2 - 3D + 3D') z = e^{3x+y} + 4$

**To find C.F**

$$((D + D')(D - D') - 3(D - D')) z = 0$$

$$(D - D')(D + D' - 3)z = 0 \text{ ----- (1)}$$

This is of the form

$$(D - m_1 D' - C_1)(D - m_2 D' - C_2)z = 0 \text{ ----- (2)}$$

Comparing (1) & (2)

$$m_1 = 1, C_1 = 0, m_2 = -1, C_2 = 3.$$

$$C.F = e^{c_1 x} f_1(y + m_1 x) + e^{c_2 x} f_2(y + m_2 x)$$

$$C.F = f_1(y + x) + e^{\frac{y-x}{3}} f_2\left(\frac{y-x}{2}\right) \quad \because e_0 = 1$$

**To find P.I**

$$P.I = \frac{1}{D^2 - D'^2 - 3D + 3D'} (e^{3x+y} + 4)$$

$$P.I = P.I_1 + P.I_2$$

$$P.I = \frac{1}{D^2 - D'^2 - 3D + 3D'} e^{3x+y} \quad \text{here } a = 3, b = 1 \quad \text{type : 1}$$

$$= \frac{1}{9 - 1 - 9 + 3} e^{3x+y} \quad \text{Rule: Replace } D = 3, D' = 1$$

$$P.I_1 = \frac{1}{2} e^{3x+y}$$

$$P.I = \frac{1}{D^2 - D'^2 - 3D + 3D'} 4e^{0x+0y} \quad \text{here } a = 0, b = 0 \quad \text{type : 1}$$

$$= \frac{1}{0} 4e^{0x+0y} \quad \text{Rule: Replace } D = 0, D' = 0$$

Introduce x in Nr. and Diff. Dr. Partially w.r.to.D in the previous step

$$= \frac{x}{2D - 0 - 3 + 0} 4e^{0x+0y}$$

$$= \frac{4x}{-3}$$

$$P.I_2 = \frac{-4x}{3}$$

$$P.I = \frac{1}{2} e^{3x+y} - \frac{4x}{3}$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y+x) + e^{3x} f_2(y-x) + \frac{1}{2} e^{3x+y} - \frac{4x}{3}$$

3.

**Solve**  $(2D^2 - DD' - D'^2 + 6D + 3D') z = xe^y$

**Solution:**

Given  $(2D^2 - DD' - D'^2 + 6D + 3D') z = xe^y$

**To find C.F**

$$\begin{aligned} (2D^2 - DD' - D'^2 + 6D + 3D') z &= 0 \\ (2D + D')(D - D') + 3(2D + D')z &= 0 \\ (2D + D')(D - D' + 3)z &= 0 \\ \boxed{D + \frac{D'}{2}} (D - D' + 3)z &= 0 \end{aligned}$$

This is of the form

$$(D - m_1 D' - C_1)(D - m_2 D' - C_2)z = 0 \text{ ----- (2)}$$

Comparing (1) & (2)

$$\boxed{m_1 = \frac{-1}{2}, C_1 = 0, m_2 = 1, C_2 = -3.}$$

$$C.F = e^{a_1 x} f_1(y + m_1 x) + e^{a_2 x} f_2(y + m_2 x)$$

$$C.F = f_1 \left( y - \frac{x}{2} \right) + e^{-3x} f_2(y+x) \quad \because e^0 = 1$$

**To find P.I**

$$P.I = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} xe^y \quad \text{Type : 4}$$

$$P.I = \frac{1}{2D^2 - DD' - D'^2 + 6D + 3D'} xe^{0x+y} \quad \text{here } a = 0, b = 1$$

Rule: replace  $D = D + a = D + 0 = D$ ;  $D' = D' + b = D' + 1$

$$\begin{aligned} P.I &= e^{0x+y} \frac{1}{2D^2 - D(D'+1) - (D'+1)^2 + 6D + 3(D'+1)} x \\ &= e^y \frac{1}{2D^2 - DD' - D - D'^2 - 2D' - 1 + 6D + 3D' + 3} x \quad \text{Type : 3} \\ &= e^y \frac{1}{2D^2 - DD' + 5D - D'^2 + D' + 2} x \\ &= e^y \frac{1}{\frac{2D^2 - DD' + 5D - D'^2 + D'}{2} + \frac{2}{2}} x \end{aligned}$$

[ normally we take out highest power term of D in the homogeneous type, but it is not necessary in the non-homogeneous type]

$$= \frac{e^y}{2} \frac{2D^2 - DD' + 5D - D'^2 + D'}{2}^{-1} x \quad \because D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$$

$$= \frac{e^y}{2} \frac{1}{2} \frac{2D^2 - DD' + 5D - D'^2 + D'}{2} x \text{ [neglect the terms } D', \text{ since } D'(x) = 0]$$

$$= \frac{e^y}{2} \frac{1}{2} \frac{5D}{2} + \dots x \quad \because D^2(x) = 0$$

$$= \frac{e^y}{2} \frac{x - 5(1)}{2} \quad \because D^2(x) = 0$$

$$= \frac{e^y}{2} \frac{2x - 5}{2}$$

$$P.I = \frac{e^y}{4} [2x - 5]$$

The general solution is

$$z = C.F + P.I$$

$$z = f_1(y + x) + e^{3x} f_2(y - x) + \frac{e^y}{4} [2x - 5]$$

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