2.1. LINEAR TRANSFORMATION

Definition

Let V and W be vector spaces over F. A function $T: V \rightarrow W$ is called a linear transformation if for all $x, y \in V$ and $\alpha \in F$,

(a) $T(x + y) = T(x) + T(y)$

(b) $T(\alpha x) = \alpha T(x)$

Properties of linear transformation

1.If T is the linear, then $T(0) = 0$

Proof

$$T(0) = T(0 + 0)$$

$$T(0) = T(0) + T(0)$$

$\therefore T(0)$ is zero element of W.

Which implies,

$T(0) = 0$

2.T is linear if and only if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$; for all $x, y \in$ V and $\alpha, \beta \in F$.

Proof

Assume T is linear.

$$T(\alpha x + \beta y) = T(\alpha x) + T(\beta y)$$

$$= \alpha T(x) + \beta T(y)$$

Conversely.

Assume $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ ... (1)

Put $\alpha = 1, \beta = 1$ in (1). Then

$T(x + y) = T(x) + T(y)$

Put $y = 0$ in (1). Then

$$T(\alpha x + 0) = \alpha T(x) + T(0)$$

$= \alpha T(x) + 0$

$= \alpha T(y)$

$\therefore T$ is linear.

3. If T is linear, then $T(x - y) = T(x) - T(y)$; for all $x, y \in V$ Given T is linear

$T(x - y) = T(x + (-y))$

$= T(x) + T(-y)$

$= T(x) - T(y)$

Example 1. $T: R^2 \to R^2$ is defined by $T(a_1, a_2) = (2a_1 + a_2, a_1)$. Verif whether T is a linear transformation

Sol: $x, y \in V$ and $\alpha \in F$

$\therefore x = (a_1, a_2)$ and $y = (b_1, b_2)$

$x + y = (a_1 + b_1, a_2 + b_2)$

Given

$T(a_1, a_2) = (2a_1 + a_2, a_1)$

To prove T is linear, we have to prove

(i) $T(x + y) = T(x) + T(y)$

(ii) $T(\alpha x) = \alpha T(y)$

Proof:

$f(x) = T(a_1, a_2)$

$=(2a_1 + a_2, a_1)$

$T'(y)= T(b_1, b_2)$

$=(2b_1 + b_2, b_1)$

(i) $r(x+y) = T(a_1+b_1, a_2+b_2))$

$= (2(a_1+b_1) + a_2 + b_2, a_1 + b_1)$

$= (2a_1 + 2b_1 + a_2 + b_2, a_1 + b_1)$

$=(2a_1 + a_2, a_1) + (2b_1 + b_2, b_1)$

$= T(a_1, a_2) + T(b_1, b_2)$

$= T(x) + T(y)$

(ii) $T(\alpha x) = T(\alpha a_1, \alpha a_2)$

$=(2\alpha a_1 + \alpha a_2, \alpha a_1)$

$= \alpha(2a_1 + a_2, a_1)$

$= \alpha T(a_1, a_2) = \alpha T(x)$

$\therefore T: R^2 \rightarrow R^2$ is a linear transformation,

Example 2) $T: V_2(R) \rightarrow V_2(R)$ is defined by $T(a_1, a_2) = (3a_1 + 2a + 4a_2$). Verify whether T is a linear transformation.

Sol: $x, y \in V$ and $\alpha \in F$

$\therefore x = (a_1, a_2)$ and $y = (b_1, b_2)$

$x + y = (a_1 + b_1, a_2 + b_2)$

Given

$T(a_1, a_2) = (3a_1 + 2a_2, 3a_1 - 4a_2)$

To prove T is linear, we have to prove

(i) $T(x+y) = T(x) + T(y)$

(ii) $T(\alpha x) = \alpha T(y)$

Proof:

$T(x) = T(a_1, a_2)$

$= (3a_1 + 2a_2, 3a_1 - 4a_2)$

$T(y) = T(b_1, b_2)$

$= (3b_1 + 2b_2, 3b_1 - 4b_2)$

(i) $T(x + y) = T(a_1 + b_1, a_2 + b_2))$

$= (3(a_1 + b_1) + 2(a_2 + b_2), 3(a_1 + b_1) - 4(a_2 + b_2))$

$= (3a_1 + 3b_1 + 2a_2 + 2b_2, 3a_1 + 3b_1 - 4a_2 - 4b_2)$

$= (3a_1 + 2a_2 + 3b_1 + 2b_2, 3a_1 - 4a_2 + 3b_1 - 4b_2)$

$= (3a_1 + 2a_2, 3a_1 - 4a_2) + (3b_1 + 2b_2, 3b_1 - 4b_2)$

$= T(a_1, a_2) + T(b_1, b_2)$

$= T(x) + T(y)$

(ii) $T(\alpha x) = T(\alpha a_1, \alpha a_2)$

$= (3(\alpha a_1) + 2(\alpha a_2), 3(\alpha a_1) - 4(\alpha a_2))$

$= (\alpha(3a_1 + 2a_2), \alpha(3a_1 - 4a_2))$

$= \alpha((3a_1 + 2a_2), \alpha(3a_1 - 4a_2))$

$= \alpha T(a_1, a_2) = \alpha T(x)$

$T: V_2(R) \to V_2(R)$ is a linear transformation.

Exmple 3. Define $T: R^3 \to R^3$ by $T(a_1, a_2, a_3) = (2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3)$.

Verify whether T is a linear transformation.

Sol: $x, y \in V$ and $\alpha \in F$

$\therefore x = (a_1, a_2, a_3)$ and $y = (b_1, b_2, b_3)$

$x + y = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

Given

$T(a_1, a_2, a_3) = (2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3)$

To prove T is linear, we have to prove

(i) $T(x+y) = T(x) + T(y)$

(ii) $T(\alpha x) = \alpha T(x)$

Proof

$T(x) = T(a_1, a_2, a_3) = (2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3) T(y) = T(b_1, b_2, b_3)$

$= (2b_1 + b_2, b_2 - b_3, 2b_2 + 4b_3)$

(i) $T(x+y) = T(a_1 + b_1, a_2 + b_2, a_3 + b_3)$

$= (2(a_1 + b_1) + (a_2 + b_2), (a_2 + b_2) - (a_3 + b_3), 2(a_2 + b_2) + 4(a_3 + b_3))$

$= (2a_1 + 2b_1 + a_2 + b_2, a_2 + b_2 - a_3 - b_3, 2a_2 + 2b_2 + 4a_3 + 4b_3)$

$= (2a_1 + a_2 + 2b_1 + b_2, a_2 - a_3 + b_2 - b_3, 2a_2 + 4a_3 + 2b_2 + 4b_3)$

$= (2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3) + (2b_1 + b_2, b_2 - b_3, 2b_2 + 4b_3)$

$= T(a_1, a_2, a_3) + T(b_1, b_2, b_3)$

$= T(x) + T(y)$

$(ii) T(\alpha x) = T(\alpha a_1, \alpha a_2, \alpha a_3)$

$=(2\alpha a_1 + \alpha a_2, \alpha a_2 - \alpha a_3, 2\alpha a_2 + 4\alpha a_3)$

$=(\pi(2a_1 + a_2), \alpha(a_2 - a_3), \alpha(2a_2 + 4a_3))$

$= \alpha(2a_1 + a_2, a_2 - a_3, 2a_2 + 4a_3)$

$= \alpha T(a_1, a_2, a_3) = \alpha T(x)$

$\therefore T$ is a linear transformation and hence a linear map on R^3.

Example 4. Define mapping $T: V_3(F) \to V_2(F)$ by $T(a_1, a_2, a_3) = (a_2, a_3)$. Verify whether T is a linear transformation.

Sol: $x, y \in V$ and $\alpha \in F$

$\therefore x = (a_1, a_2, a_3), y = (b_1, b_2, b_3)$

$x + y = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

Given

$T(a_1, a_2, a_3) = (a_2; a_3)$

To prove T is linear, we have to prove

(i) $T(x + y) = T(x) + T(y)$

(ii) $T(\alpha x) = \alpha T(x)$

Proof:

$T(x) = T(a_1, a_2, a_3) = (a_2, a_3) T(y) = T(b_1, b_2, b_3) = (b_2, b_3)$.

(i) $T(x + y) = T(a_1 + b_1, a_2 + b_2, a_3 + b_3)$

$= (a_2 + b_2, a_3 + b_3)$

$= (a_2, a_3) + (b_2, b_3)$

$= T(a_1, a_2, a_3) + T(b_1, b_2, b_3)$

$= T(x) + T(y)$

(i)$T(\alpha x) = T(\alpha a_1, \alpha a_2, \alpha a_3)$

$= (\alpha a_2, \alpha a_3)$

$= \alpha(a_2, a_3)$

$= \alpha T(a_1, a_2, a_3)$

$= \alpha T(x)$

$\therefore$ T is a linear transformation.

Example 5. Show that for $0 \leq \theta < 2\pi$,the transformation given by $T_\theta: R^2$ $R^2, T_\theta(a, b) = (a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$ is linear

Sol: $x, y \in R^2$ and $\alpha \in F$

$\therefore x = (a_1, b_1), y = (a_2, b_2) x + y = (a_1 + a_2, b_1 + b_2)$

Given

$T_\theta(a, b) = (a\cos\theta - b\sin\theta, a\sin\theta + b\cos\theta)$

To prove T is linear, we have to prove

(i) $T(x + y) = T(x) + T(y)$

(ii) $T(\alpha x) = \alpha T(x)$

Proof

$T(x) = T(a_1, b_1)$

$= (a_1\cos\theta - b_1\sin\theta, a_1\sin\theta + b_1\cos\theta)$

$T(y) = T(a_2, b_2)$

$= (a_2\cos\theta - b_2\sin\theta, a_2\sin\theta + b_2\cos\theta)$

(i) $T(x + y) = T(a_1 + a_2, b_1 + b_2)$

$= ((a_1 + a_2)\cos\theta - (b_1 + b_2)\sin\theta, (a_1 + a_2)\sin\theta + (b_1 + b_2)\cos\theta)$

$= (a_1\cos\theta + a_2\cos\theta - b_1\sin\theta - b_2\sin\theta, a_1\sin\theta + a_2\sin\theta - b_1\cos\theta - b_2\cos\theta)$

$= (a_1\cos\theta - b_1\sin\theta, a_1\sin\theta + b_1\cos\theta)$

$+ (a_2\cos\theta - b_2\sin\theta, a_2\sin\theta - b_2\cos\theta) = T(x) + T(y)$

(ii) $T(\alpha x) = T(\alpha a_1, \alpha b_1)$

$= (\alpha a_1\cos\theta - \alpha b_1\sin\theta, \alpha a_1\sin\theta + \alpha b_1\cos\theta)$

Example 9. Let $M(R)$ be the vector space of all 2×2 matrices over R and B be a fixed non-zero element of $M(R)$. Show that the mapping $T: M(R) \to M(R)$ defined by $T(A) = AB + BA, \forall A \in M(R)$ is a linear transformation.

Sol:

Let $A, C \in M(R)$ and $\alpha \in R$

Given

$T(A) = AB + BA, \forall A \in M(R)$ for a fixed non-zero element B of $M(R)$

To prove F is linear, we have to prove

(i)$F(A + C) = F(A) + F(C)$

(ii) $F(\alpha A) = \alpha F(A)$

Proof:

$T(A) = AB + BAT(C) = CB + BC$

(i)T(A+C)=(A+C)B+B(A+C)

$= AB + CB + BA + BC$

$= (AB + BA) + (CB + BC)$

$= T(A) + T(C)$

(ii) T(αA)=(αA)B+B(αA)

=α(AB+BA)

=αT(A)

$\therefore T$ is a linear transformation.

Example 14. Prove that there exists linear transformation $T: R^2 \rightarrow R^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?

Sol: Let us express (1,1) and (2,3) as a linear combination of the standar basis vectors $e_1 = (1,0)$ and $e_2 = (0,1)$ of R^2

$(1,1) = 1(1,0) + 1(0,1) = 1e_1 + 1e_2$

$= e_1 + e_2$ ... (1)

$(2,3) = 2(1,0) + 3(0,1) = 2e_1 + 3e_2$

$= 2e_1 + 3e_2$ ... (2)

Given

$T(1,1) = (1,0,2)$

$\Rightarrow T(e_1 + e_2) = (1,0,2)$ [from (1)]

$\Rightarrow T(e_1) + T(e_2) = (1,0,2)$ ... (3)

Also given

$T(2,3) = (1,-1,4)$

$\Rightarrow T(2e_1 + 3e_2) = (1, -1, 4)$ [from (2)]

$\Rightarrow 2T(e_1) + 3T(e_2) = (1, -1, 4)$ ...

Solve (3) and (4)

$$(3) \times 2 \Rightarrow 2T(e_1) + 2T(e_2) = (2,0,4)$$

$$(4) \quad => 2T(e_1) + 3T(e_2) = (1, -1, 4)$$

Subtracting $\quad - T(e_2) = (1,1,0)$

$T(e_2) = (-1, -1, 0)$

$(3) \Rightarrow T(e_1) + (-1, -1, 0) = (1, -1, 4)$

$$\Rightarrow T(e_1) = (1, -1, 4) - (-1, -1, 0)$$

$$T(e_1) = (2,0,4)$$

To find the linear transformation:

Let $(a_1, a_2) \in R^2$. Then

$$(a_1, a_2) = a_1(1,0) + a_2(0,1)$$

$$= a_1 e_1 + a_2 e_2$$

$$T(x, y) = T(a_1 e_1 + a_2 e_2)$$

$$= a_1 T(e_1) + a_2 T(e_2)$$

$$= a_1(2,0,4) + a_2(-1, -1, 0)$$

$$T(a_1, a_2) = (2a_1 - a_2, -a_2, 4a_1)$$

$T(8,11) = (5, -11, 32)$

Example 15. Is there a linear transformation $T: R^3 \to R^2$ such that $T(1,0,3) = (1,1)$ and $T(-2,0,-6) = (2,1)$?

Sol: Let us express $(1,0,3)$ and $(-2,0,-6)$ as a linear combination of the R^3

standard basis vectors $e_1 = (1,0,0)$ and $e_2 = (0,1,0)$ and $e_3 = (0,1,0)$ of R^3.

$(1,0,3)= 1(1,0,0) + 0(0,1,0) + 3(0,0,1)$

$=e_1 + 0e_2 + 3e_3$

$=e_1 + 3e_3$ ... (1)

$(-2,0,-6) = -2(1,0,0) + 0(0,1,0) - 6(0,0,1)$

$= -2e_1 + 0e_2 - 6e_3$

$=-2e_1 - 6e_3$ ... (2)

T(1,0,3) =(1,1)

=>$T(e_1 + 3e_3) = (1,1)$ [from (1)]

=>$T(e_1) + 3T(e_3) = (1,1)$ ... (3)

Also given

$T(-2,0,-6) = (2,1)$

$T(-2e_1 - 6e_3) = (2,1))$ [from (2)]

$\Rightarrow -2T(e_1) - 6T(e_2) = (2,1)) \cdots (4)$

solve (3) and (4)

$(3) \times 2 \Rightarrow \quad 2T(e_1) + 6T(e_2) = (2,2)$

$(4) \Rightarrow -2T(e_1) - 6T(e_2) = (2,1)$

Adding $\qquad 0 = (4,3)$

It is not possible.

Therefore there is no linear transtormationtion with the given data's.

Example 16. Find a linear transtormationtion $T: R^3 \to R^3$ such that T(1,1,1) = (1,1,1),T(1,2,3)=(-1,-2,3) and find T(1,1,2)=(2,2

Sol: Let us express (1,1,1), (1,2,3) and (1,1,2) as a linear combination basis vectors e_1=(1,0,0), e_2=(0,1,0) and e_3=(0,0,1) of R^3

$(1,1,1) = 1(1,0,0) + 1(0,1,0) + 1(0,0,1) = 1e_1 + 1e_2 + 1e_3$

$= e_1 + e_2 + e_3 \ldots (1)$

$(1,2,3) = 1(1,0,0) + 2(0,1,0) + 3(0,0,1) = 1e_1 + 2e_2 + 3e_3$

$= e_1 + 2e_2 + 3e_3 \ldots (2)$

$(1,1,2) = 1(1,0,0) + 1(0,1,0) + 2(0,0,1) = 1e_1 + 1e_2 + 2e_3$

$= e_1 + e_2 + e_3 \ldots (3)$

Given

$T(1,1,1) = (1,1,1)$

$\Rightarrow T(e_1 + e_2 + e_3) = (1,1,1)$ [from (1)]

$\Rightarrow T(e_1) + T(e_2) + T(e_3) = (1,1,1) \ldots (4)$

Also given

T(1,2,3) =(-1,-2,3)

$\Rightarrow T(e_1 + 2e_2 + 3e_3) = (-1,-2,3)$ [from (1)

$\Rightarrow T(e_1) + 2T(e_2) + 3T(e_3) = (-1,-2,3) \ldots (5)$

Also given

$T(1,1,2) = (2,2,4)$

$\Rightarrow T(e_1 + e_2 + 2e_3) = (2,2,4)$ [from (1)]

$\Rightarrow T(e_1) + T(e_2) + 2T(e_3) = (2,2,4) \ldots (6)$

Solve (4), (5) and (6)

$(4) \Rightarrow T(e_1) + T(e_2) + T(e_3) = (1,1,1)$

$(6) \Rightarrow \underline{T(e_1) + T(e_2) + 2T(e_3) = (2,2,4)}$

$-T(e_3) = (-1,-1,-3)$

Subtracting

$$T(e_3) = (1,1,3)$$

(4) $\Rightarrow T(e_1) + T(e_2) + (1,1,3) = (1,1,1)$

$$\Rightarrow T(e_1) + T(e_2) = (1,1,1) - (1,1,3)$$

$$= (0,0,-2) \ldots (7)$$

(5)$\Rightarrow$ $T(e_1) + 2T(e_2) + 3(1,1,3) = (-1,-2,3)$

$\Rightarrow T(e_1) + 2T(e_2) + (3,3,9) = (-1,-2,3)$

$$\Rightarrow T(e_1) + 2T(e_2) = (-1,-2,3) - (3,3,9)$$

$$= (-4,-5,-6) \ldots (8)$$

Solve (7) and (8)

$$(7) \Rightarrow \; T(e_1) + T(e_2) = (0,0,-2)$$

$$\text{Subtracting}(6) \Rightarrow \frac{T(e_1)+2T(e_2)=(-4,-5,-6)}{-T(e_2)=(4,5,4)}$$

$T(e_2) = (-4,-5,-4)$

(7) $\Rightarrow$ $T(e_1) + (-4,-5,-4) = (0,0,-2)$

$$\Rightarrow T(e_1) = (0,0,-2) - (-4,-5,-4)$$

$$T(e_1) = (4,5,2)$$

To find the linear transformation:

Let $(x, y, z) \in R^3$. Then

$(x, y, z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$$= xe_1 + ye_2 + ze_3$$

$r(x, y, z) = T(xe_1 + ye_2 + ze_3)$

$= xT(e_1) + yT(e_2) + zT(e_3)$

$= x(4,5,2) + y(-4, -5, -4) + z(1,1,3)$

$= (4x, 5x, 2x) + (-4y, -5y, -4y) + (z, z, 3z)$

$T(x, y) = (4x - 4y + z, 5x - 5y + z, 2x - 4y + 3z)$

2.1.2 NULL SPACES AND RANGES

Null space (or) Kernal

Let V and W be vector spaces and let $T: V \rightarrow W$ be linear transformation Then the set of all vectors x in V such that $T(x) = 0_W$ is called the null space (o kernel) of T. It is denoted by $(T\ )$.

$$\text{(i.e) } N(T) = \{x \in V: T(x) = 0_W\}$$

Note: 0_W is the zero element of W.

Range or Image

Let V and W be vector spaces and let $T: V \rightarrow W$ be linear transformation. Then the subset of W consisting of all images under T of vectors in V is called range o. image of T. It is denoted by $R(T)$

$$\text{(i.e) } R(T) = \{T(x): x \in V\}$$

Theorem 2.1: Let V and W be vector spaces and $T: V \rightarrow W$ be linear. Then

(a) $N(T)$ is a sub space of V and

(b) $R(T)$ is a subspace of W.

Proof: Given that V and W are vector spaces.

$T: V \rightarrow W$ is linear.

(a) To prove $N(T)$ is a subspace of V.

We have to prove for $\alpha, \beta \in F$ and $x, y \in N(T) \Rightarrow \alpha x + \beta y_6$

Since T is linear, $\mu y \in N(\gamma$

$T(0_V) = 0_W$; 0_V -zero vector of V and 0_W -zero vector of W.

$\therefore 0_V \in N(T)$

$\therefore N(T)$ is non-empty.

Let $x, y \in N(T)$ and $\alpha, \beta \in F$

$\Rightarrow T(x) = 0_W, T(y) = 0_W$

$T(\alpha x+\beta y)=\alpha T(x)+\beta T(y)$ ($\because$ Tis linear)

$$=\alpha(0_W)+\beta(0_W)=0_W$$

$\therefore T(\alpha x+\beta y)=0$

$\Rightarrow \alpha x+\beta y \in N(T)$

$\therefore N(T)$ is subspace of V

(b) To prove $R(T)$ is subspace of W.

We have to prove for $\alpha, \beta \in F$ and $x, y \in R(T) \Rightarrow \alpha x+\beta y \in N(T)$

Since $T(0_V)=0_w$ ($\because T$ is linear)

$\Rightarrow 0_w \in R(T)$

$\therefore R(T)$ is non-empty.

Let $x, y \in R(T)$ and $\alpha, \beta \in F$

Then there exits u and v in V such that

$T(u)=x$ and $T(v)=y$

$\alpha x+\beta y=\alpha T(u)+\beta T(v)$

$$=T(\alpha u+\beta v) \in R(T)[\because \alpha u+\beta v \in V]$$

$\therefore \alpha x+\beta y \in R(T)$

$\therefore R(T)$ is a subspace of W.

Theorem 2.2: Let V and W be vector spaces over a field F. Let T:V-> linear transformation which is onto. Then T maps a basis of V onto a W.

Proof Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V.

We shall prove that $T(v_1), T(v_2), \ldots, T(v_n)$ are linearly independent and that the span W.

Now , $\alpha T(v_1)+\alpha_2 T(V_2)+\cdots+\alpha_n T(v_n)=0$

$$\Rightarrow T(\alpha_1 v_1)+T(\alpha_2 v_2)+\cdots+T(\alpha_n v_n)=0. [\because T \text{ is linear}]$$

$$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots . + \alpha_n v_n) = 0$$

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$ (since $v_1, v_2, \ldots, v_n$ are linearly independent).

$\therefore T(v_1), T(v_2), \ldots . . T(v_n)$ are linearly independent.

Now, let $w \in W$. then since T is onto, there exists a vector $v \in V$ such $T(v) = w$

Since $L(S) = V$,

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

Then

$$w = T(v)$$

$$= T(\alpha_1 v_1 + \cdots + \alpha_n v_n)$$

$$= \alpha_1 T(v_1) + \cdots + \alpha_n T(v_n) [\because T \text{ is linear}]$$

Thus w is a linear combination of the vectors $T(v_1), T(v_2), \ldots, T(v_n)$. $\therefore$ $T(v_1), T(v_2), \ldots, T(v_n)$ span W an hence is a basis for W.

2.1.3. NULLITY AND RANK

Definition

Let V and W be vector spaces, and let $T: V \rightarrow W$ be a linear transformation. If $N(T)$ and $R(T)$ are finite-dimensional, then we define

$$\text{nullity } (T) = \dim[N(T)]$$

$$\text{rank}(T) = \dim[R(T)]$$

Note: If nullity $(T) = \{0\}$, then $\dim[N(T)] = 0$ i.e., nullity $(T) = 0$

Theorem 2.5: Rank-Nullity Theorem (or dimensional theorem) Let $T: V \rightarrow W$

be a linear transformation and V be a finite dimensional vector space. Then

$$\dim[R(T)] + \dim[N(T)] = \dim(V)$$

$$\text{(i.e) ran } k(T) + \text{nullity } (T) = \dim(V)$$

Proof:

Let V be a vector space of dimension m.i.e., $\dim(V) = m$. Since $N(T)$ is subspace of the finite dimensional vector space V dimension of $N(T)$ is also finite

Let $\dim(N(T)) = n$

Since $N(T)$ is a subspace of $V, n \leq m$

Let $\beta = \{v_1, v_2, \dots, v_n\}$ be a basis $N(T)$. Since $v_i \in N(T)$, for $1 \leq i \leq n$,

Then $T(v_l) = 0, 1 \leq i \leq n$

Since β is a basis of $N(T), \beta$ is linearly independent in $N(T)$.

Therefore β is linearly independent in V.

We shall extend this set β to a basis of the vector space V.

Let this basis of V be $\beta_1 = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_s\}$, where $n + s = m$.

Let $\gamma = \{T(u_1), T(u_2), \dots, T(u_s)\}$.

We shall show that this set γ is a basis of $R(T)$.

ie, to prove $L(\gamma) = R(T)$

and γ is linearly independent.

Since β_1 is a basis of V, it spans V. Hence the set

$\{T(v_1), T(v_2), \dots, T(v_n), T(u_1), T(u_2), \dots, T(u_s)\}$span $R(T)$

Since $T(v_i) = 0$, for $1 \leq i \leq n$

the set $\{T(u_1), T(u_2), \dots, T(u_s)\}$ spans $R(T)$.

$$L(\gamma) = R(T)$$

To prove is linearly independent.

Let $a_1T(u_1) + a_2T(u_2) + \cdots + a_sT(u_s) = 0.$

⇨ $T(a_1u_1 + a_2u_2 + \cdots + a_su_s) = 0[\because T \text{ is linear }]$

⇨ $a_1u_1 + a_2u_2 + \cdots + a_su_s \in N(T)$

Since $\beta = (v_1, v_2, \dots, v_n)$ is a basis in $N(T)$,

$a_1u_1 + a_2u_2 + \cdots + a_su_s = b_1v_1 + b_2v_2 + \cdots + b_nv_n$

$\Rightarrow a_1u_1 + a_2u_2 + \cdots + a_su_s - b_1v_1 - b_2v_2 - \cdots - b_nv_n = 0$

Since β_1 is a basis of $V, u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_s$ are linearly independent,

$$\therefore a_1 = a_2 = \cdots = a_s = 0, b_1 = b_2 = \cdots = b_n = 0$$

$$\therefore a_1T(u_1) + a_2T(u_2) + \cdots + a_sT(u_s) = 0$$

$$\Rightarrow a_1 = a_2 = \cdots = a_s = 0$$

$\gamma = \{T(u_1), T(u_2), \dots, T(u_s)\}$ is linearly independent

$\therefore \gamma$ is a basis of $R(T)$

$$\therefore \dim R(T) = s$$

We have $m = n + s$

$$\therefore \dim(V) = \dim[N(T)] + \dim[R(T)]$$

i.e., $\text{ran } k(T) + \text{ nullity } (T) = \dim(V)$

Theorem 2.6: Let V and W be vector space, and let $T: V \to$ W be transformation. Then T is one-to-one if and only if $N(T) = \{0\}$.

Proof:

Assume: T is $1 - 1$ (one-to-one)

Let $u \in N(T)$. Then

$$T(u) = 0 = T(0)$$

$$\therefore T(u) = T(0)$$

$$\Rightarrow u = 0 \ (\because T \text{ is } 1-1)$$

$$\therefore N(T) = \{0\}$$

Conversely, assume that $N(T) = \{0\}$

Let $T(u) = T(v)$.

$$T(u) - T(v) = 0$$

$$T(u-v) = 0 \ (\because T \text{ is linear })$$

$$\Rightarrow u - v \in N(T) = \{0\}$$

$$\therefore u - v = 0$$

$$\therefore u = v$$

$\therefore T$ is $1-1$ (one-to-one).

Theorem 2.7: Ift V and W be finite dimensional over F and $T: V \rightarrow W$ be Then the following are equivalent.

1 T is one-to-one

2 T is onto

3 rank $(T) = \dim(W)$

Proof:

By dimensional theorem we have

$\text{rank}(T) + \text{nullity}\ (T) = \dim(V)$ ... (

T is one-to-one $\Leftrightarrow N(T) = \{0\}$

$\Leftrightarrow \text{nullity}\ (T) = 0$

$\Leftrightarrow \text{rank}(T) = \dim V$ [$usin\ g(1)$]

$\Leftrightarrow \dim R(T) = \dim W$ [T is $1-1$]

$\Leftrightarrow R(T) = W[\because R(T) \subseteq W$ with same rank]

$\Leftrightarrow T'$ is onto.

2. 1.4. PROBLEMS UNDER RANK AND NULLITY

Let V and W be vector space and let $T: V \to W$ be linear map

- $N(T) = \{x \in V: T(x) = 0_W\}$
- nullity $(T) = \dim(N(T))$
- $R(T) = \{T(x): x \in V\}$
- $\text{rank}(T) = \text{image}(R(T))$

Example 17. Let $T: R^3 - R^2$ by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$. Find N and $R(T)$

Sol: To find $N(T)$:

$N(T) = \{(a_1, a_2, a_3) \in R^3: T(a_1, a_2, a_3) = 0\}$

Let $T(a_1, a_2, a_3) = 0$

$(a_1 - a_2, 2a_3) = 0$

Equating each terms to zero, we get

$2a_3 = 0$

$a_3 = 0$

$a_1 - a_2 = 0$

$a_1 = a_2$

$\text{N}(T) = \{(a_1, a_2, a_3)\}$

$=\{(a_1, a_1, 0): a_1 \in R\}$

To find $R(T)$:

The usual basis of R^3 is $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$

Given, $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$

$$T(1,0,0) = (1,0)$$

$$T(0,1,0) = (-1,0) = -1(1,0)$$

$$T(0,0,1) = (0,2) = 2(0,1)$$

$$\text{Image}(T) = \text{span}\{(1,0), -(1,0), 2(1,0)\}$$

$$= \text{span}\{(1,0), (1,0)\} \left[\begin{array}{l} * -(1,0) \text{ is depending on } (1,0 \\ 2(1,0) \text{ is multiple of } (1,0) \end{array} \right.$$

$$= \{x(1,0) + y((1,0)\}$$

$$= \{(x, y)\}$$

$$= R^2$$

Example 18. Let $T: R^2 \rightarrow R^3$ be a linear map defined by $T(a_1, a_2) = (a_1 - a_2, 0,0)$. Find nullity (T) and rank(T).

Sol: To find nullity (T) :

$N(T) = \{(a_1, a_2) \in R^2 : T(a_1, a_2) = 0\}$

Let $T(a_1, a_2) = 0$

$$(a_1 - a_2, 0,0) = (0,0,0)$$

$$a_1 - a_2 = 0$$

$$a_1 = a_2$$

$N(T) = \{(a_1, a_1)/a_1 \in R\}$

$=\{(1,1)a_1/a_1 \in R\}$

The basis of $N(T)$ is $\beta = \{(1,1)\}$

The nullity of $T = \dim[N(T)] = 1$

To find range (T) :

The usual basis of R^2 is $\beta = \{(1,0),(0,1)\}$

Given, $T(a_1, a_2) = (a_1 - a_2, 0,0)$.

$$T(1,0) = (1,0,0)$$

$$T(0,1) = (-1,0,0)$$

The image of usual basis span Image (T)

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} R_2 \to R_2 + R_1$

The basis of $R(T)$ is the non-zero row of the echelon matrix.

Therefore the basis of $R(T)$ is $\gamma = \{(1,0,0)\}$.

$\text{rank}(T) = \dim[R(T)] = 1$

Example (19) Let $T: R^2 \to R^3$ be the linear mapping defined by $T(a_1, a_2) = (a_1 + a_2, a_1 - a_2, a_2)$

Find the basis and dimension of (a) null space of T (b) Range of T

Sol: (a) To find (null space) kernel of T :

Let $T(a_1, a_2) = 0$

$(a_1 + a_2, a_1 - a_2, a_2) = (0,0,0)$

Equating the like terms

$$a_1 + a_2 = 0 \ldots (1)$$

$$a_1 - a_2 = 0 \ldots (2)$$

$$a_2 = 0$$

$(1) \Rightarrow a_2 = 0$

kernel of $T = N(T) = \{(0,0)\}$

The nullity of $T = \dim(N(T)) = 0$

(b) To find range of T :

The usual basis of R^2 is $\beta = \{(1,0),(0,1)\}$

Given, $T(a_1, a_2) = (a_1 + a_2, a_1 - a_2, a_2)$

$$T(1,0) = (1,1,0)$$

$$T(0,1) = (1,-1,0)$$

The image of usual basis span Image(T)

$$Let A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix} R_2 \rightarrow R_2 - R_1$$

The basis of $R(T)$ is the non-zero row of the echelon matrix

Thus $y = \{(1,0,1),(0,-2,0)\}$ forms a basis for $\text{Im}(T)$.

Hence $\dim[\text{lm}(T)] = 2$

i.e., Rank $(T) = 2$

Example 20. Let $T: R^3 \rightarrow R^3$ be the linear mapping defined b $T(a_1, a_2, a_3) = (a_1 + 2a_2 - a_3, a_2 + a_3, a_1 + a_2 - 2a_3)$

Find the basis and dimension of (a) Kernel(b) Image of T

Sol: (a) To find kernel of T :

Let $T(a_1, a_2, a_3) = 0$

$(a_1 + 2a_2 - a_3, a_2 + a_3, a_1 + a_2 - 2a_3) = (0,0,0)$

$$a_1 + 2a_2 - a_3 = 0 \ldots (1)$$

$$a_2 + a_3 = 0 \ldots \ (2)$$

$$a_1 + a_2 - 2a_3 = 0 \ldots (3)$$

Solve $(1), (2) \& (3)$

The matrix of the given equations is

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} R_3 \to R_3 - R_1$$

$$\sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} R_3 \to R_3 + R_2$$

$$a_1 + 2a_2 - a_3 = 0 \ldots (4)$$

$$a_2 + a_3 = 0 \ldots (5)$$

Adding (4) and (5), we get

$$a_1 + 3a_2 = 0$$

$$a_1 = -3a_2 \ldots (6)$$

a_1 is depending on a_2.

Therefore the basis of $N(T)$ contains one element

$$(6) \Rightarrow \ \frac{a_2}{-3} = \frac{a_2}{1}$$

$$a_1 = -3, a_2 = 1$$

$(5) \Rightarrow a_3 = -1$

Basis of kernel of T is $\beta = \{(-3,1,-1)\}$

$\text{nulliy}(T) = \dim(N(T)) = 1$

(b) To find Image (T) :

The basis of R^3 is $\beta = \{(1,0,0),(0,1,0),(0,0,1)\}$

Given, $T(a_1, a_2, a_3) = (a_1 + 2a_2 - a_3, a_2 + a_3, a_1 + a_2 - 2a_3)$

$$T(1,0,0) = (1,0,1)$$

$$T(0,1,0) = (2,1,1)$$

$$T(0,0,1) = (-1,1,-2)$$

The image of usual basis span Image (T)

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - R_2.$$

The basis of $R(T)$ is the non-zero row of the echelon matrix

Thus $\gamma = \{(1,0,1),(0,1,-1)\}$ form a basis form $\text{Im}(T)$.

Hence $\dim[\text{Im}(T)] = 2$

i.e, $\text{Rank}(T) = 2$

Example 21. Let $T: P_3(R) \to P_2(R)$ defined by $T[f(x)] = f'(x)$. find the nullity and rank of T.

Sol: Let $f(x) \in P_3(x)$. Then

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 \dots (1)$$

$$f'(x) = a_1 + a_2x + a_3x^2$$

To find nullity (T) :

$N(T) = \{f(x) \in P_3(R): T(f(x)) = 0\} \dots (2)$

Let $T(f(x)) = 0$. Then

$$F'(x) = 0$$

$a_1 + a_2x + a_3x^2 = 0$

$a_1 + a_2x + a_3x^2 = 0+0x+0x^2$

$$a_1 = 0, a_2 = 0, a_3 = 0$$

Example 23. Find the range space, kernel, rank and nullity of the following linear transformation. Also verify the rank-nullity theorem $T: V_2(R) \to V_2(R)$ defined by $T(x_1, x_2) = (x_1 + x_2, x_1)$

Sol: To find (T) :

Let $T(x_1, x_2) = 0$

$(x_1 + x_2, x_1) = 0$

$$x_1 + x_2 = 0 \dots (1)$$

$$x_1 = 0$$

$$(1) \Rightarrow x_2 = 0$$

$\therefore N(T)$ contain only zero element of $V_2(R)$.

$$\therefore N(T) = \{(0,0)\}$$

i.e the null space $= \{(0,0)\}$

$$\dim[N(T)] = 0$$

i.e nullity $= 0$

To find $R(T)$:

The standard basis $e_1 = (1,0), e_2 = (0,1)$ of $V_2(R)$

$$T(e_1) = T(1,0)$$

$$= (1+0,1)$$

$$T(e_2) = T(0,1)$$

$$= (0+1,0)$$

$$= (1,0)$$

The image of usual basis span Image (T)

$$\text{Let } A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} R_2 \to R_2 - R_1$$

This is in the echelon form there are two non-zero rows

Basis of Image (T) is $\gamma = \{(1,1), (0,-1)\}$

Therefore, rank of $T = 2$

Hence R(T) is the subspace generated by $(1,1)$ and $(0,-1)$

$$R(T) = x_1(1,1) + x_2(0,-1)$$

$$= (x_1, x_1) + (0, -x_2)$$

$$= (x_1, x_1 - x_2) \text{ for all } x_1, x_2 \in R$$

i.e the range space $= \{x_1, x_1 - x_2\} = V_2(R)$

Rank $+$ nullity $= 2 + 0$

$= 2$

$= \dim[V_2(R)]$

Hence the nullity theorem is verified.

Example(24) Let T: $R^3 \rightarrow R^3$ defined by Y(x,y,z)=(x+y,x-y,2x+z). Find the range space, null-space, rank and nullity of T and verify rank+nullity of T $=$ dim(R^3).

Sol: To find (T) :

$T(x, y, z) = 0(x, x - 2y, 2x, 2x - 2y + z) = (0,0,0)$

$x_1 + x_2 = 0 \ldots (1)$

$x_1 - x_2 = 0 \ldots (2)$

$2x_1 + x_3 = 0 \ldots (3)$

$(1) + (2) \Rightarrow 2x_1 = 0$

$x_1 = 0$

$(1) \Rightarrow x_2 = 0$

$(3) \Rightarrow x_3 = 0$

$\therefore N(T) = \{(0,0,0)\}$

$= \{0\}$

$\Rightarrow \dim[N(T)] = 0$

$\Rightarrow$nullity $= 0$

To find $R(T)$:

The standard basis of R^3 is $\beta = \{(1,0,0), (0,1,0), (0,0,1)\}$

Let $e_1 = (1,0,0), e_2 = (0,1,0)$ and $e_3 = (0,0,1)$

$T(x, y, z) = (x + y, x - y, 2x + z)$

$T(1,0,0) = (1 + 0,1 - 0,0 + 0)$

$= (1,1,0)$

$T(0,0,0) = (0+1, 0-1, 0+0)$

$= (1,-1,0)$

$T(0,0,1) = (0+0, 0-0, 0+1)$

$= (0,0,1)$

The image of usual basis span Image (T)

Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

This in echelon from and there are three non-zero rows.

$\dim[R(T)] = 3$

i.e rank of $T = 3$

$R(T)$ = the subspace generated by $(1,1,0), (0,-2,0), (0,0,1)$

$= x(1,1,2) + y(0,-2,-2) + z(0,0,1)$

$= (x, x, 2x) + (0, -2y, -2y) + (0,0,z)$

$R(T) = (x, x-2y, 2x, 2x-2y+z)$

Rank + nullity $= 3 = \dim(R^3)$

Fxample 25. Find the range space, kernel, rank and nullity of the following linear transformation. Also verify the rank-nullity theorem defined by

T: $V_3(R) \to V_2(R) by$

$T(e_1) = (2,1), T(e_2) = (0,1), T(e_3) = (1,1)$

Sol: To find the kernel,

Since the linear transformation is not given, first find the linear transformation

The usual basis of R^3 is $\beta = \{(1,0,0),(0,1,0),(0,0,1)\}$

$e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$

Given $T(e_1) = (2,1), T(e_2) = (0,1), T(e_3) = (1,1)$

$(x_1, x_2, x_3) = x_1(1,0,0) + x_2(0,1,0) + x_3(0,0,1)$

$= x_1 e_1 + x_2 e_2 + x_3 e_3$

$T(x_1, x_2, x_3) = T(x_1 e_1 + x_2 e_2 + x_3 e_3)$

$= x_1 T(e_1) + x_2 T(e_2) + x_3 T(e_3)$

$= x_1(2,1) + x_2(0,1) + x_3(1,1)$

$= (2x_1, x_1) + (0, x_2) + (x_3, x_3)$

$= (2x_1 + x_3, x_1 + x_2 + x_3)$

$N(T) = \{(x_1, x_2, x_3): T(x_1, x_2, x_3) = 0\}$

Put $T(x_1, x_2, x_3) = 0$

$(2x_1 + x_3, x_1 + x_2 + x_3) = (0,0)$

$2x_1 + x_3 = 0 \ldots (1)$

$x_1 + x_2 + x_3 = 0$

$(1) \Rightarrow x_3 = -2x_1$

$(2) \Rightarrow x_1 + x_2 - 2x_1 = 0$

$x_2 - x_1 = 0$

$x_1 = x_2$

$\frac{x_1}{1} = \frac{x_2}{1}$

$(3) \Rightarrow x_3 = -2$

The basis of $N(T)$ is $\beta = \{(1,1,-2)\}$

$N(T) =$ the subspace generated by $(1,1,-2)$

$=\{(1,1,-2)x_1\}$

$=\{(x_1, x_1, -2x_1)\}$

$\therefore \dim[N(T)] = 1$

i.e nullity $(T) = 1$

To find $\dim[R(T)]$

Given $T(e_1) = (2,1), T(e_2) = (0,1), T(e_3) = (1,1)$

The image of usual basis span Image (T).

$$\text{Let } A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} R_1 \leftrightarrow R_3$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} R_3 \to R_3 - 2R_1$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} R_3 \to R_3 + R_2$$

This is the echelon form and there are 2 non-zero rows in it.

The basis of $R(T)$ is $\gamma = \{(1,1), (0,1)\}$

$\therefore \dim[R(T)] = 2$

i.e rank of $T = 2$

Range space = the subspace generated by $(1,1)$ and $(0,1)$

$= x_1(1,1) + x_2(0,1)$

$= (x_1, x_1) + (0, x_2)$

$= (x_1, x_1 + x_2)$

$\therefore$ Range space $= \{(x_1, x_1 + x_2) \mid x_1, x_2 \in R\}$

Rank(T) + nullity $(T) = 2 + 1$

$= 3$

$= \dim(R^3)$

2.3 MATRIX OF LINEAR TRANSFORMATION WITH STANDARD BASES

1. **Find the matrix of the linear transformation $T: R^2 \to R^2$ given by $T(a,b) = (2a - 3b, a + b)$ relative to the basis (i) $\{(1,0),(0,1)\}$ (ii) $\{(2,3),(1,2)\}$**

Solution

Given, $T(a,b) = (2a - 3b, a + b)$

(i) The standard bases of R^2 is $\beta = \gamma = \{(1,0),(0,1)\}$

Given, $T(a,b) = (2a - 3b, a + b)$

∴the matrix of the linear transmission is $[T]_\beta^\gamma = \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}$

(ii)the basis is $\beta = \{(2,3),(1,2)\}$

$$v_1 = (2,3), v_2 = (1,2)$$

$$T(a,b) = (2a - 3b, a + b)$$

$$T(v_1) = T(2,3)$$

$$= (2(2) - 3(3), 2 + 3)$$

$$= (-5,5)$$

The first column of the matrix of T is $\begin{bmatrix} -5 \\ 5 \end{bmatrix}$

$$T(v_2) = T(1,2)$$

$$= (2(1) - 3(2), 1 + 2)$$

$$= (-4,3)$$

The second column of the matrix of T is $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$

Matrix of T is $\begin{bmatrix} -5 & -4 \\ 5 & 3 \end{bmatrix}$

2. Let $T: V_2(R) \to V_3(R)$ and $U: V_2(R) \to V_3(R)$ be the linear transformations respectively defined by $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$ and $U(a_1, a_2) = (a_1 - a_2, 2a_1, 3a_1 + 2a_2)$. Let β and γ be the standard bases of $V_2(R)$ and $V_3(R)$ respectively. Verify $[T + U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

Solution:

Given, $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$

Since β and γ be the standard bases of $V_2(R)$ and $V_3(R)$

the matrix corresponds to $\beta = [T]_\beta^\gamma = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$ …..(1)

Given, $U(a_1, a_2) = (a_1 - a_2, 2a_1, 3a_1 + 2a_2)$.

Since β and γ be the standard bases of $V_2(R)$ and $V_3(R)$

the matrix corresponds to $\beta = [U]_\beta^\gamma = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{bmatrix}$ …..(2)

$$(1) + (2) \Rightarrow [T]_{\beta}^{\gamma} + [U]_{\beta}^{\gamma} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{bmatrix} \ldots\ldots(3)$$

$$(T + U)(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2) + (a_1 - 3a_2, 2a_1, 3a_1 + 2a_2)$$

$$= (a_1 + 3a_2 + a_1 - a_2, 0 + 2a_1, 2a_1 - 4a_2 + 3a_1 + 2a_2)$$

$$= (2a_1 + 2a_2, 2a_1, 5a_1 - 2a_2)$$

Since β and γ be the standard bases of $V_2(R)$ and $V_3(R)$

the matrix corresponds to $\beta = [T + U]_{\beta}^{\gamma} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{bmatrix} \ldots\ldots(4)$

$$From\ (3)\ and\ (4) \Rightarrow [T + U]_{\beta}^{\gamma} = [T]_{\beta}^{\gamma} + [U]_{\beta}^{\gamma}$$

3. **Let $T: V_2(R) \to V_3(R)$ be the linear transformations defined by $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$. Let β and γ be the standard bases of $V_2(R)$ and $V_3(R)$ respectively. Verify $[\alpha T]_{\beta}^{\gamma} = \alpha[T]_{\beta}^{\gamma}$**

Solution

Given, $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$

Since β and γ be the standard bases of $V_2(R)$ and $V_3(R)$

the matrix corresponds to $\beta = [T]_{\beta}^{\gamma} = \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$

the matrix corresponds to $\beta = \alpha[T]_\beta^\gamma = \alpha \begin{bmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{bmatrix}$

$$= \begin{bmatrix} \alpha & 3\alpha \\ 0 & 0 \\ 2\alpha & -4\alpha \end{bmatrix} \ldots (1)$$

We have, $T(a_1, a_2) = (a_1 + 3a_2, 0, 2a_1 - 4a_2)$

$$\therefore \alpha T(a_1, a_2) = \alpha(a_1 + 3a_2, 0, 2a_1 - 4a_2)$$

$$(\alpha T)(a_1, a_2) = (\alpha a_1 + 3\alpha a_2, 0, 2\alpha a_1 - 4\alpha a_2)$$

Since β and γ be the standard bases of $V_2(R)$ and $V_3(R)$

the matrix corresponds to $\beta = \alpha[T]_\beta^\gamma = \begin{bmatrix} \alpha & 3\alpha \\ 0 & 0 \\ 2\alpha & -4\alpha \end{bmatrix} \ldots (1)$

$$From\ (1)\ and\ (2) \Rightarrow [\alpha T]_\beta^\gamma = \alpha[T]_\beta^\gamma$$

4. Let $T: P_3(R) \to P_2(R)$ be the linear transformations defined by $T(f(x)) = f'(x)$. Let β and γ be the standard bases of $P_3(R)$ and $P_2(R)$ respectively. Then find $[T]^\gamma_\beta$

Solution :

Let, $\beta = \{1, x, x^2, x^3\}$ be the standard bases of $P_3(R)$

Let, $\gamma = \{1, x, x^2\}$ be the standard bases of $P_2(R)$

Let, $w_1 = 1, w_2 = x, w_3 = x^2$

Given, $T(f(x)) = f'(x)$.

Let, $(f(x)) = 1$. Then $f'(x) = 0$

$$T(1) = T(f(x)) = f'(x) = 0 = 0.1 + 0.x + 0.x^2$$

$$= 0.w_1 + 0.w_2 + 0.w_3$$

The first column of $[T]^{\gamma}_{\beta} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Let, $(f(x)) = x$. Then $f'(x) = 1$

$$T(x) = T(f(x)) = f'(x) = 1 = 1.1 + 0.x + 0.x^2$$

$$= 1.w_1 + 0.w_2 + 0.w_3$$

The second column of $[T]^{\gamma}_{\beta} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Let, $(f(x)) = x^2$. Then $f'(x) = 2x$

$$T(x^2) = T(f(x)) = f'(x) = 2x = 0.1 + 2.x + 0.x^2$$

$$= 0.w_1 + 2.w_2 + 0.w_3$$

The third column of $[T]^{\gamma}_{\beta} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

Let, $(f(x)) = x^3$. Then $f'(x) = 3x^2$

$$T(x^3) = T(f(x)) = f'(x) = 3x^2 = 0.1 + 0.x + 3.x^2$$

$$= 0.w_1 + 0.w_2 + 3.w_3$$

The fourth column of $[T]^{\gamma}_{\beta} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

So, $[T]_\beta^\gamma = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Eigen values and Eigen vector

1. Let $T: R^2 \to R^2$ be a linear operator given given by $T(a,b) = (-2a + 3b, -10a + 9b)$. Let β be an ordered basis of R^2 with $A = [T]_B$. (i) Find the matrix A (ii)The eigen values and eigen vectors of T.

Solution

Given, $T(a,b) = (-2a + 3b, -10a + 9b)$.

Since β is the standard basis of R^2

$$A = [T]_B = \begin{bmatrix} -2 & 3 \\ -10 & 9 \end{bmatrix}$$

To find the Eigen values:

The characteristic equation is $|A - \lambda I| = 0$

$$\lambda^2 - S_1\lambda + S_2 = 0$$

S_1 =Sum of the leading diagonal elements

$$= -2 + 9 = 7$$

$$S_2 = |A| = -18 + 30 = 12$$

$$\lambda^2 - 7\lambda + 12 = 0$$

$$\lambda = 3, \lambda = 4$$

$\lambda = 3,4$ are the Eigen values of A

To find Eigen vectors:

Solve the equation $(A - \lambda I)X = 0$ we get$\begin{pmatrix} -2-\lambda & 3 \\ -10 & 9-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \ldots\ldots(a)$

Case 1: When $\lambda = 3$, from (a) we get

$$\begin{pmatrix} -5 & 3 \\ -10 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$-5x_1 + 3x_2 = 0$$

$$-10x_1 + 6x_2 = 0$$

Since the two equations are same, consider

$$-5x_1 + 3x_2 = 0$$

$$-5x_1 = -3x_2$$

$$\frac{x_1}{3} = \frac{x_2}{5}$$

$$x_1 = 3, x_2 = 5$$

Hence the Eigen vector corresponding to $\lambda = 3$ is $E_{\lambda_1} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

Case 2: When $\lambda = 4$, from (a) we get

$$\begin{pmatrix} -6 & 3 \\ -10 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$-6x_1 + 3x_2 = 0$$

$$-10x_1 + 5x_2 = 0$$

Since the two equations are same, consider

$$-6x_1 + 3x_2 = 0$$

$$-6x_1 = -3x_2$$

$$\frac{x_1}{3} = \frac{x_2}{6}$$

$$\frac{x_1}{1} = \frac{x_2}{2}$$

$$x_1 = 1, x_2 = 2$$

Hence the Eigen vector corresponding to $\lambda = 4$ is $E_{\lambda_2} = \binom{1}{2}$

2. Let $T: P_2(R) \rightarrow P_2(R)$ be the linear operator defined by $T(f(x)) = f(x) + (x+1)f'(x)$. Let $\beta = \{1, x, x^2\}$ be an ordered basis of $P_2(R)$ with $A = [T]_\beta$. Find (i) The matrix A (ii)The eigen values and eigen vectors of T.

Solution

Given, $T: P_2(R) \rightarrow P_2(R)$ be the linear operator defined by $T(f(x)) = f(x) + (x+1)f'(x)$ .... (1)

Let $\beta = \{1, x, x^2\}$ be an ordered basis of $P_2(R)$

To find $A = [T]_\beta$

Let, $(f(x)) = 1$. Then $f'(x) = 0$

$$(1) \Rightarrow T(1) = 1 + (x+1).0 = 1 = 1.1 + 0.x + 0.x^2$$

The first column of $[T]_\beta = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Let, $(f(x)) = x$. Then $f'(x) = 1$

$$(1) \Rightarrow T(x) = x + (x+1).1 = 1 + 2x = 1.1 + 2.x + 0.x^2$$

The second column of $[T]_\beta = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Let, $(f(x)) = x^2$. Then $f'(x) = 2x$

$$(1) \Rightarrow T(x^2) = x^2 + (x+1).2x = 2x + 3x^2$$

$$= 0.1 + 2.x + 3.x^2$$

The third column of $[T]_\beta = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$

$$A = [T]_\beta = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Since A is an upper triangular matrix, the eigen values are

$$\lambda = 1,2,3$$

To find Eigen vectors:

Solve the equation $(A - \lambda I)X = 0$

$$\begin{pmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \ldots\ldots (a)$$

Case 1: When $\lambda = 1$, from (a) we get

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$0x_1 + x_2 + 0x_3 = 0 \ldots . (1)$$

$$0x_1 + x_2 + 2x_3 = 0 \ldots (2)$$

$$0x_1 + 0x_2 + 2x_3 = 0 \ldots (3)$$

Solving the two distinct equations (1) and (2) by the rule of cross multiplication, we get

$$\Rightarrow \frac{x_1}{2-0} = \frac{x_2}{0-0} = \frac{x_3}{0-0}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$x_1 = 1, x_2 = 0,\ x_3 = 0$$

Hence the Eigen vector corresponding to $\lambda = 1$ is $E_{\lambda_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Case 2: When $\lambda = 2$, from (a) we get

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-x_1 + x_2 + 0x_3 = 0 \ \dots .\ (4)$$

$$0x_1 + 0x_2 + 2x_3 = 0 \ \dots (5)$$

$$0x_1 + 0x_2 + 1x_3 = 0 \ \dots (6)$$

Solving the two distinct equations (4) and (5) by the rule of cross multiplication, we get

$$\Rightarrow \frac{x_1}{2-0} = \frac{x_2}{0+2} = \frac{x_3}{0-0}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{0}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$

$$x_1 = 1, x_2 = 1,\ x_3 = 0$$

Hence the Eigen vector corresponding to $\lambda = 2$ is $E_{\lambda_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Case 3: When $\lambda = 3$, from (a) we get

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$-2x_1 + x_2 + 0x_3 = 0 \text{ ....} (7)$$

$$0x_1 - x_2 + 2x_3 = 0 \text{ ...} (8)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \text{ ...} (9)$$

Solving the two distinct equations (7) and (8) by the rule of cross multiplication, we get

$$\Rightarrow \frac{x_1}{2-0} = \frac{x_2}{0+4} = \frac{x_3}{2-0}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{4} = \frac{x_3}{2}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$x_1 = 1, x_2 = 2, x_3 = 1$

Hence the Eigen vector corresponding to $\lambda = 3$ is $E_{\lambda_3} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$