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VECTOR SPACES

Definition :

Let F be a given field and let V be a non-empty set with addition and scalar multiplication rules applicable to any u, v ∈V such as a sum u + v ∈ V and to any u ∈ V, α ∈ F a product α u ∈V . Then V is called a vector space over F if the following condition hold :

1. Closure : for all u, v ∈ V => u + v ∈ V
2. Associative : u + (v +w) = (u + v) + w ∀ u, v, w ∈ V.
3. Identity : u + 0 = 0 + u = u for all u ∈ V, there exist 0 ∈ V
4. Inverse : (-u) + u = 0 = u + (-u) there exist -u ∈V, for all u ∈ V
5. Commutative : u + v = v + u for all u, v ∈ V
6. For all α ∈ F and for all u ∈ V, α u ∈ V.
7. α(u +v)= αu + αv, for all α ∈ F for all u, v ∈V
8. (α + β)v = αv + βv, for all α,β ∈ F and for all v ∈V
9. (αβ)v = α (βv), for all α, β ∈F and for all u, v ∈V
10. 1 . v = v for all v ∈ V

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Properties of vector space :

(i) α . 0 = 0, 0 ∈ V, for all α ∈ F

(ii) 0 . v = 0, for all v ∈ V, 0 ∈ F

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(iii) $(-\alpha)v = -(\alpha v) = \alpha(-v)$ for all $v \in F$, $v \in V$

(iv) $\alpha v = 0$, $v \neq 0$, $\alpha = 0$ where $\alpha \in F$, $\alpha \in V$

(v) $\alpha(u - v) = \alpha v - \alpha v$ for all $\alpha \in F$ and $u, v \in V$

Proof :

(i) since $0+0 = 0$ where $0 \in V$

$\alpha(0 + 0) = \alpha$ for all $\alpha \in F$

$\Rightarrow \alpha 0 + \alpha 0 = \alpha 0$

$\Rightarrow \alpha 0 + \alpha 0 = \alpha 0 + 0$

Hence $\alpha 0 = 0$ [by left cancellation law]

(ii) since $0 + 0 = 0$ where $0 \in F$

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$(0 + 0)v = 0v$ for all $v \in V$

$\Rightarrow 0v + 0v = 0v$

$\Rightarrow 0v + 0v = 0v + 0$

Hence $0v = 0$ [by left cancellation law]

(iii) $(-\alpha)v = -(\alpha v) = \alpha(-v)$ for all $v \in F$, $v \in V$

Since $\alpha \in F \Rightarrow -\alpha \in F$ and $v \in V$, $-v \in V$

$\Rightarrow \alpha + (-\alpha) = 0 \in F$; $v + (-v) = 0 \in V$

$\Rightarrow \alpha v + (-\alpha)v = [\alpha + (-\alpha)]v$

For all $v \in V$; $\alpha v + \alpha(-v) = \alpha[v + (-v)]$

For all $\alpha \in F$

$\Rightarrow \alpha v + (-\alpha)v = 0v$ for all $v \in V$; $\alpha v + \alpha(-v) = \alpha 0$ for all $\alpha \in F$

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=> $\alpha v + (-\alpha)v = 0$ for all $v \in V$; $\alpha v + \alpha(-v) = 0$ for all $\alpha \in F$

=> $(-\alpha)v$ is the additive inverse of αv in V ; $\alpha(-v)$ is the additive inverse of αv in V.

$$(-\alpha)v = -(\alpha v) \quad ; \alpha(-v) = -(\alpha v)$$

(iv) $\alpha v = 0, v \neq 0$

To prove $\alpha = 0$ where $\alpha \varepsilon F, v \varepsilon V$

Let $\alpha \neq 0$ then $\alpha^{-1} \varepsilon F$

Consider $\alpha v = 0$

$\therefore \alpha^{-1} (\alpha v) = \alpha^{-1} (0)$

⇨ $(\alpha^{-1}\alpha)\, v = 0$

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⇨ $1\, v = 0$

⇨ $v = 0$ which is a contradiction.

Hence $\alpha = 0$

Note : The vector space of V over the field F is denoted as V(F).

(i) C is a vector space over a field C and ℝ

(ii) R is a vector space over a field ℝ but not in a field C

(iii) Q is a vector space over a field Q.

(iv) Z is not a vector space over a field R.

(v) The set $R^n = \{(a_1,a_2,\ldots..,a_n) \quad a_1 \varepsilon R\}$ is a vector space over R.

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(vi) The set $M_2(R)$ and $M_2(Q)$ of 2 X 2 matrices with entries from R and Q is a vector space over R.

(vii) The set $Z_p[R]$ of polynomials with coefficients from Z_p is a vector space over Z_p , where P is a prime.

(viii) Let E be a field and F be a subfield of E. Then E is a vector space over F.

(ix) Let $P_n(t)$ be the set of all polynomials P(t) over a field F, where the degree of P(t) is less than or equal to n. i.e.,

$P(t) = a_0+a_1 t +.....+a_n t^n$.

PROBLEMS UNDER VECTOR SPACE

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Example 1. Prove that $R \times R$ is a vector space aver R under addition and multiplication defined by $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and

$a(x_1, x_2) = (\alpha x_1, \alpha x_2)$

Sol: Let $x, y \in V = R \times R$

Then $x = (x_1, x_2)$

$y = (y_1, y_2)$

Where $x_1, x_2 y_1 y_2 \in R$

$$x + y = (x_1, x_2) + (y_1, y_2)$$

$$= (x_1 + y_1, x_2 + y_2) \in R \times R$$

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Let $\alpha \in F$ and $x \in \mathcal{Y}$

$$\alpha x = \alpha(x_1, x_2)$$

$$= (\alpha x_1, \alpha x_2) \in R \times R.$$

Therfore vector addition and scalar multiplications are true in $R \times R$.

1 Under addition

A_1: Commutativity: $x + y = x + y, \forall x, y \in R \times R$

$$x + y = (x_1, x_2) + (y_1, y_2)$$

$$= (x_1 + y_1, x_2 + y_2)$$

$$= (y_1 + x_1, y_2 + x_2)$$

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$$= (y_1, y_2) + (x_1, x_2)$$

$$= x + y$$

$$\therefore x + y = x + y, \forall\, x, y \in R \times R$$

A_2: Associativity: $x + (y + z) = (x + y) + z, \forall x, y, z \in R \times R$

Let $x, y, z \in R \times R$. Then

$x = (x_1, x_2), y = (y_1, y_7)_2 z = (z_1, z_2)$

Where $x_1, x_2, y_1, y_2, z_1, z_2 \in R$

$$x + (y + z) = (x_1, x_2) + [(y_1, y_z) + (z_1, z_2)]$$

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$= (x_1, x_2) + (y_1 + z_1, y_2 + z_2)$

$= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2))$

$= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2)$

$=(x_1 + y_1, x_2 + y_z) + (z_1, z_2)$

$= ((x_1, x_2) + (y_1, y_2)) + (z_1, z_2)$

$= (x + y) + z$

$x + (y + z) = (x + y) + z, \forall x, y, z \in R \times R$

A_3: Existence of Identity: There exists $0 \in R \times R$ snch that

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$x + 0 = x, \forall x \in R \times R$

Let $0 \in R$. Then $0 = (0,0) \in R \times R$.

$x + 0 = (x_1, x_2) + (0,0)$

$= (x_1 + 0, x_2 + 0)$

$= (x_1, x_2)$

$= x$

$0 = (0,0)$ is the zero clement of $R \times R$

A_4: Existence of Inverse: For all x in$R \times R$, there exists $-x \in R \times R$, such that

$x + (-x) = 0$

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$$Let\ x \in R \times R$$

$\therefore x = (x_1, x_2)$, where $x_1, x_2 \in R$

Which implies $-x_1, -x_2 \in R$

$$\Rightarrow -x = (-x_1, -x_2) \in R \times R$$

$$x + (-x) = (x_1, x_2) + (-x_1 - x_2)$$

$$= (x_1 - x_1, x_2 - x_2)$$

$$= (0,0)$$

=0

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$x + (-x) = 0$

=>Inverse of x is $-x$

ie, inverse of (x_1, x_2) is $(-x_1, -x_2)$

II Under scalar multiplicatiou:

M_1: $a(x + y) = arx + \alpha y; \forall a \in R$ and $\forall x, y \in R \times R$

$$\alpha(x + y) = \alpha(x_2 + y_1 x_2 + y_2)$$

$$= (a(x_1 + y_1), a(x_2 + y_2))$$

$$= (\alpha x_1 + \alpha y_1 - ax_2 + ay_2)$$

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$$= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2)$$

$$= \alpha(x_1, x_2) + \alpha(y_1, y_2)$$

$$= ax + ay$$

$\therefore a(x + y) = \alpha x + \alpha y \forall a \in R$ and $\forall x, y \in R \times R$

M_z: $(\alpha + \beta)x = ax + \beta x, \forall \alpha, \beta \in R, \forall x \in R \times R$

$$(\alpha + \beta)x = (u + \beta)(x_1, x_2)$$

$$= ((\alpha + \beta)x_1(\alpha + \beta)x_2)$$

$$= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2)$$

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$$= (\alpha x_1, \alpha x_2) + (\beta x_1, \beta x_2)$$

$$= \alpha(x_1, x_2) + \beta(x_1, x_2)$$

$$= \alpha x + \beta x$$

$(\boldsymbol{a} + \beta)x = ax + \beta x, \forall \alpha, \beta \in R, \forall x \in R \times R$

M_3: $a(\beta x) = (\alpha\beta)(x), \forall a_v, \beta \in R, \forall x \in R \times R$

$$\alpha(\beta x) = \alpha(\beta(x_1, x_2))$$

$$= \alpha(\beta x_1, \beta x_2)$$

$$= (\alpha(\beta x_1), \alpha(\beta x_2))$$

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$$= ((\alpha\beta)x_1, (\alpha\beta)x_2)$$

$$= (\alpha\beta)(x_1, x_2)$$

$$= (\alpha\beta)(x)$$

$$\therefore \alpha(\beta x) = (\alpha\beta)(x) \forall \alpha, \beta \in R, \forall x \in R \times R$$

M_4: $1. x = x, \forall x \in R \times R$ and $1 \in R$

$1. x = 1(x_1, x_2)$

$= (1. x_1, 1. x_2)$

$= (x_1, x_2) = x$

$1. x = x, \forall x \in R \times R$ and $1 \in R$

Therefore $V = R \times R$ is a vector space over R.

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Example 2. Prove that F^n is a vector space over a field F under addition and muitiplication defined by $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$ and $\alpha(x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$

Let $x, y \in V = F^n$

Then $x = (x_1, x_2, \dots, x_n)$

$y = (y_1, y_2, \dots, y_n)$

where $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in F$

$x + y = (x_1 + y_1, x_2 + y_2 \dots., x_n + y_n) \in F^n$

Let $\alpha \in F$ and $x \in F^n$

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$$\alpha x = \alpha(x_1, x_2, \dots, x_n)$$

$$= (\alpha x_1, \alpha x_2, \dots, \alpha x_2) \in F^n$$

Therefore vector addition and scalar multiplications are true in F^n.

I. Under addition

A_1 : Commutativity: $x + y = y + x, \forall x, y \in F^n$

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

$$= (y_1 + x_1, y_2 + x_2, \dots, y_n + x_n)$$

$$= (y_1, y_2, \dots, y_n) + (x_1, x_2, \dots, x_n)$$

$$= y + x$$

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$$x + y = y + x, \forall x, y \in F^n$$

A_2: Associativity: $x + (y + z) = (x + y) + z, \forall x, y, z \in F^n$

Let $x, y, z \in F^n$. Then

$x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n), z = (z_1, z_2, \dots, z_n)$

Where $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, z_1, z_2, \dots z_n \in F$

$$x + (y + z) = (x_1, x_2, \dots, x_n) + [(y_1, y_2, \dots, y_n) + (z_1, z_2, \dots z_n)]$$

$$= (x_1, x_2, \dots, x_n) + (y_1 + z_1, y_2 + z_2, \dots, y_n + z_n)$$

$$= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \dots, x_n + (y_n + z_n))$$

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$$= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, \dots, (x_n + y_n) + z_n)$$

$$= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) + (z_1, z_2, \dots, z_n)$$

$$= ((x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n)) + (z_1, z_2, \dots, z_n)$$

$$= (x + y) + z$$

$\therefore x + (y + z) = (x + y) + z, \forall x, y, z \in F^n$

A_3 : Existence of Identity: There exists $0 \in F^n$ such that

$$x + 0 = 0 + x = x, \forall x \in F^n$$

Let $0 \in F$. Then $0 = (0,0, \dots ,0)) \in F^n$

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$x + 0 = (x_1, x_2, \dots, x_n) + (0,0, \dots ,0)$

$= (x_1 + 0, x_2 + 0, \dots, x_n + 0)$

$= (x_1, x_2, \dots, x_0)$

$= x$

$0 = (0,0, \dots 0)$) is the zero element of F^x

A_4 : Fxistence of Inverse: For all x in F^n, there exists $-x$ in F^n such that

$$(-x) + x = 0$$

Let $x \in F^n$.

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$\therefore x = (x_1, x_2, \dots, x_n)$; where $x_1 x_2, \dots, x_n \in F_1$

Which implies $-x_1, -x_{2i} \dots, -x_n \in F$

$$-x = (-x_1 - x_2, \dots, -x_n) \in k^n$$

$$x + (-x) = (x_1, x_2, \dots, x_n) + (-x_{1+} - x_2 \dots \dots, -x_n)$$

$$= (x_1 - x_1, x_2 - x_2 + \cdots, x_n - x_n)$$

$$= (0,0, \dots ,0)$$

$$= 0$$

=>Inverse of x is - x

ie, inverse of $(x_1, x_2, \dots, x_n)$ is $(-x_1 - x_2, \dots, -x_n)$

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II Under scalar multiplication:

$M_1 = \alpha(x + y) = ax + \alpha y, \forall a \in F$ and $\forall x, y \in F^a$

$a(x + y) = \alpha(x_1 + y_1 x_2 + y_2 \dots, x_n + y_n)$

$= (\alpha(x_1 + y_1), \alpha(x_2 + y_2), \dots, \alpha(x_n + y_n))$

$= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2 \dots \dots, \alpha x_n + \alpha y_n)$

$= (\alpha x_1, a x_2) + (\alpha y_1, \alpha y_2), \dots, (\alpha x_n + \alpha y_n)$

$= \alpha(x_1, x_2) + \alpha(y_1, y_2), \dots, (\alpha x_n + \alpha y_n)$

$= \alpha x + \alpha y$

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$\therefore \alpha(x+y) = \alpha x + \alpha y, V\alpha \in F$ and $Vx, y \in F^n$

M$_2$: $(\alpha+\beta)x = ax + \beta x \ \alpha, \beta \in F, \forall x \in F''$

$$(a+\beta)x = (\alpha+\beta)(x_1, x_2, \dots, x_n)$$

$$= ((\alpha+\beta)x_1(\alpha+\beta)x_2, \dots, (\alpha+\beta)x_n)$$

$$= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2, \dots, \alpha x_n + \beta x_n)$$

$$= (\alpha x_1, \alpha x_2, \dots, \alpha x_n) + (\beta x_1, \beta x_2, \dots, \beta x_n)$$

$$= \alpha(x_1, x_2, \dots, x_n) + \beta(x_1, x_2, \dots, x_n)$$

$$= \alpha x + \beta x$$

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$\therefore (\alpha+\beta)x = \alpha x + \beta x, \forall \alpha, \beta \in F, \forall x \in F^n$

M_3: $\alpha(\beta x) = (\alpha\beta)(x), a, \beta \in F, \forall x \in F^n$

$$\alpha(\beta x) = \alpha(\beta(x_1, x_2, \dots, x_n))$$

$$= \alpha(\beta x_1, \beta x_2, \dots, \beta x_n)$$

$$= (\alpha(\beta x_1), \alpha(\beta x_2), \dots, \alpha(\beta x_n))$$

$$= ((\alpha\beta)x_1, (\alpha\beta)x_2, \dots, (\alpha\beta)\hat{x}_n)$$

$$= (\alpha\beta)(x_1, x_2, \dots, x_n)$$

$$= (\alpha\beta)(x)$$

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$\therefore \alpha(\beta x) = (\alpha\beta)(x), \forall \alpha, \beta \in F, \forall x \in F^n$

M_4: $1.x = x_1 \forall x \in F^n$ and $1 \in F$

$1.x = 1.(x_1, x_2, \dots, x_n)$

$= (1.x_1, 1.x_2, \dots, 1 \cdot x_n)$

$= (x_1, x_2, \dots, x_n) = x$

$\therefore 1.x = x, \forall x \in F^n$ and $1 \in F$

$\therefore F^n$ is a vector space over F.

Example 3. Prove that set of complex numbers is a vector space over field

R.

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Sol: $V = C = \{(x + iy)/x, y \in R\}$

Let $x, y \in C$

Then $x = x_1 + iy_1, y = x_2 + iy_2$

Where $x_1, y_1, x_2, y_2 \in R$

Addition of vectors is defined by

$$x + y = (x_1 + iy_1) + (x_2 + iy_2)$$

$$= x_1 + x_2 + i(y_1 + y_2) \in C$$

Scalar multiplication is defined by

For $\alpha \in R$ and $x \in C$

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$\alpha x = a(x_1 + iy_1)$

$= \alpha x_1 + i\alpha x_2 \in C$

Therefore vector addition and scalar multiplications are true in C.

1. Under Addition

A_1: Commutativity: $x + y = y + x, \forall x, y \in C$

$x + y = (x_1 + x_2) + i(y_1 + y_2)$

$= (x_2 + x_1) + i(y_2 + y_1)$

$= (x_2 + iy_2) + (x_1 + iy_1)$

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$= y + x$

$\therefore x + y = y + x, \forall x, y \in C$

A_2: Associativity: $x + (y + z) = (x + y) + z, \forall x, y, z \in C$

Let $x, y, z \in$ C

$\therefore x = x_1 + iy_1. y = x_2 + iy_2, z = x_3 + iy_3$

$x + (y + z) = (x_1 + iy_1) + [(x_2 + iy_2) + (x_3 + iy_3)]$

$=(x_1 + iy_1) + [(x_2 + x_3) + i(y_2 + y_3)]$

$=(x_1 + (x_2 + x_3)) + i(y_1 + (y_2 + y_3))$

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$=((x_1 + x_2) + x_3) + i((y_1 + y_2)y_3)$

$=[(x_1 + x_2) + i(y_1 + y_2)] + (x_3 + iy_3)$

$=[(x_1 + iy_1) + (x_3 + iy_2)] + (x_3 + iy_3)$

$=(x + y) + z$

$x + (y + z) = (x + y) + z, \forall x, y, z \in C$

A_3 : Existence of Identity: There exists $0 \in C$ such that

$x + 0 = x, \forall \in C$

Let $0 \in R$. Then $0 = 0 + i0 \in C$

$x + 0 = (x_1 + iy_1) + (0 + i0)$

$= x_1 + 0 + i(y_1 + 0)$

$= x_1 + iy_1$

$= x$

$0 = 0 + i0$ is the zero element of C

A_4 : Existence of Inverse: For all x in C, there exists $-x$ in C such that

$(-x) + x = 0$

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Let $x \in C$. Then

$x = x_1 + iy_1$, where $x_1, y_1 \in R$

Which implies $-x_1, y-\ _1 \in R$

$\therefore -x = -x_1 + i(-y_1) \in C$

$$x + (-x) = (x_1 + iy_1) + (-x_1 + i(-y_1))$$

$$= x_1 - x_1 + i(y_1 - y_1)$$

$$= 0 + i0$$

$$= 0$$

$\therefore$ Inverse of x is $-x$

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i.e. inverse of $x_1 + iy_1$ is $-x_1 + i(-y_1)$

II. Under scalar multiplication

$\boldsymbol{M_1}$: $\alpha(x + y) = \alpha x + \alpha y, \forall \alpha \in R$ and $\forall x, y \in C$

$$\alpha(x + y) = \alpha[(x_1 + x_2) + i(y_1 + y_2)]$$

$$= \alpha(x_1 + x_2) + i\alpha(y_1 + y_2)$$

$$= (\alpha x_1 + \alpha x_2) + i(\alpha y_1 + \alpha y_2)$$

$$= (\alpha x_1 + i\alpha y_1) + (\alpha x_2 + i\alpha y_2)$$

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$= \alpha(x_1 + iy_1) + \alpha(x_2 + iy_2)$

$= \alpha x + \alpha y$

$\therefore \alpha(x + y) = \alpha x + \alpha x, \forall \alpha \in R$ and $\forall x, y \in C$

M_2: $(\alpha + \beta)x = \alpha x + \beta x, \forall \alpha, \beta \in R, \forall x \in C$

$(\alpha + \beta)x = (\alpha + \beta)(x_1 + iy_1)$

$= (\alpha + \beta)x_1 + i(\alpha + \beta)y_1$

$= \alpha x_1 + \beta x_1 + i(\alpha y_1 + \beta y_1)$

$= (\alpha x_1 + iay_1) + (\beta x_1 + i\beta y_1)$

$= \alpha(x_1 + iy_1) + \beta(x_1 + iy_1)$

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$= \alpha x + \beta x$

$\therefore (\alpha + \beta)\, x = \alpha x + \beta x, \forall \alpha, \beta \in R, \forall x \in C$

$M_3 d\alpha(\beta x) = (\alpha\beta)(x), \forall \alpha, \beta \in R, \forall x \in C$

$\alpha(\beta x)\& = \alpha(\beta(x_1 + iy_1))$

$= \alpha(\beta x_1 + i\beta y_1)$

$= \alpha(\beta x_1) + i\alpha(\beta y_1)$

$= (\alpha\beta)x_1 + l(\alpha\beta)y_1$

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$= (\alpha\beta)(x_1 + iy_1)$

$= (\alpha\beta)x$

$\therefore\ \alpha(\beta x) = (\alpha\beta)x, \forall \alpha, \beta \in R, \forall x \in C$

M_4: $1.x = x, \forall x \in C$ and $1 \in R$

$1.x = (1 + i0)(x_1 + iy_1)$

$= x_1 + iy_1$

$= x$

$\therefore\ 1.x^n = x, \forall x \in C$ and $1 \in R$

$\therefore\ C$ is a vector space over R.

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1.2 SUBSPACES

Definition :

Let V be a vector space and U be a non-empty subset of V. If U is a vector space under the operation of addition and scalar multiplication of V, then it is said to be a subspace of V.

Note:

(i) {0} and V itself are called trivial subspaces.

(ii) All other vector subspace of V are called non-trivial subspaces.

Note :

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(i) A non-empty subset U of a vector space V over F is called subspace of V, if $u + v \in U$ and $\alpha u \in U$ for all $u, v \in U$ and $\alpha \in F$ or simply

$$\alpha u + \beta v \in U \text{ and } \alpha,\beta \in F$$

(ii) {0} is a subspace of V called zero subspace.

(iii) V is a subspace of its own.

(iv) {0} and V are called trivial subspace (or) improper subspaces.

(v) Any subspace other than{0} and V are called proper subspaces of V (or) non-trivial subspaces.

(vi) The vectors lying on a line L through the origin R^2 are subspaces of the vector space.

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(vii) A non-empty subset U of vector space V is a subspace iff $u + \alpha v \in U$ for any $v \in U$ and $\alpha \in F$.

Theorem : 1.

Let w_1 and w_2 be two subspaces of vector space V over F. Then $w_1 \cap w_2$ is a subspace of V.

Proof :

As $0 \in w_1 \cap w_2, w_1 \cap w_2$ is non-empty.

Consider $u, v \in w_1 \cap w_2, \alpha \in F$.

Then u, $v \in w_1, \alpha \in F$ and $u,v \in w_2, \alpha \in F$

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$u + \alpha v \in w_1$ and $u + \alpha v \in w_2$

So, $u + \alpha v \in w_1 \cap w_2$

Hence $w_1 \cap w_2$ is a subspace of V.

PROBLEMS BASED ON SUBSPACES

1. Let $V = R^3$. The XY-plane $w_1 = \{(x,y,0) : x, y \in R\}$ and the XZ-plane $w_2 = \{(x,0,z) : x, z \in R\}$.These are subspace of R^3 . Then $w_1 \cap w_2 = \{(x,0,0) : x \in R\}$ is the x-axis.

Solution :

Let $v \in V$, $v = (x, y,z) \in V$

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$v = (x, y, 0)+(0,0,z) \in w_1 + w_2$

So, $V \subseteq w_1 + w_2 \subseteq V$

Hence $V = w_1 + w_2$

2. Express the polynomial $3t^2 + 5t - 5$ as a linear combination of the polynomials $t^2 + 2t + 1, 2t^2 + 5t + 4, t^2 + 3t + 6$

Solution :

Let $a, b, c \in F$ such that

$$3t^2 + 5t - 5 = a(t^2 + 2t + 1) + b(2t^2 + 5t + 4) + c(t^2 + 3t + 6)$$

$$3t^2 + 5t - 5 = (a + 2b + c)t^2 + (2a + 5b + 3c)t + (a + 4b + 6c)$$

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Comparing the co-efficients, we get

$$a + 2b + c = 3 \quad \ldots(1)$$

$$2a + 5b + 3c = 5 \quad \ldots(2)$$

$$a + 4v^{\cdot} + 6c = -5 \quad \ldots(3)$$

$(3) - (1) \Rightarrow \quad 2b + 5c = -8 \quad \ldots.(4)$

Multiply (1) by 2,

$$2a + 4b + 2c = 6 \quad \ldots\ldots\ldots(5)$$

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$$(2) - (5) \Rightarrow b + c = -1 \quad \dots (6)$$

Multiply (6) by 2,

$$2b + 2c = -2 \quad \dots (7)$$

$$(4) - (7) \Rightarrow 3c = -6$$

$$\therefore c = -2$$

Substituting c in (6),

$$b - 2 = -1$$

$$b = 2 - 1 = 1$$

$$\therefore b = 1$$

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Substituting c, b in (1)

$$a + 2(1) - 2 = 3$$

$$a + 2 - 2 = 3$$

$$\Rightarrow a = 3$$

$$\therefore a = 3, b = 1, c = -2$$

Hence, $3t^2 + 5t - 5 = 3(t^2 + 2t + 1) + 1(2t^2 + 5t + 4)$

$$-2(t^2 + 3t + 6)$$

3. Let $V = R^3$, then which of the following sets is/are subspace(s) of V.

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(i) $w_1 = \{(a, b, 0); a, b \in \mathbf{R}\}$

(ii) $w_2 = \{(a, b, 0); a \geq 0\}$

Solution :

(i) $\bar{0} = (0,0,0) \in w_1$, so $w_1 \neq \phi$

Let $v_1, v_2 \in w_1, \alpha \in$ R

Then, $v_1 = (a, b, 0)$ and $v_2 = (c, d, 0)$ for some $a, b, c, d \in \mathbb{R}$

$v_1 + v_2 = (a + c, b + d, 0) \in w_1$

$\alpha v_1 = (\alpha a, \alpha b, 0) \in w_1$

Hence w_1 is a subspace of V.

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(ii) Consider $w_2 = \{(a, b, 0); a \geq 0\}$

Here we should take the value of a as zero or positive.

Let $V = (2,1,0) \in w_2$

But under scalar multiplication, the vector is not in w_2

That is $-v = (-2, -1,0) \notin w_2$

$(-1)v \notin w_2$

Hence w_2 is not a subspace of V

4. Let V be a vector space of all 2×2 matrices over real numbers. Determine whether W is a subspace of V or not, where

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(i) W consists of all matrices with non-zero determinant.

(ii) W consists of all matrices A such that $A^2 = A$.

Solution :

(i) Let $w = \{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : x, y \in \mathbb{R}\}$

Since $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in$ W, W is a non-empty subset of V.

Consider $A = \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix}, B = \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix} \in W$ and $\alpha, \beta \in R$

$$\alpha A = \begin{bmatrix} \alpha x_1 & 0 \\ 0 & \alpha y_1 \end{bmatrix} \text{ and } \alpha B = \begin{bmatrix} \beta x_2 & 0 \\ 0 & \beta y_2 \end{bmatrix}$$

$$\alpha A + \beta B = \begin{bmatrix} \alpha x_1 + \beta x_1 & 0 \\ 0 & \alpha y_1 + \beta y_2 \end{bmatrix} \in \text{W}$$

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Hence W is a subspace of V.

(ii) W is not a subspace of V because w is not closed under addition.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so that

$$\text{A}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \text{A}$$

$\therefore A \in W$

But $A + A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

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$$= \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \neq A + A$$

Thus $A + A \notin W$

7. Let $\mathbf{V} = \{\mathbf{A}/\mathbf{A} = [a_{ij}]_{n\times n}, a_{ij} \in \mathbf{R}\}$ be a vector space over $\mathbf{R}$. Show $W = \{\mathbf{A} \in \mathbf{V}/\mathbf{AX} = \mathbf{XA}$ for all $\mathbf{A} \in \mathbf{V}\}$ is a sub-space of $\mathbf{V}(\mathbf{R})$

Solution :

Since $0X = 0 = X0$ for all $X \in V$

$\Rightarrow 0 \in W$. Thus W is non-empty.

Now, let $\alpha, \beta \in R$ and $A_1, A_2 \in W$

$\Rightarrow A_1X = XA_1$ and $A_2X = XA_2$ for all $X \in V$

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$$\therefore (\alpha A_1 + \beta A_2)X = (\alpha A_1)X + (\beta A_2)X$$

$$= \alpha(A_1X) + \beta(A_2X)$$

$$= \alpha(XA_1) + \beta(XA_2)$$

$$= X(\alpha A_1) + X(\beta A_2)$$

$$= X(\alpha A_1 + \beta A_2)$$

$$= \alpha A_1 + \beta A_2 \in W$$

Hence W is a vector space of $V(R)$.

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Theorem : 3. If S is any subset of a vector space V(F), then S is a subspace of V(F) if and only if L(S) = S.

Proof:

Given S is a subspace of $V(F)$

To prove $L(S) = S$

Let $x \in \mathrm{L(S)} \Rightarrow$ there exists $x_1, \dots, x_\mathrm{n} \in \mathrm{S}$

$\alpha_1, \alpha_2, \dots, \alpha_\mathrm{n} \in \mathrm{F}$

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$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_\mathrm{n} x_\mathrm{n} \in \mathrm{S}$$

$\mathrm{L(S)} \subset \mathrm{S}$ ….. (1)

Also $\mathrm{S} \subset \mathrm{L(S)}$ ... (2) [Since S is a subspace of V(F)]

From (1) and (2), L(S) = S

Conversely, Given $L(S) = S$

To prove: S is a subspace of $V(F)$

Since $L(S)$ is a subspace of $V(F)$

∴ S is also a subspace of V(F)

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8. Let V be the set of all solutions of the differential equation $2y'' - 7y' + 3y = 0$. Then V is a vector space over R.

Solution :

Let $f, g \in V$ and $\alpha \in R$.

Then $2f'' - 7f' + 3f = 0$ and

$$2g'' - 7g' + 3g = 0$$

$$2\frac{d^2}{dx^2}(f + g) - 7\frac{d}{dx}(f + g) + 3(f + g) = 0$$

Hence $f + g \in$ V

Also $2(\alpha f)^n - 7(\alpha f)' + 3(\alpha f) = 0$

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Hence $\alpha f \in$ V

Hence V is a vector space over R.

9 Examine whether $(1, -3,5)$ belongs to the linear space generated by S, where $S = \{(1,2,1), (1,1, -1), (4,5, -2)\}$ or not?

Solution :

Suppose $(1, -3,5)$ belongs to S.

$\therefore$ There exists scalars α, β, γ such that

$$(1, -3,5) = \alpha(1,2,1) + \beta(1,1, -1) + \gamma(4,5, -2)$$

$$(1, -3,5) = (\alpha + \beta + 4\gamma, 2\alpha + \beta + 5\gamma, \alpha - \beta - 2\gamma)$$

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Comparing both sides, we get

$$\alpha + \beta + 4\gamma = 1 \quad ....(1)$$

$$2\alpha + \beta + 5\gamma = -3 \ ....(2)$$

$$\alpha - \beta - 2\gamma = 5......(3)$$

Adding (1) and (3), we get

$$2\alpha + 2\gamma = 6 \Rightarrow \alpha + \gamma = 3 \ ... \ (4)$$

Adding (2) and (3), we get

$$3\alpha + 3\gamma = 2 \Rightarrow \alpha + \gamma = \frac{2}{3} ... \ (5)$$

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Equation (4) and (5) are contradiction

Hence $(1, -3,5)$ does not belong to linear space of S.

Remark :

The union of the subspace may not be a sub-space.

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1.5 LINEARLY INDEPENDENCE AND LINEARLY DEPENDENCE

Linearly dependent set

A subset S of a vector space is called linearly dependent if there is a finite number of distinct vectors $v_1, v_2, \ldots, v_n$ in S and scalars $\alpha_1, \alpha_2, \ldots,$ zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

Linearly independent set

A subset S of a vector space that is not linearly dependent is called independent. i.e., A subset S of a vector space is called linearly independent if there exis. number of distinct vectors $v_1, v_2, \ldots, v_n$ in S and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0. \text{ Implies } \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

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Note:

- Any set of vectors which contains zero vectors is linearly dependen
- In R^2 any two straight lines which are not parallel are linearly indep
- In R^2 any two straight lines which are parallel are linearly dependen
- In R^2 any three vectors are linearly dependent therefore any set of n in the R^m are linearly dependent if $n > m$.

Theorem 1.16: $\{0\}$ is a dependent set

Proof: Let V be a vector space over F

Let $v_1 = 0$

Therefore $\alpha_1 v_1 = 0 \Rightarrow \alpha_1 \neq 0$

$\therefore \{0\}$ is linearly dependent.

Theorem 1.17: A singleton non zero vector is linearly independent set

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Proof: Let V be a vector space over F

Let $v_1 \neq 0 \in V$

Therefore $\alpha_1 v_1 = 0 \Rightarrow \alpha_1 = 0$

$\therefore \{v_1\}$ is linearly independent.

Theorem 1.18: Any subset of a linearly independent set is linearly independent.

Proof:

Let V be a vector space over a field F.

Let $S = \{v_1, v_2, \dots, v_n\}$ be a linearly independent set.

Let $S_1 = \{v_1, v_2, \dots, v_m\}$ be a subset of S, where $m < n$.

Suppose S_1 is a linearly dependent set. Then there exist $\alpha_1, \alpha_2, \dots \dots \alpha_m$ in F not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$$

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Hence $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m + 0v_{m+1} + \cdots + 0v_n = 0$ with $\alpha_1, \alpha_2, \dots \dots \alpha_m$ in F not all zero.

Therefore $\{v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n\}$ is a linearly dependent set of V i.e., S is a linearly dependent set of V, which is a contradiction.

Therefore S_1 is linearly independent.

Theorem 1.19: Any set containing a linearly dependent set is also linearly dependent

OR

Any super set of a linearly dependent set is linearly dependent set

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Proof: Let V a vector space over F.

et S be a linearly dependent set of V…

Then there exits scalar $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ not all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

now consider the super set $S_1 = \{v_1, v_2, \dots, v_n, v_{n+1}\}$

Then we have $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n + 0 v_{n+1} = 0$ with at least one $\alpha_i \neq 0$

$\therefore S_1$ is linearly dependent.

Theorem 1.20: A finite set of vectors that contains the zero vector will be linearly dependent.

Proof: Let $S = \{0, v_1, v_2, \dots, v_n\}$ be any set of vectors that contains the zero vector. Consider

$$a_1(0) + \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

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Which implies $a_1 \neq 0$

Therefore $S = \{0, v_1, v_2, \dots, v_n\}$ linearly dependent.

Theorem 1.21: Let $S = \{v_1, v_2, \dots, v_n\}$ be a linearly independent set of vectors in

a vector space V over a field F. Then every element of $L(S)$ can be uniquely written in the form $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$, where $v_i \in S$ and $\alpha_i \in F$.

Proof: By the definition, every element of $L(S)$ is of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

We prove that every element of $L(S)$ can be uniquely written in the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

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If not suppose there is linear combination $\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$ of S such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n, \quad \text{where} \quad \beta_i \in F$$

$$\Rightarrow (\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \cdots + (\alpha_n - \beta_n)v_n = 0$$

Since S is a linearly independent set, $(\alpha_i - \beta_i) = 0$ for all i.

$$\alpha_i - \beta_i = 0 \text{ for all } i$$

$$\therefore \alpha_i = \beta_i \text{ for all } i$$

Hence every element of $L(S)$ can be uniquely written in the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots \ldots + \alpha_n v_n$$

Theorem 1.22: A set $S = \{v_1, v_2, \ldots, v_n\}$; $n \geq 2$ is a linearly dependent set of vectors in V if and only if there exists a vector $v_k \in S$ such that v_k is a linear combination of the preceding vectors $v_1, v_2, \ldots, v_{k-1}$.

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1. Determine whether the following sets of vectors v_3(R) are linearly dependent or linearly independent.

- **i.** **V_1 =(0,2,-4), V_2=(1,-2,-1), V_3 =(1,-4,3)**
- **ii.** **V_1 =(1,2,-3), V_2=(1,-3,2), V_3 =(2,-1,5)**
- **iii.** **V_1 =(1,2,3), V_2=(3,1,5), V_3 =(3,-4,7)**

Solution:

(i) Let $av_1 + bv_2 + cv_3 = 0$, a, b, c ϵ R

a(0,2,-4)+ b(1,-2,-1)+c (1,-4,3) =(0,0,0)

⇨ (0, 2a, -4a)+(b, -2b, -b)+(c, -4c, -3c) =(0, 0, 0)

⇨ (b+c, 2a-2b-4c, -4a-b+3c) =(0,0,0)

b + c =0 ....(1)

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$2a - 2b - 4c \Rightarrow a - b - 2c = 0$ ....(2)

$-4a - b + 3c = 0$ .... (3)

Subtracting (3) from (2)

$5a - 5c = 0 \Rightarrow a = c$

From (1) $b = -c$

If we choose c = k, then a=k and b=-k

Hence the system is linearly dependent

(ii) $a(1,2,-3)+b(1,-3,2)+c(2,-1,5) = (0,0,0)$

$a + b + 2c = 0$ ……(1)

$2a - 3b - c = 0$ ....(2)

$-3a + 2b + 5c = 0$..................... (3)

Multiply (1) by 2,

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$2a + 2b + 4c = 0$ ....(4)

Subtracting (1) and (2),

We get $5b + 5c = 0$.................................. (5)

Multiply (1) by (3),

$3a + 3b + 6c = 0$ .... (6)

Adding (3) and (6),

$5b = 11c = 0$ .... (7)

Substituting c=0 in (5)

We get b=0

From (1), a=0

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$a = 0, b = 0, c = 0$

The given system is linearly independent.

(iii) a (1,2,3)+b(3,1,5)+c(3,-4,7) =(0,0,0)

a+3b+3c =0 …….(1)

2a +b-4c =0 ....(2)

a+ 5b+ 7c =0 ....(3)

Subtracting (3) and (1),

2b +4c =0 ....(4)

Multiply (1) by (2), 2a +6b+ 6c =0 …(5)

Subtracting (5) and (2),

5b + 10c =0

b+ 2c =0 ....(6)

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Multiplying (6) by 2,

2b +4c =0 ....(7)

From (4) and (7),

B= -2c

Substituting b in (2)

2a -2c -4c =0

2a = 6c

a= 3c

The given system is linearly dependent.

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2.If V_1 =(2, -1,0), V_2 =(1,2,1) and V_3= (0,2,-1). Show V_1, V_2, V_3 are linearly independent. Is it possible (3,2,1) as a linear combination of V_1, V_2, V_3.

Solution:

Let $av_1 + bv_2 + cv_3 = 0$, a, b, c ϵ F

a(2,-1,0)+b(1,2,1)+c(0,2,-1) =(0,0,0)

2a+b =0 …….(1)

-a +2b+2c =0................ (2)

b-c =0 ....(3)

these equation can be put in the form AX =0

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\text{Det } A = \det \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \det \begin{bmatrix} 2 & 1 & 0 \\ -1 & 4 & 2 \\ 0 & 0 & -1 \end{bmatrix} \; C_1\text{->} \; C_2 + C_3$$

$$= -\det \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} = -9 \neq 0$$

a= b = c =0

hence the system is linearly independent.

Let $v = a_1v_1 + a_2v_2 + a_3v_3$ where a_1, a_2,a_3 ϵ F

(3,2,1) = a_1(2,-1,0)+a_2(1,2,1)+a_3(0,2,-1)

(3,2,1) = (2 a_1 + a_2,- a_1+ 2 a_2+2 a_3, a_2- a_3)

Comparing

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$3=2a_1+a_2$ ....(4)

$2=-a_1+2a_2+a_3$ ....(5)

$1=a_2-a_3$ ....(6)

Multiplying (5) by 2,

$4=2-a_1+4a_2+4a_3$.(7)

Adding (4) and (7)

$7=5a_2+4a_3$ ......(8)

Multiplying (6) by 5,

$5=5a_2+5a_3$ ....(9)

Subtracting (8) and (9)

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$2=9a_3 \Rightarrow a_3=2/9$

Substituting a_3 in (6)

$1= a_2-2/9 \Rightarrow 1+2/9$

$a_2=\frac{11}{9}$

Substituting a_2 in (4)

$3=2a_1+\frac{11}{9}$

$2a_1=3-\frac{11}{9}$

$2a_1=\frac{27-11}{9}$

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$\Rightarrow a_1 = \frac{16}{2*9} = \frac{8}{9}$

$a_1 = \frac{8}{9}$, $a_2 \frac{11}{9}$, $a_3 = \frac{2}{9}$

hence $(3,2,1) = \frac{8}{9}(2,-1,0) + \frac{11}{9}(1,2,1) + \frac{2}{9}(0,2,-1)$

which is the required linear combination.

1. If x,y,z are linearly independent vectors in a vector space V then prove that all linearly independent x+y,x-y,x-2y+2

Solution:

Let a, b, c ϵ F such that

A(x+y) +b(x-y)+c(x-2y-z)=0

⇨ (a+b+c)x+(a- b- 2c)y + cz =0, x+0, y+0,z

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Comparing a + b+ c =0 … (1)

a – b-2c =0 … (2),

c=0 …. (3)

Note:

1. Any matrix with distint eigen values can be diagonalizable.
2. All matrices donot posses n linearly independent eigen vectors. Therefore all matrices are not diagonalizable.
3. Similar matrices have the same eigen values.
4. If A is diagonalizable then it has n linearly indebendent eigen vectors.
5. Symmetric matrices are always diagonalizable.
6. Let A be a square matirix, A is orthogonally diagonalizable iff it is a symmetric matrix.

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Definition:

A square matrix A is said to be orthogonally diagonalizable if there exists an orthogonal matrix N such that D $=N^{T}AN$ is a diagonal matrix.

1.Show that the following matrix A= $\begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ is diagonalizable hence find A^{9}.

Solution:

The characteristic equation I sgiven by $|A - \lambda I|$=0

(i.e.,) $\begin{vmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{vmatrix} = 0$

=>$(-4-\lambda)(5-\lambda)$ -3(-6) = 0

=> $-20+4\lambda-5\lambda+\lambda^2+18=0$

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=> $\lambda^2-\lambda-2=0$

$(\lambda+1)(\lambda-2)=0$

$\lambda=-1,2$

The eigen values are λ=-1,2

To find eigen vectors :

$(A-\lambda I)v=0$

$\begin{vmatrix} -4-\lambda & -6 \\ 3 & 5-\lambda \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ .....(1)

Case (i)

Substituting λ =2 in we get

$\begin{vmatrix} -4-2 & -6 \\ 3 & 5-2 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

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$$\begin{bmatrix} -6 & -6 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-6x_1 - 6x_2 = 0$

$3x_1 + 3x_2 = 0 \Rightarrow 3x_1 = -3x_2$

$\Rightarrow x_1 = -x_2$

Let $x_2 = t$, then $x_1 = t$

$V_1 = t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Case (ii)

Substituting $\lambda = -1$ in we get

$$\begin{bmatrix} -4+1 & -6 \\ 3 & 5+1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -6 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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$-3x_1 - 6x_2 = 0$

$3x_1 + 6x_2 = 0 \Rightarrow 3x_1 = -6x_2$

$\Rightarrow x_1 = -2x_2$

Let $x_2 = s$, then $x_1 = -2s$

$V_2 = s\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Since A has two linearly indebendent eigen vectors it is diagonalizable.

Modal matrix is the column vectors of the diagonalizing matrix M.

$$M = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$M^{-1}AM = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

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$M^{-1} = \frac{1}{|M|}$ (cofactor matrix)^T

$= \frac{1}{(-1+2)}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

Substituting M^{-1} in (2),

$M^{-1}AM = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -4+6 & -6+10 \\ 4-3 & 6-5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -2+4 & -4+4 \\ -1+1 & -2+1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = D$

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$M^{-1}AM = D$ .... (3)

Pre-multiply (3) by M and postmultiply (3) by M^{-1} on both

$MM^{-1}AMM^{-1} = MDM^{-1}$

$A = MDM^{-1}$

$A^9 = MD^9M^{-1}$.................................(4)

$D^9 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^9 == \begin{bmatrix} 2^9 & 0 \\ 0 & (-1)^9 \end{bmatrix}$

$== \begin{bmatrix} 512 & 0 \\ 0 & -1 \end{bmatrix}$

$A^9 = MD^9M^{-1}$

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$$=\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}\begin{bmatrix}512 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}-1 & -2\\ 1 & 1\end{bmatrix}^{-1}$$

$$=\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}\begin{bmatrix}512 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\ -1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}-512+0 & 0+2\\ 512+0 & 0-1\end{bmatrix}\begin{bmatrix}1 & 2\\ -1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}-514 & +1026\\ 513 & 1025\end{bmatrix}$$

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LINEAR COMBINATIONS

Definition :

Let $v_1, v_2, \ldots, v_m$ be vectors of vector space V. The vector v in V is a linear combination of $v_1, \ldots, v_m$ if there exist scalars $a_1, \ldots, a_m$ such that v can be written as $v = a_1v_1 + a_2v_2 + \ldots + a_mv_m$

Span

Definition :

Let $v_1, v_2, \ldots, v_m$ be vector of vector space V. These vector span V if every vector in V can be expressed as a linear combination of them.

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THE SYSTEM OF HOMOGENOUS EQUATIONS

The system of homogenous equations is $AX = 0$

where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n1} & a_{n2} & \ldots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}, 0 = \begin{bmatrix} 0 \\ 0 \\ . \\ 0 \\ 0 \end{bmatrix}$

Evidently $X = 0$ is a solution of $AX = 0$ in which $X = 0$, called trivial solution.

There are solutions to $AX = 0$ in which $X \neq 0$, called non-trivial solution.

Note: For $AX = 0$, there is more than one solution.

We have the following two theorems without proof.

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Theorem 1 : The system of homogenous equations $AX = 0$ has trivial |ution

$(X = 0)$ if and only if $|A| \neq 0$

Theorem 2 :The system of homogenous equations $AX = 0$ has non-trivial ution

$(X \neq 0)$ if and only if $|A| = 0$.

Find the non-trivial solutions of the equations

$$\boldsymbol{x_1 + 2x_2 - x_3 = 0, 3x_1 + x_2 - x_3 = 0, 2x_1 - x_2 = 0}$$

Sol:

The system is equivalent to

$$AX = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\therefore |A| = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{vmatrix}$$

$$= 1(0 - 1) - 2(0 + 2) - 1(-3 - 2)$$

$$= -5 + 5 = 0$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \neq 0$$

Hence rank of A is $r = 2$.

n = number of unknown $= 3$

Therefore, $n - r = 3 - 2 = 1$.

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There is only one linearly independent non-zero solution.

Solving actually, by rule of cross multiplication, the equation

$x_1 + 2x_2 - x_3 = 0$

$3x_1 + x_2 - x_3 = 0$ we get,

$$\frac{x_1}{-2+1} = \frac{x_2}{-3+1} = \frac{x_3}{1-6}$$

$$\frac{x_1}{-1} = \frac{x_2}{-2} = \frac{x_3}{-5}\ \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{5}.$$

$$x_1 = 1, x_2 = 2, x_3 = 5$$

Solve the system of homogeneous equations

$x_1 + x_2 + 2x_3 = 0, 2x_1 - 3x_2 - x_3 = 0, -3x_1 + 2x_2 + 5x_3 = 0$

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The system is equivalent to

$$AX = \begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -3 & -1 \\ -3 & 2 & 5 \end{vmatrix}$$

$$= 1(-15+2) - 1(10-3) + 2(4-9)$$

$$= -30 \neq 0$$

Therefore the system has a trivial solution

$$x_1 = 0, x_2 = 0, x_3 = 0$$

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THE SYSTEM OF NON-HOMOGENOUS EQUATIONS

The system of non-homogenous equations is $AX = B$

$$\text{where } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ 0 & \cdot & . & 0 \\ . & . & . & . \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ . \\ \vdots \\ b_n \end{bmatrix}$$

The system $AX = B$ is said to be consistent if it has a solution. Otherwise it is inconsistence.

Roaches' theorem :

The system $AX = B$ is consistent if and only if $r(A, B) = r(A)$

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Note

- If $r(A, B) = r(A) =$ number of unknowns, then the system has unique solution.
- If $r(A, B) = r(A) <$ number of unknowns, then the system has an infinite number of solutions.
- If $\tau(A, B) \neq r(A)$, then the system has no solution.

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Show that the equations $+y+z=6, x-y+2z=5$ $3x+y+z=8,$ and, $2x-2y+3z=7$ are consistent and solve them.

Sol:

The system of the given equations is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \\ 2 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 8 \\ 7 \end{bmatrix}$$

$$[A \quad , \quad B] = \begin{pmatrix} 1 & 1 & 1 & 6 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 8 \\ 2 & -2 & 3 & 7 \end{pmatrix}$$

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$$\sim \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & -2 & -2 & -10 \\ 0 & -4 & 1 & -5 \end{pmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 2R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 1 & 3 \end{pmatrix} \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 2R_2 \end{matrix}$$

$$[A \quad , \quad B] = \begin{pmatrix} 1 & 1 & 1 & 6 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{pmatrix} R_4 \to 3R_4 - R_3$$

$$\text{Now } A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

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$r(A) =$ number of non-zero rows of A

$= 3$

$r(A, B) =$ number of non-zero rows of $[A, B]$

$= 3$

Since $r(A, B) = r(A) = 3 =$ number of unknowns, the system is consistent unique solution.

$3z = 9$

$\therefore z = 3$

$-2y + z = -1$

$-2y + 3 = -1$

$-2y = -4$

$\therefore y = 2$

$x + y + z = 6$

$x + 2 + 3 = 6$

$\therefore x = 1$

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Examine if the following system of equations is consistent and find the solution if it exists. The system of the given equations is $+y + z = 1, 2x -$

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$\mathbf{2y + 3z = 1, x - y + 2z = 5, \text{ and, } 3x + y + z = 2}$

Sol: The system of the given equations is

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -2 & 3 \\ 1 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 2 \end{bmatrix}$$

The augmented matrix is given by

$$[A, B] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & -2 & 3 & 1 \\ 1 & -1 & 2 & 5 \\ 3 & 1 & 1 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & -2 & 1 & 4 \\ 0 & -2 & -2 & -1 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

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$$\sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & -5 & -1 \end{pmatrix} \begin{matrix} R_3 \to 2R_3 - R_2 \\ R_4 \to 2R_4 - R_2 \end{matrix}$$

$$[A, B] \sim \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -4 & 1 & -1 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 44 \end{pmatrix} R_4 \to R_4 + 5R_3$$

$\sim (A) =$ number of non-zeru rows of $[A, B]$

$= 4$

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Now $A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$r(A)$ = number of non-zero rows of A

$= 3$

$r(A,B)$ = number of non-zero rows of $[A,B]$

$= 3$

Since $r(A,B) \neq r(A)$, the system is inconsistent and has no solution.

Solve the system of equations if consistent

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$$\boldsymbol{x_1 + 2x_2 - x_3 - 5x_4 = 4}$$

$$\boldsymbol{x_1 + 3x_2 - 2x_3 - 7x_4 = 5}$$

$$\boldsymbol{2x_1 - x_2 + 3x_3 = 3}$$

Sol: The system of the given equations is

$$\begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 1 & 3 & -2 & -7 & 5 \\ 2 & -1 & 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

The augmented matrix is given by

$$[A,B] = \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 1 & 3 & -2 & -7 & 5 \\ 2 & -1 & 3 & 0 & 3 \end{bmatrix}$$

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$$\sim \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & -5 & 5 & 10 & -5 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & -5 & 4 \\ 0 & 1 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 + 5R_2$$

$r(A, B)$ = number of non-zero rows of $[A, B]$

= 2

$$A \sim \begin{bmatrix} 1 & 2 & -1 & -5 & | & 4 \\ 0 & 1 & -1 & -2 & | & 1 \end{bmatrix}$$

$r(A)$ = number of non-zero rows of A

= 2

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$(A, B) = r(A) = 2 <$ number of unknowns $= 4$,

The system is consistent and has many solution.

To find the solutions

we have,

$$x_1 + 2x_2 - x_3 - 5x_4 = 4 \quad \ldots(1)$$

and

$$x_2 - x_3 - 2x_4 = 1 \quad \ldots\ldots(2)$$

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As there are 2 equations, we can solve for only two unknown. Hence other two variables are treated as parameters

Let $x_3 = k_1$, $x_4 = k_2$

$$(2) \Rightarrow x_2 - k_1 - 2k_2 = -1$$

$$x_2 = k_1 + 2k_2 + 1$$

$$(1) \Rightarrow x_1 + 2(k_1 + 2k_2 + 1) - k_1 - 5k_2 = 4$$

$$x_1 + 2k_1 + 4k_2 + \underline{2} - k_1 - 5k_2 = 4$$

$$x_1 + k_1 - k_2 = 2$$

$$x_1 = 2 - k_1 + k_2$$

∴ The given system possess a two parameters family of solution.

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LINEAR COMBINATION

Definition : Let V be a vector space over F and $v_1, v_2, \ldots . . v_n \in V$. Any vector of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$, is called a linear combination of the vectors $v_1, v_2, \ldots,$

If $w_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $w_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, what is the linear combination $w_1 y_1 + w_2 y_2$?

Sol:

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$$w_1y_1 + w_2y_2 = \begin{pmatrix}1\\0\\1\end{pmatrix} y_1 + \begin{pmatrix}1\\2\\0\end{pmatrix} y_2$$

$$= \begin{pmatrix}y_1\\0\\y_1\end{pmatrix} + \begin{pmatrix}y_2\\2y_2\\0\end{pmatrix}$$

$$= \begin{pmatrix}y_1 + y_2\\2y_2\\y_1\end{pmatrix}$$

In R^3, determine whether $(5, 1, -5)$ is expressed as a line combination of $(1, -2, -3)$ and $(-2, 3, -4)$.

Sol: Given $v = (5,1,-5), v_1 = (1,-2,-3)$ and $v_2 = (-2,3,-4)$

The linear combination of v_1 and v_2 is

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$$v = a_1v_1 + \alpha_2v_2$$

$$(5,1,-5) = a_1(1,-2,-3) + \alpha_2(-2,3,-4) \ldots (1)$$

$$= (\alpha_1, -2\alpha_1, -3\alpha_1) + (-2\alpha_2, 3\alpha_2, -4\alpha_2)$$

$$= (\alpha_1 - 2\alpha_2, -2\alpha_1 + 3\alpha_2, -3\alpha_1 - 4\alpha_2)$$

From the equivalent system of equations by setting corresponding components equal to each other and then reduce to echelon form

$$a_1 - 2\alpha_2 = 5 \ldots (2)$$

$$-2\alpha_1 + 3\alpha_2 = 1 \ldots (3)$$

$$-3\alpha_1 - 4a_2 = -5 \ldots .(4)$$

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sol ve(2) and (3)

$$(1) \times 2 \Rightarrow 2\alpha_1 - 4\alpha_2 = 10$$

$$(3) \Rightarrow -2\alpha_1 + 3\alpha_2 = 1$$

$$\alpha_2 = -11$$

$(3) \Rightarrow \alpha_1 - 2(-11) = 5$

$$a_1 = -17$$

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Substitute the values in (1), we get

$$(5,1,-5) = -17(1,-2,-3) - 11(-2,3,-4)$$

$$(5,1,-5) = (5,1,95), \text{ which is false}$$

$\therefore v$ is not a linear combination of v_1 and v_2

In R^3, determine whether $(1, 7, -4)$ is expressed as a linear ;ombination of $u = (1, -3, 2)$ and $v = (2, -1, 1)$ in R^3.

Sol: We wish to write

$(1,7,-4) = \alpha_1 u + \alpha_2 v$

$$= \alpha_1(1,-3,2) + \alpha_2(2,-1,1) \dots (1)$$

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$$= (\alpha_1 + 2\alpha_2, -3\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2)$$

From the equivalent system of equations by setting corresponding component equal to each other, and then reduce to echelon form

$$\alpha_1 + 2\alpha_2 = 1 \ldots (2)$$

$$-3\alpha_1 - \alpha_2 = 7 \ldots (3)$$

$$2\alpha_1 + \alpha_2 = -4 \ldots \ldots (4)$$

Verify $2x^3 - 2x^2 + 12x - 6$ is a linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$ in $P_3(R)$.

Sol: $P(x) = 2x^3 - 2x^2 + 12x - 6, Q(x) = x^3 - 2x^2 - 5x - 3$

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and $R(x) = 3x^3 - 5x^2 - 4x - 9$

We wish to write $P(x) = \alpha_1 Q(x) + a_2 R(x)$, with α_1 and α_2 as unknown scalars. Thus

$$2x^3 - 2x^2 + 12x - 6$$

$$= a_1(x^3 - 2x^2 - 5x - 3) + a_2(3x^3 - 5x^2 - 4x - 9) \ldots (1)$$

$$2x^3 - 2x^2 + 12x - 6$$

$$= (\alpha_1 + 3a_2)x^3 + (-2\alpha_1 - 5\alpha_2)x^2 + (-5\alpha_1 - 4\alpha_2)x + (-3a_1 - 9\alpha_2)$$

Equating the co-efficient on both sides, we get

$$a_1 + 3a_2 = 2 \ldots (2)$$

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$$-2\alpha_1 - 5\alpha_2 = -2 \ldots (3)$$

$$-5\alpha_1 - 4\alpha_2 = 12 \ldots (4)$$

$$-3\alpha_1 - 9\alpha_2 = -6 \ldots (5)$$

Solve (2) and (3)

$$(2) \times 2 \Rightarrow 2\alpha_1 + 6\alpha_2 = 4$$

Adding $(3) \Rightarrow \underline{-2\alpha_1 - 5\alpha_2 = -2}$

$$\alpha_2 = 2$$

From (2), we get $\alpha_1 + 3(2) = 2$

$$\alpha_1 = 2 - 6$$

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$$\therefore \alpha_1 = -4.$$

From (4), $-5\alpha_1 - 4\alpha_2 = 12$

$$-5(-4) - 4(2) = 12$$

20-8=12

12=12

(4) holds good.

From (5), $-3\alpha_1 - 9\alpha_2 = -6$

$$-3(-4) - 9(2) = -6$$

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12-18=-6

(5) holds good.

∴ P(x) is a linear combination of Q(x) and R(x).

Seample (44) Verify $3x^3 - 2x^2 + 7x + 8$ is a linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$ in $P_3(R)$

Sol: $P(x) = 3x^3 - 2x^2 + 7x + 8, Q(x) = x^3 - 2x^2 - 5x - 3$

and $R(x) = 3x^3 - 5x^2 - 4x - 9$

We wish to write $P(x) = \alpha_1 Q(x) + \alpha_2 R(x)$, with α_1 and α_2 as unknown scalars. Thus

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$$3x^3 - 2x^2 + 7x + 8$$

$$= \alpha_1(x^3 - 2x^2 - 5x - 3) + \alpha_2(3x^3 - 5x^2 - 4x - 9) \dots (1)$$

$$3x^3 - 2x^2 + 7x + 8$$

$$= (\alpha_1 + 3\alpha_2)x^3 + (-2\alpha_1 - 5\alpha_2)x^2 + (-5\alpha_1 - 4\alpha_2)x + (-3\alpha_1 - 9\alpha_2)$$

Equating the co-efficient on both sides, we get

$$\alpha_1 + 3\alpha_2 = 3 \dots (2)$$

$$-2\alpha_1 - 5\alpha_2 = -2 \dots (3)$$

$$-5\alpha_1 - 4\alpha_2 = 7 \dots (4)$$

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$$-3\alpha_1 - 9\alpha_2 = 8 \ldots (5)$$

Solve (2) and (3)

$$(2) \times 2 \Rightarrow 2\alpha_1 + 6\alpha_2 = 6$$

$$(3) \Rightarrow -2\alpha_1 - 5\alpha_2 = -2$$

Adding $\alpha_2 = 4$

From (2), we get $\alpha_1 + 3(4) = 3$

$$\alpha_1 = 3 - 12$$

$$\therefore \alpha_1 = -9$$

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From (4), $-5\alpha_1 - 4\alpha_2 = 7$

$$(-9) - 4(4) = 7$$

$$45 - 16 = 7$$

$$29 = 7$$

(4) does not holds good.

$\therefore$ P(x) cannot be written as a linear combination of Q(x) and R(x).

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LINEAR SPAN

Definition:

Let V be a vector space over F and S be a non-empty subset of V. Then the set of all linear combination of the finite subset of S is called the linear span of set of and is denoted by $L(S)$.

i.e., $L(S) = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n / \alpha_i \in F, v_i \in S\}$

Note:

- $L(S) \subseteq V$
- If $S = \emptyset$, then $L(S) = 0$.

Definition:

A subset S of a vector space V generates (or span) V, if $L(S) = V$

Theorem 1.13: Let S be a nonempty subset of a vector space $V(F)$.

i) $L(S)$ is a subspace of V and $S \subseteq L(S)$

ii) if W is a subspace of V such that $S \subseteq W$, then $L(S) \subseteq W$

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Proof:

i) Let S be a nonempty subset of a vector space $V(F)$.

Let $u, v \in L(S)$ and $\alpha, \beta \in F$.

Then $u = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m$ and $v = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$

where $\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_n \in F$ and

$u_1, u_2, \dots, u_m, v_1, v_2 \dots, v_n \in S$ and also m and n are finite.

$\text{u} + \beta v = \alpha(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m) + \beta(\beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n)$

$\alpha\alpha_1 u_1 + \alpha\alpha_2 u_2 + \cdots + \alpha\alpha_m u_m + \beta\beta_1 v_1 + \beta\beta_2 v_2 + \cdots + \beta\beta_n v_n$ ... (1)

assume$\alpha\alpha_i = \gamma_i; \beta\beta_i = \gamma_{m+i}$ and $v_i = u_{m+i}$ in (1), we get

$u + \beta v$

$\gamma_1 u_1 + \gamma_2 u_2 + \cdots + \gamma_m u_m + \gamma_{m+1} u_{m+1} + \gamma_{m+1} u_{m+1} + \cdots + \gamma_{m+n} u_{m+n}$

$\in L(S)$

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$u + \beta v \in L(S)$

hence $L(S)$ is a subspace of V.

Let W be a subspace of V such that $S \subseteq W$

have to prove $L(S) \subseteq W$

v $\mathcal{E} L(S)$. Then $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ where the $\alpha_i \in F$ and $v_i \in S$

Since $S \subseteq W, v_1, v_2, \ldots, v_n \in W$

Since W is a subspace of V, m

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in W$

$\Rightarrow v \in W$

$v \in L(S) \Rightarrow v \in W$

$\therefore L(S) \subseteq W$

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Theorem 1.14: Let V be a vector space over a field $\boldsymbol{F}$.

Let $S, T \subseteq V$. Then

(a) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

(b) $L(S \cup T) = L(S) + L(T)$

(C) $L(S) = S$ if and only if S is a subspace of V.

Proof:

(a) Let $S \subseteq T$ and $v \in L(S)$,

Then $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ where $v_i \in S$ and $\alpha_i \in F$.

Now, since $S \subseteq T, v_1, v_2, \ldots, v_n \in T$

$\therefore \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in L(T)$

$\therefore v \in L(T)$

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$v \in L(S) \Rightarrow v \in L(T)$

$\Rightarrow L(S) \subseteq L(T)$

(ii) Let $v \in L(S \cup T)$

Then $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \ldots + \alpha_n v_n$ where $v_1, v_2, \ldots, v_n \in S \cup T$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$. Without loss of generality, we shall assume that $v_1, v_2, \ldots, v_m \in S$ and $v_{m+1}, v_{m+2}, \ldots v_n \in T$

Hence

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m \in L(S)$ and $\alpha_{m+1} v_{m+1} + \alpha_{m+2} v_{m+2} + \cdots + \alpha_n v_n \in L(T)$.

$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \ldots + \alpha_n v_n$

$= \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m + \alpha_{m+1} v_{m+1} + \alpha_{m+2} v_{m+2} + \cdots + \alpha_n v_n$

$v \in L(S) + L(T)$

$v \in L(S \cup T) \Rightarrow v \in L(S) + L(T)$

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$\therefore L(S \cup T) \subseteq L(S) + L(T) \ldots (1)$

Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, we have $L(S) \subseteq L(S \cup T)$ and $L(T) \subseteq L(S \cup T)$.

$\therefore$ their linear sum $L(S) + L(T) \subseteq L(S \cup T) \ldots$ (2)

From (1) and (2),

$$L(S \cup T) = L(S) + L(T)$$

(C) Let $L(S) = S$.

Since $L(S)$ is a subspace of V. we get S is a subspace $V(F)$.

Conversely let S is a subspace $V(F)$.

We know that $S \subseteq L(S) \ldots$ (3).

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Let $v \in L(S)$. Then $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots .. + \alpha_n v_n$ where $v_1, v_2, \dots, v_n \in S$ and

$\alpha_1, \alpha_2, \dots, \alpha_n \in F$

Since S is a subspace of $V, \alpha_1 v_1 + \alpha_2 v_2 + \cdots, + \alpha_n v_n \in S$

i.e., $v \in S$

$v \in L(S) \Rightarrow v \in S$

$\therefore L(S) \subseteq S$ ... (4)

From (3) and (4), we get

Hence $L(S) = S$.

Corollary 1.15: $L[L(S)] = L(S)$

Proof: If S is a subspace of V, then $L(S) = S$ ... (1)

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Since $L(S)$ is a subspace of V, then $L[L(S)] = L(S) = S$[From (1)]

$\therefore L[L(S)] = L(S)$

Example 46. Let $S = \{(1,2), (2,1)\}$; $V = R^2$. Prove that V is a linear span of S.

Sol: We know that $L(S) \subseteq V$… (1)

Let us consider $(x, y) \in V$

$$(x, y) = \alpha_1(1,2) + \alpha_2(2,1) \dots (2)$$

$$= (\alpha_1, 2\alpha_1) + (2\alpha_2, \alpha_2)$$

$$(x, y) = (\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2)$$

$\alpha_1 + 2\alpha_2 = x$ ... (3)

$2\alpha_1 + \alpha_2 = y$ ... (4)

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(3) × 2 => $2a_1 + 4a_2 = 2x$

(4) => $2a_1 + a_2 = y$

$$3a_2 = 2x - y$$

$$a_2 = \frac{2x-y}{3}$$

From equation (4)

$$2a_1 = y - a_2$$

$$2a_1 = y - (\frac{2x-y}{3})$$

$$= \frac{3y-2x+y}{3}$$

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$$2a_1 = \frac{4y-2x}{3}$$

$$a_1 = \frac{2y-x}{3}$$

Sabtitute the values of a_4 and a_2 in (2), we get

$$x(x,y) = (\frac{2y-x}{3})(1,2) + (\frac{2x-y}{3})(2,1)$$

Hence (x, y) is a lincar combination of S

$$s(x,y) \in L(S)$$

We have $(x,y) \in V \Rightarrow (x,y) \in L(S)$

$$\therefore V \subset L(S) - (5)$$

From (1) and (5), we get

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$$L(S) = V$$

Therefore S generates V.

Example 47. Prove that in $V_2(R)$, $(3,7)$ belongs to the linear space ((1,2), (0,1))

sol: Let $S = ((1,2), (0,1))$

$$v_1 = (1,2), v_2 = (0,1)$$

Lat $v = (x, y) \in L(S)$

$v = a_1 v_1 + a_z v_2$

$(x, y) = \alpha_1(1,2) + a_2(0,1) \dots (1)$

$$=(a_1, 2a_2 + a_2)$$

$$a_1 = x$$

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$$2a_1 + a_2 = y$$

$$2x + a_2 = y$$

$$a_2 = y - 2x$$

(1)⇒ $(x, y) = x(1,2) + (y - 2x)(0,1)$

we check(3,7) $\in L(S)$

Here $x = 3, y = 7$

(1) ⇒ $(3,7) = 3(1,2) + (7 - 6)(0,1)$

$$= (3,6) + (0,1)$$

$$= (3,7)$$

which is true.

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$(3,7) \in L(S$am

Example48. Prove that the vectors $(1,1,0), (1,0,1), (0,1,1)$ generates R^3.

Sol:Let $S = \{(1,1,0), (1,0,1), (0,1,1)\}$

We know that $L(S) \subseteq R^3 \dots (1)$

Let v $\in R^3$. Then $v = (a, b, c)$

Let $v = \alpha_1(1,1,0) + \alpha_2(1,0,1) + \alpha_3(0,1,1)$

$(a, b, c) = (\alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3)$

$\alpha_1 + \alpha_2 = a \dots (1)$

$\alpha_1 + \alpha_3 = b \dots (2)$

$\alpha_2 + \alpha_3 = c \dots (3)$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

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$$[A, B] = \begin{bmatrix} 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & 1 & c \end{bmatrix}$$

$$[A, B] \begin{bmatrix} 1 & 1 & 0 & a \\ 0 & -1 & 1 \mid & b - a \\ 0 & 1 & 1 & c \end{bmatrix} \; R_2 \to R_2 - R_1$$

$$\begin{bmatrix} 1 & 1 & 0 & \\ 0 & -1 & 1 & b - a \\ 0 & 0 & 2 & b - a + c \end{bmatrix} \; R_3 \to R_3 + R_2$$

$$2\alpha_3 = b - a + c$$

$$\alpha_3 = \frac{1}{2}(b - a + c)$$

$$-\alpha_2 + \alpha_3 = b - a$$

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$$-\alpha_2 = b - a - \alpha_3$$

$$= b - a - \frac{1}{2}(b - a + c)$$

$$= \frac{1}{2}(2b - 2a - b + a - c)$$

$$= \frac{1}{2}(b - a + c)$$

$$\alpha_2 = \frac{1}{2}(a - b + c)$$

$$\alpha_1 + \alpha_2 = a$$

$$\alpha_1 = a - \alpha_2$$

$$\alpha_1 = a - \frac{1}{2}(a - b + c)$$

$$= \frac{1}{2}(2a - a + b - c)$$

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$$= \frac{1}{2}(a + b - c)$$

Substitute the values of $\alpha_1, \alpha_2, \alpha_3$ in (1), we get

$$v = \frac{1}{2}(a + b - c)(1,1,0) + \frac{1}{2}(a - b + c)(1,0,1) + \frac{1}{2}(b - a + c)(0,1,1)$$

$$\therefore v \in L(S)$$

$$\therefore R^3 \subseteq L(S) \ldots (5)$$

From (1) and (5), we get

$$L(S) = R^3$$

Therefore S generates R^3.

Example 49. Prove that the polynomials $x^2 + 3x - 2, 2x^2 + 5x - 3$ and $-x^2 -$

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$4x + 4$ generates $P_2(R)$

Let $p(x) = x^2 + 3x - 2, q(x) = 2x^2 + 5x - 3$ and $r(x) = -x^2 - 4x + 4$

Let $S = \{p(x), q(x), r(x)\}$. Then

$$L(S) \subseteq P_2(R) \ldots (1)$$

Let $t(x) \in P_2(R)$. Then

$$t(x) = ax^2 + bx + c; a, b, c \in R$$

Let $t(x) = \alpha_1 p(x) + \alpha_2 q(x) + \alpha_3 r(x)$

$$= \alpha_1(x^3 + 3x - 2) + \alpha_2(2x^2 + 5x - 3) + \alpha_3(-x^2 - 4x + 4) \ldots.(1)$$

$ax^2 + bx + c$

$$= (\alpha_1 + 2\alpha_2 - \alpha_3)x^2 + (3\alpha_1 + 5\alpha_2 - 4\alpha_3)x + (-2\alpha_1 - 3\alpha_2 + 4\alpha_3)$$

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$$\alpha_1 + 2\alpha_2 - \alpha_3 = a \ldots (2)$$

$$3\alpha_1 + 5\alpha_2 - 4\alpha_3 = b \ldots (3)$$

$$-2\alpha_1 - 3\alpha_2 + 4\alpha_3 = c \ldots (4)$$

$$(A, B) \sim \begin{pmatrix} 1 & 2 & -1 & a \\ 3 & 5 & -4 & b| \\ -2 & -3 & & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & -1 & a \\ 0 & -1 & -1 & b - 3a \\ 0 & 1 & 2 & c + 2a \end{pmatrix} R_2 \to R_2 - 3R_1, R_3 \to R_3 + 2R_1$$

$$\sim \begin{pmatrix} 1 & 2 & & -1 \\ 0 & -1 & & -1 \\ 0 & 0 & 1 & b - 3a \\ & & & c + b - a \end{pmatrix} R_3 \to R_3 + R_2$$

$$\alpha_3 = c + b - a$$

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$-\alpha_2 - \alpha_3 = b - 3a$

$-\alpha_2 = b - 3a + \alpha_3$

$= b - 3a + c + b - a$

$= 2b - 4a + c$

$\alpha_2 = 4a - 2b - c$

$\alpha_1 + 2\alpha_2 - \alpha_3 = a$

$\alpha_1 = a - 2\alpha_2 + \alpha_3$

$= a - 2(4a - 2b - c) + (c + b - a)$

$= -8a + 5b + 3c$

Substitute the values of $\alpha_1, \alpha_2, \alpha_3$ in (1), we get

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$t(x) = (-8a + 5b + 3c)(x^3 + 3x - 2) + (4a - 2b - c)(2x^2 + 5x - 3)$

$+(c + b - a)(-x^2 - 4x + 4) \in L(S)$

$\therefore P_2(R) \subseteq L(S) \ldots (5)$

From (1) and (5), we get

$L(S) = P_2(R)$

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1.6.1. PROBLEMS UNDER BASIS

Let V be a vector space with $\dim(V) = n$. Then any basis of V contains n elements.

Let β be a set with cardinality(number of elements) $|\beta|$.

- If $|\beta| < n$ or $|\beta| > n$, then S does not form a basis of V.
- If β is a linearly independent set in V with $|\beta| = n$, then β forms a basis in V.

Example. Determine whether $(1,1,1), (1,0,1)$ forms a basis of R^3

Sol: Since $\dim(R^3) = 3$, any basis of R^3 contains three elements. Let $\beta = \{(1,1,1), (1,0,1)\}$. Since β contains two elements, β does not form a basis of R^3.

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Example 80. Show that the sets of vectors

$\{(1,2,1), (3,1,5), (-1,0,1), (1,-1,2)\}$ do not form a basis for $V_3(R)$.

Sol: Since $\dim(V_3(R)) = 3$, any basis of $V_3(R)$ contains three elements.

Let $\beta = \{(1,2,1), (3,1,5), (-1,0,1), (1,-1,2)\}$. Since β contains four elements, does not form a basis of $V_3(R)$.

Example Verify the vectors $(1,-1,2), (1,-2,1), (1,1,4)$ in R° forms a basis of R^3.

Sol: Let $\beta = \{(1,-1,2), (1,-2,1), (1,1,4)\}$

$\dim(R^3) = 3$, which is finite.

In R^3, any independent set with three elements is a basis of R^3.

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

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$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & 4 \end{vmatrix}$$

$= 1(-8-1) + 1(4-1) + 2(1+2) = 0$

$\therefore \beta$ is a linearly dependent set in R^3.

$\therefore \beta$ does not form a basis of R^3.

Example. Verify the vectors $(1,2,0), (2,3,0), (8,13,0)$ of R^3 is a basis of $\boldsymbol{R}^3$

Sol: Let $\beta = \{(1,2,0), (2,3,0), (8,13,0)\}$

$\dim(R^3) = 3$, which is finite.

In R^3, any independent set with three elements is a basis of $\mathbb{R}^3$.

Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 8 & 13 & 0 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 8 & 13 & 0 \end{vmatrix} = 0$$

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$\therefore \beta$ is a linearly dependent set in R^3.

$\therefore \beta$ is not a basis of R^3

Example Verify the vectors $(2,1,0), (-3,-3,1), (-2,1,-1)$ in R^3 basis of R^3

Sol: Let $\beta = \{(2,1,0), (-3,-3,1), (-2,1,-1)\}$.

$\dim(R^3) = 3$, which is finite.

In R^3, any independent set with three elements is a basis of R^3.

Let $A = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -3 & 1 \\ -2 & 1 & -1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ -3 & -3 & 1 \\ -2 & 1 & -1 \end{vmatrix} = -1 \neq 0$$

$\therefore \beta$ is a linearly independent set in R^3.

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$\therefore \beta$ is a basis of R^3.

Example. Check whether the following are basis for the space R^3

(a) $\{(1,1,-1), (2,3,4), (4,1,-1), (0,1,-1)\}$

(b) $\{(1,1,-1), (0,3,4), (0,0,-1)\}$

(C) $\{(1,2,0), (0,1,-1)\}$

Sol:

$\dim(R^3) = 3$, which is finite.

In R^3, any independent set with three elements is a basis for R^3.

(a) $\beta = \{(1,1,-1), (2,3,4), (4,1,-1), (0,1,-1)\}$

Since β is contains four elements, it is not a basis for R^3.

(b) $\beta = \{(1,1,-1), (0,3,4), (0,0,-1)\}$

The set contains three elements

Let $v_1 = (1,1,-1), v_2 = (0,3,4), v_3 = (0,0,-1)$

To prove S is a basis we have to prove S is a linearly independent.

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Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{bmatrix}$

$|A| = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{bmatrix} = -3 \neq 0$

$\therefore \beta$ is linearly independent in R^3

$\Rightarrow \beta$ is a basis in R^3

(C) $\beta = \{(1,2,0), (0,1,-1)\}$

Since the set contains two elements, it does not form a basis in R^3.

Example 85. Determine $\{1 + 2x + x^2, 3 + x^2, x + x^2\}$ is a basis for $P_2(R)$. Sol:

$\dim P_2(R) = 3$, which is finite. In $P_2(R)$, any independent set with three elements is a basis.

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Given $v_1 = 1 + 2x + x^2, v_2 = 3 + x^2, v_3 = x + x^2$

The vector equation is

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\alpha_1(1 + 2x + x^2) + \alpha_2(3 + x^2) + \alpha_3(x + x^2) = 0$$

$$(\alpha_1 + 3\alpha_2) + (2\alpha_1 + \alpha_3)x + (\alpha_1 + \alpha_2 + \alpha_3)x^2 = 0$$

Equating the like terms, we get

$$\alpha_1 + 3\alpha_2 = 0$$

$$2\alpha_1 + \alpha_3 = 0$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 0$$

Let A be the coefficients matrix,

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$$\therefore A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A| = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = -4 \neq 0$$

the system of homogenous equations have only the trivial solution

$\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$

$\therefore v_1, v_2, v_3$ are linearly independent

Hence v_1, v_2, v_3 is a basis of $P_2(R)$

Therefore $\{1 + 2x + x^2, 3 + x^2, x + x^2\}$ is a basis over R.

Example 86. Let $V = P_2(R)$ and. $\beta = \{1, 1 + x, 1 + x + x^2\}$. Check whether S forms a basis in V.

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Sol: $\dim P_2(R) = 3$, which is finite.

In $P_2(R)$, any independent set with three elements is a basis.

Given $v_1 = 1, v_2 = 1 + x, v_3 = 1 + x + x^2$

The vector equation is

$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

$\alpha_1(1) + \alpha_2(1 + x) + \alpha_3(1 + x + x^2) = 0$

$\alpha_3 + \alpha_1 + \alpha_2 + \alpha_2 x + \alpha_3 x + \alpha_3 x^2 = 0x^2 + 0x + 0$

$(\alpha_3 + \alpha_1 + \alpha_2) + (\alpha_2 + \alpha_3)x + \alpha_3 x^2 = 0x^2 + 0x + 0$

Equating the like terms, we get

$\alpha_3 + \alpha_1 + \alpha_2 = 0 \ldots (1)$

$\alpha_2 + \alpha_3 = 0 \ldots (2)$

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$\alpha_3 = 0$

$(2) \Rightarrow \alpha_2 = 0$

$(1) \Rightarrow \alpha_1 = 0$

$\therefore \beta$ is linearly independent set in $P_2(R)$,

Therefore β is a basis in $P_2(R)$,

Example 87. If the vectors $\{u, v, w\}$ form a basis for R^3, show that the vectors $\{u, u - w, u + v - 2w\}$ also forms a basis for R^3.

Sol: $\dim(R^3) = 3$, which is finite.

In R^3, any independent set with three elements is a basis for R^3.

Let $\beta = \{u, v, w\}$ and $\beta_1 = \{uu - w, u + v - 2w\}$

Given β forms a basic for R^3.

$\therefore \beta$ is a linearly independent set in R^3.

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In a finite dimensional vector space, any two bases has same number of elements.

Also in a finite dimensional vector space, any independent set with number elements $\dim(V)$ is a basis.

To prove β_1 is a basis for R^3, it is enough to prove β_1 is a linearly independent set. The vector equation is

$$\alpha_1 u + \alpha_2(u - w) + \alpha_3(u + v - 2w) = 0$$

$$\alpha_1 u + \alpha_2 u - \alpha_2 w + \alpha_3 u + \alpha_3 v - 2\alpha_3 w = 0$$

$$(\alpha_1 + \alpha_2 + \alpha_3)u + \alpha_3 v + (-\alpha_2 - 2\alpha_3)w = 0$$

Since u, v and w are linearly independent,

$\alpha_1 + \alpha_2 + \alpha_3 = 0$ … … … … (1)

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$\alpha_3 = 0$

$\alpha_2 - 2\alpha_3 = 0$ … … … … (2)

$(2) \Rightarrow -\alpha_2 - 2(0) = 0$

$\alpha_2 = 0$

$(1) \Rightarrow \alpha_1 = 0$

$\therefore \alpha_1 u + \alpha_2(u - w) + \alpha_3(u + v - 2w) = 0 \Rightarrow \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$

$\therefore \beta_1$ is a linearly independent set.

Hence β_1 is a basis of R^3.

$$= \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}$$

Equating the like terms, we get

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$\alpha_1 = 2$

$\alpha_2 = 3$

$\alpha_3 = 4$

$\alpha_4 = -7$

The coordinate of A relative to the usual basis is $(2,3,4,-7)$.

1.6.2. PROBLEMS UNDER BASIS AND DIMENSION OF A SUBSPACE

Let W be a subspace of a vector space V over F. To find the basis dimension of :

- From W, find linear span of W. Let it be β.
- Check β is linearly independent or not.
- If β is linearly independent set, then β forms a basis in W.
- $\dim(W) = |\beta|$

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Example 91. Find the dimension of the subspace W of the vector space R^3 over R if $W = \{(a,0,0)/a \in R\}$

Sol: Let $v \in W$. Then

$v = (a,0,0) = a(1,0,0)$

$\therefore \beta = \{(1,0,0)\}$ spans $\underline{W}$.

Any set with one element is linearly independent

$\therefore B$ is a linearly independent set in W.

$\therefore B = \{(1,0,0)\}$ is a basis of W.

$\therefore \dim(W) = 1$

Example 92. Find the dimension of the subspace W of the vector space R^3 over

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R, if $W = \{(a_1, a_2, a_3)/(2a_1 - 7a_2 + a_3 = 0)\}$

Sol: $W = \{(a_1, a_2, a_3)/(2a_1 - 7a_2 + a_3 = 0)\}$

Given $2a_1 - 7a_2 + a_3 = 0$

$$\Rightarrow a_3 = -2a_1 + 7a_2$$

Let $v \in W$. Then

$v = (a_1, a_2, a_3)$

$(a_1, a_2, a_3) = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$

$= a_1(1,0,0) + a_2(0,1,0) + (-2a_1 + 7a_2)(0,0,1)$

$= a_1(1,0,0) + a_2(0,1,0) - 2a_1(0,0,1) + 7a_2(0,0,1)$

$= (a_1, 0,0) + (0, a_2, 0) + (0,0,-2a_1) + (0,0,7a_2)$

$= (a_1, 0, -2a_1) + (0, a_2, 7a_2)$

$= a_1(1,0,-2) + a_2(0,1,7)$

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$\therefore \beta = \{(1,0,-2), (0,1,7)\}$ spans W i.e., $L(\beta) = W$

Next we prove that B is a linearly independent set in W.

Consider the vector equation

$a_1 v_1 + a_2 v_2 = 0$

$a_1(1,0,-2) + a_2(0,1,7) = 0$

$(a_1, a_2, -2a_1 + 7a_2) = 0$

$\Rightarrow a_1 = a_2 = 0$

$\therefore \beta$ is a linearly independent set in W.

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$\therefore \beta = \{(1,0,-2),(0,1,7)\}$ is a basis of W

Since the basis contains two elements, $\dim(W) = 2$

Example 93. Find the dimension of the subspace W of the vector space F^5 over F, if $W = \{(a_1, a_2, a_3, a_4, a_5)/a_1 - a_3 + a_4 = 0\}$

Sol: $W = \{(a_1, a_2, a_3, a_4, a_5)/a_1 - a_3 + a_4 = 0\}$

Given $a_1 - a_3 + a_4 = 0$

$$\Rightarrow a_4 = a_3 - a_1$$

Let $v \in W$. Then

$v = (a_1, a_2, a_3, a_4, a_5)$

$(a_1, a_2, a_3, a_4, a_5)$

$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_3(0,0,1,0,0) + a_4(0,0,0,1,0)$

$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_3(0,0,1,0,0) + (a_3 - a_1)(0,0,0,1,0) + a_5(0,0,0,0,1)$

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$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_3(0,0,1,0,0) + a_3(0,0,0,1,0) - a_1(0,0,0,1,0)$

$+a_5(0,0,0,0,1)$

$= a_1(1,0,0,-1,0) + a_2(0,1,0,0,0) + a_3(0,0,1,1,0) + a_5(0,0,0,0,1)$

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$\therefore \beta = a_1(1,0,0,-1,0), (0,0,-1,0), (0,1,0,0,0), (0,0,1,1,0), (0,0,0,0,1)\}$ spans W

i.e., $L(\beta) = W$

Next we prove that β is a linearly independent set in W.

Consider the vector equation

$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$

$a_1(1,0,0,-1,0) + a_2(0,1,0,0,0) + a_3(0,0,1,1,0) + a_4(0,0,0,0,1) = 0$

$(a_1, a_2, a_3, -a_1 + a_3, a_4) = 0$

$\Rightarrow a_1 = a_2 = a_3 = a_4 = 0$

$\therefore \beta$ is a linearly independent set in W.

$\therefore \beta = \{(1,0,0,-1,0), (0,1,0,0,0), (0,0,1,1,0), (0,0,0,0,1)\}$ is a basis of W.

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Since the basis contains four elements, $\dim(W) = 4$.

Example 94. Find the dimension of the subspace W of the vector space F^5

over$R$, if $W = \{(a_1, a_2, a_3, a_4, a_5)/a_2 = a_3 = a_4, a_1 + a_5 = 0\}$

Sol: $W = \{(a_1, a_2, a_3, a_4, a_5)/a_2 = a_3 = a_4 = 0, a_1 + a_5 = 0\}$

Given $a_1 + a_5 = 0$

$\Rightarrow a_5 = -a_1$

Also given $a_2 = a_3 = a_4$

$\therefore a_3 = a_2$ and $a_4 = a_2$

Let $v \in W$. Then

$v = (a_1, a_2, a_3, a_4, a_5)$

$(a_1, a_2, a_3, a_4, a_5)$

$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_3(0,0,1,0,0) + a_3(0,0,0,1,0) + a_5(0,0,0,0,1)$

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$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_2(0,0,1,0,0) + a_2(0,0,0,1,0) - a_1(0,0,0,0,1) = a_1(1,0,0,0,-1) + a_2(0,1,1,1,0)$

$\beta = \{(1,0,0,0,-1), (0,1,1,1,0)\}$ spans W

i.e., $L(\beta) = W$

Next we prove that β is a linearly independent set in W.

Consider the vector equation

$$a_1v_1 + a_2v_2 = 0$$

$$a_1(1,0,0,0,-1) + a_2(0,1,1,1,0) = 0$$

$$(a_1, a_2, a_2, a_2, -a_1) = 0$$

$$\Rightarrow a_1 = a_2 = 0$$

$\therefore$ β is a linearly independent set in W.

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$\therefore$ $\beta = \{(1,0,0,0,-1), (0,1,1,1,0)\}$ is a basis of W. Since the basis contains two elements, $\dim(W) = 2$

Example 95. Find the dimension of the subspace W of the vector space R^3 over R, if $W = \{(a,b,c): 2a + 3b = c; 7c + 9b = a\}$

Sol:

$$W = \{(a,b,c): 2a + 3b = c; 7c + 9b = a\}$$

Given

$$2a + 3b = c$$

$$2a + 3b - c = 0$$

Also given

$$7c + 9b = a$$

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$a - 9b - 7c = 0 \ldots (2)$

Solve (1) and (2)

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -9 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

Let $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -9 & -7 \end{bmatrix}$

$$\sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & -21 & -12 \end{bmatrix} R_2 \to R_2 - R_1$$

$$\begin{vmatrix} 2 & 3 \\ 0 & -21 \end{vmatrix} = -42 \neq 0$$

$R(A) = 2 <$ the number of unknowns $= 3$

Therefore the system has an infinite number of solutions.

From the last row, we get

$-21b - 12c = 0$

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$-21b = 12c$

$b = -\frac{4}{7}c$

Let $c = k$

$\therefore b = -\frac{4}{7}k$

From the first equation, we get

$2a + 3b - c = 0$

$2a - \frac{12}{7}k - k = 0$

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$2a = \frac{19}{7}k$

$a = \frac{19}{14}k$

where k is a parameter

$W = \{(\frac{19}{14}k, -\frac{4}{7}k, k)\} : k \in R\}$

$= \{(\frac{19}{14}, -\frac{4}{7}, 1)\ k\} : k \in R\}$

$\therefore \beta = \{(\frac{19}{14}, -\frac{4}{7}, 1)\}$ spans W.

i.e., $L(\beta) = W$

Any set with one non vector is linearly independent

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$\therefore \beta$ is a linearly independent set in W. $\therefore \beta = \{(\frac{19}{14}, -\frac{4}{7}, 1)\}$ is a basis of W.

Since the basis contains one element, $\dim(W) = 1$

Example(96) Find the dimension of the subspace W of the vector space $M_{2\times2}(R)$ over R, if $W = \{\begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} : \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d}\} = \boldsymbol{0}\}$

Sol: $W = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b + c + d\}$

Given

$a + b + c + d = 0$

$d = -a - b - c \ldots (1)$

Let $v \in W$. Then

$v = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

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$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

[ ] [[ ] [0
0 0 0 0 1 0 0 1

[ ] [[ ] [ ] [ ] [0
0 0 0 0 1 0 0 1 0 1 0 1

$$= a\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

$\therefore \beta = \left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\right\}$ spans W.

i.e., $L(\beta) = W$

Next we prove that β is a linearly independent set in W.

Consider the vector equation

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$$a_1v_1 + a_2v_2 + a_3v_3 = 0$$

$$a_1\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + a_2\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + a_3\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = 0$$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 \quad -a_1 - a_2 & -a_3 \end{bmatrix} = 0$$

$$\Rightarrow a_1 = a_2 = a_3 = 0$$

$\therefore \beta$ is a a linearly independent set in W.

$\therefore \beta = \left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\right\}$ is a basis of W.

Since the basis contains three elements, $\dim(W) = 3$

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