Hasse Diagram:

Pictorial representation of a Poset is called Hasse Diagram.

Example:

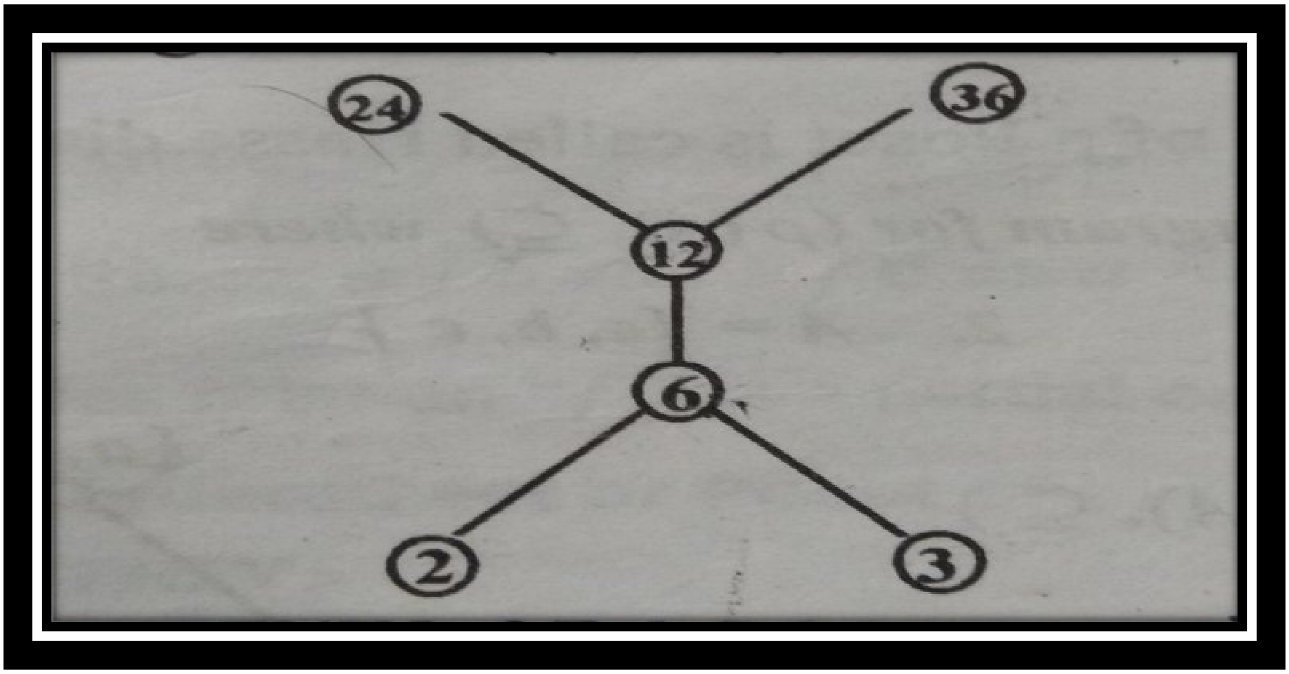
If $X = \{2, 3, 6, 12, 24, 36\}$ and the relation R defined on X by $R = \{\langle a, b\rangle / a/\ b\}$. Draw the Hasse diagram for (X, R).

Solution:

The relation

$$R = \left\{ \begin{matrix} \langle 2, 6\rangle\ \langle 2, 12\ \rangle\ \langle 2, 24\rangle\ \langle 2, 36\rangle\ \langle 3, 6\rangle\ \langle 3, 12\rangle\ \langle 3, 24\rangle\ \langle 3, 36\rangle\ \langle 6, 12\rangle \\ \langle 6, 24\rangle\ \langle 6, 36\rangle\ \langle 12, 24\rangle\ \langle 2,36\rangle \end{matrix} \right\}$$

The Hasse Diagram for (X, R) is



Special Elements of a Poset:

Let $(P, \leq)$ be a Poset. An element $a \in P$ is called least element in P, if $a \leq x$ for all $x \in P$.

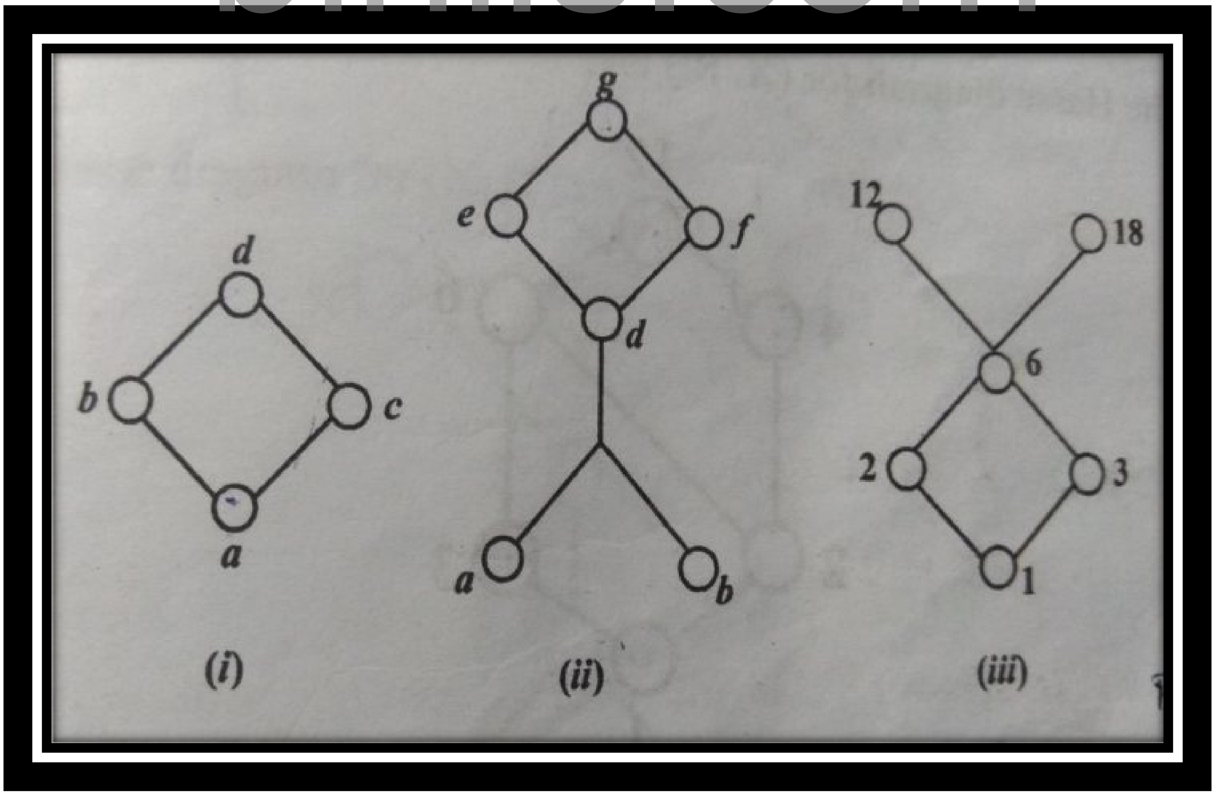
An element $b \in P$ is called greatest element in P, if $b \geq x$ for all $x \in P$

Note:

The least element is called "0" element and the greatest element is called "1" element.

Example:

Consider the following Hasse Diagram



In (i) “a” is the least element and “d” is the greatest element.

In (ii) “g” is the greatest element and there is no least element.

In (iii) “1” is the least element and there is no greatest element.

Definition:

Let $(P, \leq)$ be a Poset an A be any non - empty subset of P. An element $a \in P$ is an upper bound of A, if $a \geq x$ for all $x \in A$.

An element $b \in P$ is said to be lower bound in P, if $b \leq x$ for all $x \in A$.

Least Upper Bound: (LUB)

Let $(P, \leq)$be a Poset and $A \subseteq P$. An element $a \in P$ is said to be least upper bound (LUB) or supremum (sup) of A, if a is a upper bound of A.

$a \leq c$, where c is any other upper bound of A.

Greatest Lower Bound: (GLB)

Let $(P, \leq)$be a Poset and $A \subseteq P$. An element $b \in P$ is said to be least upper bound (GLB) or infimum (inf) of A, if b is a lower bound of A.

$b \geq d$, where d is any other lower bound of A.

Examples:

1. If $X = \{1, 2, 3, 4, 6, 12\}$ and the relation R defined on X by $R = \{\langle a, b\rangle / a/\ b\}$. Find LUB and GLB for the Poset (X, R).

Solution:

The relation

$$R = \{\langle 1,2\rangle\ \langle 1,3\ \rangle\ \langle 1,4\rangle\ \langle 1,6\rangle\ \langle 1,12\rangle\ \langle 2,4\rangle\ \langle 2,6\rangle\ \langle 2,12\rangle\ \langle 3,6\rangle\ \langle 3,12\rangle\ \langle 4,12\rangle\ \}$$

The Hasse Diagram for (X, R) is

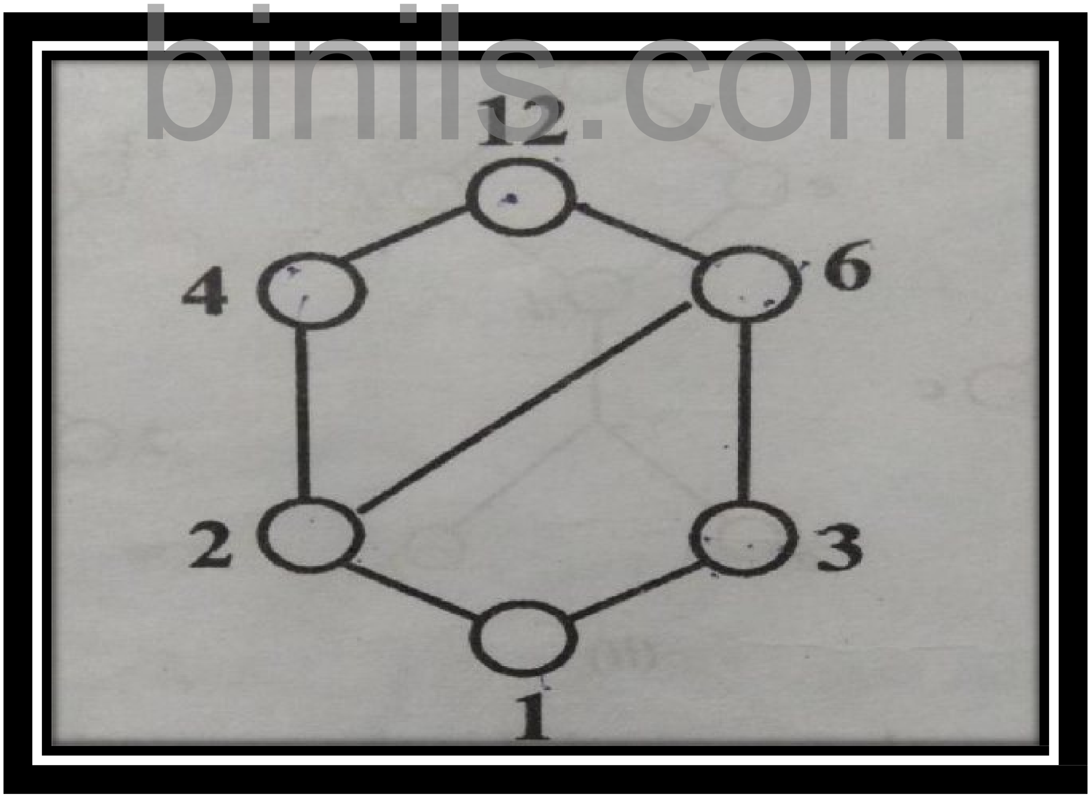


Table for LUB and GLB

1	$UB\{1, 3\} = \{3, 6, 12\}$ $LUB\{1, 3\} = 3$	**1**	$LB\{1, 3\} = \{1\}$ $GLB\{1, 3\} = 1$
2	$UB\{1, 2, 3\} = \{6, 12\}$ $LUB\{1,2, 3\} = 6$	**2**	$LB\{1, 2, 3\} = \{1\}$ $GLB\{1,2, 3\} = 1$
3	$UB\{2, 3\} = \{3, 6, 12\}$ $LUB\{2, 3\} = 6$	**3**	$LB\{2, 3\} = \{1\}$ $GLB\{2, 3\} = 1$
4	$UB\{2, 3, 6\} = \{6, 12\}$ $LUB\{2, 3, 6\} = 6$	**4**	$LB\{2, 3, 6\} = \{1\}$ $GLB\{2, 3, 6\} = 1$
5	$UB\{4, 6\} = \{12\}$ $LUB\{4, 6\} = 12$	**5**	$LB\{4, 6\} = \{1, 2\}$ $GLB\{4, 6\} = 2$

2. If $X = \{2, 3, 6, 12, 24, 36\}$ and the relation R defined on X by $R = \{\langle a, b\rangle / a / b\}$. Draw the Hasse diagram for (X, R).

Solution:

The relation

$$R = \left\{ \begin{matrix} \langle 2, 6\rangle \langle 2, 12 \rangle \langle 2, 24\rangle \langle 2, 36\rangle \langle 3, 6\rangle \langle 3, 12\rangle \langle 3, 24\rangle \langle 3, 36\rangle \langle 6, 12\rangle \\ \langle 6, 24\rangle \langle 6, 36\rangle \langle 12, 24\rangle \langle 2,36\rangle \end{matrix} \right\}$$

The Hasse Diagram for (X, R) is

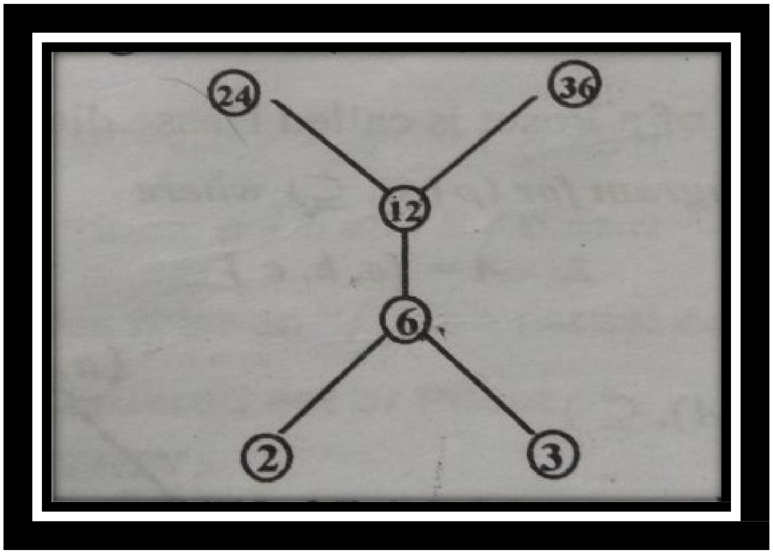


Table of LUB and GLB

1	$UB\{2, 3\} = \{6, 12, 24, 36\}$ $LUB\{2, 3\} = 6$	**1**	$LB\{2, 3\}$ =does not exists $GLB\{2, 3\}$ =does not exists
2	$UB\{24, 36\}$ = does not exists $LUB\{24, 36\}$ = does not exists	**2**	$LB\{24, 36\} = \{2, 3, 6, 12\}$ $GLB\{24, 36\} = 12$

Lattice:

A Lattice is a partially ordered set(Poset) $(L, \leq)$ in which for every pair of elements $a, b \in L$, both greatest lower bound (GLB) and least upper bound (LUB) exists.

Note:

(i) GLB $\{a, b\} = a * b \ (or) \ a \wedge b \ (or) \ a \cdot b$

(ii) LUB $\{a, b\} = a \oplus b \ (or) a \vee b \ (or) a + b$

Properties of lattice:

Some important laws and its proof:

(i) Idempotent law:

$a \vee a = a, a \wedge a = a$

(ii) Commutative law:

$a \vee b = b \vee a \ and \ \ a \wedge b = b \wedge a$

(iii) Associative law:

$a \vee (b \vee c) = (a \vee b) \vee c \ and \ a \wedge (b \wedge c) = (a \wedge b) \wedge c$

(iv) Absorption law:

$a \vee (a \wedge b) = a \ and \ a \wedge (a \vee b) = a$

(v) Distributive law:

$a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

Note:

i) $a \leq a \vee b \text{ and } b \leq a \vee b$

$a \vee b$ is the upper bound of a and b.

If $a \leq c \text{ and } b \leq c$ then $a \vee b \leq c$

Hence $a \vee b$ is the lub of a and b.

(ii) $a \wedge b \leq a \text{ and } a \wedge b \leq b$

$a \wedge b$ is the lower bound of a and b.

If $c \leq a \text{ and } c \leq b$ then $c \leq a \wedge b$

Hence $a \wedge b$ is the glb of a and b.

Note:

If $a \leq b \text{ and } a \leq c$ then $a \leq b \vee c$

If $a \leq b \text{ and } a \leq c$ then $a \leq b \wedge c$

Problems:

1. State and prove Idempotent law:

Let $(L, \wedge, \vee)$ **be given lattice. Then, for any** $a, b, c \in L$**,**

$a \vee a = a, a \wedge a = a.$

Proof:

Given $a \vee a = \text{LUB}(a, a) = \text{LUB}(a) = a$

Hence $a \vee a = a$

Now, $a \wedge a = \text{GLB}(a, a) = \text{GLB}(a) = a$

Hence $a \wedge a = a$

Hence the proof.

2. State and prove Commutative law:

Let $(L, \wedge, \vee)$ **be given lattice. Then, for any** $a, b, c \in L$**,**

$a \vee b = b \vee a$ *and* $a \wedge b = b \wedge a$

Proof:

Given $a \vee b = \text{LUB}(a, b) = \text{LUB}(b, a) = b \vee a$

Hence $a \vee b = b \vee a$

Now, $a \wedge b = \text{GLB}(a, b) = \text{GLB}(b, a) = b \wedge a$

Hence $a \wedge b = b \wedge a$

Hence the proof.

3. State and prove Absorption law.

(or)

Prove that $\boldsymbol{a \vee (a \wedge b) = a \text{ and } a \wedge (a \vee b) = a}$

Proof:

We have $a \wedge b \leq a \text{ and } a \leq a$

$\Rightarrow$ a is the upper bound of $a \wedge b$ and a.

$\Rightarrow a \vee (a \wedge b) \leq a \ldots (1)$

From the definition of lub we have

$\Rightarrow a \leq a \vee (a \wedge b) \ldots (2)$

From (1) and (2) we have $a \vee (a \wedge b) = a$

Similarly we can prove that $a \wedge (a \vee b) = a$

Hence the proof.

4. Every finite Lattice is bounded.

Proof:

Let $(L, \wedge, \vee)$ be a given lattice.

Since L is a Lattice both GLB and LUB exist.

Let "a" be GLB of L and "b" be LUB of L.

Then for any $x \in L$, we have $a \leq x \leq b$ . . . (1)

From (1)

GLB $\{a, x\} = a \wedge x = a$

LUB $\{a, x\} = a \vee x = x$

And

GLB $\{x, b\} = x \wedge b = x$

LUB $\{x, b\} = x \vee b = b$

Therefore any finite lattice is bounded.

Hence the proof.

5. State and prove Isotonicity property.

Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$ then $b \leq c = \begin{cases} a \wedge b \leq a \wedge c \\ a \vee b \leq a \vee c \end{cases}$

Proof:

By consistency law we have, $a \leq b \Leftrightarrow a \wedge b = a \; and \; a \vee b = a$

Let $b \leq c \Rightarrow b \wedge c = b$ and $b \vee c = c$

Consider $(a \wedge b) \wedge (a \wedge c) = a \wedge [(b \wedge a) \wedge c]$ by Associative law

$= a \wedge [(a \wedge b) \wedge c]$ by Commutative law

$= (a \wedge a) \wedge (b \wedge c)$ by Associative law

$= a \wedge (b \wedge c)$ by Idempotent law

$= a \wedge b$ by $[b \wedge c = b]$

Hence $(a \wedge b) \wedge (a \wedge c) = a \wedge b$

$\Rightarrow a \wedge b \leq a \wedge c \quad \ldots (1)$

Consider $(a \vee b) \wedge (a \vee c) = a \vee [(b \vee a) \vee c]$ by Associative law

$= a \vee [(a \vee b) \vee c]$ by Commutative law

$= (a \vee a) \vee (b \vee c)$ by Associative law

$= a \vee (b \vee c)$ by Idempotent law

$= a \vee b$ by $[b \vee c = b]$

Hence $(a \vee b) \wedge (a \vee c) = a \vee b$

$\Rightarrow a \vee b \leq a \vee c \quad \ldots (2)$

Hence the proof.

6. State and prove Distributive law.

$\boldsymbol{a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)}$

$\boldsymbol{a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)}$

Proof:

We know that $a \wedge b \leq a$ and $a \wedge b \leq b$

Also $b \leq b \vee c$

Hence $a \wedge b \leq a$ and $a \wedge b \leq b \leq b \vee c$

Hence $a \wedge b$ is the lower bound of a and $b \vee c$.

$\Rightarrow a \wedge b \leq a \wedge (b \vee c) \ldots (1)$

Again $a \wedge c \leq a$ and $a \wedge c \leq c$

Also $c \leq b \vee c$

Hence $a \wedge c \leq a$ and $a \wedge c \leq c \leq b \vee c$

Hence $a \wedge c$ is the lower bound of a and $b \vee c$.

$\Rightarrow a \wedge c \leq a \wedge (b \vee c) \ldots (2)$

From (1) and (2) we have

$a \wedge (b \vee c)$ is the upper bound of $a \wedge b$ and $a \wedge c$

Hence $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$

$\Rightarrow a \wedge (b \vee c) \geq (a \wedge b) \vee (a \wedge c)$ ... (I)

We know that $a \leq a \vee b$ and $a \leq a \vee b$

Also $b \wedge c \leq b$

Hence $a \leq a \vee b$ and $b \wedge c \leq b \leq a \vee b$

Hence $a \vee b$ is the lower bound of a and $b \wedge c$.

$\Rightarrow a \vee (b \wedge c) \leq a \vee b$ ... (3)

Again $a \leq a \vee a$ and $c \leq a \vee c$

Also $b \wedge c \leq c$

Hence $a \leq a \vee c$ and $b \wedge c \leq c \leq a \vee c$

Hence $a \vee c$ is the upper bound of a and $b \wedge c$.

$\Rightarrow a \vee (b \wedge c) \leq a \vee c$ ... (4)

From (3) and (4) we have

$a \vee (b \wedge c)$ is the lower bound of $a \vee b$ and $a \vee c$

$\Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ ... (II)

Hence the proof.

7. State and prove Cancellation law.

Let $(L, \leq)$ be a distributive lattice. Then $a \vee b = a \vee c$ and $a \wedge b = a \wedge c \Rightarrow$ $b = c\ \forall\ a, b, c \in L$

Proof:

By absorption law, we have $a \vee (a \wedge b) = a$

Consider $b = b \vee (a \wedge b)$

$= b \vee (a \wedge c)$

$= (a \vee b) \wedge (b \vee c)$

$= (a \vee c) \wedge (b \vee c)$

$= (a \wedge b) \vee c$

$= (a \wedge c) \vee c$

$= c$

Hence the proof.

8. State and prove Consistency Law.

Let $(L, \leq)$ be a lattice. Then $a \leq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b\ \ \forall\ a, b, c \in L$

Proof:

First we prove that $a \leq b \Leftrightarrow a \wedge b = a$

We assume that $a \leq b$

To prove $a \wedge b = a$

We have $a \leq b$ and $a \leq a$

$\Rightarrow$ a is the lower bound of a and b.

$\Rightarrow a \leq a \wedge b \quad \quad ...(1)$

By the definition of greatest lower bound

$\Rightarrow a \wedge b \leq a \quad \quad ...(2)$

From (1) and (2) we have, $a = a \wedge b$

Conversely assume that $a = a \wedge b$

To prove $a \leq b$

This is possible only when $a \leq b$

Hence $a \leq b \Leftrightarrow a \wedge b = a$

Next we prove that $a \wedge b = a \Leftrightarrow a \vee b = b$

Assume that $a \wedge b = a$

To prove $a \vee b = b$

By absorption law $a \vee (a \wedge b) = a \; and \; a \wedge (a \vee b) = a$

Consider $b = b \vee (a \wedge b)$

$= b \vee a$

Hence $a \vee b = b$

Conversely assume that $a \vee b = b$

To prove $a \wedge b = a$

By absorption law $a \wedge (a \vee b) = a$

Consider $a = a \wedge (a \vee b)$

$= a \wedge b$

Hence $a \wedge b = a \Leftrightarrow a \vee b = b$

9. Show that a chain is a lattice.

Proof:

Let $(L, \leq)$ be a lattice.

If $a, b \in L$ then $a \leq b$ or $b \leq a$

If $a \leq b$ then $a \wedge b = a$ and $a \vee b = b$

Therefore GLB and LUB of a and b exists.

If $b \leq a$ then $b \wedge a = b$ and $b \vee a = a$

Therefore GLB and LUB of a and b exists.

Hence every pair of elements has a GLB and LUB.

Hence chain is lattice.

Duality in Lattice:

When " $\leq$ " is a partial order relation on a set S, then its converse " $\geq$ " is also a partial order relation on S.

Distributive lattice:

A lattice $(L, \wedge, \vee)$ is said to be distributive lattice if $\wedge$ and $\vee$ satisfies the following conditions $\forall a, b, c \in L$

$$D_1: a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$D_2: a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

Modular Inequality:

If $(L, \wedge, \vee)$ is a Lattice, then for any $\boldsymbol{a, b, c} \in \boldsymbol{L}$, $\boldsymbol{a} \leq \boldsymbol{c} \Leftrightarrow \boldsymbol{a} \vee (\boldsymbol{b} \wedge \boldsymbol{c}) \leq (\boldsymbol{a} \vee \boldsymbol{b}) \wedge \boldsymbol{c}$.

Proof:

Assume $a \leq c$

$\Rightarrow a \vee c = c \qquad ...(1)$

By, distributive inequality, we have

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$

$\Rightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c \qquad$ (Using (1))

Therefore, $a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$.............(2)

Conversely, assume $a \vee (b \wedge c) \leq (a \vee b) \wedge c$

Now, by the definition of LUB and GLB, we have

$a \leq a \vee (b \wedge c) \leq (a \vee b) \wedge c \leq c$

$\Rightarrow a \leq c$

Hence $a \vee (b \wedge c) \leq (a \vee b) \wedge c \Rightarrow a \leq c$.............. (3)

From (2) and (3), we have $a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$.

Hence the proof.

Modular Lattice:

A Lattice $(L, \wedge, \vee)$ is said to be Modular lattice if it satisfies the following condition.

M_1: if $a \leq c$ then $a \vee (b \wedge c) = (a \vee b) \wedge c$

Theorem: 1

Every distributive Lattice is Modular, but not conversely.

Proof:

Let $(L, \wedge, \vee)$ be the given distributive lattice

$D_1: a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ . . . (1)

Now, if $a \leq c$ then $a \vee c = c$ . . . (2)

(1)(1) $\Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$= (a \vee b) \wedge c$ (using (2))

If $a \leq c$ then $a \vee (b \wedge c) = (a \vee b) \wedge c$

Therefore every distributive lattice is Modular.

But, converse is not true.

i.e., Every Modular Lattice need not be distributive.

For example, M_5 Lattice is Modular but it is not distributive.

Hence the proof.

Theorem: 2

In any distributive lattice $(L, \wedge, \vee)$ $\forall a, b, c \in L$. Prove that

$\boldsymbol{a \vee b = a \vee c, a \wedge b = a \wedge c \Rightarrow b = c}$

Proof:

Consider $b = b \vee (b \wedge a)$ (Absorption law)

$= b \vee (a \wedge b)$ (Commutative law)

$= b \vee (a \wedge c)$ (Given condition)

$= (b \vee a) \wedge (b \vee c)$ (D1 – Condition)

$= (a \vee b) \wedge (b \vee c)$ (Commutative law)

$= (a \vee c) \wedge (b \vee c)$ (Using given condition)

$= (c \vee a) \wedge (c \vee b)$ (Commutative law)

$= c \vee (a \wedge b)$ (By D1- condition)

$= c \vee (a \wedge c)$ (Given Condition)

$= c \vee (c \wedge a)$ (Commutative law)

$= c$ (Absorption law)

Lattice as a Algebraic system

A Lattice is an algebraic system $(L, \wedge, \vee)$ with two binary operation $\wedge$ and $\vee$ on L which are both commutative, associative and satisfies absorption laws.

SubLattice:

Let $(L, \wedge, \vee)$ be a lattice and let $S \subseteq L$ be a subset of L. Then $(S, \wedge, \vee)$ is a sublattice of $(L, \wedge, \vee)$ iff S is closed under both operation $\wedge$ and $\vee$.

$\forall a, b \in S \Rightarrow a \wedge b \in S$ and $a \vee b \in S$

Lattice Homomorphism:

Let $(L_1, \wedge, \vee)$ and $(L_2, *, \oplus)$ be two given lattices.

A mapping $f: L_1 \to L_2$ is called Lattice homomorphism if $\forall a, b \in L_1$

$f(a \wedge b) = f(a) * f(b)$

$f(a \vee b) = f(a) \oplus f(b)$

A homomorphism which is also 1 – 1 is called an isomorphism.

Bounded lattice:

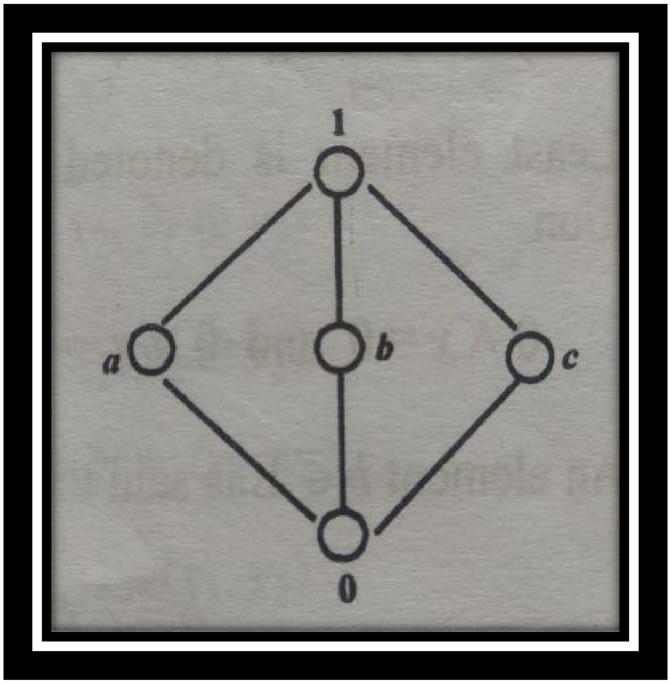
Let $(L, \wedge, \vee)$ be a given Lattice. If it has both "0" element and "1" element then it is said to be bounded Lattice. It is denoted by $(L, \wedge, \vee, 0, 1)$

Complement:

Let $(L, \wedge, \vee, 0, 1)$ be given bounded lattices. Let "a" be any element of L. We say that "b" is complement of a, if $a \wedge b = 0$ and $a \vee b = 1$ and "b" is denoted by the symbol a'. i.e., $(b = a')$. Therefore $a \wedge a' = 0$ and $a \vee a' = 1$.

Note: An element may have no complement or may have more than 1 complement.

Example for a complement.

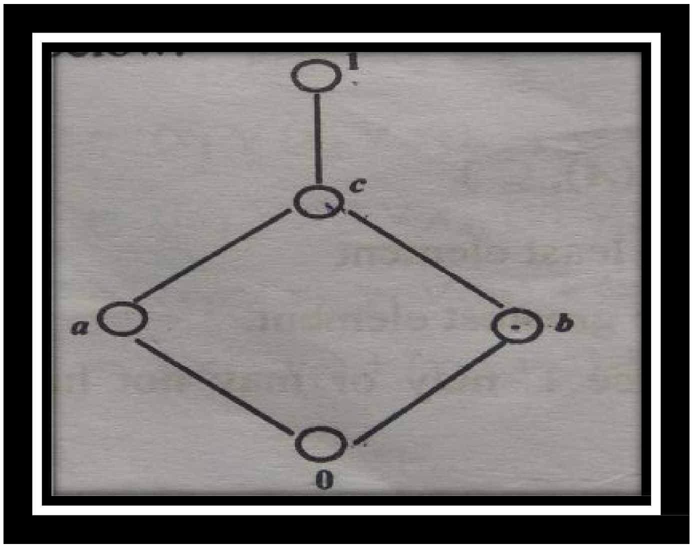


Complement of $a = a'$ is b and c.

Complement of $b = b'$ is a and c.

Complement of $c = c'$ is a and b.

In the example given below:



Complement of does not exist.

Complement of b does not exist.

Complement of c does not exist.

Complemented Lattice:

A bounded lattice $(L, \wedge, \vee, 0, 1)$ is said to be a complemented lattice if every element of L has atleast one complement.

Complete Lattice:

A lattice $(L, \wedge, \vee)$ is said to be complete lattice if every non empty subsets of L has both glb &lub.

1. Prove that in a bounded distributive lattice, the complement of any element is unique.

Proof:

Let L be a bounded distributive lattice.

Let b and c be complements of an element $a \in L$.

To prove $b = c$

Since b and c are complements of a we have

$a \wedge b = 0, a \vee b = 1, a \wedge c = 0, a \vee c = 1$

Now $b = b \wedge 1$

$= b \wedge (a \vee c)$

$= (b \wedge a) \vee (b \wedge c)$

$= (a \wedge b) \vee (b \wedge c)$

$= 0 \vee (b \wedge c)$

$= (a \wedge c) \vee (b \wedge c)$

$= (a \wedge b) \wedge c$

$= 1 \wedge c$

$= c$

Hence the proof.

2. Prove that every distributive lattice is modular.

Proof:

Let $(L, \leq)$ be a distributive lattice.

Let $a, b, c \in L$ such that $a \leq c$

To prove that $a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$

Assume that $a \leq c$

To prove that $a \vee (b \wedge c) = (a \vee b) \wedge c$

When $a \leq c \Rightarrow a \vee c = c$

Therefore $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$$= (a \vee b) \wedge c$$

Hence $a \vee (b \wedge c) = (a \vee b) \wedge c$

Hence the proof.

3. Show that in a complemented distributive lattice, $\boldsymbol{a \leq b \Leftrightarrow a * b' = 0 \Leftrightarrow}$

$\boldsymbol{a' \oplus b = 1 \Leftrightarrow b' \leq a'}$ (or) , $\boldsymbol{a \leq b \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = 1 \Leftrightarrow b' \leq a'}$

Proof:

To prove $(\boldsymbol{i}) \Rightarrow (\boldsymbol{ii})$

We assume that $a \leq b$

To prove that $a \wedge b' = 0$

We know that $a \leq b \Rightarrow a \wedge b = a$ and $a \vee b = b$

We take $a \vee b = b$

$\Rightarrow (a \vee b) \wedge b' = b \wedge b' = 0$

$\Rightarrow (a \wedge b') \vee (b \wedge b') = 0$

$\Rightarrow (a \wedge b') \vee 0 = 0$

$\Rightarrow (a \wedge b') = 0$

Hence $(i) \Rightarrow (ii)$

To prove $(ii) \Rightarrow (iii)$

We assume that $a \wedge b' = 0$

To prove that $a' \vee b = 1$

Taking complement on both sides

$\Rightarrow (a \wedge b')' = 0'$

$\Rightarrow a' \vee b = 1$

Therefore $a \wedge b' = 0 \Rightarrow a' \vee b = 1$

Hence $(ii) \Rightarrow (iii)$

To prove $(iii) \Rightarrow (iv)$

Assume that $a' \vee b = 1$

To prove that $b' \leq a'$

Now $a' \vee b = 1$

$\Rightarrow (a' \vee b) \wedge b' = 1 \cdot b'$

$\Rightarrow (a' \vee b) \wedge b' = b'$

$\Rightarrow (a' \wedge b') \wedge (b \wedge b') = b'$

$\Rightarrow (a' \wedge b') \vee 0 = b'$

$\Rightarrow (a' \wedge b') = b'$

$\Rightarrow (b' \wedge a') = b'$ by Commutative law

Therefore $a' \vee b = 1 \Rightarrow b' \leq a'$

Hence $(iii) \Rightarrow (iv)$

To prove $(iv) \Rightarrow (i)$

Assume that $b' \leq a'$

To prove that $a \leq b$

We have $(b' \wedge a') = b'$

Taking complement on both sides

$\Rightarrow (b' \wedge a')' = (b')'$

$\Rightarrow b \vee a = b$

Therefore $a \vee b = b \Rightarrow a \leq b$

Hence $(iv) \Rightarrow (i)$

Hence $a \leq b \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = 1 \Leftrightarrow b' \leq a'$

Hence the proof.

4. State and prove DeMorgan's law of lattice.

(OR)

Let $(L, \wedge, \vee, 0, 1)$ is a complemented lattice, then prove that

1. $(a \wedge b)' = a' \vee b'$

2. $(a \vee b)' = a' \wedge b'$

Proof:

1. Claim: $(a \wedge b)' = a' \vee b'$

To prove the above, it is enough to prove that

(i) $(a \wedge b) \wedge (a' \vee b') = 0$

(ii) $(a \wedge b) \vee (a' \vee b') = 1$

(i) Let $(a \wedge b) \wedge (a' \vee b')$

$\Rightarrow ((a \wedge b) \wedge a') \vee ((a \wedge b) \wedge b')$ (Distributive law)

$\Rightarrow (a \wedge b \wedge a') \vee (a \wedge b \wedge b')$ (Associative law)

$\Rightarrow (0 \wedge b) \vee (a \wedge 0)$ $(b \wedge b' = 0)$

$\Rightarrow 0 \vee 0$ $\qquad (a \wedge 0 = 0)$

Hence $(a \wedge b) \wedge (a' \vee b') = 0$ $\qquad ...(1)$

(ii) Let $(a \wedge b) \wedge (a' \vee b')$

$\Rightarrow (a \vee (a' \vee b')) \wedge (b \vee (a' \vee b'))$ $\qquad$ (Distributive law)

$\Rightarrow (a \vee b \vee a') \wedge (a \vee b \vee b')$ $\qquad$ (Associative law)

$\Rightarrow (1 \vee b) \wedge (a \vee 1)$ $\qquad (b \vee b' = 1)$

$\Rightarrow 1 \wedge 1 = 1$ $\qquad (a \wedge 0 = 0)$

Hence $(a \wedge b) \wedge (a' \vee b') = 1$ $\qquad ...(2)$

From (1) and (2) we have, $(a \wedge b)' = a' \vee b'$

2. Claim: $(\boldsymbol{a} \vee \boldsymbol{b})' = \boldsymbol{a}' \wedge \boldsymbol{b}'$

To prove the above, it is enough to prove that

(i) $(a \vee b) \wedge (a' \wedge b') = 0$

(ii) $(a \vee b) \vee (a' \wedge b') = 1$

(i) Let $(a \vee b) \wedge (a' \wedge b')$

$\Rightarrow (a \wedge (a' \wedge b')) \vee (b \wedge (a' \wedge b'))$ $\qquad$ (Distributive law)

$\Rightarrow (a \wedge a' \wedge b') \vee (b \wedge b' \wedge a')$ $\qquad$ (Associative law)

$\Rightarrow (0 \wedge b') \vee (0 \wedge a')$ $\qquad (b \wedge b' = 0)$

$\Rightarrow 0 \vee 0$ $\qquad (a \wedge 0 = 0)$

Hence $(a \vee b) \wedge (a' \wedge b') = 0$ $\qquad ...(3)$

(ii) Let $(a \vee b) \vee (a' \wedge b')$

$\Rightarrow ((a \vee b) \vee a') \wedge ((a \vee b) \vee b')$ $\qquad$ (Distributive law)

$\Rightarrow (a \vee b \vee a') \wedge (a \vee b \vee b')$ $\qquad$ (Associative law)

$\Rightarrow (1 \vee b) \wedge (a \vee 1)$ $\qquad (b \vee b' = 0)$

$\Rightarrow 1 \wedge 1 = 1$ $\qquad$ (Idempotent law)

Hence $(a \vee b) \vee (a' \wedge b') = 1$ $\qquad ...(4)$

From (3) and (4) we have, $(a \vee b)' = a' \wedge b'$

Boolean Algebra:

A complemented distributive lattice is called Boolean Algebra.

A Boolean algebra is distributive lattice with "0" element and "1" element in which every element has a complement.

A Boolean algebra is a non empty set with 2 binary operations $\wedge$ and $\vee$ and is satisfied by the following conditions. $\forall a, b, c \in L$

1. L_1: $a \wedge a = a$ and $a \vee a = a$

2. L_2: $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$

3. L_3: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$

4. L_4: $a \wedge (a \vee b) = a$ and $a \vee (a \wedge b) = a$

5. D_1: $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

6. D_2: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

7. There exist between "0" and "1" such that $a \wedge 0 = 0, a \vee 0 = a, a \wedge 1 = a$ and $a \vee 1 = 1 \forall a$

8. $\forall a \in L$, there exist corresponding element a' in L such that $a \wedge a' = 0$ and $a \vee a' = 1$

Note:

Boolean Sum is defined as $1+1=1, 1+0=1,\ 0+1=1,\ 0+0=0$

Boolean Product is defined as $1\cdot 1=1, 1\cdot 0=0,\ 0\cdot 1=0,\ 0\cdot 0=0$

Absorption law in Boolean Algebra

1. Prove that $\boldsymbol{a+ab=a}$

Solution:

$\text{LHS} = a + ab$

$= a(1+b)$ (Distributive law)

$= a(1)$ $(1+a)=1$

$a + ab = a$ $(a \cdot 1 = a)$

2. Prove that $\boldsymbol{a+\bar{a}b=a+b}$

Solution:

$\text{LHS} = a + \bar{a}b$

$= a + ab + \bar{a}b$ $(a = a + ab)$

$= a + b(a+\bar{a})$ (Distributive law)

$= a + b(1)$ $(a+\bar{a}) = 1$ $(a \cdot 1 = a)$

= RHS

3. Prove that $(\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{a}+\boldsymbol{c}) = \boldsymbol{a}+\boldsymbol{bc}$

Solution:

LHS = $(a+b)(a+c)$

$= aa + ac + ab + bc$ (Distributive law)

$= a + ac + ab + bc$ $(a \cdot a = a)$

$= a(1+c) + ab + bc$ (Distributive law)

$= a + ab + bc$ $(1 + a = 1)$

$= a + bc$ $(a + ab = a)$

= RHS

4. In any Boolean Algebra, show that $\boldsymbol{a} = \boldsymbol{b} \Leftrightarrow \boldsymbol{a}\bar{\boldsymbol{b}} + \bar{\boldsymbol{a}}\boldsymbol{b} = \mathbf{0}$

Proof:

Let $(B, \cdot, +, 0, 1)$ be any Boolean Algebra.

Let $a, b \in B$ and $a = b$ . . . (1)

Claim: $a\bar{b} + \bar{a}b = 0$

Now $a\bar{b} + \bar{a}b = a \cdot \bar{b} + \bar{a}b$

$= a\cdot\bar{a} + \bar{a}a$ using (1)

$= 0 + 0$ (since $a\cdot\bar{a} = 0$)

$= 0$

Conversely, assume $a\bar{b} + \bar{a}b = 0$

$\Rightarrow a + a\bar{b} + \bar{a}b = a$ (Left Cancellation law)

$\Rightarrow a + \bar{a}b = a$ (Absorption law)

$\Rightarrow (a + \bar{a})\cdot(a + b) = a$ (Distributive law)

$\Rightarrow 1\cdot(a + b) = a$ $(a + \bar{a} = 1)$

$\Rightarrow (a + b) = a$ $(a\cdot 1 = a)$ . . . (a)

Consider $a\bar{b} + \bar{a}b = 0$

$\Rightarrow a\bar{b} + \bar{a}b + b = b$ (Right Cancellation law)

$\Rightarrow a\bar{b} + b = b$ (Absorption law)

$\Rightarrow (a + b)\cdot(b + \bar{b}) = b$ (Distributive law)

$\Rightarrow (a + b)\cdot 1 = a$ $(b + \bar{b} = 1)$

$\Rightarrow (a + b) = b$ $(b\cdot 1 = b)$ . . . (b)

From (a) and (b) we get $a = a + b = b$

Hence $a = b$

5. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are two elements of a Boolean algebra, then prove that

$$\boldsymbol{a + (a \cdot b) = a, a \cdot (a + b) = a}$$

Proof:

Consider $a + (a \cdot b) = a = a \cdot 1 + (a \cdot b)$

$= a \cdot (1 + b)$

$= a \cdot 1 \qquad [a + 1 = 1, 1 + a = 1]$

$= a$

Consider $a \cdot (a + b) = a = a \cdot a + (a \cdot b)$

$= a + (a \cdot b)$

$= a \cdot 1 + a \cdot b$

$= a \cdot (1 + b)$

$= a \cdot 1 \qquad [a \cdot a = a, a \cdot 0 = 0]$

$= a$

Hence the proof.

6. Prove that in a Boolean algebra, the complement of any element is unique.

Proof:

Let b and c be the complements of the element a.

Then $b + a = 1, b \cdot a = 0$

$a + c = 1, a \cdot c = 0$

Consider $b = 1 \cdot b$

$= (a + c) \cdot b$

$= a \cdot b + c \cdot b$

$= 0 + c \cdot b$

$= a \cdot c + c \cdot b$

$= c \cdot (a + b)$

$= 1 \cdot c$

$= c$

Hence the complement is unique.

7. In a Boolean algebra show that the following statements are equivalent. For any a and b (i) $a + b = b$ (ii) $a \cdot b = a$ (iii) $a' + b = 1$ (iv) $a \cdot b' = 0$ (v) $a \leq b$

Proof:

To prove $(i) \Rightarrow (ii)$

Assume that $a + b = b$

To prove that $a \cdot b = a$

Now $a = a \cdot (a + b)$

$= a \cdot b$

Hence $(i) \Rightarrow (ii)$

To prove $(ii) \Rightarrow (iii)$

Assume that $a \cdot b = a$

To prove that $a' + b = 1$

Now $a' + b = (a \cdot b') + b$

$= a' + b' + b$

$= a' + 1$

$= 1$

Hence $(ii) \Rightarrow (iii)$

To prove $(iii) \Rightarrow (iv)$

Assume that $a' + b = 1$

To prove that $a \cdot b' = 0$

Taking complement on both sides

$\Rightarrow (a' + b)' = 1'$

$\Rightarrow a \cdot b' = 0$

Hence $(iii) \Rightarrow (iv)$

To prove $(iv) \Rightarrow (v)$

Assume that $a \cdot b' = 0$

To prove that $a \leq b$

Then $a \cdot b = a \cdot b + 0$

$= a \cdot b + a \cdot b'$

$= a(b + b')$

$= a \cdot 1$

$= a$

Hence $(iv) \Rightarrow (v)$

To prove $(v) \Rightarrow (i)$

Assume that $a \leq b$

To prove that $a+b=b$

We have $a \cdot b = b$

$\Rightarrow a+b=(a\cdot b)+b$

$= a\cdot b + 1\cdot b$

$= (a+1)\cdot b$

$= 1\cdot b$

$= b$

Hence the proof.

8. Prove that in a Boolean algebra

$$(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}'+\boldsymbol{c}) = \boldsymbol{ac}+\boldsymbol{a}'\boldsymbol{b} = \boldsymbol{ac}+\boldsymbol{a}'\boldsymbol{b}+\boldsymbol{bc}$$

Proof:

Now, $(a+b)\cdot(a'+c) = (a+b)\cdot a' + (a+b)\cdot c$

$= a'\cdot(a+b)+(a+b)\cdot c$

$= aa' + a'b + ac + bc$

$= 0 + a'b + ac + bc$

$= a'b + ac + bc$

$= ac(b + b') + a'b(c + c') + bc(a + a')$

$= abc + ab'c + a'bc + a'bc' + abc + a'bc$

$= abc + ab'c + a'bc + a'bc'$

$= abc + ab'c + a'b(c + c')$

$= ac(b + b') + a'b(c + c')$

$= ac(1) + a'b(1)$

$= ac + a'b$

= RHS

9. Show that in a Boolean algebra the law of the double complement holds.

(or) Prove the involution law $(a')' = a$

Solution:

It is enough to prove that $a' + a = 1$ and $a \cdot a' = 0$

By domination laws of Boolean algebra, we get

$a' + a = 1$ and $a \cdot a' = 0$

By commutative law, we get $a' + a = 1$ and $a \cdot a' = 0$

Therefore complement of a' is a

$\Rightarrow (a')' = a$

$\Rightarrow a'' = a$

Hence the proof.

Relation

A relation R is a well - defined rule, which tells whether given 2 elements x and y of A are related or not.

If x is related to y, we write xRy, otherwise x does not related to y.

Equivalence Relation

Let X be any set. R be a relation defined on X. If R satisfies Reflexive, Symmetric and Transitive then the relation R is said to be an Equivalence relation.

Partial Order Relation

Let X be any set. R be a relation defined on X. Then R is said to be a partial order relation if it satisfies reflexive, antisymmetric and transitive relation.

Example:

Subset relation ⊆ is a Partial order relation.

Solution:

Consider any three sets A, B, C

Since any set is a subset to itself, $A \subseteq A$, therefore ⊆ is reflexive.

If $A \subseteq B$ and $B \subseteq A$, then $A = B$, therefore ⊆ is antisymmetric.

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$, therefore ⊆ is transitive.

Hence $\subseteq$ is a Partial order relation.

Example:2

Divides relation is a Partial order relation.

Solution:

For Z_+ be the set of positive integer $a, b, c \in Z_+$

Since a/a ,/ is reflexive.

Since $a/b \text{ and } b/a \Rightarrow a = b$,/ is antisymmetric.

Since $a/b \text{ and } b/c \Rightarrow a \,/\, c$ is transitive.

Therefore, Divides relation " / " is a partial order relation.

Hence the proof.

Partially Ordered Set or Poset:

A set together with a partial order relation defined on it is called partially ordered set or Poset.

Usually, a partial order relation is defined by the symbol " $\leq$ ", this symbol does not necessarily mean "less than or equal to" as we use for real numbers.

For example,

Let $\mathbb{R}$ be the set of real numbers. The relation "less than or equal to" or " $\leq$ " is a partial order on $\mathbb{R}$. Therefore $(\mathbb{R}, \leq)$ is a Poset.

Comparable Property:

In a Poset for any 2 elements a, b either $a \leq b$ or $b \leq a$ is called comparable property. Otherwise it is called incomparable property.

Totally Ordered Set or Linearly Ordered Set or Chain:

A partially ordered set $(\rho, \leq)$ is said to be totally ordered set or linearly ordered set or chain if any 2 elements are comparable.

i.e., given any 2 elements x and y of a Poset either x y or $y \leq x$

Example:

aRb if $a \leq b$ is a total order.

aRb if a/b is not a total order.

For, Given elements 2 and 3 neither $2/3$ nor $3/2$.

(i.e., 2 and 3 are not comparable).

Problems:

1. Show that the "greater than or equal to" relation is a Partial ordering on the set of integers.

Solution:

Since $a \geq a$ for every integer a, $\geq$ is reflexive.

If $a \geq b$ and $b \geq a$ then $a = b$. Hence $\geq$ is antisymmetric.

Since $a \geq b$ and $b \geq c$ imply $a \geq c$. Hence $\geq$ is transitive.

Therefore, $\geq$ is a partial order relation on the set of integers.

2. In the Poset $(\boldsymbol{Z}^{+}, /)$ are the integers 3 and 9 comparable? Are 5 and 7 are comparable?

Solution:

Since $3/9$, the integers 3 and 9 are comparable.

For 5, 7 neither $5/7$ nor $7/5$

Therefore, the integers 5 and 7 are not comparable (incomparable).

3. Check the following Posets are totally orders set (or linearly ordered set or chain) (i) $(\boldsymbol{Z}, \leq)$ (ii) $(\boldsymbol{Z}^{+}, /)$

Solution:

(i) Consider, the Poset $(Z, \leq)$

If a and b are integer then either $a \leq b$ or $b \leq a$, for all a, b

Therefore, the Poset $(Z, \leq)$ satisfies comparable property.

$(Z, \leq)$ is a totally ordered set.

(ii) Consider, the Poset $(Z^+, /)$

Take 5 and 7.

Since, neither $5/7$ nor $7/5$

$(Z^+, /)$ does not satisfies the comparable property.

Therefore, $(Z^+, /)$ is not a totally ordered set.

4. Show that $(N, \leq)$ is a partially ordered set where N is set of all positive integers and $\leq$ is defined by $m \leq n$ iff $n - m$ is a non – negative integer.

Solution:

Give N is the set of all positive integer.

The given relation is $m \leq n$ iff $n - m$ is a non – negative integer.

(i) To prove R is reflexive

Now, $\forall x \in N$, $x - x = 0$ is a non – negative integer.

Therefore, $xRx \ \forall x \in N$.

Therefore R is reflexive.

(ii) To prove R is Antisymmetric.

Consider xRy & yRx

Since $xRy \Rightarrow x - y$ is a non – negative integer.

$yRx \Rightarrow y - x$ is a non – negative integer.

$\Rightarrow -(x - y)$ is a non – negative integer.

$\Rightarrow x = y$

Therefore R is Antisymmetric.

(iii) To prove R is Transitive.

Assume xRy & yRz

Since $xRy \Rightarrow x - y$ is a non – negative integer.

$yRz \Rightarrow y - z$ is a non – negative integer.

$\Rightarrow (x - y) + (y - z)$ is a non – negative integer.

$\Rightarrow x - z$ is a non – negative integer.

$\Rightarrow xRz$

xRy & $yRz \Rightarrow xRz$

Therefore R is transitive.

Hence R is partial order relation.

5. Is the Poset $(Z^+, /)$ a lattice.

Solution:

Let a and b be any two positive integer.

Then LUB $\{a, b\}$ =LCM $\{a, b\}$

GLB $\{a, b\}$ =GCD $\{a, b\}$

Should exist in Z^+.

For, example let $a = 4, b = 20$

Then LUB $\{a, b\}$ =LCM $\{4, 20\} = 1$

GLB $\{a, b\}$ =GCD $\{4, 20\} = 4$

Hence both GLB and LUB exist.

Therefore, the Poset $(Z^+, /)$ a lattice.