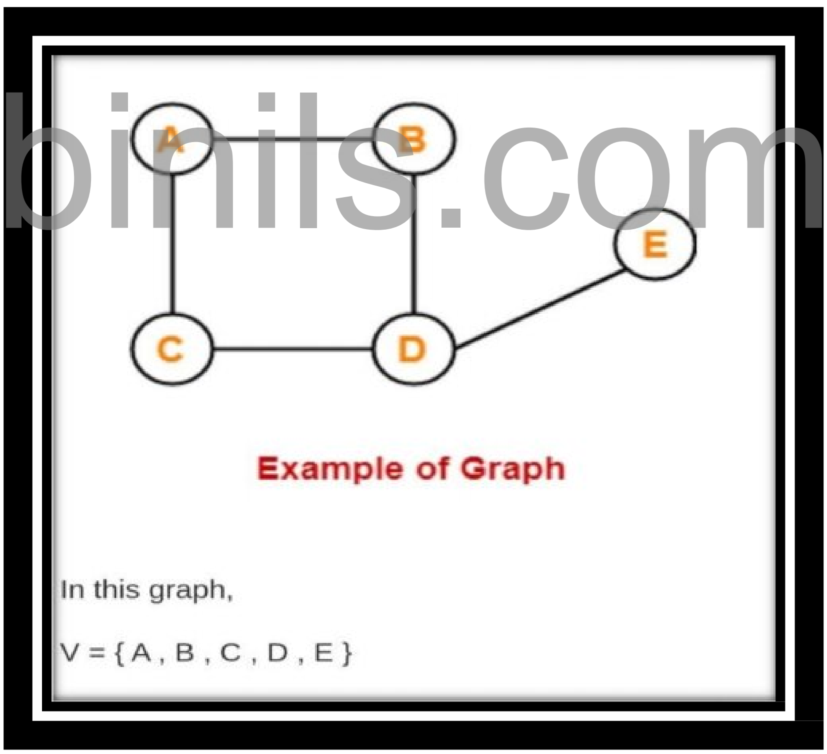
Catalog

Graph:

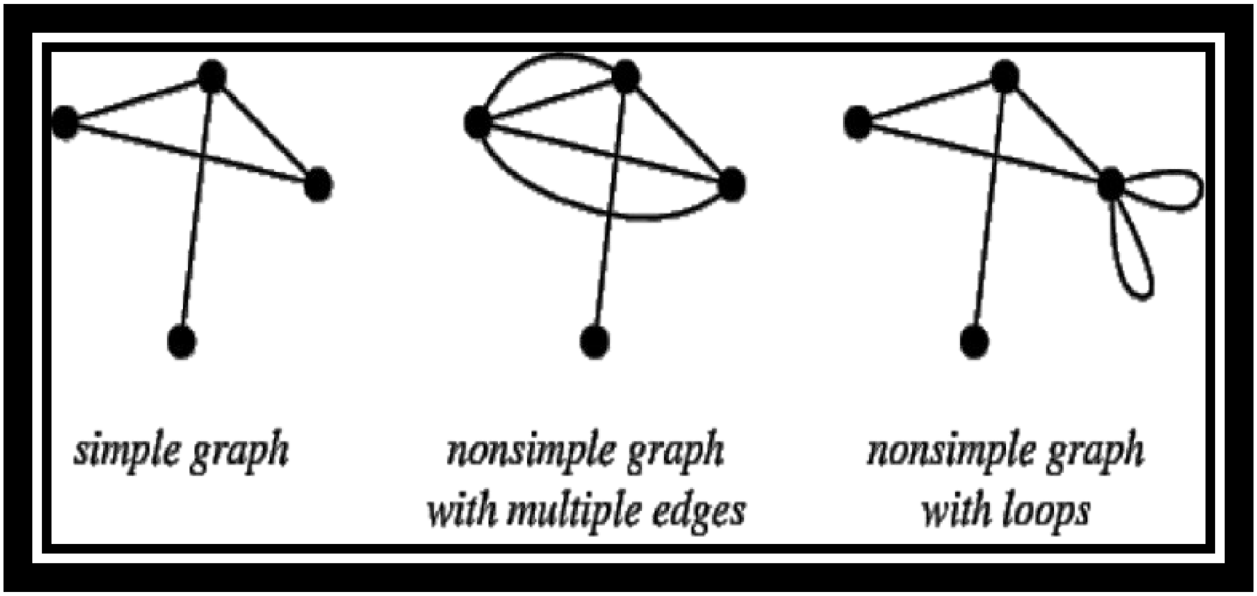
A graph $G = (V, E, \phi)$ consists of a non – empty set $V = \{V_1, V_2, \ldots\}$ called the set of nodes (Points, Vertices) of the graph, $E = \{e_1, e_2, \ldots\}$ is said to be the set of edges of the graph, and , ϕ is a mapping from the set of edges E to set of ordered or unordered pairs of elements of V.

The vertices are represented by points and each edge is represented by a line digrammatically.



Self Loop:

If there is an edge from v_i to v_i then that edge is called self loop or simply loop.



Parallel edges:

If two edges have same end points then the edges are called parallel edges.

Incident:

If the vertex v_i is an end vertex of some edge e_k then e_k is said to be incident with v_i.

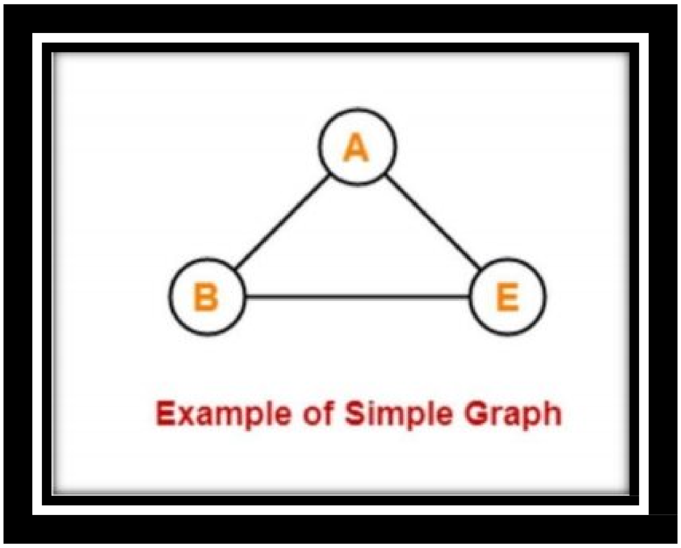
Adjacent edges and vertices:

Two edges are said to be adjacent if they are incident on a common vertex.

Two vertices v_i and v_j are said to be adjacent if v_iv_j is an edge of the graph.

Simple Graph:

A graph which has neither self loops nor parallel edges is called a simple graph.



Isolated vertex:

A vertex having no edge incident on it is called an isolated vertex. It is obvious that for an isolated vertex degree is zero.

Pendent vertex:

If the degree of any vertex is one, then that vertex is called pendent vertex

Directed edges:

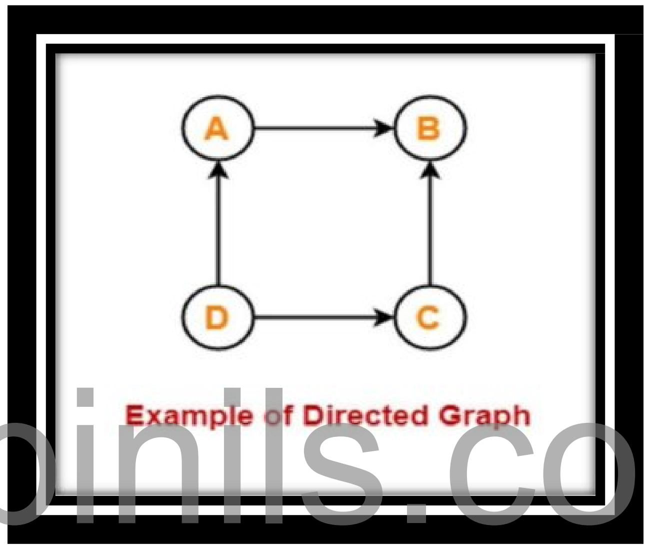
In a graph $G = (V, E)$, on edge which is associated with an ordered pair of $V \times V$ is called a directed edge of G.

Undirected edge:

If an edge which is associated with an unordered pair of nodes is called an undirected edge.

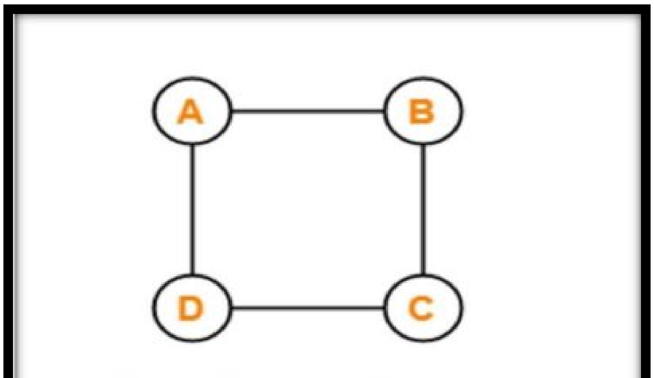
Digraph:

A graph in which every edge is directed edge is called a digraph or directed graph.



Undirected graph:

A graph in which every edge is undirected is called an undirected graph.



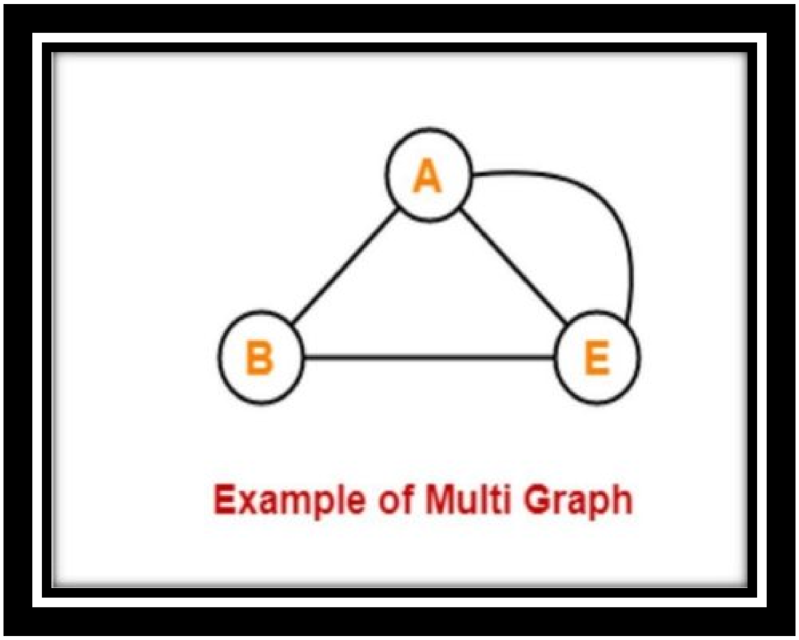
Example of Non-Directed Graph

Mixed graph:

If some edges are directed and some are undirected in a graph, the graph is called mixed graph.

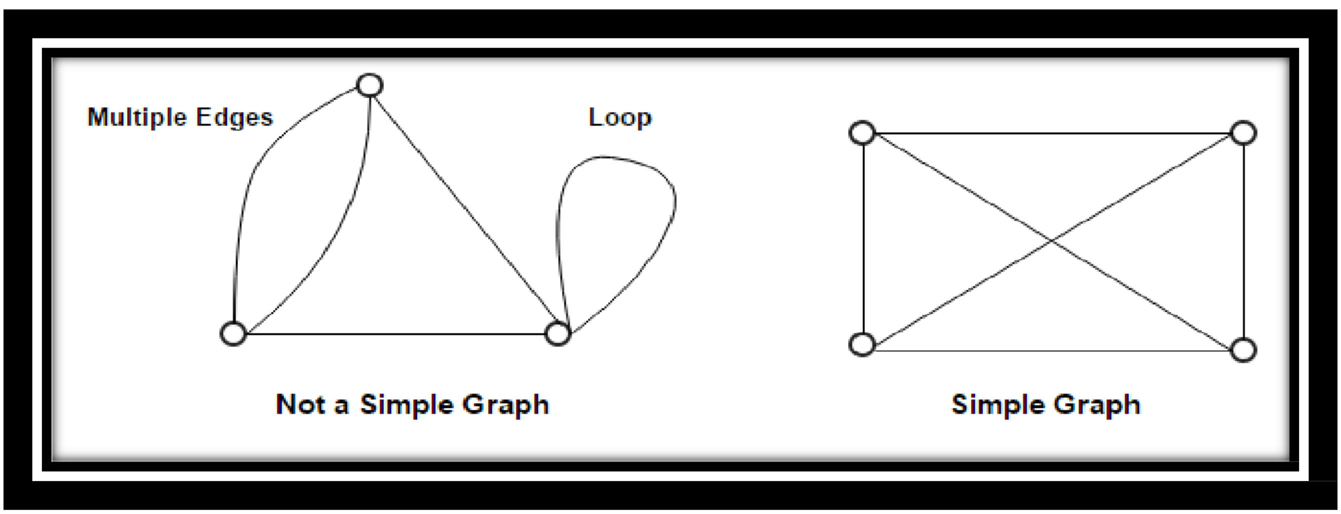
Multigraph:

A graph which contains some parallel edges is called a multigraph.



Pseudograph:

A graph in which loops are parallel edges are allowed is called a Pseudo graph.

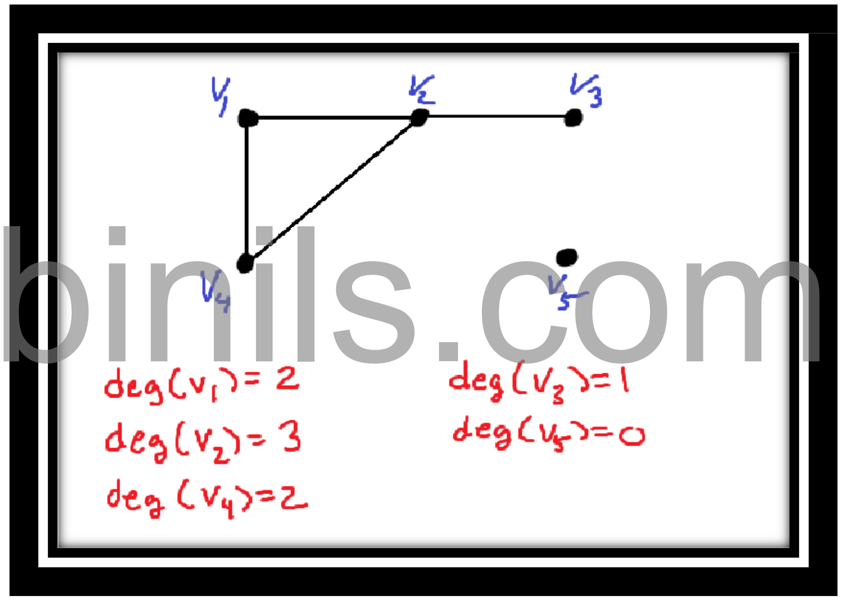


Graph Terminology:

Degree of a vertex:

The number of edges incident at the vertex v_i is called the degree of the vertex with self loops counted twice and it is denoted by $d(v_i)$.

Example:



(i) $d(v_1) = 2$

(ii) $d(v_2) = 3$

(iii) $d(v_3) = 1$

(iv) $d(v_4) = 2$

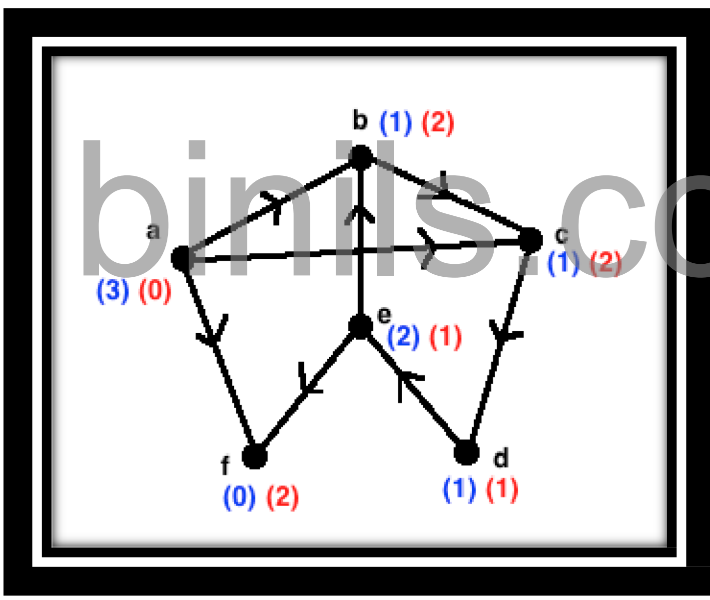
(v) $d(v_5) = 0$

In – degree and out – degree of a directed graph:

In a directed graph, the in – degree of a vertex V, denoted by $deg^{-}(V)$and defined by the number of edges with V as their terminal vertex.

The out – degree of V, denoted by $deg^{+}(V)$, is the number of edges with V as their initial vertex.

Example:



In – degree	Out – degree	Total degree
$deg^{-}(a) = 0$	$deg^{+}(a) = 3$	$deg(a) = 3$
$deg^{-}(b) = 2$	$deg^{+}(b) = 1$	$deg(b) = 3$
$deg^{-}(c) = 2$	$deg^{+}(c) = 1$	$deg(c) = 3$

$deg^{-}(d) = 1$	$deg^{+}(d) = 1$	$deg(d) = 2$
$deg^{-}(e) = 1$	$deg^{+}(e) = 2$	$deg(e) = 3$
$deg^{-}(f) = 2$	$deg^{+}(f) = 0$	$deg(f) = 2$

Note:

A loop at a vertex contributes 1 to both the in – degree and the out – degree of this vertex.

Theorem: 1(The Handshaking Theorem)

Let $G = (V, E)$ be an undirected graph with e edges then $\sum_{v \in V} deg(v) = 2e$.

The sum of degrees of all the vertices of an undirected graph is twice the number of edges of the graph and hence even.

Proof:

Since every edge is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertices.

All the 'e' edges contribute $(2e)$ to the sum of the degrees of vertices.

Hence $\sum_{v \in V} deg(v) = 2e$

Hence the proof.

Theorem: 2

In a undirected graph, the number of odd degree vertices are even.Proof:

Let V_1 and V_2 be the set of all vertices of even degree and set of all vertices of odd degree, respectively, in a graph $G = (V, E)$.

$\Rightarrow \sum d(v) = \sum_{v_i \in V_1} d(v_i) + \sum_{v_j \in V_2} d(v_j)$

By Handshaking theorem, we have

$$\Rightarrow 2e = \sum_{v_i \in V_1} d(v_i) + \sum_{v_j \in V_2} d(v_j)$$

Since each deg(v_i) is even, $\sum_{v_i \in V_1} d(v_i)$ is even

As left hand side of equation (1) is even and the first expression of the RHS of (1) is even, we have the second expression on the RHS must be even.

$\sum_{v_j \in V_2} d(v_j)$ is even.

Since each $d(v_j)$ is odd, the number of terms contained in $\sum_{v_j \in V_2} d(v_j)$ is even.

i.e., The number of vertices of odd degree is even.

Hence the proof.

Theorem: 3

The maximum number of edges in a simple graph with "n" vertices is $\frac{n(n-1)}{2}$

Proof:

We prove this theorem by the principle of Mathematical induction.

For $n = 1$, a graph with one vertex has no edges.

The result is true for $n = 1$.

For $n = 2$, a graph with 2 vertices may have atmost one edge.

$\Rightarrow \frac{2(2-1)}{2} = 1$

The result is true for $n = 2$.

Assume that the result is true for $n = k$.

i.e., a graph with k vertices has atmost $\frac{k(k-1)}{2}$ edges.

When $n = k + 1$, let G be a graph having "n" vertices and G' be the graph obtained from G by deleting one vertex say $v \in V(G)$.

Since G' has k vertices, then by the hypothesis G' has atmost $\frac{k(k-1)}{2}$ edges.

Now add the vertex "v" to G'. Such that "v" may be adjacent to all the k vertices of G'.

The total number of edges in G are,

$$\frac{k(k-1)}{2} + k = \frac{k^2-k+2k}{2}$$

$$= \frac{k^2+k}{2}$$

$$= \frac{k(k+1)}{2}$$

$$= \frac{(k+1)(k+1-1)}{2}$$

The result is true for $n = k+1$

Hence the maximum number of edges in a simple graph with "n" vertices is $\frac{n(n-1)}{2}$.

Hence the proof.

Theorem: 4

If all the vertices of an undirected graph are each of degree k, show that the number of edges of the graph is a multiple of k.

Proof:

Let $2n$ be the number of vertices of the given graph.......... (1)

Let n_e be the number of edges of the given graph.

By Handshaking theorem, we have $\sum_{i=1}^{2n} \deg V_i = 2\, n_e$

$\Rightarrow 2nk = 2n_e$ using (1)

$\Rightarrow nk = n_e$

$\Rightarrow$ number of edges = multiple of k.

The number of edges of the given graph is a multiple of k.

Example:1

How many edges are there in a graph with ten vertices each of degree six.

Solution:

Let e be the number of edges of the graph.

$\Rightarrow 2e =$ Sum of all degrees

$= 10 \, X \, 6 \, = \, 60$

$\Rightarrow 2e = \, 60$

$\Rightarrow e = \, 30$

There are 30 edges.

Example: 2

Can a simple graph exist with 15 vertices of degree 5.

Solution:

$\Rightarrow 2e = \sum d(v)$

$\Rightarrow 2e = 15 X 5 = 75$

$\Rightarrow e = \frac{75}{2}$

Which is not an integer.

Such a graph does not exist.

(or) By theorem (2) in a graph the number of odd degree vertices is even.

Therefore, it is not possible to have 15 vertices, which is of odd degree.

Such a graph does not exist.

Example: 3

For the following degree sequences 4, 4, 4, 3, 2 find if there exist a graph or not.

Solution:

Sum of the degree of all vertices $= 4 + 4 + 4 + 3 + 2 = 17$

Which is an odd number.

Such a graph does not exist.

Example: 4

Does there exist a simple graph with five vertices of the following degrees? If so draw such graph (a) 1, 1, 1, 1, 1 (b) 3, 3, 3, 3, 2

Solution:

We know that in any graph the number of odd degree vertices is always a even.In case (a) number of odd degree vertices is 5 (not an even)

Such graph does not exist.

For case (b)

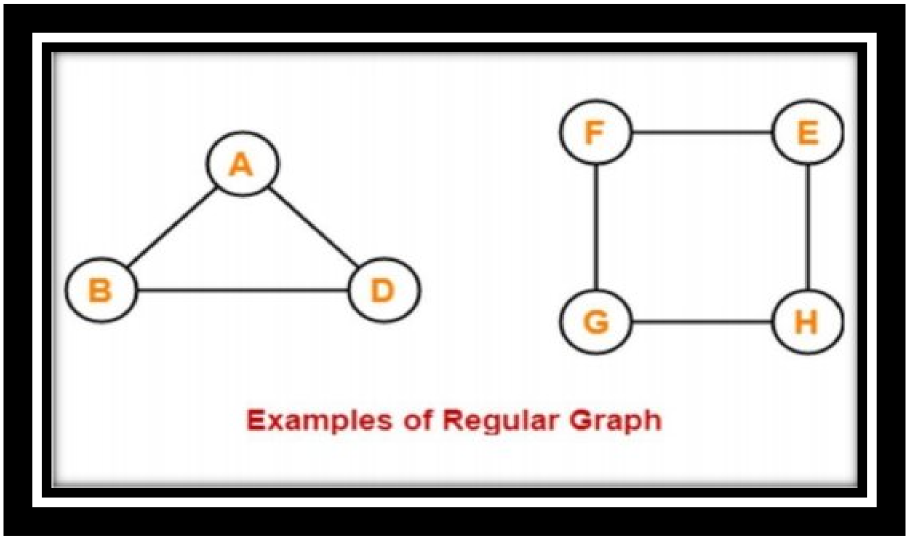
Sum of degree = 14 = even

The graph exist.

Special Types of Graphs

Regular Graph

If every vertex of a simple graph has the same degree, then the graph is called a regular graph.



If every vertex in a regular graph has degree k, then the graph is called k – regular.

Complete Graph

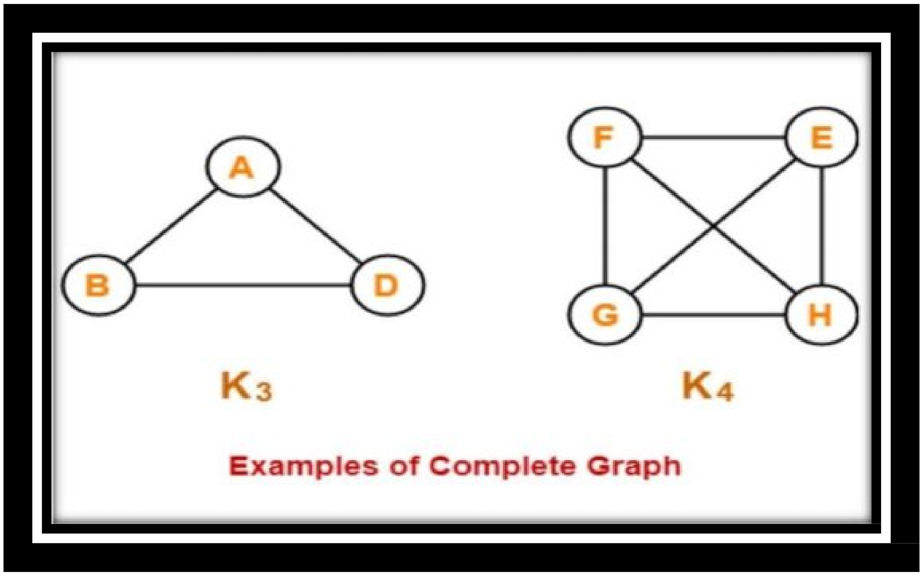
In a graph, if there exist an edge between every pair of vertices, then such a graph is called complete graph.

In a graph if every pair of vertices are adjacent then such a graph is called complete graph.

If is noted that, every complete graph is a regular graph. In fact every complete graph with n vertices is a $(n-1)$ regular graph.

The complete graph on n vertices is denoted by k_n. The graphs k_n for

$n = 1, 2, 3, 4, 5$ are



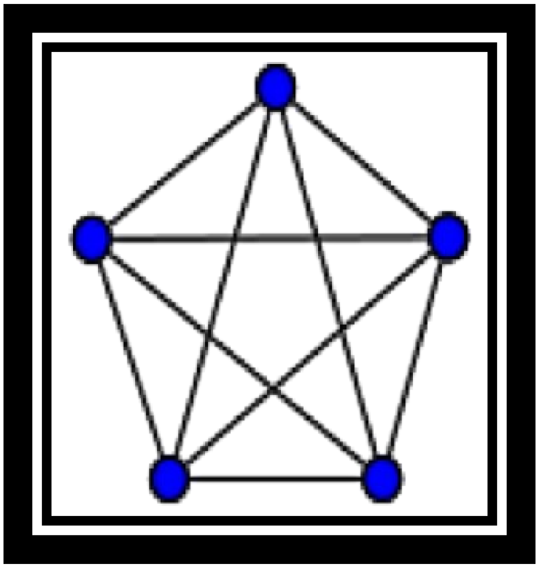
Example: 1

Draw the complete graph k_5 with vertices A, B, C, D, E. Draw all complete sub graph of k_5 with 4 vertices.

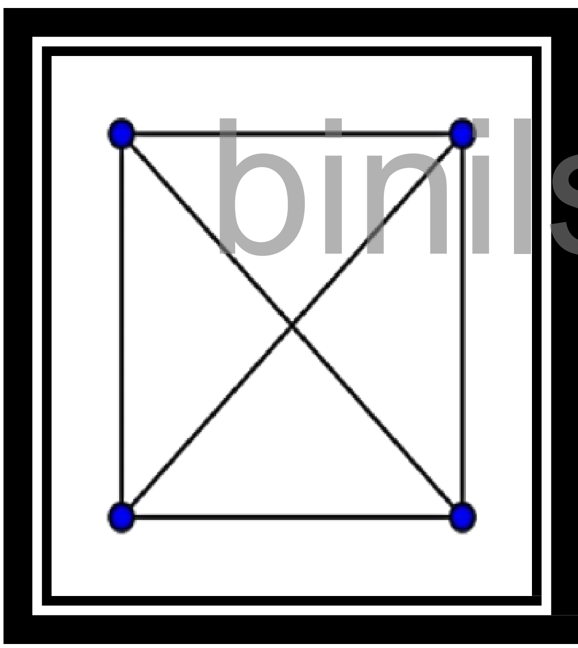
Solution:

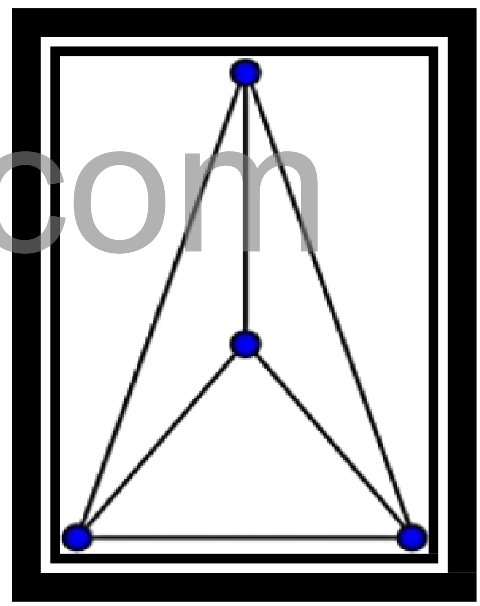
In a graph, if there exist an edge between every pair of vertices, then such a graph is called complete graph.

(i) e., In a graph if every pair of vertices are adjacent, then such a graph is calledcomplete graph. Complete graph k_5 is



Now, complete subgraph of k_5 with 4 vertices are





Bipartite Graph

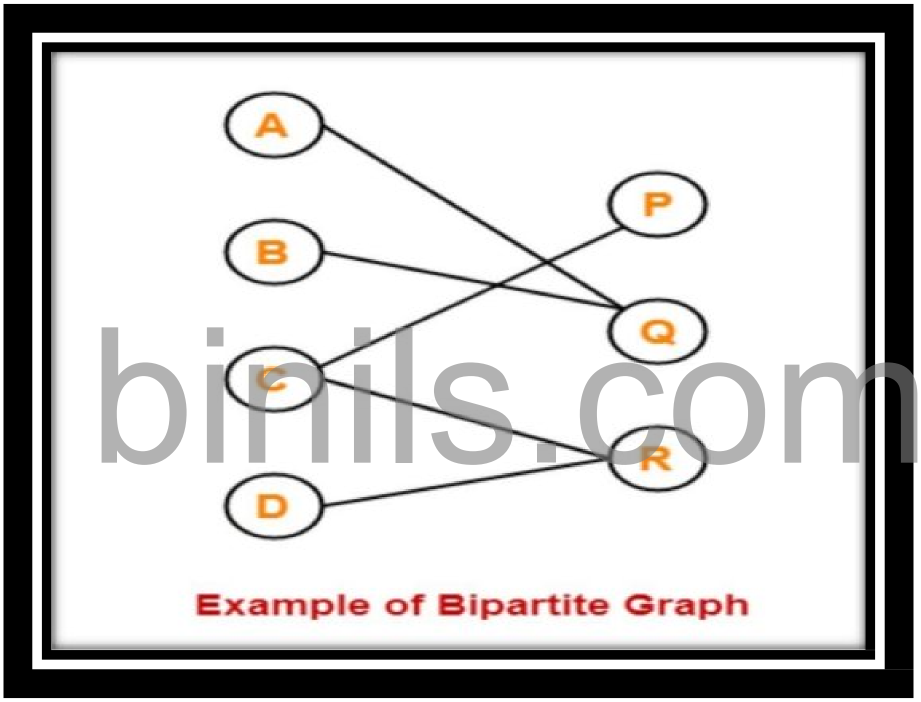
A graph G is said to be bipartite if its vertex set $V(G)$ can be partitioned into two disjoint non empty sets V_1 and V_2, $V_1 \cup V_2 = V(G)$, such that every edge in $E(G)$

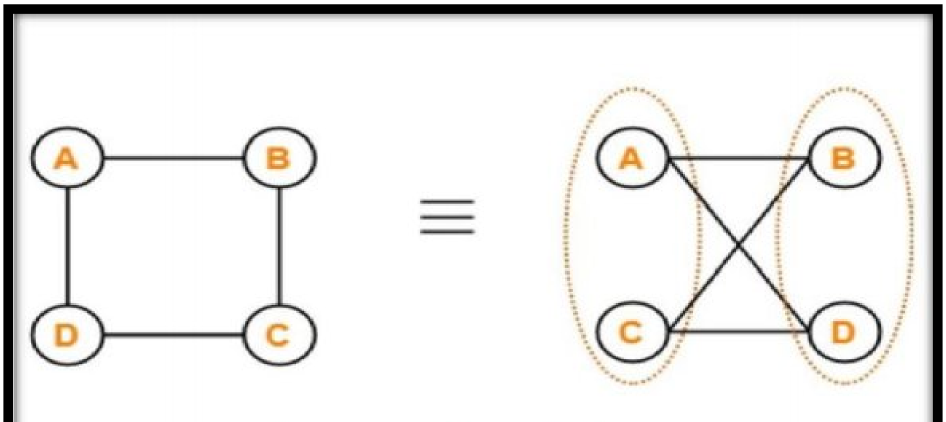
has one end vertex in V_1 and another end vertex in V_2 . (So that no edges in ,

connects either two vertices in V_1 or two vertices in V_2.)

For example, consider the graph G

Then G is a Bipartite graph.



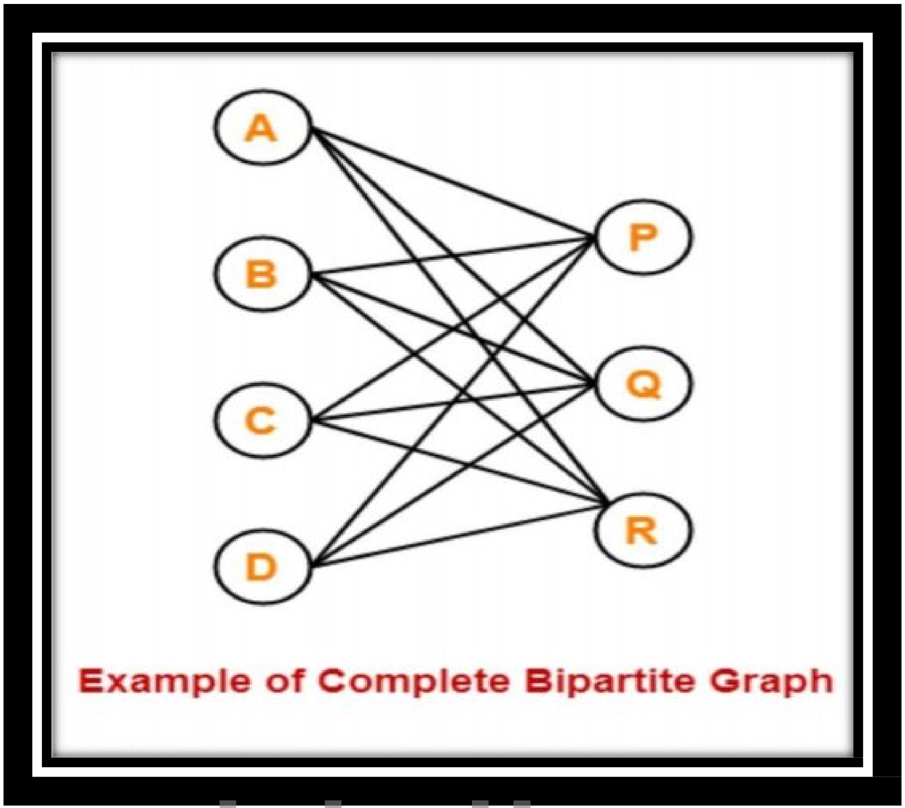


Example of Bipartite Graph

Complete Bipartite Graph:

A bipartite graph G, with the partition V_1 and V_2, is called complete bipartite graph, if every vertex in V_1 is adjacent to every vertex in V_2. Clearly, every vertex in V_2 is adjacent to every vertex in V_1.

A complete bipartite graph with 'm' and 'n' vertices in the bipartition is denoted by $k_{m,n}$.



Subgraph:

A graph $H = (V', E')$ is called a subgraph of $G = (V, E)$, if $V' \subseteq V$ and $E' \subseteq E$.

In other words, a graph H is said to be a subgraph is said to be a subgraph of G, if all the vertices and all the edges of H are in G and if the adjacency is preserved in H exactly as in G.

Hence, we have the following

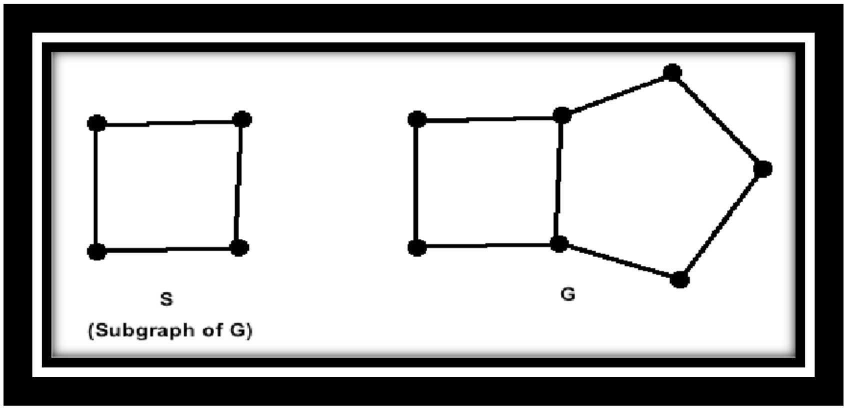
(i)Each graph has its own subgraph.

(ii) A single vertex in a graph G is a subgraph of G.

(iii) A single edge in G, together with its end vertices is also a subgraph of G.

(iv) A subgraph of a subgraph of G is also a subgraph of G.

(v)H is a proper subgraph of G if $H \neq G$.



Graph representation:

Adjacency Matrix of a simple graph:

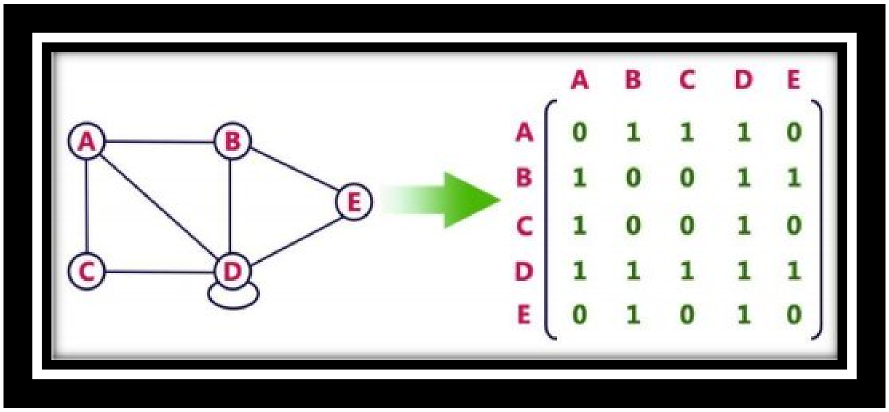
Let $G = (V, E)$ be a simple graph with n – vertices $\{v_1, v_2, \ldots, v_n\}$. Its adjacency matrix is denoted by $A = [a_{ij}]$ and defined by

$$A = [a_{ij}] = \begin{cases} 1, \text{if there exist an edge between } v_i \text{ and } v_j \\ 0, \text{otherwise} \end{cases}$$

Example: 1

Find adjacency matrix of the graph given below.

Solution:

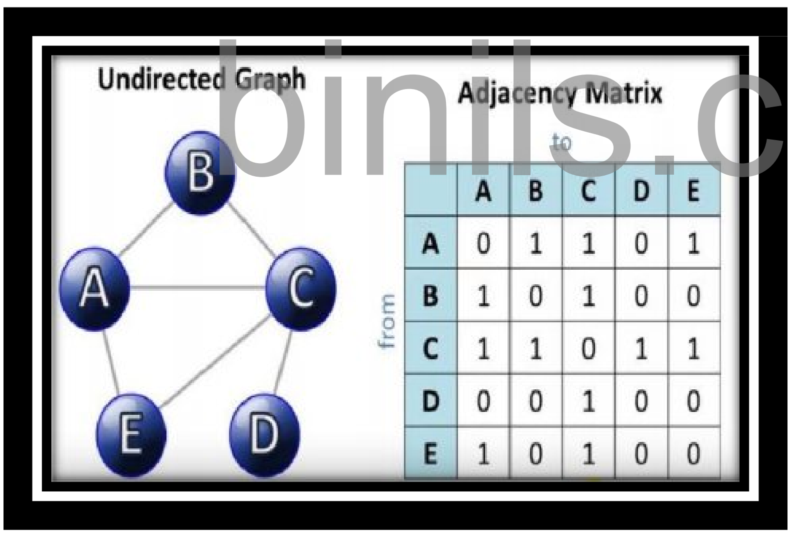


	A	B	C	D	E
A	0	1	1	1	0
B	1	0	0	1	1
C	1	0	0	1	0
D	1	1	1	1	1
E	0	1	0	1	0

Example: 2

Find adjacency matrix of the graph given below.

Solution:



	A	B	C	D	E
A	0	1	1	0	1
B	1	0	1	0	0
C	1	1	0	1	1
D	0	0	1	0	0
E	1	0	1	0	0

Incidence matrices:

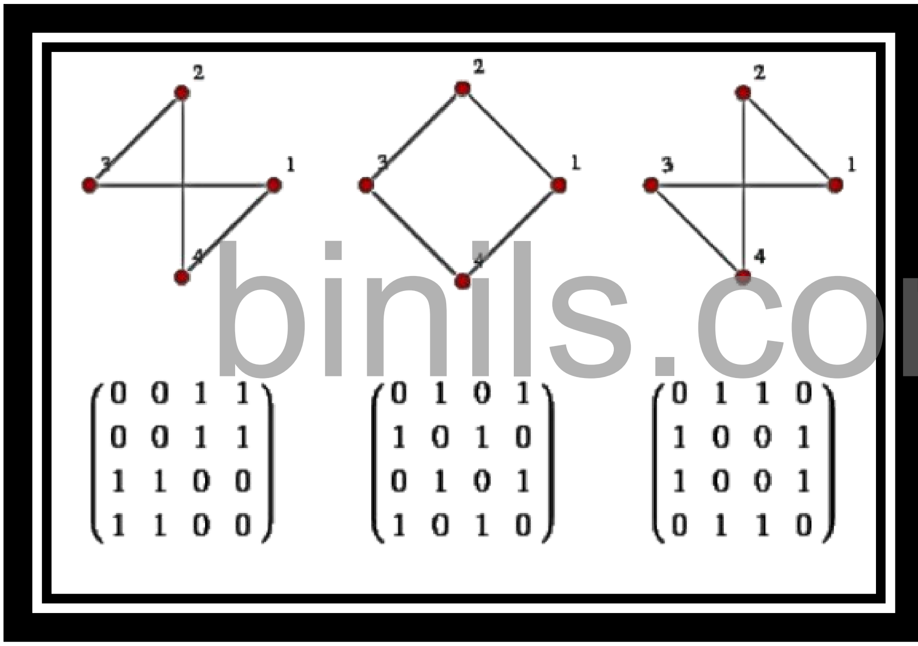
Let $G = (V, E)$ be an undirected graph with n vertices $\{V_1, V_2, \ldots, V_n\}$ and m edges $\{e_1, e, \ldots, e_m\}$. Then the $(n \times m)$ matrix $B = [b_{ij}]$, where

$$B = [b_{ij}] = \{ \begin{matrix} 1, when\ edge\ e_j\ incident\ on\ V_i \\ 0, otherwise \end{matrix}$$

Example: 1

Find incidence matrix of the following graph and your observations regarding the entries of B.

Solution:



Path Matrix:

Let $G = (V, E)$ be a simple digraph in which $|V| = n$ and the nodes of g are assumed to be ordered. An $n \times n$ matrix P whose elements are given by

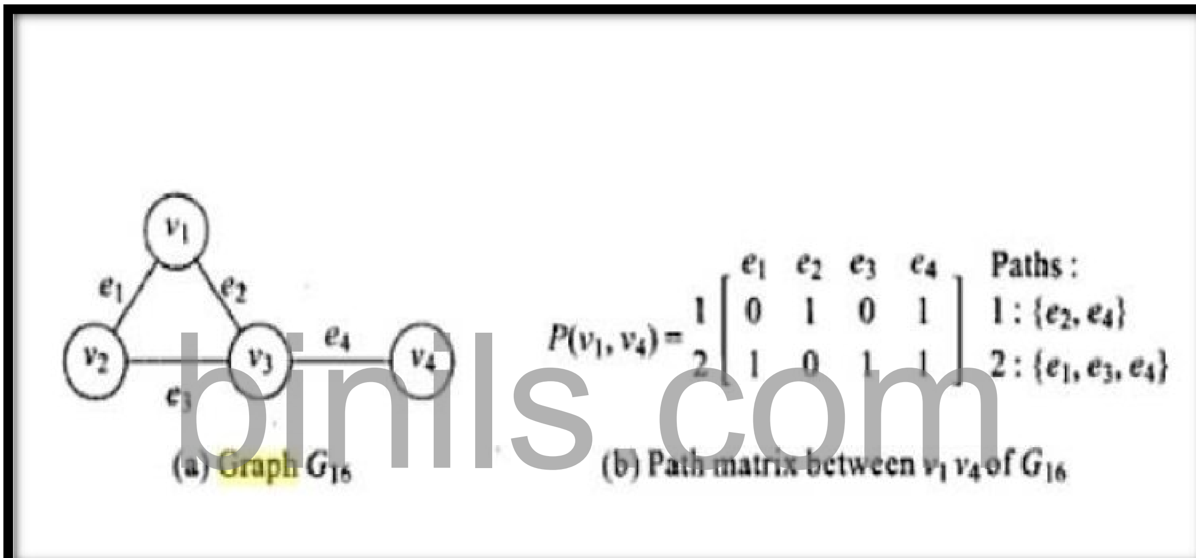
$$P_{ij} = \{ \begin{matrix} 1, If\ there\ exists\ a\ path\ from\ V_i\ to\ V_j \\ 0, otherwise \end{matrix}$$

is called a path matrix (reachability matrix) of the graph G.

Example: 1

Find path matrix

Solution:



(a) Graph G_{16} (b) Path matrix between v_1 v_4 of G_{16}

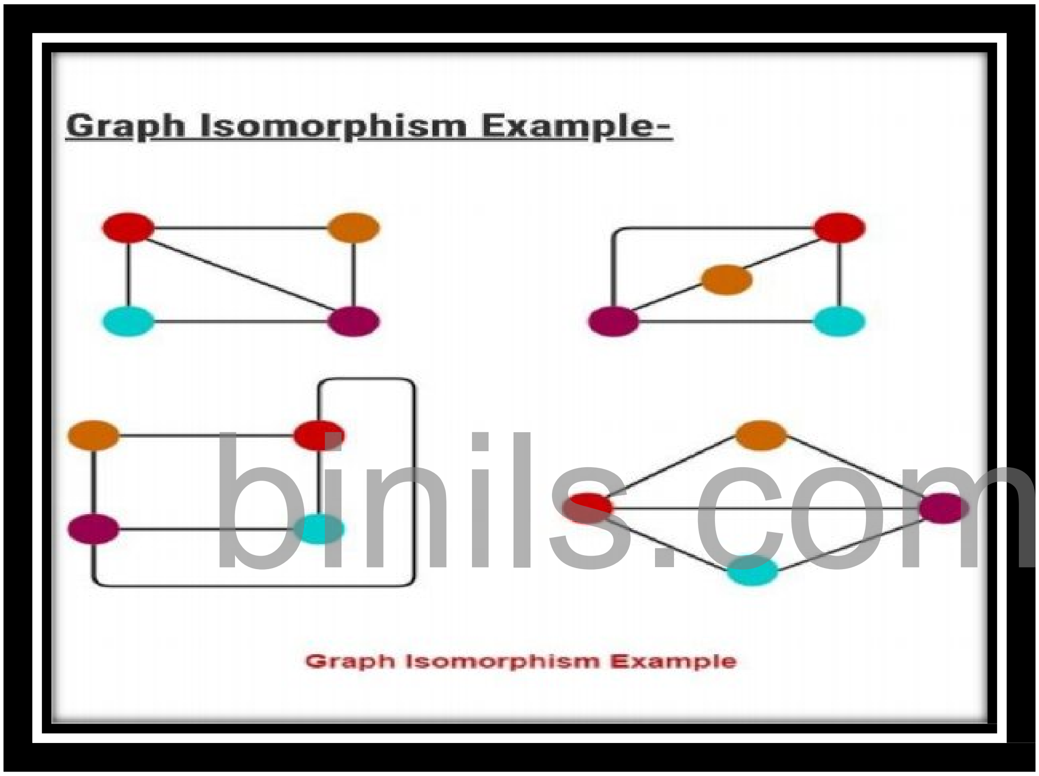
Note:

Path Matrix is very useful in communications and transportation networks.

Graph Isomorphism:

Two graphs G_1 and G_2 are said to be isomorphic to each other, if there exist a one – to –one correspondence between the vertex sets which preserves adjacency of the vertices.

The Graph $G_1 = (V_1, E_1)$ is isomorphic to the graph $G_2 = (V_2, E_2)$ if there is a one – to – one correspondence between the vertex sets V_1 and V_2 and the edge sets $E_1$and E_2 in such a way that if e_1 is incident on u_1 and V_1 in G_1, then the corresponding edge e_2 in G_2 is incident on u_2 and V_2 which correspondence is called graph isomorphism.



However, the definition of isomorphism of two graphs were easy, but the given graph having "n" vertices itself has n! ways of one – to – one correspondence.

So, before going to isomorphism, we can verify whether they have the same number of vertices and edges and if the degree sequence of the graphs are same. If not, the we can say the graphs are not isomorphic.

Note:

If G_1 and G_2 are isomorphic then G_1 and G_2 have

(i)The same number of vertices.

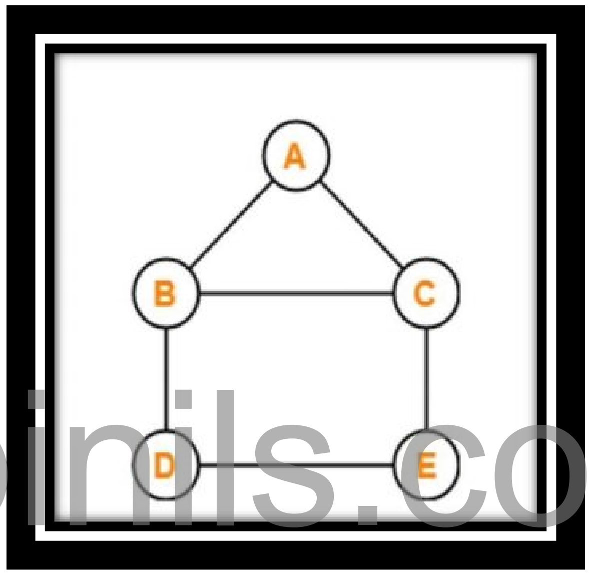
(ii) The same number of edges.

(iii) An equal number of vertices with a given degree.

However, these conditions are not sufficient for graph isomorphism.

Paths, Reachability and Connectedness:

A path is a graph is a sequence $v_1, v_2, v_3, \ldots, v_k$ of vertices each adjacent to the next. In other words, starting with the vertex v_1, one can travel along edges $(v_1, v_2), (v_2, v_3)$, ... and reach the vertex v_k.



Length of the path:

The number of edges appearing in the sequence of a path is called the length of path.

Cycle or Circuit:

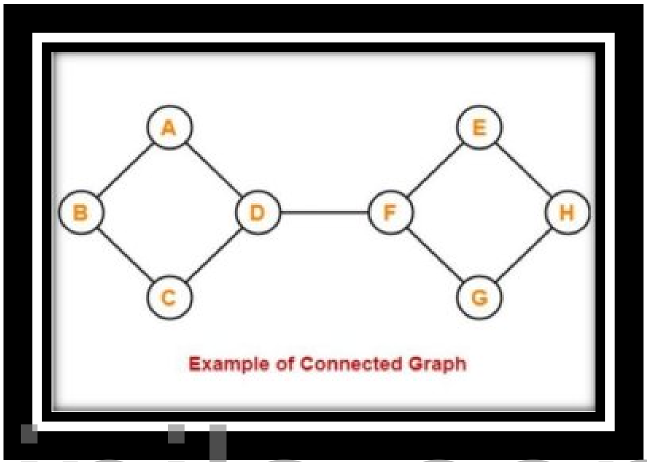
A path which originates and ends in the same node is called a cycle of circuit.

A path is said to be simple if all the edges in the path are distinct.

A path in which all the vertices are traversed only once is called an elementary path.

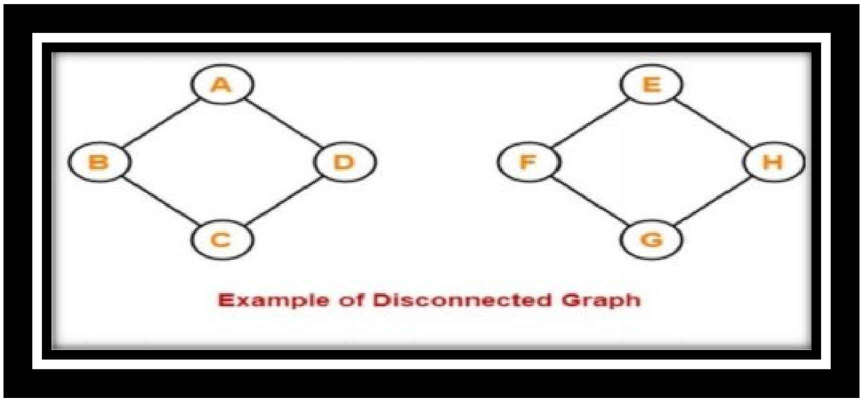
Connected Graph:

An directed graph is said to be connected if any pair of nodes are reachable from one another. That is, there is a path between any pair of nodes.



Disconnected graph:

A graph which is not connected is called disconnected graph.



Theorem: 1

If a graph has n vertices and a vertex v is connected to a vertex w, then there exists a path from v to w of length not more than $(n-1)$.

Proof:

Let $v, u_1, u_2, \ldots, u_{m-1}, w$ be a path in G from v to w.

By definition pf path, the vertices $v, u_1, u_2, \ldots, u_{m-1}$ and w all are distinct.

As G, contains only "n" vertices, it follows that $m + 1 \leq n$

$$\Rightarrow m \leq n - 1$$

Hence the proof.

Theorem: 2

Prove that a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Proof:

Let G be a simple graph with n vertices and more than $\frac{(n-1)(n-2)}{2}$ edges.

Suppose if G is not connected, then G must have atleast two components. Let it be G_1 and G_2.

Let V_1 be the vertex set of G_1 with $|V_1| = m$. If V_2 is the vertex set of G_2, then

$|V_2| = n - m$.

Then (i) $1 \leq m \leq n - 1$

(ii) There is no edge joining a vertex of V_1 and a vertex of V_2.

(iii) $|V_2| = n - m \geq 1$

Now, $|E(G)| = |E(G_1 \cup G_2)|$

$$= |E(G_1)| + |E(G_2)|$$

$$\leq \frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2}$$

$$= \frac{1}{2}[m^2 - m + n(n - m - 1) - m(n - m - 1)]$$

$$= \frac{1}{2}[n(n - 1) - nm - m(n - m - 1) + m^2 - m]$$

$$= \frac{1}{2}[(n - 1)(n - 2) + 2(n - 1) - 2nm + m^2 + m + m^2 - m]$$

Adding and Subtracting $2n - 2$

$$= \frac{1}{2}[(n - 1)(n - 2) + 2n - 2 - 2nm + 2m^2]$$

$$= \frac{1}{2}[(n - 1)(n - 2) + 2n(1 - m) + 2(m^2 - 1)]$$

$$= \frac{1}{2}[(n - 1)(n - 2) - 2n(m - 1) + 2(m - 1)(m + 1)]$$

$$= \frac{1}{2}[(n-1)(n-2) - 2(m-1)(n-m-1)]$$

$|E(G)| \leq \frac{(n-1)(n-2)}{2}$, Since $(m-1)(n-m-1) \geq 0$ for $1 \leq m \leq n-1$

Which is a contradiction as G has more than $\frac{(n-1)(n-2)}{2}$ edges.

Hence G is a connected graph.

Hence the proof.

Theorem: 3

Let G be a simple graph with n vertices. Show that if $\delta(G) \geq [\frac{n}{2}]$, then G is connected where $\delta(G)$ is minimum degree of the graph G.

Proof:

Let u and v be any two distinct vertices in the graph G.

We claim that there is a $u - v$ path in G.

Suppose uv is not an edge of G. Then, X be the set of all vertices which are adjacent to u and Y be the set of all vertices which are adjacent to v.

Then $u, v \notin X \cup Y$. (Since G is a simple graph)

And hence $|X \cup Y| \leq n - 2$

We have $|X| = deg\ (u) \geq \delta(G) \geq [\frac{n}{2}]$ and $|Y| = deg\ (v) \geq \delta(G) \geq [\frac{n}{2}]$

Now, $|X| + |Y| \geq [\frac{n}{2}] + [\frac{n}{2}] = n \geq n - 1$

We know that $|X \cup Y| = |X| + |Y| - |X \cap Y|$

$n - 2 \geq |X \cup Y| \geq n - 1 - |X \cap Y|$

We have, $|X \cap Y| \geq 1 \Rightarrow X \cap Y \neq \emptyset$

Now, take a vertex $w \in X \cap Y$. Then uvw is a $u - v$ path in G.

Thus for every pair of distinct vertices of G there is a path between them.

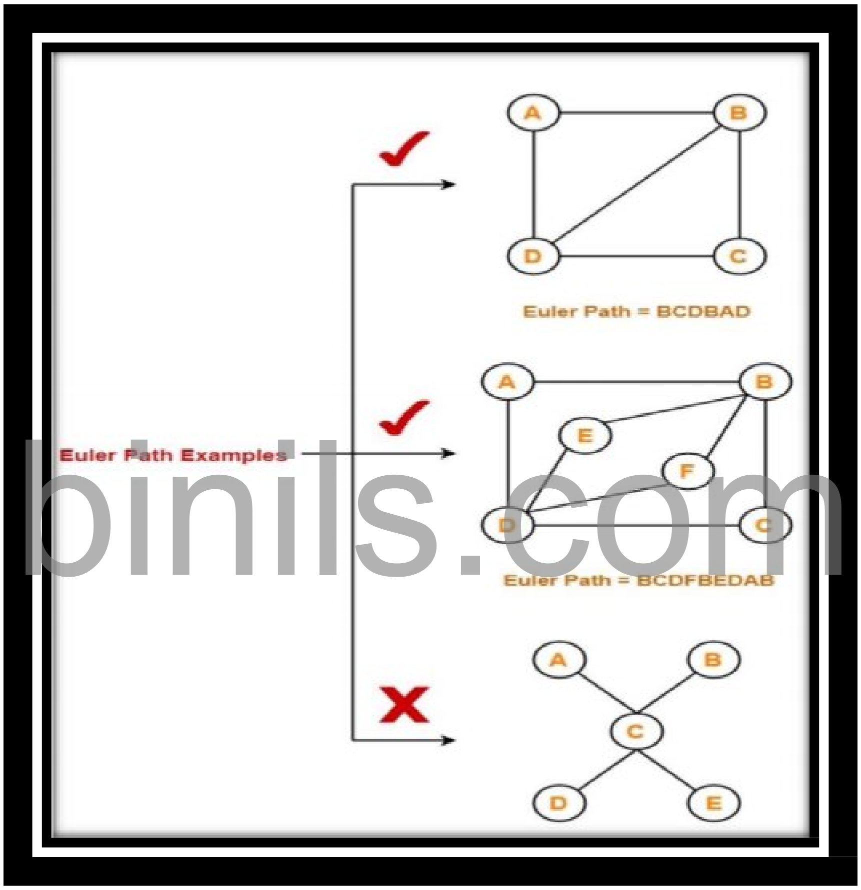
Hence G is connected.

Hence the proof.

Euler graph and Hamilton graph:

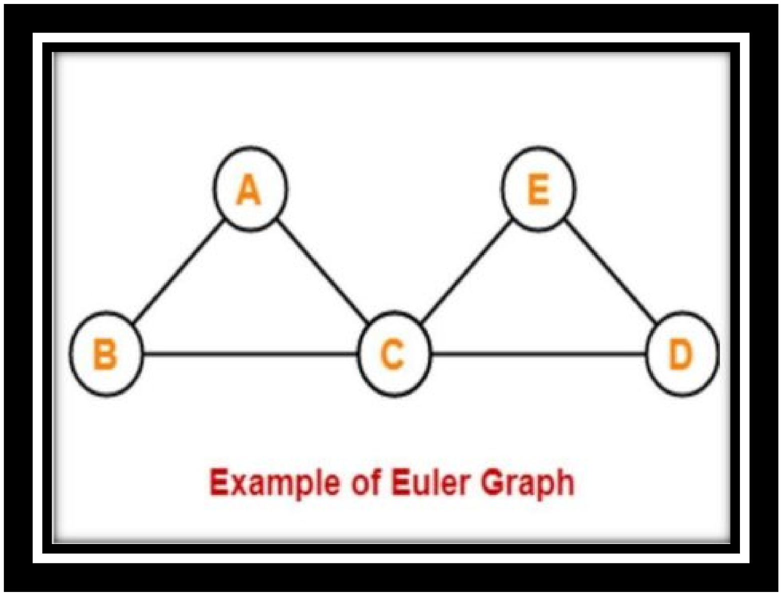
Euler path:

A path of a graph G is called an Eulerian path, if it contains each edge of the graph exactly once.



Euler graph:

A path of a graph G is called an Eulerian path, if it contains each edge of the graph exactly once.



Eulerian Circuit or Eulerian Cycle:

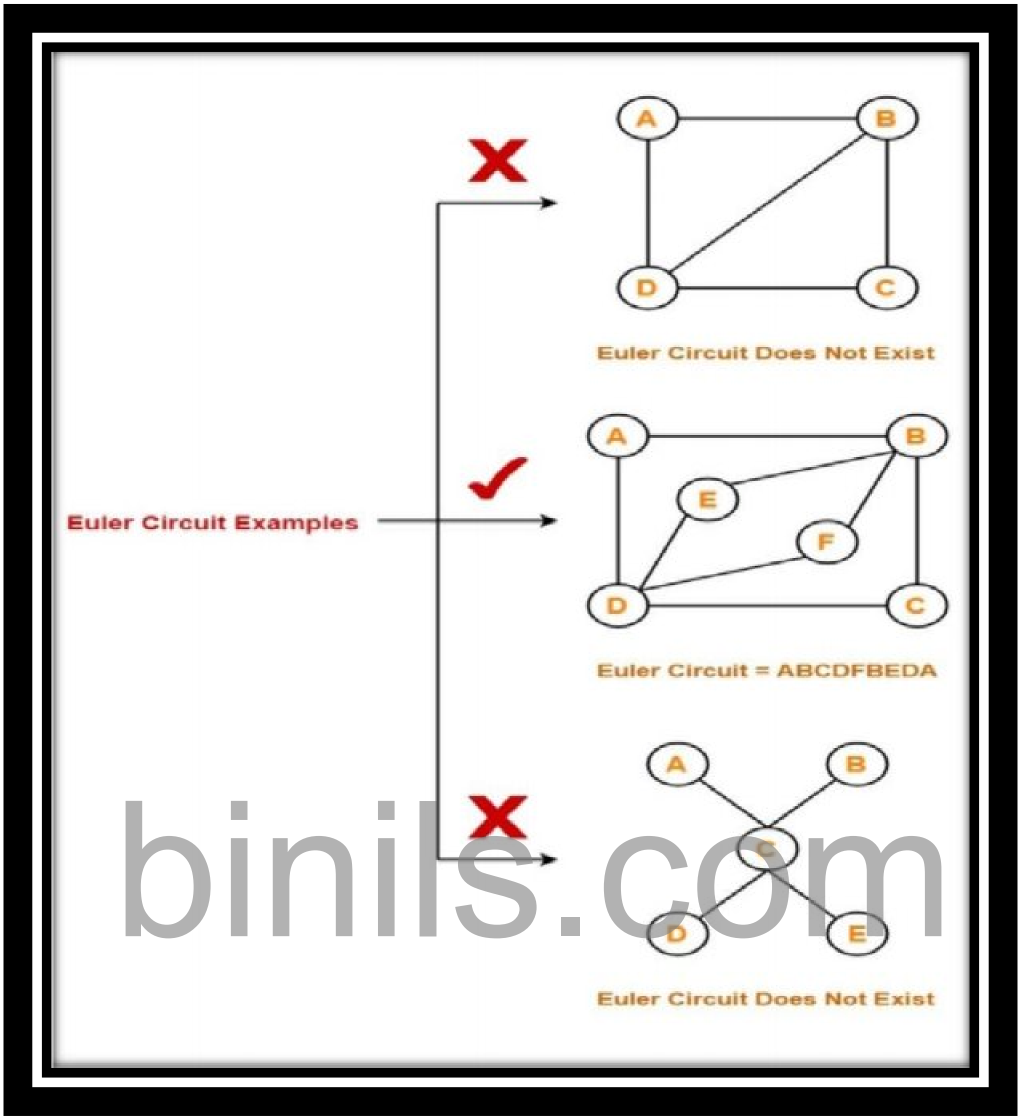
A circuit or cycle of a graph G is called Eulerian circuit or cycle, if it includes each edge of G exactly once.

Here starting and ending vertex are same.

An Eulerian circuit or cycle should satisfies the following conditions:

Starting and ending points (vertices) are same.

Cycle should contain all the edges of graph but exactly once.



Eulerian Graph or Euler graph:

Any graph containing an Eulerian circuit or cycle is called an Eulerian graph.

Theorem:1

A connected graph is Euler graph (contains Eulerian circuit) if and only if each of its vertices is of even degree.

Proof:

Let G be any graph having an Eulerian circuit and let "C" be an Eulerian circuit of G with origin vertex as u. Each time a vertex occurs as an internal vertex of C, then two of the edges incident with v are accounted for degree.

We, get, for internal vertex $v \in v(G)$

$d(v) = 2 \times$ number of times v occur inside the Euler circuit C

= even degree

And, since an Euler circuit C contains every edge of G and C starts and ends at u.

$d(u) = 2 + 2 + \times$ number of times u occur inside C.

= even degree Hence G has all the vertices of even degree.

Conversely, assume each of its vertices has an even degree.

Claim:

G has an Eulerian circuit.

Assume G be a connected graph which is not having an Euler circuit, with all vertices of even degree and less number of edges. That is, any graph having less number of edges than G, then it has an Eulerian circuit. Since each vertex of G has atleast two, therefore G contains closed path. Let C be a closed path of maximum possible length in G. If C itself has all the edges of G, then G itself an Euler circuit in G.

By assumption, C is not an Euler circuit of G and $G - E(C)$ has some component G' with $|E(G')| > 0$. C has less number of edges than G, therefore C itself is an Eulerian, and C has all the vertices of even degree.

Since $|E(G')| < |E(G)|$, therefore G' has an Euler circuit C'. Because G is connected, there is a vertex v in both C and C'. Now join C and C' and traverse all the edges of C and C' with common vertex v, we get CC' is a closed path in G and $E(CC') > E(C)$ which is not possible choices of C.

Hence G has an Eulerian circuit.

Hence G is a Euler graph.

Hence the proof.

Theorem:2

Prove that if a graph G has not more than two vertices of odd degree, then there can be Euler path in G.

Proof:

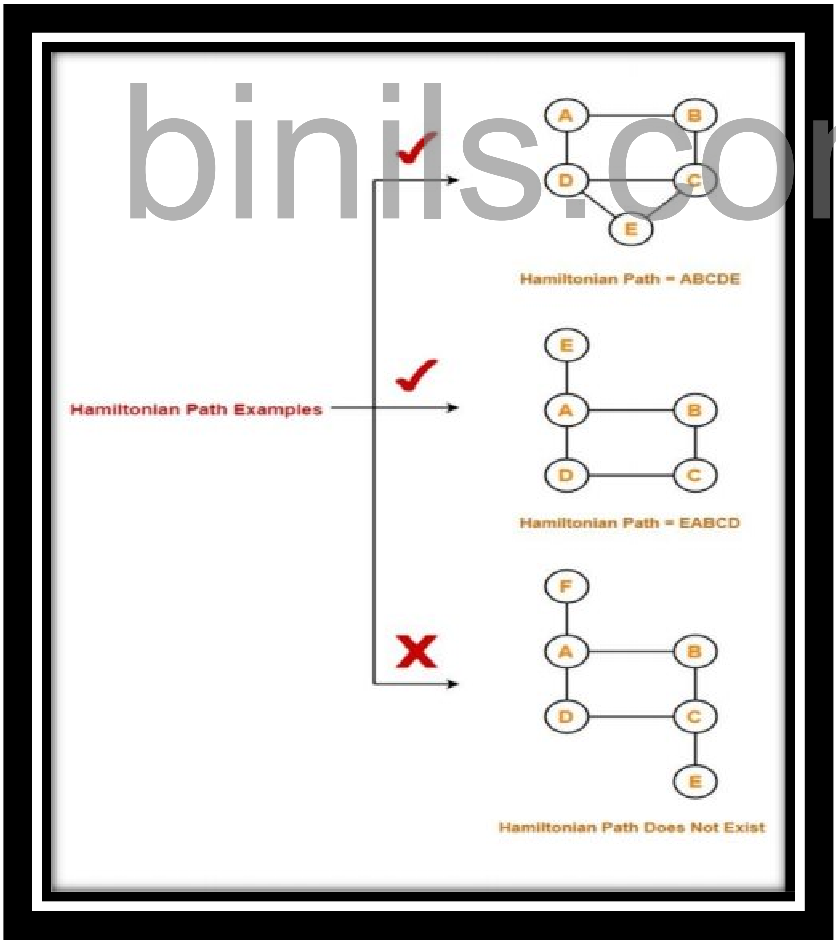
Let the odd degree vertices be labelled as V and W in any arbitrary order. Add an edge of G between the vertex pair (V, W) to form a new graph G'.

Now every vertex of G' is of even degree and hence G' has an Eulerian trial T. If the edge that we added to G is now removed from T, it will split into an open trail containing all edges of G which is nothing but an Euler path in G.

Hamiltonian Graph:

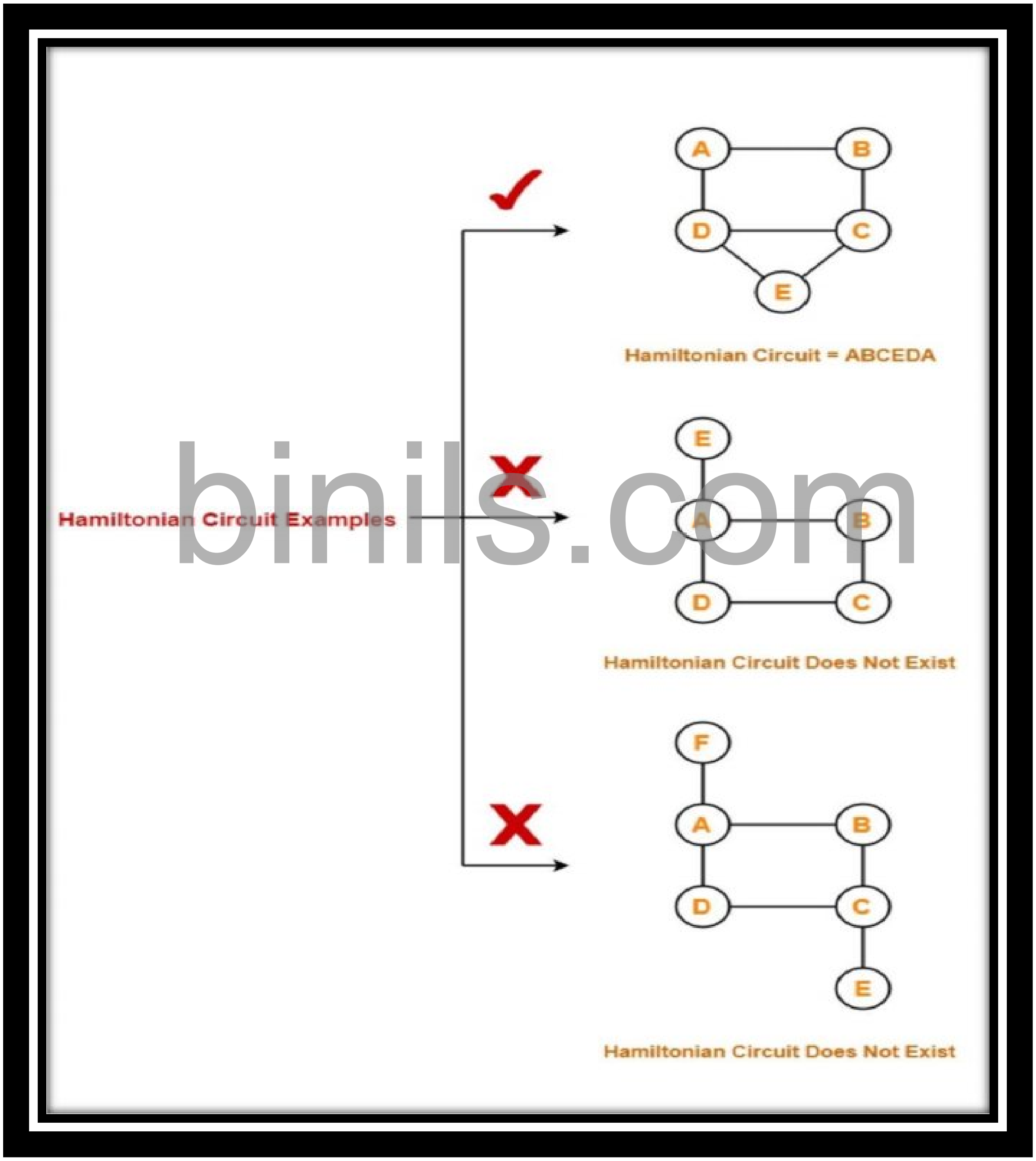
Hamiltonian Path:

A path of a graph G is called a Hamiltonian path, if it includes each vertex of G exactly once.



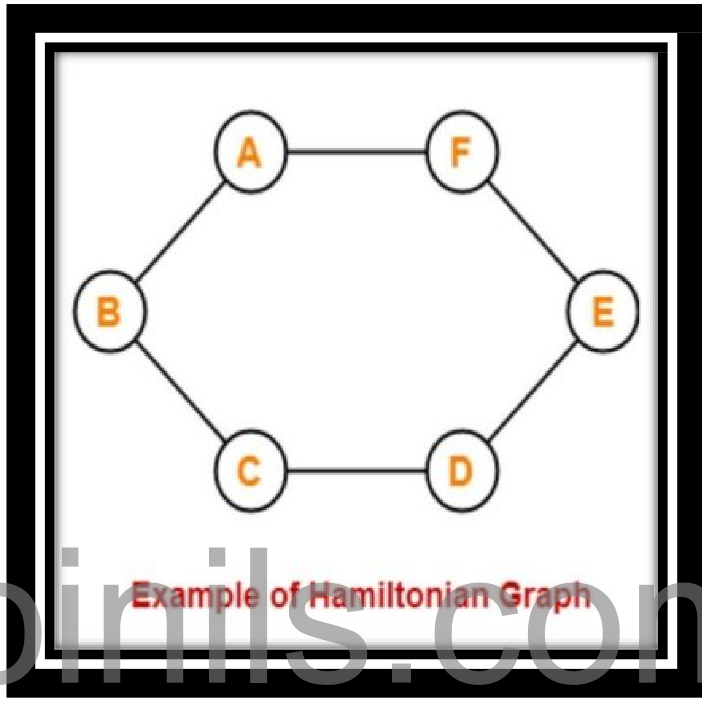
Hamiltonian Circuit or Cycle:

A circuit of a graph G is called a Hamiltonian circuit, if it includes each vertex of G exactly once, except the starting and ending vertices.



Hamiltonian graph:

Any graph containing a Hamiltonian circuit or cycle is called a Hamiltonian graph.



Properties:

(i) A Hamiltonian circuit contains a Hamiltonian path, but a graph containing a Hamiltonian path need not have a Hamiltonian cycle.

(ii) By deleting any one edge from Hamiltonian cycle, we can get Hamiltonian path.

(iii) A graph may contain more than one Hamiltonian cycle.

(iv) A complete graph k_n, will always have a Hamiltonian cycle, when $n \geq 3$.

(v) A graph with a vertex of degree one cannot have a Hamiltonian cycle.

Theorem: 1

Let G be a simple indirected graph with n vertices. Let u and v be two nonadjacent vertices in G such that $deg(u) + deg(v) \geq n$ in G. Show that G is Hamiltonian if and only if $G + uv$ is Hamiltonian.

Proof:

If G is Hamiltonian, then obviously $G + uv$ is Hamiltonian.

Conversely, suppose that $G + uv$ is Hamiltonian, but G is not.

Then by Dirac theorem, we have $deg(u) + deg(v) < n$

Which is a contradiction to our assumption.

Thus $G + uv$ is Hamiltonian implies G is Hamiltonian.

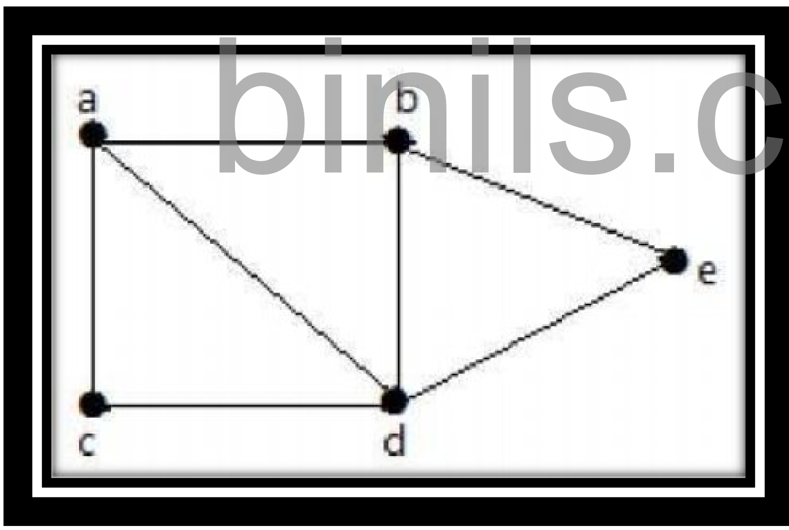
Hence the proof.

Connectivity:

A graph is said to be connected if there is a path between every pair of vertex. From every vertex to any other vertex, there should be some path to traverse. That is called the connectivity of a graph. A graph with multiple disconnected vertices and edges is said to be disconnected.

Example 1

In the following graph, it is possible to travel from one vertex to any other vertex. For example, one can traverse from vertex 'a' to vertex 'e' using the path 'a-b-e'.



Theorem: 1

Show that graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in V_1 and the other in V_2.

Proof:

Suppose that such a partitioning exists. Consider two arbitrary vertices a and b of G such that $a \in V_1$ and $b \in V_2$.

No path can exist between vertices a and b.

Otherwise, there would be atleast one edge whose one end vertex be in V_1 and the other in V_2.

Hence if partition exists, G is not connected.

Conversely, let G be a disconnected graph.

Consider a vertex a in G.

Let V_1 be the set of all vertices that are joined by paths to a.

Since G is disconnected, V_1 does not include all vertices of G.

The remaining vertices will form a set V_2.

No vertex in V_1 is joined to any in V_2 by an edge.

Hence the partition.

Hence the proof.

Components of a graph:

The connected subgraphs of a graph G are called components of the graph G.

Theorem: 1

A simple graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof:

Let $n_1, n_2, \ldots, n_k$ be the number of vertices in each of k components of the graph G.

Then $n_1 + n_2 + \ldots + n_k = n = |V(G)|$

$\sum_{i=1}^{k} n_i = n \quad \ldots (1)$

Now, $\sum_{i=1}^{k} (n_i - 1) = (n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1)$

$$= \sum_{i=1}^{k} n_i - k$$

$$\Rightarrow \sum_{i=1}^{k} (n_i - 1) = n - k$$

Squaring on both sides

$$\Rightarrow \left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n - k)^2$$

$$\Rightarrow (n_1 - 1)^2 + (n_2 - 1)^2 + \ldots + (n_k - 1)^2 \leq n^2 + k^2 - 2nk$$

$$\Rightarrow {n_1}^2 + 1 - 2n_1 + {n_2}^2 + 1 - 2n_2 + \ldots + {n_k}^2 + 1 - 2n_k \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} {n_i}^2 + k - 2n \leq n^2 + k^2 - 2nk$$

$$\Rightarrow \sum_{i=1}^{k} {n_i}^2 \leq n^2 + k^2 - 2nk + 2n - k$$

$$\Rightarrow \sum_{i=1}^{k} {n_i}^2 = n^2 + k^2 - k - 2nk + 2n$$

$$= n^2 + k(k-1) - 2n(k-1)$$

$$= n^2 + (k-1)(k-2n) \quad \ldots (2)$$

Since, G is simple, the maximum number of edges of G in its components is $\frac{n_i(n_i-1)}{2}$.

Maximum number of edges of $G = \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$

$$= \sum_{i=1}^{k} \left[\frac{{n_i}^2 - n_i}{2}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{k} {n_i}^2 - \frac{1}{2}\sum_{i=1}^{k} n_i$$

$$\leq \frac{1}{2}[n^2 + (k-1)(k-2n)] - \frac{n}{2} \quad \text{(Using (1) and (2))}$$

$$= \frac{1}{2}[n^2 - 2nk + k^2 + 2n - k - n]$$

$$= \frac{1}{2}[n^2 - 2nk + k^2 + n - k]$$

$$= \frac{1}{2}[(n-k)^2 + (n-k)]$$

$$= \frac{1}{2}[(n-k)(n-k+1)]$$

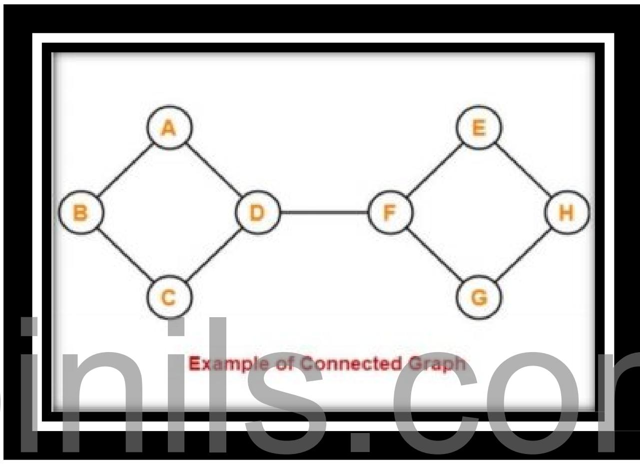
Maximum number of edges of $G \leq \frac{(n-k)(n-k+1)}{2}$

Hence the proof.

Directed Graphs:

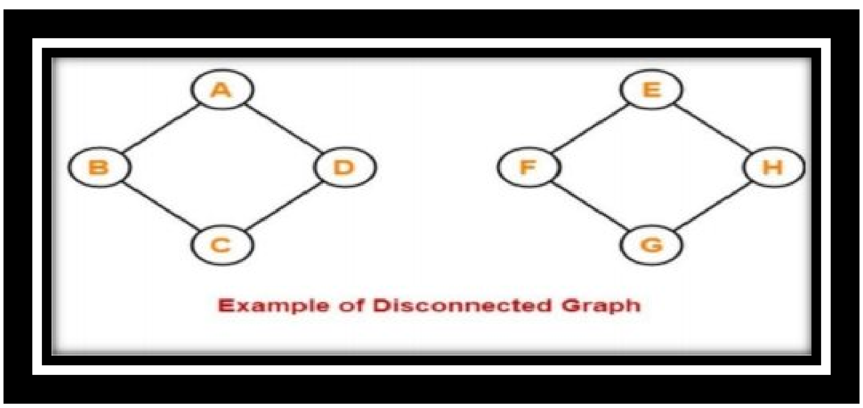
Connected Graph:

An directed graph is said to be connected if any pair of nodes are reachable from one another. That is, there is a path between any pair of nodes.



Disconnected graph:

A graph which is not connected is called disconnected graph.



Unilaterally connected:

A simple digraph is said to be unilaterally connected, if for any pair of nodes of the graph atleast one of the nodes of the pair is reachable from the other node.

Strongly connected:

A simple digraph is said to be strongly connected, if for any pair of nodes of the graph both the nodes of the pair are reachable from one to another.

Weakly connected:

A digraph is weakly connected, if it is connected as an undirected graph in which the direction of the edges is neglected.

Note:

A unilaterally connected digraph is weakly connected, but a weakly connected digraph is not necessarily unilaterally connected.

A strongly connected digraph is both unilaterally and weakly connected.

Theorem: 1

In a simple digraph $G = (V, E)$, every node of the digraph lies in exactly one strong component.

Proof:

Let $v \in V(G)$ and S be the set of all vertices of G which are mutually reachable with v.

Then $v \in S$, and S is a strong component of G. This shows that every vertex of G is contained in a strong component.

Assume that the vertex v is in two strong components S_1 and S_2.

Since $v \in S$, and any pair of vertices are mutually reachable with v, and also any pair of vertices of S_2

Are mutually reachable with v, we get any pair of vertices $S_1 \cup S_2$ are mutually reachable through v.

Therefore, $S_1 \cup S_2$ becomes one strong component of G.

This is impossible.

Therefore every vertex of G lies in exactly one strong component.

Hence the proof.