Catalog

2.1 Mathematical induction:

Statement of the principle of Mathematical Induction

Let $P(n)$be statement involving the natural number "n".

If P(1) is true.

Under the assumption that when P(k) is true, P(k+1) is true, then we conclude that a statement P(n) is true for all natural number "n".

Steps to prove that a statement P(n) is true for all natural numbers

Step:1 We must prove that P(1) is true.

Step:2 By assuming P(k) is true, we must prove that P(k+1) is also true.

NOTE:

Step:1 is known as the basic step.

Step:2 is known as inductive step.

Problems on Mathematical Induction:

1. Show that $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ using mathematical induction.

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

RHS $\Rightarrow \frac{n(n+1)}{2} = \frac{1(1+1)}{2} = 1 = \text{LHS}$

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$$1 + 2 + \ldots + k = \frac{k(k+1)}{2} \qquad \ldots (1)$$

To prove $p(k+1)$ is true.

Adding $k + 1$ on both sides

$$\Rightarrow 1 + 2 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

Hence $p(k+1)$ is true.

2. Show that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

RHS $\Rightarrow \frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2+1)}{6} = 1 =$ LHS

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \qquad \ldots (1)$$

To prove $p(k+1)$ is true.

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Adding $(k+1)^2$ on both sides

$$\Rightarrow 1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right]$$

$$= \frac{(k+1)}{6}[k(2k+1) + 6(k+1)]$$

$$= \frac{(k+1)}{6}[2k^2 + k + 6k + 6]$$

$$= \frac{(k+1)}{6}[2k^2 + 7k + 6]$$

$$= \frac{(k+1)}{6}[2k^2 + 4k + 3k + 6]$$

$$= \frac{(k+1)}{6}[2k(k + 2) + 3(k + 2)]$$

$$= \frac{(k+1)}{6}[(k + 2) + (2k + 3)]$$

Hence $p(k + 1)$ is true.

3. Show that $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

RHS $\Rightarrow \frac{n^2(n+1)^2}{4} = \frac{1^2(1+1)^2}{4} = 1 = $ LHS

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4} \qquad \ldots (1)$$

To prove $p(k+1)$ is true.

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

Adding $(k+1)^3$ on both sides

$$\Rightarrow 1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2}{4}[k^2 + 4(k+1)]$$

$$= \frac{(k+1)^2}{4}[k^2 + 4k + 4]$$

$$= \frac{(k+1)^2}{4}[k^2 + 2k + 2k + 4]$$

$$= \frac{(k+1)^2}{4}[k(k+2) + 2(k+2)]$$

$$= \frac{(k+1)^2}{4}[(k+2) + (k+2)]$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

Hence $p(k+1)$ is true.

4.Prove that $n^3 - n$ is divisible by 3, using mathematical induction .

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

RHS $\Rightarrow n^3 - n = 1^3 - 1 = 0$ is divisible by 3.

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$k^3 - k$ is divisible by 3.

$\Rightarrow k^3 - k = 3m$

$\Rightarrow k^3 = 3m + k \; . \; . \; . \; (1)$

To prove $p(k + 1)$ is true.

$(k + 1)^3 - (k + 1)$ is divisible by 3.

$\Rightarrow k^3 + 1 + 3k^2 + 3k - k - 1$

$\Rightarrow k^3 + 3k^2 + 2k$

$\Rightarrow (3m + k\ ) + 3k^2 + 2k$

$\Rightarrow 3m + 3k^2 + 3k$

$\Rightarrow 3(m + k^2 + k)$ is divisible by 3.

Hence $p(k+1)$ is true.

5. Prove that $8^n - 3^n$ is a multiple of 5. .

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

RHS $\Rightarrow 8^n - 3^n \quad = 8^1 - 3^1 \quad = 5$ is a multiple of 5 which is true.

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$8^k - 3^k$ is a multiple of 5.

$\Rightarrow 8^k - 3^k = 5m$

$\Rightarrow 8^k = 5m + 3^k \ldots (1)$

To prove $p(k+1)$ is true.

$8^{k+1} - 3^{k+1}$ is a multiple of 5.

$$\Rightarrow 8 \cdot 8^k - 3 \cdot 3^k$$

$$\Rightarrow (5m + 3^k) \cdot 8 - 3 \cdot 3^k$$

$$\Rightarrow 5 \cdot 8m + 8 \cdot 3^k - 3 \cdot 3k$$

$$\Rightarrow 5 \cdot 8m + 5 \cdot 3^k$$

$\Rightarrow 5(8m + 3^k)$ is a multiple of 5.

Hence $p(k+1)$ is true.

6. State and prove Handshaking theorem.

Suppose there are "n" people in a room, $n \geq 1$ and that they all shake hands with one another, prove that $\frac{n(n-1)}{2}$ handshakes will have accured.

Solution:

Let S be the set of positive integers.

To prove $p(1)$ is true.

When $n = 1$

$$p(1) = \frac{n(n-1)}{2} = \frac{1(1-1)}{2} = 0$$

$\Rightarrow$ there is no handshake accured which means there is only one person.

Hence $p(1)$ is true.

Assume that $p(k)$ is true.

$$p(k) = \frac{k(k-1)}{2} \qquad \ldots (1)$$

To prove $p(k+1)$ is true.

$$p(k+1) = \frac{(k+1)k}{2}$$

Suppose if one person entered into the room then he will shake his hand with "k" other person whenever $p(k)$ is true.

Hence $p(k+1)$ is true by mathematical induction.

The Well – Ordering Property:

The validity of mathematical induction follows from the following fundamental axioms about the set of integers.

Every non – empty set of non – negative integers has a least element.

The well ordering property can often be used directly in the proof.

2.2 Pigeon Hole Principle and Generalized Pigeon Hole Principle

Pigeonhole Principle:

The pigeonhole principle in its simplest incarnation, states the following

If you have more pigeons than pigeonholes, and you try to stuff the pigeons into the holes, then Atleast one hole must contain at least two pigeons.

Basic Pigeonhole Principle:

If $k + 1$ or more objects are placed into k boxes, then there is Atleast one box containing two or more of the objects.

Pigeonhole Principle:

If $(\boldsymbol{n} + \mathbf{1})$ Pigeon occupies "$\boldsymbol{n}$" holes then atleast one hole has more than one pigeon.

Proof:

Assume $(n + 1)$ pigeon occupies "n" holes.

Claim: Atleast one hole has more than one pigeon.

Suppose not,

Atleast one hole has not more than one pigeon.

Therefore each and every hole has exactly one pigeon.

Since, there are “n” hole, which implies, we have totally “n” pigeon.

Which is a contradiction to our assumption that there are $(n + 1)$ pigeon.

Therefore atleast one hole has more than one pigeon.

Hence the proof.

Generalized Pigeon Hole Principle

If m pigeon occupies “n” holes $(m > n)$ then atleast one hole has more than $[\frac{m-1}{n}] + 1$ pigeon.Here $[x]$ denotes the greatest integer less than or equal to x, which is a real number.

Proof:

Assume “m” pigeon occupy “n” holes $(m > n)$

Claim: Atleast on hole has more than $[\frac{m-1}{n}] + 1$ pigeon.

Suppose not, i.e., Atleast one hole has not more than $[\frac{m-1}{n}] + 1$ pigeon.

Each and every hole has exactly $[\frac{m-1}{n}] + 1$ pigeon.

Since we have n holes, totally there are $n\ [[\frac{m-1}{n}] + 1\ ]$ pigeon.

$\Rightarrow m - 1 + n$ pigeons

$\Rightarrow m + n - 1$ pigeons

Which is a contradiction to the assumption, that there are m pigeons.

Therefore, Atleast one hole has more than $[\frac{m-1}{n}]+1$ pigeon.

Problems under Pigeonhole and Generalized pigeonhole principle

1. Show that, among 100 people, atleast 9 of them were born in the same month.

Solution:

Here, Number of Pigeon = Number of people = 100

Number of holes = Number of month = 12

Then by generalized pigeon hole principle,

$$[\frac{m-1}{n}]+1=[\frac{100-1}{12}]+1=9$$

Were born in the same month.

2. Show that, if seven colors are used to paint 50 bicycles, atleast 8 bicycles will be the same.

Solution:

Here, Number of Pigeon = Number of bicycle = 50

Number of holes = Number of colors = 7

Then by generalized pigeon hole principle,

$$[\frac{m-1}{n}]+1=[\frac{50-1}{7}]+1=8$$

Therefore atleast 8 bicycles will have the same color.

3. Show that, if 25 dictionaries in a library contain a total of 40,325 pages, then one of the dictionaries must have atleast 1614 pages.

Solution:

Here, Number of Pigeon = Number of bicycle = 40325

Number of holes = Number of colors = 25

Then by generalized pigeon hole principle,

$$[\frac{m-1}{n}]+1=[\frac{40325-1}{25}]+1=1614$$

Here, Number of Pigeon = Number of grades = $n \;\; = 5$

Let k be number of students (pigeon) in discrete mathematics class.

$$\Rightarrow k+1=6$$

$$\Rightarrow k=5$$

The total number of students $= kn+1$

$$= 5\times 5+1=26$$

Minimum number of students = 26.

2.3 Permutation and Combination

The process of selecting things is called combination and that of arranging things is called permutation.

Examples of combinations and permutations:

(i) Formation of a team from a number of players.

(ii) Formation of a 3 member committee from 10members.

(iii) Arrangement of books on a shelf.

(iv) Formation of word with the given letters.

Permutation:

Each of the different arrangements which can be made by taking some or all of a number of things at a time is called a permutation.

The number of permutations of "n" things taken "r" at a time is denoted by nP_r

Examples:

$6P_2$ means the number of permutations of 6 things taken 2 at a time.

Formulae:

(i) $nP_r = \mathrm{n}(n-1)(n-2)...(n-r+1)$

(ii) The number of permutations of "n" things taken all at a time is

$nP_n = \mathrm{n}(n-1)(n-2)...3 \cdot 2 \cdot 1$

$\Rightarrow nP_n = \mathrm{n}!$

Problems based on Permutations:

1. In how many ways can 6 persons occupy 3 vacant seats?

Solution:

Given $n = 6, r = 3$

Total number of ways = $nP_r = 6P_3$ ways

$= 6 \times 5 \times 4 = 120$ ways

2. How many permutations of the letters ABCDEFGH contain the string ABC.

Solution:

Given $n = 6$

No of arrangements = $nP_r = 6P_6 = 6!$ Ways

$= 720$ ways

3. In how many ways can letters of the word "INDIA" be arranged.

Solution:

The word INDIA contains 5 letters of which 2 are I's.

The number of word possible $= \frac{5!}{2!} = \frac{5\times4\times3\times2\times1}{2\times1}$

$= 60$ ways

4. There are 6 books on Economics, 3 on Commerce and 2 on History. In how many ways can these be placed on a shelf if books on the same subject are to be together.

Solution:

6 Economics books can be arranged in $6P_6$ ways or $6!$ Ways.

3 Commerce books can be arranged in $3P_3$ ways or $3!$ Ways.

2 History books can be arranged in $2P_2$ ways or 2! Ways.

The three books can be arranged in $3P_3$ ways

The total number of required arrangements

$= 6! \times 3! \times 2! \times 3!$ Ways

$= 51840$ ways

5. Out of 7 consonants and 4 vowels, how many words of 3 consonants and 2 vowels can be formed?

Solution:

Number of ways of selecting 3 consonants from 7 = $^7C_3$

Number of ways of selecting 2 vowels from 4 = $^4C_2$

Number of ways of selecting 3 consonants from 7 and 2 vowels from 4

= $^7C_3 \times {}^4C_2$

$$= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \times \frac{4 \times 3}{2 \times 1} = 210$$

6. Find the number of distinct permutations that can be formed from all the letters of each word (i) RADAR (ii) UNUSUAL

Solution:

The word contains 5 letters of which 2 are A's and 2 are R's.

The number of possible words $= \frac{5!}{2!2!} = 30$

(ii) The word contains 7 letters of which 3 U's are there

The number of possible words $= \frac{7!}{3!} = 40$

7. Find the value of n if $nP_2 = 20$

Solution:

We know that $nP_r = \frac{n!}{(n-r)!}$

$$nP_2 = \frac{n!}{(n-2)!} = \frac{n(n-1)(n-2)!}{(n-2)!}$$

$$\Rightarrow n(n-1) = 20$$

$$\Rightarrow n = 20\ (or)\ n - 1 = 20$$

$$\Rightarrow n = 21$$

Combinations:

Each of the different groups or selections which can be made by taking some or all of a number of things at a time is called a combination.

The number of combinations of "n" things taken "r" at a time is denoted by nC_r.

Formula:

$$nC_r = \frac{n!}{r!\ (n-r)!}$$

Problems based on Combinations:

1. In how many ways can 5 persons be selected from amongst 10 persons?

Solution:

The selection can be done in $10C_5$ ways.

$$= \frac{10\times9\times8\times7\times6}{1\times2\times3\times4\times5}$$

$$= 252 \text{ ways}$$

2. How many ways are there to select five players from 10 member tennis team to make a trip to match to another school.

Solution:

5 members can be selected from 10 members in $10C_5$ ways.

Now $10C_5 = \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1}$

$= 252$ ways

3. Find the number of diagonals that can be drawn by joining the angular points of a heptagon.

Solution:

A heptagon has seven angular points and seven sides.

The join of two angular points is either a side or a diagonal.

The number of lines joining the angular points

$= 7C_2$

$= \frac{7\times6}{2\times1} = 21$

But the number of sides = 7

Hence the number of diagonals $= 21 - 7 = 14$

4. A team of 11 players is to be chosen from 15members. In how many ways can this be done if (i)one particular player is always included? (ii) Two such players have always to be included?

Solution:

(i) Let one player be fixed.

The remaining players are 14.

Out of these 14 players, we have to select 10 players in $14C_{10}$ ways.

$$14C_{10} = \frac{n!}{r!\,(n-r)!} = \frac{14!}{10!\,(14-10)!} = 1001 \text{ ways.}$$

(ii) Let 2 players be fixed.

The remaining players are 13.

Out of 13 players, we have to select 9 players in $13C_9$ ways.

$$13C_9 = \frac{n!}{r!\,(n-r)!} = \frac{13!}{9!\,(13-9)!} = 715 \text{ ways.}$$

5. If $nC_5 = 20nC_4$, find the value of n.

Solution:

Given $nC_5 = 20nC_4$

$$\frac{n!}{5!\,(n-5)!} = \frac{20n!}{4!\,(n-4)!}$$

$\Rightarrow (n-4)!\,4! = 20 \times (n-5)!\,5!$

$\Rightarrow (n-4-1)!\,(n-4)4! = 20 \times (n-5)!\,5!$

$\Rightarrow (n-5)!\,(n-4)4! = 20 \times (n-5)!\,4! \times 5$

$\Rightarrow (n-4) = 100$

$\Rightarrow n = 100 + 4 = 104$

$\Rightarrow n = 104$

6. A question paper has 3 parts, Part A, Part B and Part C having 12, 4 and 4 questions respectively. A student has to answer 10 questions from Part A and 5 questions from Part B and Part C put together selecting atleast 2 from each one of these two parts. In how many ways the selection of questions can be done.

Solution:

12	4	4
Part A	Part B	Part C
10	2	3
10	3	2

The selection of questions can be done in

$$12C_{10} \times 4C_2 \times 4C_3 + 12C_{10} \times 4C_3 \times 4C_2$$

$$= 3168 \text{ ways}$$

2.4 Recurrence Relations:

An equation that expresses a_n, the general term of the sequence $\{a_n\}$ in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a non – negative integer is called a recurrence relation for $\{a_n\}$ or a difference equation.

If the terms of a sequence satisfies a recurrence relation, then the sequence is called a solution of the recurrence relation.

For example, we consider the famous Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, . . .

Which can be represented by the recurrence relation.

$$F_n = F_{n-1} + F_{n-2}, n \geq 2$$

and $F_0 = 0, F_1 = 1$

Here, $F_0 = 0, F_1 = 1$ are called initial conditions.

It is a second order recurrence relation.

Definition:

A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \ldots + C_K k a_{n-k}$$

Where $C_1, C_2, \ldots, C_k$ are real numbers, and $C_k \neq 0$.

The recurrence relation in the definition is linear since the right – hand side is a sum of multiplies of the previous terms of the sequence.

The recurrence relation is homogeneous, since no terms occur that are not multiplies of the a_j's.

The coefficients of the terms of the sequence are all constants, rather than function that depend on "n".

The degree is k because a_n is expressed in terms of the previous k terms of the sequence.

Solving Linear Homogeneous Recurrence Relations With Constant Coefficients:

Step: 1 Write down the characteristic equation for the given recurrence relation. Here, the degree of character equation is 1 less than the number of terms in recurrence relation.

Step: 2 By solving the characteristic equation find out the characteristic roots.

Step: 3 Depends upon the nature of roots, find out the solution a_n as follows:

Case (i) Let the roots be real and distinct say $r_1, r_2, \ldots, r_n$.

Then $a_n = \alpha_1 {r_1}^n + \alpha_2 {r_2}^n + \alpha_3 {r_3}^n + \ldots + \alpha_n {r_n}^n$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrary constants.

Case (ii) Let the roots be real and equal say $r_1 = r_2 = \ldots = r_n$.

Then $a_n = \alpha_1 {r_1}^n + n\alpha_2 {r_2}^n + n^2\alpha_3 {r_3}^n + \ldots + n^n \alpha_n {r_n}^n$ where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are arbitrary constants.

Case (iii) When the roots are complex conjugate, then

$a_n = r^n(\alpha_1 \cos n\theta + \alpha_2 \sin n\theta)$

Step: 4 Apply initial conditions and find out arbitrary constants.

Note:

There is no single method or technique to solve all recurrence relations. There exist some recurrence relations which cannot be solved. The recurrence relation

$S(k) = 2[S(k-1)]^2 - kS(k-3)$ cannot be solved.

1. If the sequence $a_n = 3 \cdot 2^n, n \geq 1$, then find the corresponding recurrence relation.

Solution:

Given $a_n = 3 \cdot 2^n$

$\Rightarrow a_{n-1} = 3 \cdot 2^{n-1}$

$= 3 \cdot \frac{2^n}{2}$

$\Rightarrow a_{n-1} = \frac{a^n}{2}$

$\Rightarrow a_n = 2(a_{n-1})$

Hence $a_n = 2a_{n-1}, n \geq 1$ with $a_0 = 3$

2. Find the recurrence relation for $S(n) = 6(-5)^n, n \geq 0$

Solution:

Given $S(n) = 6(-5)^n$

$\Rightarrow S(n-1) = 6(-5)^{n-1}$

$= 6\frac{(-5)^n}{-5}$

$= \frac{S(n)}{-5}$

$\Rightarrow S(n) = -5 \cdot S(n-1), n \geq 0$ with $S(0) = 6$

3. Find the recurrence relation from $y_k = A \cdot 2^k + B \cdot 3^k$

Solution:

Given $y_k = A \cdot 2^k + B \cdot 3^k \quad \ldots(1)$

$\Rightarrow y_{k+1} = A \cdot 2^{k+1} + B \cdot 3^{k+1}$

$= A \cdot 2^k \cdot 2 + B \cdot 3^k \cdot 3$

$= 2A \cdot 2^k + 3B \cdot 3^k \qquad \ldots (2)$

$\Rightarrow y_{k+2} = 4A \cdot 2^k + 9B \cdot 3^k \qquad \ldots (3)$

$(3) - 5(2) + 6(1)$

$\Rightarrow y_{k+2} - 5y_{k+1} + 6y_k = 4A \cdot 2^k + 9B \cdot 3^k - 10A \cdot 2^k - 15B \cdot 3^k + 6A \cdot 2^k + 6B \cdot 3^k = 0$

$\Rightarrow y_{k+2} - 5y_{k+1} + 6y_k = 0$

4. Find the recurrence relation from $y_n = A3^n + B(-4)^n$

Solution:

Given $y_n = A3^n + B(-4)^n \qquad \ldots (1)$

$\Rightarrow y_{n+1} = y_n = A3^{n+1} + B(-4)^{n+1}$

$= A3^n \cdot 3 + B(-4)^n \cdot (-4)$

$= 3A \cdot 3^n - 4B \cdot (-4)^n \qquad \ldots (2)$

$\Rightarrow y_{n+2} = 9A \cdot 3^n + 16B \cdot (-4)^n \qquad \ldots (3)$

$(3) + (2) - 12(1)$

$\Rightarrow y_{n+2} + y_{n+1} - 12y_n = 9A3^n + 16B(-4)^n + 3A3^n - 4B(-4)^n - 12A3^n - 12B(-4)^n = 0$

$\Rightarrow y_{n+2} + y_{n+1} - y_n = 0$

5. Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 2, a_1 = 5, a_2 = 15$

Solution:

The recurrence relation can be written as $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$

The characteristic equation is $r^3 - 6r^2 + 11r - 6 = 0$

By solving, we get the characteristic roots, $r = 1, 2, 3$

Solution is $a_n = \alpha_1 \cdot 1^n + \alpha_2 2^n + \alpha_3 3^n \quad \ldots (A)$

Given $a_0 = 2$, Put $n = 0$ in (A)

$$a_0 = \alpha_1 \cdot (1)^0 + \alpha_2(2)^0 + \alpha_3(3)^0$$

(A) $\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = 2 \quad \ldots (1)$

Given $a_1 = 5$, Put $n = 1$ in (A)

$$a_1 = \alpha_1 \cdot (1)^1 + \alpha_2(2)^1 + \alpha_3(3)^1$$

(A) $\Rightarrow \alpha_1 + 2\alpha_2 + 3\alpha_3 = 5 \quad \ldots (2)$

Given $a_2 = 15$, Put $n = 2$ in (A)

$$a_2 = \alpha_1 \cdot (1)^2 + \alpha_2(2)^2 + \alpha_3(3)^2$$

(A) $\Rightarrow \alpha_1 + 4\alpha_2 + 9\alpha_3 = 15 \qquad \ldots(3)$

To solve (1), (2) and (3)

(1) $\Rightarrow \alpha_3 = 2 - \alpha_1 - \alpha_2 \qquad \ldots(4)$

Using (4) in (2)

(2) $\Rightarrow 2\alpha_1 + \alpha_2 = 1 \qquad \ldots(5)$

Using (4) in (3)

(3) $\Rightarrow 8\alpha_1 + 5\alpha_3 = 3 \qquad \ldots(6)$

Solving (5) and (6), we get $\alpha_1 = 1$ and $\alpha_2 = -1$

Using $\alpha_1 = 1$ and $\alpha_2 = -1$ in (1) we get $\alpha_3 = 2$

Substituting $\alpha_1 = 1$ and $\alpha_2 = -1$ and $\alpha_3 = 2$ in (A), we get

Solution is $a_n = 1 \cdot 1^n - 1 \cdot 2^n + 2 \cdot 3^n$

2.5 Generating Function:

The generating function for the sequence "s" with terms $a_0, a_1, \ldots, a_n$ of real numbers is the infinite sum.

$$G(x) = G(s, x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$$

$$= \sum_{n=0}^{\infty} a_nx^n$$

For example, (i) The generating function for the sequence "s" with the terms $1, 1, 1, \ldots$ is given by

$$G(x) = G(s, x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

(ii) The generating function for the sequence "s" with terms $1, 2, 3, 4, \ldots$ is given by

$$G(x) = G(s, x) = \sum_{n=0}^{\infty}(n+1)x^n$$

$$= 1 + 2x + 3x^2 + \ldots$$

$$= (1-x)^{-2}$$

$$= \frac{1}{(1-x)^2}$$

Problems:

1. Write the generating function for the sequence $1, a, a^2, a^3, a^4, \ldots$

Solution:

Generating function $G(x) = 1 + a + a^2 + a^3 + a^4 + \ldots$

$$= \frac{1}{1-ax} \text{ for } |ax| < 1$$

Solution for Recurrence Relations using Generating Functions:

Procedure for solving Recurrence Relation using Generating Function:

Step: 1 Rewrite the recurrence relation as an equation on RHS

Step: 2 Multiply the equation in step: 1 by x^n and summing it from 1 to ∞ or

(0 to ∞) or (2 to ∞)

Step: 3 Put $G(x) = \sum_{n=0}^{\infty} a_n x^n$ and write $G(x)$ as a function of x.

Step: 4 Decompose $G(x)$ into partial fraction.

Step: 5 Express $G(x)$ as a sum of familiar series.

Step: 6 Express a_n as the coefficient of x^n in $G(x)$.

The following table represents some sequences and their generating functions.

S. no	Sequence	Generating Function
1	1	$\frac{1}{1-z}$
2	$(-1)^n$	$\frac{1}{1+z}$
3	a^n	$\frac{1}{1-az}$
4	$(-a)^n$	$\frac{1}{1+az}$
5	$n+1$	$\frac{1}{1-(z)^2}$
6	n	$\frac{1}{(1-z)^2}$
7	n^2	$\frac{z(1+2)}{(1-z)^3}$
8	na^n	$\frac{az}{(1-za)^2}$

1. Using generating function solve the recurrence relation $a_n = 3a_{n-1}$ for $n \geq 1$ with $a_0 = 2$

Solution:

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$

Given $a_n - 3a_{n-1} = 0$

Multiply the above equation by x^n and summing from 1 to ∞, we get

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} 3a_{n-1}x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - 3x \sum_{n=1}^{\infty} 3a_{n-1}x^{n-1} = 0$$

$$\Rightarrow (G(x) - a_0) - 3xG(x) = 0$$

$$\Rightarrow G(x)(1 - 3x) = a_0$$

$$\Rightarrow G(x)(1 - 3x) = 2$$

$$\Rightarrow G(x) = \frac{2}{(1-3x)} = 2(1 - 3x)^{-1}$$

$$= 2(1 + 3x + (3x)^2 + \ldots)$$

$$= 2\sum_{n=0}^{\infty} 3^n x^n$$

Consequently, $a_n = 2$. coefficient of x^n in $G(x)$

$$a_n = 2 \cdot 3^n$$

2. Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$ given that $a_0 = 10, a_1 = 41$ using generating function.

Solution:

The given recurrence relation is $a_n - 7a_{n-1} + 10a_{n-2} = 0$

Multiply the above equation by x^n and summing from 2 to ∞, we get

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1}x^n + 10\sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 7x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} + 10x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0$$

$\Rightarrow [G(x) - a_0 - a_1x] - 7x[G(x) - a_0] + 10x^2G(x) = 0$

$\Rightarrow G(x) - 10 - 41x - 7x[G(x) - 10] + 10x^2G(x) = 0$

$\Rightarrow G(x)(1 - 7x + 10x^2) + 29x - 10 = 0$

$\Rightarrow G(x) = \frac{10-29x}{10x^2-7x+1}$

$\Rightarrow G(x) = \frac{10-29x}{(1-2x)(1-5x)}$

$\Rightarrow G(x) = \frac{A}{(1-2x)} + \frac{B}{(1-5x)}$

$= A(1-2x)^{-1} + B(1-5x)^{-1}$

$= A[1 + 2x + (2x)^2 + \dots] + B[1 + 5x + (5x)^2 + \dots]$

$= A\sum_{n=2}^{\infty} 2^n x^n + B\sum_{n=2}^{\infty} 5^n x^n$

a_n = coefficient of x^n in $G(x)$

$a_n = A2^n + B5^n, n \geq 2 \quad ...(A)$

Given $a_0 = 10$, Put $n = 0$ in (A), we get

$\Rightarrow a_0 = A2^0 + B5^0$

$\Rightarrow 10 = A + B \quad ...(1)$

Given $a_1 = 41$, Put $n = 1$ in (A), we get

$\Rightarrow a_1 = A2^1 + B5^1$

$\Rightarrow 41 = 2A + 5B \quad ...(2)$

Solving (1) and (2) we get $A = 3, B = 7$

Hence $a_n = 3 \cdot 2^n + 7 \cdot 5^n$

3. Using generating function solve the recurrence relation corresponding to the Fibonacci sequence $a_n = a_{n-1} + a_{n-2}, n \geq 2$ with $a_0 = 1, a_1 = 1$

Solution:

Given recurrence relation $a_n - a_{n-1} - a_{n-2} = 0$

Multiply the above recurrence relation by x^n and summing from 2 to ∞, we get

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} - x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0$$

$$\Rightarrow [G(x) - a_0 - a_1x] - x[G(x) - a_0] - x^2G(x) = 0$$

$$\Rightarrow G(x) - 10 - 41x - 7x[G(x) - 10] + 10x^2G(x) = 0$$

$$\Rightarrow G(x)(1 - x - x^2) = a_0 - a_0x + a_1x$$

$$\Rightarrow G(x) = \frac{1}{1-x-x^2}$$

$$= \frac{1}{(1-\frac{1+\sqrt{5}}{2}x)(1-\frac{1-\sqrt{5}}{2}x)}$$

$$= \frac{A}{(1-\frac{1+\sqrt{5}}{2}x)} + \frac{B}{(1-\frac{1-\sqrt{5}}{2}x)}$$

Now $\frac{1}{1-x-x^2} = \frac{A}{(1-\frac{1+\sqrt{5}}{2}x)} + \frac{B}{(1-\frac{1-\sqrt{5}}{2}x)}$ . . . (1)

$1 = A\,(1 - \frac{1-\sqrt{5}}{2}x) + B\,(1 - \frac{1+\sqrt{5}}{2}x)$ . . . (2)

Put $x = 0$ in (2)

(2) $\Rightarrow A + B = 1$ . . . (3)

Put $x = \frac{2}{1-\sqrt{5}}$ in (2)

(2) $\Rightarrow 1 = B\,(1 - \frac{1+\sqrt{5}}{1-\sqrt{5}})$

$$\Rightarrow 1 = B\left(\frac{1-\sqrt{5}-1-\sqrt{5}}{1-\sqrt{5}}\right)$$

$$\Rightarrow 1 = B\left(\frac{-2\sqrt{5}}{1-\sqrt{5}}\right) \quad \text{(Using B value in (3))}$$

$$\Rightarrow B = \frac{1-\sqrt{5}}{-2\sqrt{5}}$$

$$(3) \Rightarrow A = \frac{1}{2\sqrt{5}}(1+\sqrt{5})$$

Sub A and B in (1), we get

$$G(x) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\left(1-\left(\frac{1+\sqrt{5}}{2}\right)x\right)^{-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)\left(1-\left(\frac{1-\sqrt{5}}{2}\right)x\right)^{-1}$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\left[1+\frac{1+\sqrt{5}}{2}x+\left(\frac{1+\sqrt{5}}{2}\right)^2+\dots\right]$$

$$-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)\left[1+\frac{1-\sqrt{5}}{2}x+\left\{\left(\frac{1-\sqrt{5}}{2}\right)x\right\}^2+\dots\right]$$

a_n = coefficient of x^n in $G(x)$

Solving, we get

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

4. Identify the sequence having the expression $\frac{5+2x}{1-4x^2}$ as a generating function.

Solution:

Given $G(x) = \frac{5+2x}{1-4x^2}$ $\quad\ldots(1)$

$$= \frac{5+2x}{(1+2x)(1-2x)}$$

Now, $\frac{5+2x}{(1+2x)(1-2x)} = \frac{A}{(1+2x)} + \frac{B}{(1-2x)}$ $\quad\ldots(2)$

Put $x = \frac{1}{2}$

$\Rightarrow 5 + 1 = 2B$

$\Rightarrow B = 3$

Put $x = -\frac{1}{2}$

$\Rightarrow 5 - 1 = 2A$

$\Rightarrow A = 2$

$(2) \Rightarrow \frac{5+2x}{(1+2x)(1-2x)} = \frac{2}{(1+2x)} + \frac{3}{(1-2x)}$

$$= 2(1-2x)^{-1} + 3(1-2x)^{-1}$$

$$= A[1 - 2x - (2x)^2 + \ldots] + B[1 + 2x + (2x)^2 + \ldots]$$

$$= 2\sum_{n=2}^{\infty}(-1)^n 2^n x^n + 3\sum_{n=2}^{\infty} 2^n x^n$$

$$= 2\sum_{n=2}^{\infty}(-2)^n x^n + 3\sum_{n=2}^{\infty} 2^n x^n$$

The required sequence is the coefficient of x^n in $G(x)$

Hence $S(n) = 2(-2)^n + 3(2)^n$

5. Identify the sequence having the expression $\frac{3-5x}{1-2x-3x^2}$ as a generating function.

Solution:

Given $G(x) = \frac{3-5x}{1-2x-3x^2}$ $\quad ...(1)$

$= \frac{3-5x}{(1-3x)(1+x)}$

Now, $\frac{3-5x}{(1+2x)(1-2x)} = \frac{A}{(1-3x)} + \frac{B}{(1+x)}$ $\quad ...(2)$

$3 - 5x = A(1 + x) + B(1 - 3x)$

Put $x = -1$

$\Rightarrow 3 + 5 = 4B$

$\Rightarrow B = 2$

Put $x = \frac{1}{3}$

$\Rightarrow 3 - \frac{5}{3} = A\,(1 + \frac{1}{3})$

$\Rightarrow \frac{4}{3} = \frac{4}{3}A$

$\Rightarrow A = 1$

$$(2) \Rightarrow \frac{3-5x}{(1+2x)(1-2x)} = \frac{1}{(1-3x)} + \frac{2}{(1+x)}$$

$$= (1-3x)^{-1} + 2(1+x)^{-1}$$

$$= A[1+3x+(3x)^2+\ldots] + B[1-x+(x)^2+\ldots]$$

$$= \sum_{n=2}^{\infty} 3^n x^n + 3\sum_{n=2}^{\infty} (-1)^n x^n$$

The required sequence is the coefficient of x^n in $G(x)$

Hence $S(n) = 3^n + 2(-1)^n$

2.6 The Principle of Inclusion – Exclusion:

Formula

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| + |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup A_4| &= |A_1| + |A_2| + |A_3| + |A_4| + |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| \\ &- |A_1 \cap A_4| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3| \\ &+ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4| \\ &- |A_1 \cap A_2 \cap A_3 \cap A_4| \end{aligned}$$

Problems under Inclusion and Exclusion:

1. How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution:

Let A denote the set of positive integers not exceeding 1000 that are divisible by 7.

Let B denote the set of positive integers not exceeding 1000 that are divisible by 11.

Then, $|A| = [\frac{1000}{7}] = [142.8] = 142$

$$|B| = [\frac{1000}{11}] = [90.9] = 90$$

$|A \cap B| = [\frac{1000}{7\times11}] = [12.9] = 12$

The number of positive integer not exceeding 1000 that are divisible either 7 or 77 is $|A \cup B|$

By principle of inclusion – exclusion,

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 142 + 90 - 12 = 220$

There are 220 positive integer not exceeding 1000 divisible by either 7 or 11.

2. Determine n such that $1 \leq n \leq 100$ which are not divisible by 5 or by 7.

Solution:

Let A denote the number n, $1 \leq n \leq 100$ which is divisible by 5.

Let B denote the number n, $1 \leq n \leq 100$ which is divisible by 7.

Then, $|A| = [\frac{100}{5}] = [20] = 20$

$|B| = [\frac{100}{7}] = [14.3] = 14$

$|A \cap B| = [\frac{1000}{5\times7}] = [2.8] = 2$

Now, the number n, $1 \leq n \leq 100$ which is divisible by either 5 or 7 is $|A \cup B|$

By principle of inclusion – exclusion,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 20 + 14 - 2 = 32$$

The number n, $1 \leq n \leq 100$ which is divisible by either 5 or 7 is

$$= 100 - 32 = 68$$

There are 68 number not exceeding 100 that are not divisible by either 5 or 7.

3. Find the number of integers between 1 to 250 that are not divisible by any of the integers 2, 3, 5 and 7

Solution:

Let A denote the integer from 1 to 250 that are divisible by 2.

Let B denote the integer from 1 to 250 that are divisible by 3.

Let C denote the integer from 1 to 250 that are divisible by 5.

Let D denote the integer from 1 to 250 that are divisible by 7.

Then, $|A| = [\frac{250}{2}] = 125$

$$|B| = [\frac{250}{3}] = 83$$

$$|C| = [\frac{250}{5}] = 50$$

$|D| = [\frac{250}{7}] = 35$

The number of integer between 1 – 250 that are divisible by 2 & 3

$|A \cap B| = [\frac{250}{2\times3}] = 41$

$|A \cap C| = [\frac{250}{2\times5}] = 25$

$|A \cap D| = [\frac{250}{2\times7}] = 17$

$|B \cap C| = [\frac{250}{3\times5}] = 16$

$|B \cap D| = [\frac{250}{3\times7}] = 11$

$|C \cap D| = [\frac{250}{5\times7}] = 7$

The number of integer between 1 – 250 that are divisible by 2, 3 & 5

$|A \cap B \cap C| = [\frac{250}{2\times3\times5}] = 8$

$|A \cap B \cap D| = [\frac{250}{2\times3\times7}] = 5$

$|A \cap C \cap D| = [\frac{250}{2\times5\times7}] = 3$

$|B \cap C \cap D| = [\frac{250}{3\times5\times7}] = 2$

$$|A \cap B \cap C \cap D| = [\frac{250}{2\times3\times5\times7}] = 1$$

The number of integer between 1 – 250 that are divisible by 2, 3, 5 & 7 is

$|A \cap B \cap C \cap D|$

By principle of inclusion and exclusion,

$$|A \cup B \cup C \cup D|$$
$$= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C|$$
$$- |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D|$$
$$+ |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

$$= (125 + 83 + 50 + 35) - (41 + 25 + 17 + 16 + 11 + 7) + (8 + 5 + 3 + 2)$$
$$- 1$$

$$= 293 - 117 + 18 - 1 = 193$$

Now, the number of integer not divisible by any of 2, 3, 5 and 7

$$= 250 - 193 = 57$$

4. How many integers between 1 to 100 that are (i) not divisible by 7, 11 or 13 (ii) divisible by 3 but not by 7.

Solution:

Let A, B and C denote respectively the number of integer between 1 to 100 that are divisible by 7, 11 and 13 respectively.

Then, $|A| = [\frac{100}{7}] = 14$

$|B| = [\frac{100}{11}] = 9$

$|C| = [\frac{100}{13}] = 7$

$|A \cap B| = [\frac{100}{7 \times 11}] = 1$

$|A \cap C| = [\frac{100}{7 \times 13}] = 1$

$|B \cap C| = [\frac{100}{11 \times 13}] = 0$

$|A \cap B \cap C| = [\frac{100}{7 \times 11 \times 13}] = 0$

The number of integers between 1 – 100 that are divisible by 7, 11 and 13 is

$|A \cup B \cup C \cup D|$

By principle of inclusion and exclusion,

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| - |B \cap D| +$

$|A \cap B \cap C| - |A \cap B \cap C|$

$= (14 + 9 + 7) - (1 + 1 + 0) + (0)$

$= 30 - 2 = 28$

Now, the number of integer not divisible by any of 7, 11 and 13

$$= 100 - 28 = 72$$

Let A and B denote the number between 1 – 100that are divisible by 3 and 7 respectively.

Then, $|A| = [\frac{100}{3}] = 33$

$|B| = [\frac{100}{7}] = 14$

$|A \cap B| = [\frac{100}{3\times7}] = 4$

The number of integers divisible by 3 but not by 7.

$$= |A| - |A \cap B| = 33 - 4 = 29$$

5. Find the number of integers between 1 to 100 that are divisible by (i) 2, 3, 5 and 7 (ii) 2, 3, 5 but not by 7

Solution:

Let A, B, C and D denote the number of positive integers between 1 to 100 that are divisible by 2, 3, 5 and 7 respectively.

Then, $|A| = [\frac{100}{2}] = 50$

$|B| = [\frac{100}{3}] = 33$

$|C| = [\frac{100}{5}] = 20$

$|D| = [\frac{100}{7}] = 14$

$|A \cap B| = [\frac{100}{2\times3}] = 16$

$|A \cap C| = [\frac{100}{2\times5}] = 10$

$|A \cap D| = [\frac{100}{2\times7}] = 7$

$|B \cap C| = [\frac{100}{3\times5}] = 6$

$|B \cap D| = [\frac{100}{3\times7}] = 4$

$|C \cap D| = [\frac{100}{5\times7}] = 2$

$|A \cap B \cap C| = [\frac{100}{7\times11\times13}] = 3$

$|A \cap B \cap D| = [\frac{100}{2\times3\times7}] = 2$

$|A \cap C \cap D| = [\frac{100}{2\times5\times7}] = 1$

$|B \cap C \cap D| = [\frac{100}{3\times5\times7}] = 0$

$|A \cap B \cap C \cap D| = [\frac{100}{2\times3\times7\times11}] = 0$

By principle of inclusion and exclusion,

$$|A \cup B \cup C \cup D|$$

$$= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C|$$

$$- |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D|$$

$$+ |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

$$= (50 + 33 + 20 + 14) - (16 + 10 + 7 + 6 + 4 + 2) + (3 + 2 + 1 + 0) - 0$$

$$= 117 - 45 + 6 = 78$$

(ii) The number of integers between 1 – 100 that are divisible by 2, 3, 5 but not by 7 $= |A \cap B \cap C| - |A \cap B \cap C \cap D|$

$$= 3 - 0 = 3$$

6. Determine the number of positive integers n, $1 \le n \le 1000$, that are not divisible by 2, 3 or 5 but are divisible by 7.

Solution:

Let A, B, C and D denote the number of positive integers between 1 to 1000 that are divisible by 2, 3, 5 and 7 respectively.

$|D| = [\frac{1000}{7}] = [142.8] = 142$

$|A \cap B \cap C \cap D| = [\frac{1000}{2\times3\times5\times7}] = [4.7] = 4$

The number between 1 – 1000 that are divisible by 7 but not divisible by 2, 3, 5 and 7 $= |D| - |A \cap B \cap C \cap D|$

$= 42 - 4 = 138$

7. In a survey of 100 students, it was found that 30 studied Mathematics, 54 studied Statistics, 25 studied Operation research, 1 studied all the three subjects. 20 studied Mathematics and Statistics, 3 studied Mathematics and Operation Research and 15 studied Statistics and Operations Research, (i) How many students studied none of these subjects?(ii) How many students studied only Mathematics.

Solution:

Let A denote the students who studied Mathematics.

Let B denote the students who studied Statistics.

Let B denote the students who studied Operations Research.

It is given that $|A| = 30, |B| = 54, \ |C| = 25, |A \cap B| = 20, |A \cap C| = 3,$
$|B \cap C| = 15, |A \cap B \cap C| = 1$

By principle of inclusion – exclusion, the number of students playing either volleyball or hockey is

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 30 + 54 + 25 - 20 - 3 - 15 + 1$

Students who studied none of these 3 subjects $= 100 - 72 = 28$

The number of students only studied Mathematics and Statistics $= 20 - 1 = 19$

The number of students only studied Mathematics and Operations Research

$= 3 - 1 = 2$

The number of students only studied Mathematics $= 30 - 19 - 2 - 1 = 8$

8. A survey of 500 from a school produced the following information. 200 play volleyball, 120 play hockey, 60 play both volleyball and hockey. How many are not playing either volleyball or hockey?

Solution:

Let A denote the students who play volleyball.

Let B denote the students who play hockey.

It is given that $n = 500, |A| = 200, |B| = 120, |A \cap B| = 60$

By principle of inclusion – exclusion, the number of students playing either volleyball or hockey is

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 200 + 120 - 60 = 260$

The number of students not playing either volleyball or hockey is

$= 500 - 260 = 240$

9. Out of 100 students in a college, 38 play tennis, 57 play cricket and 31 play hockey, 9 play cricket and hockey, 10 play hockey and tennis, 12 play tennis and cricket. How many play (i) all three games (ii) just one game (iii) tennis and cricket but not hockey. (Assume that each student plays atleast one game)

Solution:

Let T, C and H denote the set of students playing Tennis, Cricket and Hockey respectively.

Given that $|T| = 38, |C| = 57, \ |H| = 31, |T \cap C| = 12, |T \cap H| = 10,$

$|C \cap H| = 9, |T \cup C \cup H| = 100$

Now, the number of integer who play all three games $= |T \cap C \cap H|$

By principle of inclusion – exclusion,

$$|T \cup C \cup H| = |T| + |C| + |H| - |T \cap C| - |T \cap H| - |C \cap H| + |T \cap C \cap H|$$

$$100 = 38 + 5 + 31 - 103 - 12 - 10 + 9 + |T \cap C \cap H|$$

$$|T \cap C \cap H| = 100 - 126 + 31 = 5$$

Number of students who play all 3 games = 5

Number of students playing just one game = number of students Tennis only + number of students playing cricket only + number of students playing Hockey only

$$= 21 + 41 + 17 = 79$$

The number of students playing Tennis and Cricket but not Hockey

$$= |T \cap C| - |T \cap C \cap H|$$

$$= 12 - 5 = 7$$

10. A survey of 500 television watchers produced the following information. 285 watch Hockey games. 195 watch Football games. 115 watch basketball games.70 watch football and hockey games. 50 watch hockey and basketball games and 30 watch football and basketball games. 50 do not watch any of the three games. How many people watch exactly one of the three games.

Solution:

Let H denotes the television watchers who watch Hockey.

Let F denotes the television watchers who watch Football.

Let B denotes the television watchers who watch Basket Ball.

Given that $|H| = 285, |F| = 195, |B| = 115, |H \cap F| = 70, |H \cap B| = 50,$

$|F \cap B| = 30$

Let x be the number of television watchers who watch all three games.

Now, we have

Given 50 members does not watch any of the three games.

Hence, $(165 + x) + (95 + x) + (35 + x) + (70 - x) + (50 - x) + (30 - x) + x = 500$

$\Rightarrow 445 + x = 500$

$\Rightarrow x = 55$

Number of students who watches exactly one game is

$= 165 + x + 95 + x + 35 + x$

$= 295 + 3 \times 55 = 295 + 165 = 460$

11. A total of 1232 have taken a course in Tamil, 879 have taken a course in Telugu, and 114 have taken a course in Hindi. Further 103 have taken a course in both Tamil and Telugu, 23 have taken a course in Tamil and Hindi, and 14 have taken a course in Telugu and Hindi. If 2092 students have taken

atleast one of the Tamil, Telugu and Hindi, how many students have taken a course in all three languages.

Solution:

Let A denote the students who have taken a course in Tamil.

Let B denote the students who have taken a course in Telugu.

Let C denote the students who have taken a course in Hindi.

It is given that $|A| = 1232, |B| = 879, \ |C| = 114, |A \cap B| = 103, |A \cap C| = 23,$ $|B \cap C| = 14, |A \cup B \cup C| = 2092$

By principle of inclusion – exclusion, the number of students playing either volleyball or hockey is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |A \cap B \cap C|$$

$$|A \cap B \cap C| = 2232 - 2225 = 7$$

Therefore, there are 7 students who have taken the course in Tamil, Telugu and Hindi.