Catalog

1. 1. Proportional Logic

Proposition:

A proposition (or statement) is a declarative sentence which is either true or false but not both.

Notations:

If a proposition is true then it's truth value is denoted by T.

If a proposition is false then it's truth value is denoted by F.

P, Q, R, S, . . . are used to denote propositions.

Connectives:

Connective is an operation which is used to connect two or more than two statements. Simply it is called sentential connectives. It is also known as Logical Connectives or Logical Operators.

Compound Statement:

Statements which contain one or more primary statements and some connectives are called compound or molecular or composite statements.

Example:

Let $p: 5 + 10 = 20$ be the statement

$\neg p$: It is false that 5 + 10 = 20

Hence $\neg p$ is a compound statement with primary statement as p and connective as $\neg p$

Five Basic Connectives

	Logical Connectives	Name	Symbols	Type of Operator
1	Not	Negation	$\neg$	Unary
2	And	Conjunction	$\wedge$	Binary
3	Or	Disjunction	$\vee$	Binary
4	If . . . then	Conditional (or) Implication	$\rightarrow$	Binary
5	If and only if (iff)	Biconditional	$\leftrightarrow (or) \rightleftharpoons$	Binary

Statement Formula:

A statement formula is an expression which is a string consisting of variables (capital letters with or without subscripts), parenthesis and connective symbols.

Truth Tables:

The truth value of proposition is either true (T) or false (F).

A truth table is a table that shows the truth value of a compound proposition for all possible cases.

Negation:

If a statement is **TRUE**, then its negation is **FALSE**. (And if a statement is **FALSE**, then its negation is **TRUE**).

P	**¬p**
T	F
F	T

Conjunction:

A conjunction is a compound statement formed by joining two statements with the connector AND. The conjunction "*p* and *q*" is symbolized by p, q.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction:

A conjunction is a compound statement formed by joining two statements with the connector OR. The disjunction “p or q” is symbolized by p, q. A disjunction is FALSE if and only if (iff) both statements are FALSE; otherwise it is TRUE.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional:

A conditional statement, symbolized by $p \rightarrow q$ is an if – then statement in which p is a hypothesis and q is a conclusion. The logical connector in a conditional statement is denoted by the symbol. The conditional is defined to be TRUE unless a TRUE hypothesis leads to a FALSE conclusion.

P	Q	$P \rightarrow Q$
T	T	T
T	F	F

F	T	T
F	F	T

Bi conditional:

A bi -conditional statement is defined to be TRUE whenever both parts have the same truth value. The bi-conditional operator is denoted by a double – headed arrow. The bi-conditional p↔ q represents "p if and only if", where p is a hypothesis and q is a conclusion.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Problems under logical connectives

1. Write the following statements in symbolic form, "If either S.Pavithra takes calculus or S. Sharnika takes sociology, then Malathy will take English.

Solution:

P: S.Pavithra takes calculus.

Q: S.Sharnika takes sociology.

R: Malathy takes English.

∴ The logical expression is $(P \vee Q) \rightarrow R$

2. S. Pavithra can accesses the internet from campus only if she is a computer science major or she is not a fresh girl.

Solution:

P: S. Pavithra can accesses the internet from campus.

Q: S. Pavithra is a computer science major.

R: S. Pavithra is a fresh girl.

$\neg P$: S. Pavithra is not a fresh girl.

∴ The logical expression is $P \rightarrow (Q \vee \neg R)$

3. How can this English sentence be translated into logical expression.

"You can access the internet from campus only if you are computer science major or you are not a freshman".

Solution:

P: You can access the internet from campus.

Q: You are computer science major.

R: You are a freshman.

$\neg$R: you are not a freshman.

$\therefore$ The logical expression is $P \rightarrow (Q \vee \neg R)$

4. Write the logical expression for "If tigers have wings then the earth travels round the sun."

Solution:

P: Tigers have wings. (F)

Q: Earth travels round the sun. (F)

The logical expression is $P \rightarrow Q$ (T)

5. Construct the truth table for a) $\neg (P \wedge Q)$ and b) $(\neg P) \vee (\neg Q)$

Solution:

To prove $\neg (P \wedge Q)$ and $(\neg P) \vee (\neg Q)$

P	Q	$(P \wedge Q)$	$\neg (P \wedge Q)$
T	T	T	F
T	F	F	T

F	T	F	T
F	F	F	T

P	Q	¬P	¬Q	(¬P) ∨ (¬Q)
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

6. Construct the truth table for (P ∨ Q) ∨ ¬Q.

Solution:

P	Q	P ∨ Q	¬Q	(P ∨ Q) ∨ ¬Q.
T	T	T	F	T
T	F	T	T	T
F	T	T	F	T
F	F	F	T	T

7. Construct the truth table for ¬(¬*P* ∨ ¬Q).

Solution:

P	Q	¬P	¬Q	¬*P* ∨ ¬Q	¬(¬*P* ∨ ¬Q)
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

8. Construct the truth table for S: (P∧Q) ∨ (¬P∧Q) ∨ (P∧ ¬Q)

Solution:

P	Q	P ∧ Q	¬P	¬Q	¬P∧Q	P∧ ¬Q	(¬P∧Q) ∨ (P∧ ¬Q)	S
T	T	T	F	F	F	F	F	T
T	F	F	F	T	F	T	T	T
F	T	F	T	F	T	F	T	T
F	F	F	T	T	F	F	F	F

9. Construct the truth table for i) R: ¬(¬*P* ∨ ¬Q). ii) ¬(¬*P* ∧ ¬Q).

Solution:

P	Q	¬P	¬Q	¬*P* ∨ ¬Q	¬*P* ∧ ¬Q	R	S

T	T	F	F	F	F	T	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	T	T	F	F

10. Construct the truth table for (P →Q) ∧ (Q→ P).

Solution:

P	Q	P →Q	Q→ P	(P →Q) ∧ (Q→ P)
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Tautology:

A Tautology is a statement that is always TRUE, no matter what. If you construct a truth table for a statement and all the column values for the statement are TRUE, then the statement is a Tautology because it’s always TRUE.

Contradiction:

A statement that is always FALSE is known as a Contradiction.

i.e, The last column values contains all FALSE values.

Results:

Tautology	**Contradiction**
In the result column all the entries are T	In the result column all the entries are F
T	F
T	F
T	F
T	F

Problems under Tautology and contradiction

1.Show that the proposition P ∨ ¬ (P ∧ Q) is a tautology.

Solution:

P	Q	P ∧ Q	¬ (P ∧ Q)	P ∨ ¬ (P ∧ Q)
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

2.Show that (Q ∨ (P∧ ¬Q)) ∨ (¬P ∧ ¬Q) is a tautology.

Solution:

Let S = (Q ∨ (P∧ ¬Q)) ∨ (¬P ∧ ¬Q)

P	Q	¬P	¬Q	P∧ ¬Q	¬P ∧ ¬Q	Q ∨ (P∧ ¬Q)	S
T	T	F	F	F	F	T	T
T	F	F	T	T	F	T	T
F	T	T	F	F	F	T	T
F	F	T	T	F	T	F	T

3. Show that ¬P → (P →Q) is a tautology.

Solution:

P	Q	¬P	P →Q	¬P → (P →Q)
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

4. Show that (P ∧ Q) ∧ ¬ (P ∨ Q) is a contradiction.

Solution:

P	Q	P ∧ Q	P ∨ Q	¬ (P ∨ Q)	(P ∧ Q) ∧ ¬ (P ∨ Q)
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

1. 2. Propositional Equivalences

Logical Equivalences or Equivalence Rules

Laws	Formulae
Idempotent Laws	$p \wedge p \Leftrightarrow p, p \vee p \Leftrightarrow p$
Associative Laws	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
Commutative Laws	$p \wedge q \Leftrightarrow q \wedge p$ $p \vee p \Leftrightarrow q \vee p$
DeMorgan's Laws	$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
Distributive Laws	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
Complement Laws	$p \wedge \neg p \Leftrightarrow F, p \vee \neg p \Leftrightarrow T$
Dominance Laws	$p \vee T \Leftrightarrow T, p \wedge F \Leftrightarrow F$
Identity Laws	$p \wedge T \Leftrightarrow p, p \vee F \Leftrightarrow p$
Absorption Laws	$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$
Double Negation Laws	$\neg(\neg p) \Leftrightarrow p$
Contra Positive Laws	$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$

Conditional as Disjunction	$p \to q \Leftrightarrow \neg p \vee q$
Biconditional as Conditional	$p \to q \Leftrightarrow (p \to q) \wedge (q \to p)$
Exportations laws	$p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

1. Determine whether $(\neg Q \wedge (P \to Q)) \to \neg P$ is a tautology.

Solution:

$(\neg Q \wedge (P \to Q)) \to \neg P$	Reason
$\Rightarrow (\neg Q \wedge (\neg P \vee Q)) \vee \neg P$	$P \to Q \Leftrightarrow \neg P \vee Q$
$\Rightarrow \neg(\neg Q \wedge (\neg P \vee Q)) \vee \neg P$	$P \to Q \Leftrightarrow \neg P \vee Q$
$\Rightarrow (Q \vee (P \wedge \neg Q)) \vee \neg P$	(DeMorgan's law)
$\Rightarrow ((Q \vee P) \wedge (Q \vee \neg Q)) \vee \neg P$	(Distributive law)
$\Rightarrow ((Q \vee P) \wedge T) \vee \neg P$	$P \vee \neg P \Leftrightarrow T$
$\Rightarrow (Q \vee P) \vee \neg P$	$P \wedge T \Leftrightarrow P$
$\Rightarrow (Q \vee P \vee \neg P)$	(Associative law)
$\Rightarrow (Q \vee T)$	$P \vee \neg P \Leftrightarrow T$
$\Rightarrow T$	$P \vee T \Leftrightarrow T$

2. Show that the formula $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

Solution:

$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$	Reason
$\Rightarrow Q \vee (P \vee \neg P) \wedge \neg Q$ $\Rightarrow (Q \vee (P \vee \neg P)) \vee (Q \vee \neg Q)$ $\Rightarrow (Q \vee T) \wedge T$ $\Rightarrow T \wedge T$	(Distributive law) (Distributive law) $P \vee \neg P \Leftrightarrow T$ $P \vee T \Leftrightarrow P$

3. Show that the formula $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology.

Solution:

$(P \wedge Q) \rightarrow (P \vee Q)$	Reason
$\Rightarrow \neg(P \wedge Q) \vee (P \vee Q)$ $\Rightarrow (\neg P \vee \neg Q) \vee (P \vee Q)$ $\Rightarrow (P \vee \neg P) \vee (Q \vee \neg Q)$ $\Rightarrow T \vee T = T$	$P \rightarrow Q \Leftrightarrow \neg P \vee Q$ (DeMorgan's law) (Associative law) (Negation law)

4. Show that the formula $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$

Solution:

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$	Reason
$\Rightarrow (\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R)$	(Distributive law)
$\Rightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$	(Associative law)
$\Rightarrow [(P \vee \neg Q) \vee (Q \vee P)] \wedge R$	(Distributive law)
$\Rightarrow [\neg(P \vee Q) \vee (P \vee Q)] \wedge R$	(DeMorgan's law)
$\Rightarrow T \wedge$ R	$P \vee \neg P \Leftrightarrow T$
$\Rightarrow$ R	$P \wedge T \Leftrightarrow P$

Equivalence

Two statement formulas A and B are equivalent iff $A \leftrightarrow B$ or $A \leftrightarrows B$ is a tautology.

It is denoted by the symbol $A \Leftrightarrow B$ which is read as "A is equivalent to B."

Remark:

To prove two statement formulas A and B are equivalent, we can use any one of the following method:

(i) using Truth Table, we show that truth values of both statements formulas A and B are same for each 2^n combinations.

(ii) Assume A. By applying various equivalence rules try to derive B and vice versa.

(iii) Prove $A \leftrightharpoons B$ is a tautology.

1. Show that $\neg(P \vee (\neg P \wedge Q))$ & $\neg P \wedge \neg Q$ are logically equivalent.

Solution:

$\neg(P \vee (\neg P \wedge Q))$	Reason
$\Leftrightarrow \neg P \wedge (\neg(\neg P \wedge \neg Q\ ))$	(DeMorgan's law)
$\Leftrightarrow \neg P \wedge [\neg(\neg P) \vee \neg Q]$	(DeMorgan's law)
$\Leftrightarrow \neg P \wedge (P \vee \neg Q\ )$	(Double Negation law)
$\Leftrightarrow (\neg P \wedge P) \vee (\neg P \wedge \neg Q)$	(Distributive law)
$\Leftrightarrow F \vee\ (\neg P \wedge \neg Q\ )$	$\neg P \wedge P \Leftrightarrow F$
$\Leftrightarrow (\neg P \wedge \neg Q\ ) \vee F$	(Commutative law)
$\Leftrightarrow \neg P \wedge \neg Q$	(identity law)

Hence $\neg(P \vee (\neg P \wedge Q))$ & $\neg P \wedge \neg Q$ are logically equivalent.

2. Prove that $P \to Q \Leftrightarrow P \to (P \wedge Q)$

Solution:

$P \to (P \wedge Q)$	Reason

$\Leftrightarrow \neg P \vee (P \wedge Q)$	(Conditional as disjunction)
$\Leftrightarrow (\neg P \vee P) \wedge (\neg P \wedge Q)$	(Distributive law)
$\Leftrightarrow T \wedge (\neg P \wedge Q)$	$\neg P \wedge P \Leftrightarrow F$
$\Leftrightarrow \neg P \wedge Q$	(Identity law)
$\Leftrightarrow P \rightarrow Q$	(Conditional as disjunction)

3. Prove that $(P \rightarrow R) \wedge (Q \rightarrow R) \Leftrightarrow (P \vee Q) \rightarrow R$

Solution:

$(P \rightarrow R) \wedge (Q \rightarrow R)$	Reason
$\Leftrightarrow (\neg P \wedge R) \wedge (\neg Q \wedge R)$	(Conditional as disjunction)
$\Leftrightarrow (\neg P \wedge \neg Q) \vee R$	(Distributive law)
$\Leftrightarrow \neg(P \vee Q) \vee R$	(DeMorgan's law)
$\Leftrightarrow (P \vee Q) \rightarrow R$	(Conditional as disjunction)

4. Prove that $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

Solution:

$P \to (Q \to R)$	Reason
$\Leftrightarrow \neg P \vee (Q \to R)$	(Conditional as disjunction)
$\Leftrightarrow \neg P \vee (\neg Q \vee R)$	(Conditional as disjunction)
$\Leftrightarrow \neg(\neg P \vee \neg Q) \vee R$	(Associative law)
$\Leftrightarrow \neg(P \wedge Q) \vee R$	(DeMorgan's law)
$\Leftrightarrow (P \wedge Q) \to R$	(Conditional as disjunction)

1. 3. Quantifiers

Normal Forms

Elementary Product

A product of the statement variables and their negations in the formula is called Elementary Product.

The possible elementary products are

$P, Q, \neg P \wedge Q, \neg Q \wedge P, P \wedge \neg P, Q \wedge \neg Q, P \wedge \neg P \wedge Q$

Elementary Sum

A sum of the two statement variables and their negations is called Elementary Sum.

The possible elementary sums are

$P, Q, \neg P \vee Q, \neg Q \vee P, P \vee \neg P \vee Q, P \vee Q$

Disjunctive Normal Forms (DNF)

A statement formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form of the given formula,

DNF = (Elementary product) ∨ (Elementary product) ∨ . . . ∨ (Elementary product)

Conjunctive Normal Forms (DNF)

A statement formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form of the given formula,

DNF = (Elementary product) ∨ (Elementary product) ∨ . . . ∨ (Elementary product)

Remark:

(i) Note that DNF and CNF of given statement formula need not be unique.

(ii) In DNF and CNF, the number of variables in each term need not be same.

1. Obtain a disjunctive Normal form $P \wedge (P \rightarrow Q)$

Solution:

$P \wedge (P \rightarrow Q)$	Reason
$\Rightarrow P \wedge (\neg P \vee Q)$ $\Rightarrow (P \wedge \neg P) \vee (P \wedge Q)$	$(P \rightarrow Q \Rightarrow \neg P \vee Q)$ (Distributive law)

Since the given statement formula is written in terms of sum of elementary product.

DNF of $P \wedge (P \rightarrow Q)$ is $(P \wedge \neg P) \vee (P \wedge Q)$

2. Obtain DNF of $\neg(P \vee Q) \rightleftharpoons (P \wedge Q)$

Solution:

$\neg(P \vee Q) \rightleftharpoons (P \wedge Q)$	Reason
$\Rightarrow [\neg(P \vee Q) \wedge (P \wedge Q)]$ $\vee [(P \vee Q) \wedge \neg(P \wedge Q)]$	$(R \rightleftharpoons S \Leftrightarrow (R \wedge S) \vee (\neg R \wedge \neg S))$
$\Rightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee [(P \vee Q) \wedge (\neg P \vee \neg Q)]$	(DeMorgan's law & Associative law)
$\Rightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee [P \wedge (\neg P \vee \neg Q) \vee (Q \wedge (\neg P \vee \neg Q))]$	(Distributive law)
$\Rightarrow (P \wedge Q \wedge \neg P \wedge \neg Q) \vee (P \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg P) \vee (Q \wedge \neg Q)$	(Distributive law)

Which is the required DNF.

Principal Normal Forms:

Min terms:

Let P and Q be 2 statement variables. Let us construct all possible formulas which consist of conjunction of P or its negation and conjunction of Q or its negation.

None of the formulas should contain both a variable and its negation. Using commutative law, if any two terms are equivalent choose any one of the term. Collect the remaining terms. They are called minterms.

For example, let P and Q be two variables, then the minterms are

$$P \wedge Q, P \wedge \neg Q, \neg P \wedge Q, \neg P \wedge \neg Q$$

Remark: 1

1. If there are "n" variables then the number of minterms is 2^n.

2. In elementary product a variable and its negation exist. But in minterms such things does not exist.

3. Let P, Q and R be 3 variables. The possible minterms are

1. $P \wedge Q \wedge R$

2. $P \wedge Q \wedge \neg R$

3. $P \wedge \neg Q \wedge R$

4. $\neg P \wedge Q \wedge R$

5. $\neg P \wedge \neg Q \wedge R$

6. $\neg P \wedge Q \wedge \neg R$

7. $P \wedge \neg Q \wedge \neg R$

8. $\neg P \wedge \neg Q \wedge \neg R$

Max terms:

Let P and Q be 2 statement variables. Let us construct all possible conjunction of disjunction P or its negation and Q or its negation. None of the formulas should contain both a variable and its negation. Using commutative law, if any two terms are equivalent choose any one of the term. Collect the remaining terms. They are called maxterms.

The possible maxterms with 2 variables are

$$P \vee Q, P \vee \neg Q, \neg P \vee Q, \neg P \vee \neg Q$$

Principal Disjunctive Normal Forms (PDNF)

For a given statement formula, an equivalent formula consisting of disjunction of minterms only is known as its Principal Disjunctive Normal Forms (PDNF)

PDNF = (minterms) ∨ (minterms) ∨ . . . ∨ (minterms)

Principal Conjunctive Normal Forms (PCNF)

For a given statement formula, an equivalent formula consisting of conjunction of maxterms only is known as its Principal Conjunctive Normal Forms (PCNF)

PCNF = (maxterms) ∧ (maxterms) ∧ . . . ∧ (maxterms)

Note:

1. PDNF is also called sum –of – products canonical form. PCNF is also called product – of – sums canonical form.

2. PDNF and PCNF of a given statement formula need not be unique.

PDNF and PCNF using Truth table

Using truth table, we can easily find PDNF and PCNF of given statement formulas.

Working rule to find PDNF:

1. Construct truth table for the given statement formula.

2. Choose each and every row in which the final column value is "TRUE"

3. In the selected row, if the truth value of each individual variable value is TRUE select that variable and truth value is FALSE then select the negation of that variable. In such a way collect all possible minterms.

4. Sum of all minterms gives the required PDNF.

Working rule to find PCNF:

1. Construct truth table for the given statement formula.

2. Choose each and every row in which the final column value is "FALSE"

3. In the selected row, if the truth value of each individual variable value is FALSE select that variable and truth value is TRUE then select the negation of that variable. In such a way collect all possible maxterms.

4. Product of all maxterms gives the required PCNF.

Problems under PDNF and PCNF using Truth table

1. Obtain PDNF of $P \to Q$

Solution:

P	Q	$P \to Q$	Min term
T	T	T	$P \wedge Q$
T	F	F	-
F	T	T	$\neg P \wedge Q$
F	F	T	$\neg P \wedge \neg Q$

2. Obtain the PDNF of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

Solution:

P	Q	R	$P \wedge Q$	$\neg P \wedge R$	$Q \wedge R$	$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$	Min term
T	T	T	T	F	T	T	$P \wedge Q \wedge R$
T	T	F	T	F	F	T	$P \wedge Q \wedge \neg R$
T	F	T	F	F	F	F	
T	F	F	F	F	F	F	
F	T	T	F	T	T	T	$\neg P \wedge Q \wedge R$
F	T	F	F	F	F	F	
F	F	T	F	T	F	T	$\neg P \wedge \neg Q \wedge R$
F	F	F	F	F	F	F	

The PDNF is $(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$

3. Find the PCNF and PDNF of the proposition $P \wedge (Q \rightarrow R)$

Solution:

P	Q	R	$Q \rightarrow R$	$P \wedge (Q \rightarrow R)$	Min term	Max term
T	T	T	T	F		$P \vee Q \vee R$
T	T	F	T	F		$P \vee Q \vee \neg R$
T	F	T	F	F		$P \vee \neg Q \vee R$

T	F	F	T	F		$P \vee \neg Q \vee \neg R$
F	T	T	T	T	$P \wedge \neg Q \wedge \neg R$	
F	T	F	T	T	$P \wedge \neg Q \wedge R$	
F	F	T	F	F		$\neg P \vee \neg Q \vee R$
F	F	F	T	T	$P \wedge Q \wedge R$	

The PDNF is $(P \wedge \neg Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge R)$

The PCNF is $(P \vee Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge$ $(\neg P \vee \neg Q \vee R)$

4. Find the Principal Conjunctive normal form of $(P \wedge A) \wedge (\neg P \wedge R)$

Solution:

P	Q	R	$\neg P$	$P \wedge Q$	$\neg P \wedge R$	$(P \wedge Q) \vee (\neg P \wedge R)$	Min term
T	T	T	F	T	F	T	
T	T	F	F	T	F	T	
T	F	T	F	F	F	F	$\neg P \vee Q \vee \neg R$
T	F	F	F	F	F	F	$\neg P \vee Q \vee R$
F	T	T	T	F	T	T	
F	T	F	T	F	F	F	$P \vee \neg Q \vee R$

F	F	T	T	F	T	T	
F	F	F	T	F	T	T	

The required PCNF is $(\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee R)$

The Theory of Inference

The main aim of logic is to provide rules of inference, or principles of reasoning. Here, we are concerned with the inferring of a conclusion from given premises.

We are going to check the logical validity of the conclusion, from the given set of premises by making use of Equivalence rule and implication rule, the theory associated with such things is called inference theory.

Direct Method

When a conclusion is derived from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called a direct proof.

Indirect method of proof:

(i) Method of Contradiction:

In order to show that a conclusion C follows logically from the premises

$H_1, H_2, \ldots, H_m$, we assume that C is false and consider $\neg C$ as an additional premises. If the new set of premises gives contradict value, then the assumption $\neg C$ is true does not hold simultaneously with $H_1 \wedge H_2, \wedge \ldots \wedge H_m$.

Therefore, C is true whenever $H_1 \wedge H_2, \wedge \ldots \wedge H_m$ ids true. Thus C follows logically from the premises $H_1, H_2, \ldots, H_m$.

(ii) Method of contrapositive:

In order to prove $H_1 \wedge H_2, \wedge \ldots \wedge H_m \Rightarrow C$, if we prove

$\neg C \Rightarrow \neg(H_1 \wedge H_2, \wedge \ldots \wedge H_m)$ then the original problem follows. This method is called contrapositive method.

Rules of Inference

Rule P: A premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced at any point in a derivation if S is tautologically implied by any one or more of the preceding formulas.

Rule CP: If S can be derived from R and set of premises, then $R \to S$ can be derived from the set of premises alone.

Remark:

(i) Rule CP means Rule of Conditional Proof.

(ii) Rule CP is also called the deduction theorem.

Implication Rule:

$P, P \to Q \Rightarrow Q$	Modus Phones
$\neg Q, P \to Q \Rightarrow \neg P$	Modus Tollens
$\neg P, P \vee Q \Rightarrow Q$	Disjunctive syllogism
$P \to Q, Q \to R \Rightarrow P \to R$	Hypothetical syllogism (or) chain rule

$P, Q \Rightarrow P \wedge Q$	Simplification rule
$P, Q \Rightarrow P \vee Q$	Addition rule
$P \wedge \neg Q \Rightarrow \neg(P \to Q)$	Equivalence rule

Note:

In the derivation, we should use all the rules but exactly once. Also, the order is immaterial.

1.Demonstrate that R is a valid inference from the premises $P \to Q, Q \to R$ & P

Solution:

$\{1\}$	$1) P \to Q$	Rule P
$\{2\}$	$2) Q \to R$	Rule P
$\{1,2\}$	$3) P \to R$	Rule T $(P \to Q, Q \to R \Rightarrow P \to R)$
$\{4\}$	$4) P$	Rule P
$\{1,2,4\}$	$5) R$	Rule T $(P, P \to Q \Rightarrow Q)$

2. Show that $\neg P$ follows logically from the premises $\neg(P \wedge \neg Q), (\neg Q \vee R)$ &$\neg R$

Solution:

Given premises are $\neg(P \wedge \neg Q), (\neg Q \vee R), \neg R$

Conclusion: &$\neg R$

$\{1\}$	1)$\neg(P \wedge \neg Q)$	Rule P
$\{2\}$	2))$\neg P \vee Q$	Rule T (Demorgan's law)
$\{1\}$	3)$P \rightarrow Q$	Rule T $(P \rightarrow Q \Leftrightarrow \neg P \vee R)$
$\{4\}$	4)$\neg Q \vee R$	Rule P
$\{4\}$	5)$Q \rightarrow R$	Rule T $(P \rightarrow Q \Leftrightarrow \neg P \vee R)$
$\{1,4\}$	6) $P \rightarrow R$	Rule T $(P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R)$
$\{7\}$	7) $\neg R$	Rule P
$\{1,4,7\}$	8) $\neg P$	Rule T $\neg Q, P \rightarrow Q \Rightarrow \neg P$

Consistency and Inconsistency of Premises

A set of formulae $H_1, H_2, \ldots, H_m$ is said to be inconsistent if their conjunction implies contradiction.

i.e., $H_1 \wedge H_2 \wedge \ldots \wedge H_m \Rightarrow R \wedge \neg R$ for some formulae R.

Note:$R \wedge \neg R \Leftrightarrow F$

Consistent:

A set of formulae $H_1, H_2, \ldots, H_m$ is said to be consistent if their conjunction implies tautology.

Inconsistent:

A set of formula $H_1, H_2, \ldots, H_m$ is said to be consistent if it is not inconsistent.

1. Show that $P \to Q, P \to R, Q \to \neg R$ & P are inconsistent.

Solution:

$\{1\}$	1) $P \to Q$	Rule P
$\{2\}$	2) $Q \to \neg R$	Rule P
$\{1, 2\}$	3) $P \to \neg R$	Rule T
$\{4\}$	4) P	Rule P
$\{1, 2, 4\}$	5) $\neg R$	Rule T
$\{6\}$	6) $P \to R$	Rule P
$\{1, 2, 4, 6\}$	7) $\neg P$	Rule T
$\{1, 2, 4, 6\}$	8) $P \wedge \neg P$	Rule T

Which is nothing but false value.

Hence given set of premises are inconsistent.

2. Prove that $P \to Q, Q \to R, S \to \neg R$ & $P \wedge S$ are inconsistent.

Solution:

$\{1\}$	$1)P \to Q$	Rule P
$\{2\}$	$2)Q \to R$	Rule P
$\{1,2\}$	$3)P \to R$	Rule T
$\{4\}$	$4)S \to \neg R$	Rule P
$\{4\}$	$5)R \to \neg S$	Rule T
$\{1,2,4\}$	$6)P \to \neg S$	Rule T
$\{1,2,4\}$	$7)\neg P \vee \neg S$	Rule T
$\{1,2,4\}$	$8)\neg(P \wedge S)$	Rule T
$\{9\}$	$9)P \wedge S$	Rule P
$\{1,2,4,9\}$	$10)(P \wedge S) \wedge \neg(P \wedge S)$	Rule T

Which is nothing but false value.

Hence given set of premises are inconsistent.

3. Prove that $a \to (b \to c), d \to (b \wedge \neg c),$ & $a \wedge d$ are inconsistent.

Solution:

$\{1\}$	$1)a \wedge d$	Rule P
$\{1\}$	$2)a, d$	Rule T

$\{3\}$	3)$a \rightarrow (b \rightarrow c)$	Rule P
$\{1,3\}$	4)$b \rightarrow c$	Rule T
$\{1,3\}$	5)$\neg b \vee c$	Rule T
$\{6\}$	6)$d \rightarrow (b \wedge \neg c)$	Rule P
$\{6\}$	7)$\neg(b \wedge \neg c) \rightarrow \neg d$	Rule T
$\{6\}$	8))$\neg b \vee c \rightarrow \neg d$	Rule T
$\{1,3,6\}$	9) $\neg d$	Rule T
$\{1,3,6\}$	10) $d \wedge \neg d$	Rule T

Which is nothing but false value.

Hence given set of premises are inconsistent.

1.6 The Predicate Calculus

The predicate calculus deals with the study of predicates.

Consider the following statement.

"Ram is a boy"

In the above statement, **"is a boy"** is the predicate and the name **"Ram"** is the subject.

If we denote **"is a boy"** by B and subject **"Ram"** by r, then the statement **"Ram is a boy"** can be represented as B(r).

Some examples

1." x is a man"

Here, Predicate is **"is a man"** and it is denoted by M. Subject is **"x"** and it is denoted by x.

Hence the given statement **"x is a man"** can be denoted by **M(x).**

2. "Sam is poor and Ram is intelligent"

The statement **"Sam is poor"** can be represented by **P(s)** where P represents predicate **"is poor"** and s represents subject **"Sam"**

The statement "Ram is intelligent" can be represented by I(r) where I represents predicate **"is intelligent"** and r represents subject **"Ram".**

Hence the given statement **"Sam is poor and Ram is intelligent"** can be symbolized as $\boldsymbol{P(s) \wedge I(r)}$**.**

The Theory of Inference for Predicate Calculus

Universal Specification (UG): $A(y) \Rightarrow (x)A(x)$

Existential Generalization (EG): $A(y) \Rightarrow (\exists x)A(x)$

Universal Specification (US): $(x)A(x) \Rightarrow A(y)$

Existential Specification (ES): $(\exists x)A(x) \Rightarrow A(y)$

Problems:

1.Show that $\boldsymbol{(x)(H(x) \rightarrow M(x)) \wedge H(s) \Rightarrow M(s)}$

Solution:

$\{1\}$	1) $(x)(H(x) \rightarrow M(x))$	Rule P
$\{1\}$	2)$H(s) \rightarrow M(s)$	Rule US
$\{3\}$	3 $H(s)$	Rule P
$\{1, 3\}$	4)$M(s)$	Rule T $(P, P \rightarrow Q \Rightarrow Q)$

2. Show that $(x)(P(x) \rightarrow Q(x)) \wedge (x)(Q(x) \rightarrow R(x)) \Rightarrow (x)(P(x) \rightarrow R(x))$

Solution:

$\{1\}$	1) $(x)(P(x) \rightarrow Q(x))$	Rule P
$\{1\}$	2)$P(y) \rightarrow Q(y)$	Rule US
$\{3\}$	3 $(x)(Q(x) \rightarrow R(x))$	Rule P
$\{1,3\}$	4)$Q(y) \rightarrow R(y)$	Rule US
$\{1,3\}$	5) $P(y) \rightarrow R(y)$	Rule T $(P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R)$
$\{1,3\}$	6)$(x)(P(x) \rightarrow R(x))$	Rule UG

3. Show that $(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$

Solution:

$\{1\}$	1) $(\exists x)(P(x) \wedge Q(x))$	Rule P
$\{1\}$	2)$P(y) \wedge Q(y)$	Rule ES
$\{3\}$	3 $P(y)$	Rule T $(P \wedge Q \Rightarrow P)$
$\{1,3\}$	4)$Q(y)$	Rule T $(P \wedge Q \Rightarrow P)$
$\{1,3\}$	5) $(\exists x)P(x)$	Rule EG
$\{1,3\}$	6)$(\exists x)Q(x)$	Rule EG
$\{1\}$	7)$(\exists x)P(x) \wedge (\exists x)Q(x)$	Rule T$(P, Q \Rightarrow P \wedge Q)$

4.Show that $(x)(P(x) \vee Q(x)) \Rightarrow (x)P(x) \vee (\exists x)Q(x)$

Solution:

We shall use the indirect method of proof.

Method of contradiction:

Assume $\neg((x)P(x) \vee (\exists x)Q(x))$ as an additional premises.

{1}	1) $\neg((x)P(x) \vee (\exists x)Q(x))$	Assumed Premises
{1}	2) $(\exists x)\neg P(x) \wedge (x)Q(x)$	Rule T (D'Morgan's law)
{1}	3) $(\exists x)\neg P(x)$	Rule T $(P \wedge Q \Rightarrow P)$
{1}	4) $(x)Q(x)$	Rule T $(P \wedge Q \Rightarrow P)$
{1}	5) $\neg P(y)$	Rule ES
{1}	6) $\neg Q(y)$	Rule US
{1}	7) $\neg P(y) \wedge \neg Q(y)$	Rule T$(P, Q \Rightarrow P \wedge Q)$
{1}	8) $\neg(P(y) \vee Q(y))$	Rule T (D'Morgan's law)
{1}	9) $(x)(P(x) \vee Q(x))$	Rule P
{1}	10) $P(y) \vee Q(y)$	Rule US
{1}	11) $(P(y) \vee Q(y)) \wedge \neg(P(y) \vee Q(y))$	Rule T$(P, Q \Rightarrow P \wedge Q)$

which is nothing but false value.

5.Show that $(x)(P(x) \to Q(x)) \Rightarrow (x)P(x) \to (x)Q(x)$

Solution:

Assume $\neg((x)P(x) \to (x)Q(x))$

$\{1\}$	1) $\neg((x)P(x) \to (x)Q(x))$	Assumed Premises
$\{1\}$	2) $(x)P(x) \wedge \neg(x)Q(x)$	Rule T $(P \to Q \Rightarrow \neg P \vee Q)$
$\{1\}$	3) $(x)P(x)$	Rule T $(P \wedge Q \Rightarrow P)$
$\{1\}$	4) $\neg((x)Q(x))$	Rule T $(P \wedge Q \Rightarrow P)$
$\{1\}$	5) $(\exists x)\neg Q(x)$	Rule T(Taking $\neg$)
$\{1\}$	6) $P(y)$	Rule US
$\{1\}$	7) $\neg Q(y)$	Rule ES
$\{1\}$	8) $P(y) \wedge \neg Q(y)$	Rule T $(P, Q \Rightarrow P \wedge Q)$
$\{9\}$	9) $\neg(P(y) \to Q(y))$	Rule T$((P \wedge \neg Q) \Leftrightarrow \neg(P \to Q))$
$\{9\}$	10) $(\exists x)\neg(P(x) \to Q(x))$	Rule EG
$\{1, 9\}$	11) $\neg((x)P(x) \to Q(x))$	Rule T(Taking $\neg$)