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EXISTENCE CONDITIONS-LAPLACE TRANSFORM

Let $f(t)$ be a function of t defined for all $t \geq 0$ .then the Laplace transform of $f(t)$, denoted by $L[f(t)]$ is defined by

$$L[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

Provided that the integral exists, "s" is a parameter which may be real or complex. Clearly $L[f(t)]$ is a function of s and is briefly written as $F(s)$ $(i.e.)$ $L[f(t)] = F(s)$

Piecewise continuous function

A function $f(t)$ is said to be piecewise continuous is an interval $a \leq t \leq b$, if the interval can be sub divided into a finite number of intervals in each of which the function is continuous and has finite right and left hand limits.

Exponential order

A function $f(t)$ is said to be exponential order if $\lim_{t\to\infty} e^{-st} f(t)$ is a finite quantity, where $s > 0$(exists).

Example: Show that the function $f(t) = e^{t^3}$ is not of exponential order.

Solution:

$$\lim_{t\to\infty} e^{-st} e^{t^3} = \lim_{t\to\infty} e^{-st+t^3} = \lim_{t\to\infty} e^{t^3-st}$$

$= e^\infty = \infty$, not a finite quantity.

Hence $f(t) = e^{t^3}$ is not of exponential order.

Sufficient conditions for the existence of the Laplace transform

The Laplace transform of $f(t)$ exists if

i) $f(t)$ is piecewise continuous in the interval $a \leq t \leq b$

ii) $f(t)$ is of exponential order.

Note: The above conditions are only sufficient conditions and not a necessary condition.

Example: Prove that Laplace transform of e^{t^2} does not exist.

Solution:

$$\lim_{t\to\infty} e^{-st} e^{t^2} = \lim_{t\to\infty} e^{-st+t^2} = \lim_{t\to\infty} e^{t^2-st}$$

$= e^\infty = \infty$,not a finite quantity.

$\therefore$ e^{t^2} is not of exponential order.

Hence Laplace transform of e^{t^2} does not exist.

Laplace transform of elementary functions

Result: 1 Prove that $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[t^n] = \int_0^\infty e^{-st}\, t^n\, dt$$

$$L[t^n] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \int_0^\infty e^{-u} \frac{u^n}{s^{n+1}}\, du$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n\, du$$

$$\therefore L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$$

$\because \int_0^\infty e^{-u} u^n\, du$

Let $st = u \cdots\cdots (1)$

$t = \frac{u}{s}$

$dt = \frac{du}{s}$

When $t \to 0 (1) => u \to 0$

,

$t \to \infty, (1) => u \to \infty$

Note: If n is an integer, then $\Gamma(n+1) = n!$

$\therefore L[t^n] = \frac{n!}{s^{n+1}}$ if n is an integer

If $n = 0$, then $L[1] = \frac{1}{s}$

If $n = 1$, then $L[t] = \frac{1}{s^2}$

Similarly $L[t^2] = \frac{2!}{s^3}$

$L[t^3] = \frac{3!}{s^4}$

Result: 2 Prove that $L(e^{at}) = \frac{1}{s-a}$, $s > a$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$\therefore L(e^{at}) = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-t(s-a)}\, f(t)\, dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty$$

$$= -\left[0 - \left(\frac{1}{s-a}\right)\right]$$

$$\therefore L(e^{at}) = \frac{1}{s-a}$$

Result: 3 Prove that $L(e^{-at}) = \frac{1}{s+a}$, $s > a$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$\therefore L(e^{-at}) = \int_0^\infty e^{-st} e^{-at}\; dt$$

$$= \int_0^\infty e^{-t(s+a)} f(t)\, dt$$

$$= \left[\frac{e^{-t(s+a)}}{-(s+a)}\right]_0^\infty$$

$$= -\left[0 - \left(\frac{1}{s+a}\right)\right]$$

$$\therefore L(e^{at}) = \frac{1}{s+a}$$

Result: 4 Prove that $L[sinat] = \frac{a}{s^2+a^2}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[sinat] = \int_0^\infty e^{-st} sinat\, dt$$

$$\therefore L[sinat] = \frac{a}{s^2+a^2}, s > |a| \qquad \left[\because \int_0^\infty e^{-at} sinbt\, dt = \frac{b}{a^2+b^2}\right]$$

Result: 5 Prove that $L[cosat] = \frac{s}{s^2+a^2}$

Proof:

We know that $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$L[cosat] = \int_0^\infty e^{-st} cosat\, dt$$

$$\therefore L[cosat] = \frac{s}{s^2+a^2}, s > |a| \qquad \because \int_0^\infty e^{-at} cosbt\, dt = \frac{a}{a^2+b^2}$$

Result: 6 Prove that $L[sinhat] = \frac{a}{s^2-a^2}$ $, s > |a|$

Proof:

We have $L[sinhat] = L\left[\frac{e^{at}-e^{-at}}{2}\right]$

$$= \frac{1}{2}[L(e^{at}) - L(e^{-at})]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$$

$$\therefore L[sinhat] = \frac{a}{s^2-a^2}, s > |a|$$

Result: 7 Prove that $L[coshat] = \frac{s}{s^2-a^2}$ $, s > |a|$

Proo

We have $L[coshat] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$

$= \frac{1}{2}[L(e^{at}) + L(e^{-at})]$

$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$

$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$

$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]$

$\therefore L[coshat] = \frac{s}{s^2-a^2}\ , s > |a|$

Example: Find $L[t^{\frac{1}{2}}]$

Solution:

We have $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Put $n = \frac{1}{2}$

$\therefore L[t^{\frac{1}{2}}] = \frac{\Gamma(\frac{1}{2}+1)}{s^{\frac{1}{2}+1}}$ $\qquad \because \Gamma(n+1) = n\Gamma n$

$= \frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{\frac{1}{2}+1}}$ $\qquad \because \Gamma(\frac{1}{2}) = \sqrt{\pi}$

$= \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$

$\therefore L[t^{\frac{1}{2}}] = \frac{\sqrt{\pi}}{2s\sqrt{s}}$

Example: Find the Laplace transform of $t^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{t}}$

Solution:

We have $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

Put $n = -\frac{1}{2}$

$\therefore L[t^{-\frac{1}{2}}] = \frac{\Gamma(-\frac{1}{2}+1)}{s^{-\frac{1}{2}+1}}$ $\qquad \because \Gamma(n+1) = n\Gamma n$

$= \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}}$ $\qquad \because \Gamma(\frac{1}{2}) = \sqrt{\pi}$

$= \frac{\sqrt{\pi}}{\sqrt{s}}$

$\therefore L\left[\frac{1}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}}$

FORMULA

$L[f(t)] = F(s)$	$L[f(t)] = F(s)$
$L[1] = \frac{1}{s}$	$L[sinat] = \frac{a}{s^2 + a^2}$
$L[t] = \frac{1}{s^2}$	$L[cosat] = \frac{s}{s^2 + a^2}$
$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$ if n is not an integer	$L[coshat] = \frac{s}{s^2 - a^2}$
$L[t^n] = \frac{n!}{s^{n+1}}$ if n is an integer	$L[sinhat] = \frac{a}{s^2 - a^2}$
$L(e^{at}) = \frac{1}{s-a}$	
$L(e^{at}) = \frac{1}{s+a}$	

Problems using Linear property

Example:Find the Laplace transform for the following

i.	$3t^2 + 2t + 1$	v.	$sin\sqrt{2}\,t$	ix.	sin^2t
ii.	$(t+2)^3$	vi.	$sin(at+b)$	x.	$cos^2 2t$
iii.	a^t	vii.	$cos^3 2t$	xi.	$cos5tcos4t$
iv.	e^{2t+3}	viii.	sin^3t		

Solution:

(i) Given $f(t) = 3t^2 + 2t + 1$

$$L[f(t)] = L[3t^2 + 2t + 1]$$
$$= L[3t^2] + L[2t] + L[1]$$
$$= L[3t^2] + L[2t] + L[1]$$
$$= 3L[t^2] + 2L[t] + L[1]$$
$$= 3\frac{2}{s^3} + 2\frac{1}{s^2} + \frac{1}{s}$$
$$\therefore L[3t^2 + 2t + 1] = \frac{6}{s^3} + \frac{2}{s^2} + \frac{1}{s}$$

(ii) Given $f(t) = (t+2)^3 = t^3 + 3t^2(2) + 3t2^2 + 2^3$

$$L[f(t)] = L[t^3 + 3t^2(2) + 3t2^2 + 2^3]$$
$$= L[t^3] + L[6t^2] + L[12t] + L[8]$$
$$= L[t^3] + 6L[t^2] + 12L[t] + 8L[1]$$

$$= \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{12}{s}$$

(iii) Given $f(t) = a^t$

$$L[f(t)] = L[a^t] = L[e^{tloga}]$$

$$L[a^t] = \frac{1}{s-loga}$$

(iv)Given $f(t) = e^{2t+3}$

$$L[f(t)] = L[e^{2t+3}] = L[e^{2t}. e^3]$$

$$= e^3 L[e^{2t}]$$

$$= e^3 [\frac{1}{s-2}]$$

$$\therefore L[e^{2t+3}] = e^3 [\frac{1}{s-2}]$$

(v) $L[sin\sqrt{2}t] = \frac{\sqrt{2}}{s^2+2}$

(vi)Given $f(t) = \sin(at+b) = sinatcosb + cosatsinb$

$$L[f(t)] = L[\sin(at+b)]$$

$$= L[sinatcosb + cosatsinb]$$

$$= cosb\, L[sinat] + sinb\, L[cosat]$$

$$L[\sin(at+b)] = cosb \frac{s}{s^2+a^2} + sinb \frac{s}{s^2+a^2}$$

(vii) Given $f(t) = \cos^3 2t = \frac{1}{4}[3cos2t + cos6t]$

$$\because \cos^3\theta = \frac{3cos\theta + cos3\theta}{4}$$

$$L[f(t)] = \frac{1}{4} L[3cos2t + cos6t]$$

$$= \frac{1}{4}[3L(cos2t) + L(cos6t)]$$

$$= \frac{1}{4}[3 \frac{s}{s^2+4} + \frac{s}{s^2+36}]$$

$$L[\cos^3 2t] = \frac{1}{4}[3 \frac{s}{s^2+4} + \frac{s}{s^2+36}]$$

(viii) Given $f(t) = \sin^3 t = \frac{1}{4}[3sint - sin3t]$

$$L[f(t)] = \frac{1}{4} L[3sint - sin3t]$$

$$= \frac{1}{4}[3L(sint) - L(sin3t)]$$

$$= \frac{1}{4}[3 \frac{1}{s^2+1} - \frac{3}{s^2+9}]$$

$$L[\sin^3 t] = \frac{3}{4}[\frac{1}{s^2+1} - \frac{1}{s^2+9}]$$

(ix) Given $f(t) = \sin^2 t = \frac{1-cos2t}{2}$

$$L[f(t)] = L [\frac{1-cos2t}{2}]$$

$$= \frac{1}{2}[L(1) - L(cos2t)]$$

$$= \frac{1}{2}[\frac{1}{s} - \frac{s}{s^2+4}]$$

$$L[\cos^2 2t] = \frac{1}{2}[\frac{1}{s} - \frac{s}{s^2+4}]$$

(x) Given $f(t) = \cos^2 2t = \frac{1+cos4t}{2}$

$$L[f(t)] = L\left[\frac{1+cos4t}{2}\right]$$

$$= \frac{1}{2}[L(1) + L(cos4t)]$$

$$= \frac{1}{2}[\frac{1}{s} + \frac{s}{s^2+16}]$$

$$L[\cos^2 2t] = \frac{1}{2}[\frac{1}{s} + \frac{s}{s^2+16}]$$

(xi) Given $f(t) = cos5tcos4t$

$$L[f(t)] = L[cos5tcos4t]$$

$$= \frac{1}{2}[L(cos9t) + L(cost)]$$

$$= \frac{1}{2}[\frac{s}{s^2+81} + \frac{s}{s^2+1}]$$

PROPERTIES OF LAPLACE TRANSFORM

Property: 1 Linear property

$L[af(t) \pm bg(t)] = aL[f(t)] \pm bL[g(t)]$ **, where a and b are constants.**

Proof:

$$L[af(t) \pm bg(t)] = \int_0^\infty [af(t) \pm bg(t)]\, e^{-st} dt$$

$$= a\int_0^\infty f(t) e^{-st} dt \pm b\int_0^\infty g(t)\, e^{-st} dt$$

$$\boxed{L[af(t) \pm bg(t)] = a\, L[f(t)] \pm b\, L[g(t)]}$$

Property: 2 Change of scale property.

If $L[f(t)] = F(s)$, **then** $L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$; $a > 0$

Proof:

Given $L[f(t)] = F(s)$

$$\therefore \int_0^\infty e^{-st} f(t)\, dt = F(s) \cdots\cdots (1)$$

By the definition of Laplace transform, we have

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\, dt \cdots\cdots (2)$$

Put at= x ie., $t = \frac{x}{a} \Rightarrow dt = \frac{dx}{a}$

$$(2) \Rightarrow L[f(at)] = \int_0^\infty e^{\frac{-sx}{a}} f(x) \frac{dx}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{\frac{-sx}{a}} f(x) dx$$

Replace x by t, $L[f(at)] = \frac{1}{a}\int_0^\infty e^{\frac{-st}{a}} f(t) dt$

$$\boxed{L[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right) ;\, a > 0}$$

Property: 3 First shifting property.

If $L[f(t)] = F(s)$, **then i)** $L[e^{-at} f(t)] = F(s + a)$

ii) $L[e^{at} f(t)] = F(s - a)$

Proof:

(i) $L[e^{-at} f(t)] = F(s + a)$

Given $L[f(t)] = F(s)$

$$\therefore \int_0^\infty e^{-st} f(t)\, dt = F(s) \cdots (1)$$

By the definition of Laplace transform, we have

$$L[e^{-at} f(at)] = \int_0^\infty e^{-st}\, e^{-at} f(t)\, dt$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\, dt$$

$$= F(s+a) \quad \text{by (1)}$$

(ii) $L[e^{at} f(at)] = \int_0^\infty e^{-st}\, e^{at} f(t)\, dt$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt$$

$$= F(s-a) \quad \text{by (1)}$$

Property: 4 Laplace transforms of derivatives $L[f'(t)] = sL[f(t)] - f(0)$

Proof:

$$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\, dt = \int_0^\infty u\,dv$$

$$= [uv]_0^\infty - \int u\,dv$$

$$= [e^{-st} f(t)]_0^\infty - \int_0^\infty f(t)\,(-s)e^{-st}\, dt$$

$$= 0 - f(0) + sL[f(t)]$$

$$= sL[f(t)] - f(0)$$

$$L[f'(t)] = sL[f(t)] - f(0)$$

$u = e^{-st}$ $\therefore du = -se^{-st}dt$ $dv = f'(t)dt$ $\therefore v = \int f'(t)dt$ $= f(t)$

Property: 5 Laplace transform of derivative of order n

$$L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - s^{n-3} f''(0) - \cdots f^{n-1}(0)$$

Proof:

We know that $L[f'(t)] = sL[f(t)] - f(0) \cdots\cdots (1)$

$$L[f^n(t)] = L[[f'(t)]']$$

$$= sL[f'(t)] - f'(0)$$

$$= s[sL[f(t)] - f(0)] - f'(0)$$

$$= s^2 L[f(t)] - sf(0) - f'(0)$$

Similarly, $L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$

In general, $L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - s^{n-3} f''(0) - \cdots f^{n-1}(0)$

Laplace transform of integrals

Theorem: 1 If $L[f(t)] = F(s)$, then $L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$

Proof:

Let $g(t) = \int_0^t f(t)dt$

$\therefore g'(t) = f(t)$

And $g(0) = \int_0^0 f(t)dt = 0$

Now $L[g'(t)] = L[f(t)]$

$$sL[g(t)] - g(0) = L[f(t)]$$

$$sL[g(t)] = L[f(t)] \quad \therefore g(0) = 0$$

$$L[g(t)] = \frac{L[f(t)]}{s}$$

$$\therefore \boldsymbol{L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}}$$

Theorem: 2 If $\boldsymbol{L[f(t)] = F(s)}$**, then** $\boldsymbol{L[tf(t)] = -\frac{d}{ds}F(s)}$

Proof:

Given $L[f(t)] = F(s)$

$$\therefore \int_0^\infty e^{-st} f(t)\, dt = F(s) \cdots\cdots (1)$$

Differentiating (1) with respect to s, we get

$$\frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt = \frac{d}{ds} F(s)$$

$$\int_0^\infty \frac{\partial}{\partial s} (e^{-st})f(t)\, dt = \frac{d}{ds}F(s)$$

$$\int_0^\infty (-t)e^{-st} f(t)\, dt = \frac{d}{ds}F(s)$$

$$-\int_0^\infty e^{-st} f(t)\, dt = \frac{d}{ds} F(s)$$

$$-L[tf(t)] = \frac{d}{ds}F(s)$$

$$\therefore \boldsymbol{L[tf(t)] = -\frac{d}{ds}F(s)}$$

Note: In general $\boldsymbol{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)}$

Example: If $\boldsymbol{L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)}}$ **then find** $\boldsymbol{L[f(2t)]}$**.**

Solution:

Given $L[f(t)] = \frac{s^2-s+1}{(2s+1)^2(s-1)} = F(s)$

$$L[f(2t)] = \frac{1}{2} F\left(\frac{s}{2}\right)$$

$$= \frac{1}{2}\frac{\left(\frac{s}{2}\right)^2 - \frac{s}{2}+1}{\left(2\frac{s}{2}+1\right)^2\left(\frac{s}{2}-1\right)}$$

$$= \frac{1}{2}\frac{\left[\frac{s^2}{4} - \frac{s}{2}+4\right]}{(s+1)^2\left(\frac{s-2}{2}\right)}$$

$$= \frac{s^2-2s+1}{4(s+1)^2(s-2)}$$

Shifting theorem

$$L[e^{-at}f(t)] = L[f(t)]_{s\to s+a}$$

$$L[e^{at}f(t)] = L[f(t)]_{s\to s-a}$$

Example: 5.7 Find the Laplace transform for the following:

i. te^{-3t}	vii. $t^2 2^t$
ii. t^3e^{2t}	viii. $t^3 2^{-t}$
iii. $e^{4t}sin2t$	ix. $e^{-2t}sin3tcos2t$
iv. $e^{-5t}cos3t$	x. $e^{-3t}cos4tcos2t$
v. $sinh2tcos3t$	xi. $e^{4t}cos3tsin2t$
vi. cosh3tsin2t	

(i) te^{-3t}

$$L[te^{-3t}] = L[t]_{s\to s+3}$$

$$= \left(\frac{1}{s^2}\right)_{s\to s+3} \qquad \because L(t) = \frac{1}{s^2}$$

$$\therefore L[te^{-3t}] = \frac{1}{(s+3)^2}$$

(ii) t^3e^{2t}

$$L[t^3e^{2t}] = L[t^3]_{s\to s-2}$$

$$= \left(\frac{3!}{s^4}\right)_{s\to s-2} \qquad \because L(t) = \frac{3!}{s^{3+1}}$$

$$\therefore L[t^3e^{2t}] = \frac{6}{(s-2)^4}$$

(iii) $e^{4t}sin2t$

$$L[e^{4t}sin2t] = L[sin2t]_{s\to s-4}$$

$$= \left(\frac{2}{s^2+2^2}\right)_{s\to s-4}$$

$$= \frac{2}{(s-4)^2+4}$$

$$= \frac{2}{s^2-8s+16+4}$$

$$\therefore L[e^{4t}sin2t] = \frac{2}{s^2-8s+20}$$

(iv) $L[e^{-5t}cos3t]$

$$L[e^{-5t}cos3t] = L[cos3t]_{s\to s+5}$$

$$= \left(\frac{s}{s^2+3^2}\right)_{s\to s+5}$$

$$= \frac{s+5}{(s+5)^2+9}$$

$$= \frac{s+5}{s^2+10s+25+9}$$

$$\therefore \ L[e^{-5t}cos3t] = \frac{s+5}{s^2+10s+34}$$

(v) $L[sinh2tcos3t]$

$$L[sinh2tcos3t] = L\left[\left(\frac{e^{2t}-e^{-2t}}{2}\right)cos3t\right]$$

$$= \frac{1}{2}[L(e^{2t}cos3t) - L(e^{-2t}cos3t)]$$

$$= \frac{1}{2}[L(cos3t)_{s\to s-2} - L(cos3t)_{s\to s+2}]$$

$$= \frac{1}{2}\left[\left(\frac{s}{s^2+3^2}\right)_{s\to s-2} - \left(\frac{s}{s^2+3^2}\right)_{s\to s+2}\right]$$

$$\therefore L[sinh2tcos3t] = \frac{1}{2}\left[\frac{s-2}{(s-2)^2+9} - \frac{s+2}{(s+2)^2+9}\right]$$

(vi) $L[cosh3tsin2t]$

$$L[cosh3tsin2t] = L\left[\left(\frac{e^{3t}+e^{-3t}}{2}\right)sin2t\right]$$

$$= \frac{1}{2}[L(e^{3t}sin2t) + L(e^{-3t}sin2t)]$$

$$= \frac{1}{2}[L(sin2t)_{s\to s-3} + L(sin2t)_{s\to s+3}]$$

$$= \frac{1}{2}\left[\left(\frac{2}{s^2+2^2}\right)_{s\to s-3} + \left(\frac{2}{s^2+2^2}\right)_{s\to s+3}\right]$$

$$\therefore \ L[cosh3tsin2t] = \frac{1}{2}\left[\frac{2}{(s-3)^2+4} + \frac{2}{(s+3)^2+4}\right]$$

(vii) $t^2 2^t$

$$L[t^2 2^t] = L[t^2 e^{log2^t}]$$

$$= L[t^2 e^{tlog2}] = L[t^2]_{s\to s-log2}$$

$$= \left(\frac{2!}{s^3}\right)_{s\to s-log2}$$

$$= \frac{2}{(s-log2)^3}$$

$$\therefore L[t^2 2^t] = \frac{2}{(s-log2)^3}$$

(viii) $t^3 2^{-t}$

$$L[t^3 2^{-t}] = L[t^3 e^{log2^{-t}}]$$

$$= L[t^3 e^{-tlog2}] = L[t^3]_{s\to s+log2}$$

$$= \left(\frac{3!}{s^4}\right)_{s\to s+\log 2}$$

$$= \frac{6}{(s+\log 2)^4}$$

$$\therefore L[t^3 2^{-t}] = \frac{6}{(s+\log 2)^4}$$

(ix) $L[e^{-2t}\sin 3t\cos 2t]$

$$L[e^{-2t}\sin 3t\cos 2t] = L[\sin 3t\cos 2t]_{s\to s+2}$$

$$= \frac{1}{2}L[\sin(3t+2t)+\sin(3t-2t)]_{s\to s+2}$$

$$= \frac{1}{2}L[\sin 5t+\sin t]_{s\to s+2}$$

$$= \frac{1}{2}[L(\sin 5t)+L(\sin t)]_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2}+\frac{1}{s^2+1^2}\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\frac{5}{(s+2)^2+25}+\frac{1}{(s+2)^2+1}\right]$$

$$\therefore L[e^{-2t}\sin 3t\cos 2t] == \frac{1}{2}\left[\frac{5}{(s+2)^2+25}+\frac{1}{(s+2)^2+1}\right]$$

(x) $L[e^{-3t}\cos 4t\cos 2t]$

$$L[e^{-3t}\cos 4t\cos 2t] = L[\cos 4t\cos 2t]_{s\to s+3}$$

$$= \frac{1}{2}L[\cos(4t+2t)+\cos(4t-2t)]_{s\to s+3}$$

$$= \frac{1}{2}L[\cos 6t+\cos 2t]_{s\to s+3}$$

$$= \frac{1}{2}[L(\cos 6t)+L(\cos 2t)]_{s\to s+3}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+6^2}+\frac{s}{s^2+2^2}\right]_{s\to s+3}$$

$$= \frac{1}{2}\left[\frac{s+3}{(s+3)^2+36}+\frac{s+3}{(s+3)^2+4}\right]$$

$$\therefore L[e^{-3t}\cos 4t\cos 2t] = \frac{1}{2}\left[\frac{s+3}{(s+3)^2+36}+\frac{s+3}{(s+3)^2+4}\right]$$

(xi) $L[e^{4t}\cos 3t\sin 2t]$

$$L[e^{4t}\cos 3t\sin 2t] = L[\cos 3t\sin 2t]_{s\to s-4}$$

$$= \frac{1}{2}L[\sin(3t+2t)-\sin(3t-2t)]_{s\to s-4}$$

$$= \frac{1}{2}L[\sin 5t-\sin t]_{s\to s-4}$$

$$= \frac{1}{2}[L(\sin 5t)-L(\sin t)]_{s\to s-4}$$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2}-\frac{1}{s^2+1^2}\right]_{s\to s-4}$$

$$= \frac{1}{2}\left[\frac{5}{(s-4)^2+25}+\frac{1}{(s-4)^2+1}\right]$$

$$\therefore L[e^{4t}cos3tsin2t] = \frac{1}{2}[\frac{5}{(s-4)^2+25} + \frac{1}{(s-4)^2+1}]$$

LAPLACE TRANSFORM OF DERIVATIVES AND INTEGRALS

Problems using the formula

$$L[tf(t)] = \frac{-d}{ds} L[f(t)]$$

Example: Find the Laplace transform for $tsin4t$

Solution:

$$L[tsin4t] = \frac{-d}{ds} L[tsin4t]$$

$$= \frac{-d}{ds}\left[\frac{4}{s^2+4}\right]$$

$$= \frac{-[(s^2+16)0-4(2s)]}{(s^2+16)^2}$$

$$\therefore L[tsin4t] = \frac{8s}{(s^2+16)^2}$$

Example: Find $L[tsin^2t]$

Solution:

$$L[tsin^2t] = \frac{-d}{ds} L[\sin^2 t] = \frac{-d}{ds} L\left[\frac{(1-cos2t)}{2}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}[L(1) - L(cos2t)]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right]$$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{4}{s(s^2+4)}\right]$$

$$= -\frac{4}{2}\frac{d}{ds}\left[\frac{1}{s(s^2+4)}\right]$$

$$= -2\left[\frac{0-(3s^2+4)}{(s^3+4s)^2}\right]$$

$$\therefore L[tsin^2t] = \frac{2(3s^2+4)}{(s^3+4s)^2}$$

Example: Find the Laplace transform for $f(t) = sinat - atcosat$

Solution:

$$L[sinat - atcosat] = \mathrm{L}(\sin at) - a\,\mathrm{L}(\mathrm{tcosat})$$

$$= \frac{a}{s^2+a^2} - a\left(\frac{-d}{ds} L[cosat]\right)$$

$$= \frac{a}{s^2+a^2} + a\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{(s^2+a^2)1-s(2s)}{(s^2+a^2)^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{s^2+a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{a}{s^2+a^2} + a\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$= \frac{a(s^2+a^2)+a(a^2-s^2)}{(s^2+a^2)^2}$$

$$= \frac{as^2+a^3+a^3-as^2)}{(s^2+a^2)^2}$$

$$\therefore L[sinat - atcosat] = \frac{2a^3}{(s^2+a^2)^2}$$

Example: Find the Laplace transform for the following

(i) $te^{-3t}sin2t$ (ii) $te^{-t}cosat$ 9iii) $tsinhtcos2t$

Solution:

(i) $L[te^{-3t}sin2t] = L[tsin2t]_{s\to s+3} = \frac{-d}{ds}L[sin2t]_{s\to s+3}$

$$= \frac{-d}{ds}\left(\frac{2}{s^2+2^2}\right)_{s\to s+3}$$

$$= \left[\frac{(s^2+4)0-2(2s)}{(s^2+4)^2}\right]_{s\to s+3}$$

$$= \left[\frac{4s}{(s^2+4)^2}\right]_{s\to s+3}$$

$$\therefore \quad L[te^{-3t}sin2t] = \frac{4(s+3)}{((s+3)^2+4)^2}$$

(ii) $L[te^{-t}cosat] = L[tcosat]_{s\to s+1} = \frac{-d}{ds}L[cosat]_{s\to s+1}$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+a^2}\right)_{s\to s+1}$$

$$= -\left[\frac{(s^2+a^2)1-s(2s)}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$= \left[\frac{s^2-a^2}{(s^2+a^2)^2}\right]_{s\to s+1}$$

$$\therefore \quad L[te^{-t}cosat] = \frac{(s+1)^2-a^2}{((s+1)^2+a^2)^2}$$

(iii) $L[tsinhtcos2t]$

$$L[tsinhtcos2t] = L\left[t\left(\frac{e^t-e^{-t}}{2}\right)cos2t\right]$$

$$= \frac{1}{2}[L(te^tcos2t) - L(te^{-t}cos2t)]$$

$$= \frac{1}{2}\left[\frac{-d}{ds}L[cos2t]_{s\to s-1} + \frac{d}{ds}L[cos2t]_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[\frac{-d}{ds}\left(\frac{s}{s^2+4}\right)_{s\to s-1} + \frac{d}{ds}\left(\frac{s}{s^2+4}\right)_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[-\left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s\to s-1} + \left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[-\left[\frac{4-s^2}{(s^2+4)^2}\right]_{s\to s-1} + \left[\frac{4-s^2}{(s^2+4)^2}\right]_{s\to s+1}\right]$$

$$\therefore L[tsinhtcos2t] = \frac{1}{2}\left[\frac{(s-1)^2-4}{((s-1)^2+4\,)^2} + \frac{4-(s+1)^2}{((s+1)^2+4\,)^2}\right]$$

Problems using the formula

$$L[t^2 f(t)] = \frac{d^2}{ds^2}L[f(t)]$$

Example: Find the Laplace transform for (i) $t^2 sint$ (ii) $t^2 cos2t$

Solution:

(i) $L[t^2 sint] = \frac{d^2}{ds^2}L[sint]$

$$= \frac{d^2}{ds^2}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{d}{ds}\left(\frac{[(s^2+1)0-1(2s)]}{(s^2+1)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-2s}{(s^2+1)^2}\right)$$

$$= -2\frac{d}{ds}\left(\frac{s}{(s^2+1)^2}\right)$$

$$= \frac{-2[(s^2+1)^2(1)-s(2)(s^2+1)(2s)]}{(s^2+1)^4}$$

$$= \frac{-2(s^2+1)[(s^2+1)-4s^2]}{(s^2+1)^4}$$

$$= \frac{-2[1-3s^2]}{(s^2+1)^3}$$

$$\therefore L[t^2 sint] = \frac{6s^2-2}{(s^2+1)^3}$$

(ii) $L[t^2 cos2t] = \frac{d^2}{ds^2}L[cos2t]$

$$= \frac{d^2}{ds^2}\left[\frac{s}{s^2+4}\right]$$

$$= \frac{d}{ds}\left(\frac{[(s^2+4)1-s(2s)]}{(s^2+4)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{4-s^2}{(s^2+4)^2}\right)$$

$$= \frac{[(s^2+4)^2(-2s)-(4-s^2)2(s^2+4)(2s)]}{(s^2+4)^4}$$

$$= \frac{2s(s^2+4)[(s^2+4)(-1)-(4-s^2)2]}{(s^2+4)^4}$$

$$= \frac{2s[s^2-12]}{(s^2+4)^3}$$

$$\therefore L[t^2 \cos 2t] = \frac{2s[s^2-12]}{(s^2+4)^3}$$

Example: Find the Laplace transform for (i) $t^2e^{-2t}cost$ (ii) $t^2e^{4t}sin3t$

Solution:

(i) $L[t^2e^{-2t}cost] = L[t^2cost]_{s\to s+2} = \frac{d^2}{ds^2}L[cost]_{s\to s+2}$

$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)_{s\to s+2}$

$= \frac{d}{ds}\left[\frac{(s^2+1)1-s(2s)}{(s^2+1)^2}\right]_{s\to s+2}$

$= \frac{d}{ds}\left[\frac{1-s^2}{(s^2+1)^2}\right]_{s\to s+2}$

$= \left[\frac{[(s^2+1)^2(-2s)-(1-s^2)2(s^2+1)(2s)]}{(s^2+1)^4}\right]_{s\to s+2}$

$= (s^2+1)\left[\frac{[(s^2+1)(-2s)-4s(1-s^2)]}{(s^2+1)^4}\right]_{s\to s+2}$

$= \left[\frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3}\right]_{s\to s+2}$

$= \left[\frac{2s^3-6s}{(s^2+1)^3}\right]_{s\to s+2}$

$\therefore\ L[t^2e^{-2t}cost] = \frac{2(s+2)^3-6(s+2)}{((s+2)^2+1)^3}$

(ii) $L[t^2e^{4t}sin3t] = L[t^2sin3t]_{s\to s-4} = \frac{d^2}{ds^2}L[sin3t]_{s\to s-4}$

$= \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)_{s\to s-4}$

$= \frac{d}{ds}\left[\frac{(s^2+9)0-3(2s)}{(s^2+9)^2}\right]_{s\to s-4}$

$= \frac{d}{ds}\left[\frac{-6s}{(s^2+9)^2}\right]_{s\to s-4} = -6\frac{d}{ds}\left[\frac{s}{(s^2+9)^2}\right]_{s\to s-4}$

$= -6\left[\frac{[(s^2+9)^2(1)-(s)2(s^2+9)(2s)]}{(s^2+9)^4}\right]_{s\to s-4}$

$= -6(s^2+9)\left[\frac{[(s^2+9)-4s^2]}{(s^2+9)^4}\right]_{s\to s-4}$

$= -6\left[\frac{9-3s^2}{(s^2+9)^3}\right]_{s\to s-4}$

$= \left[\frac{18s^2-54}{(s^2+9)^3}\right]_{s\to s-4}$

$\therefore\ L[t^2e^{4t}sin3t] = \frac{18(s-4)^2-54}{((s-4)^2+9)^3}$

Problems using the formula

$$L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} L[f(t)]ds$$

This formula is valid if $\lim_{t\to 0}\frac{f(t)}{t}$ is finite.

The following formula is very useful in this section

$$\int \frac{ds}{s} = logs$$

$$\int \frac{ds}{s+a} = \log(s+a)$$

$$\int \frac{s\,ds}{s^2+a^2} = \frac{1}{2}\log(s^2+a^2)$$

$$\int \frac{a\,ds}{s^2+a^2} = \tan^{-1}\frac{s}{a}$$

Example: Find $L\left[\frac{cosat}{t}\right]$

Solution:

$$\lim_{t\to 0}\frac{cosat}{t} = \frac{cosa(0)}{0} = \frac{1}{0} = \infty$$

∴ Laplace transform does not exists.

Example: Find $L\left[\frac{sin^3t}{t}\right]$

Solution:

$$\frac{sin^3t}{t} = \frac{3sint - sin3t}{4t}$$

$$\lim_{t\to 0}\frac{sin^3t}{t} = \lim_{t\to 0}\frac{3sint - sin3t}{4t}$$

$$= \frac{0-0}{0} = \frac{0}{0}$$ (by applying L−Hospital rule)

$$= \lim_{t\to 0}\frac{3sint - sin3t}{4t} = 0$$

Hence Laplace transform exists

$$L\left[\frac{sin^3t}{t}\right] = L\left[\frac{3sint - sin3t}{4t}\right]$$

$$= \frac{1}{4}\int_s^{\infty} L[(3sint - sin3t)]ds$$

$$= \frac{1}{4}\int_s^{\infty}\left(3\frac{1}{s^2+1} - \frac{3}{s^2+9}\right)ds$$

$$= \frac{1}{4}\left[3\tan^{-1}s - \tan^{-1}\frac{s}{3}\right]_s^{\infty}$$

$$= \frac{1}{4}\left[3(\tan^{-1}\infty - \tan^{-1}s) - (\tan^{-1}\infty - \tan^{-1}\frac{s}{3})\right]$$

$$= \frac{1}{4}\left[\left(\frac{\pi}{2} - \tan^{-1}s\right) - \left(\frac{\pi}{2} - \tan^{-1}\frac{s}{3}\right)\right]$$

$$= \frac{1}{4}\left[\cot^{-1}s - cot^{-1}\frac{s}{3}\right]$$

Example: Find $L\left[e^{-2t}\frac{sin2tcos3t}{t}\right]$

Solution:

$$L\left[e^{-2t}\frac{sin2tcos3t}{t}\right] = L\left[\frac{sin2tcos3t}{t}\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty L(\sin(3t+2t) - \sin(3t-2t))ds\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty L((\sin 5t) - L(\sin t))ds\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\int_s^\infty \left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right]\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left[\tan^{-1}\frac{s}{5} - \tan^{-1}s\right]_s^\infty\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left[\left(\tan^{-1}\infty - \tan^{-1}\frac{s}{5}\right) - \left(\tan^{-1}\infty - \tan^{-1}s\right)\right]\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\left(\frac{\pi}{2} - \tan^{-1}\frac{s}{5}\right) - \left(\frac{\pi}{2} - \tan^{-1}s\right)\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\cot^{-1}\frac{s}{5} - cot^{-1}s\right]_{s\to s+2}$$

$$= \frac{1}{2}\left[\cot^{-1}\frac{(s+2)}{5} - cot^{-1}(s+2)\right]$$

Problems using $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)]$

Example: Find the Laplace transform for (i) $\int_0^t e^{-2t}dt$ (ii) $\int_0^t cos2tdt$ (iii) $\int_0^t tsin3tdt$ (iv) $t\int_0^t costdt$

Solution:

(i) $L\left[\int_0^t e^{-2t}dt\right] = \frac{1}{s}L[e^{-2t}] = \frac{1}{s}\left(\frac{1}{s+2}\right)$

$$\therefore L\left[\int_0^t e^{-2t}dt\right] = \frac{1}{s(s+2)}$$

(ii) $L\left[\int_0^t cos2tdt\right] = \frac{1}{s}L[cos2t] = \frac{1}{s}\left(\frac{s}{s^2+4}\right)$

$$\therefore L\left[\int_0^t cos2tdt\right] = \frac{1}{s^2+4}$$

(iii) $L\left[\int_0^t tsin3tdt\right] = \frac{1}{s}L[tsin3t]$

$$= \frac{1}{s}\left[\frac{-d}{ds}[L[sin3t]]\right]$$

$$= \frac{-1}{s}\left[\frac{d}{ds}\left[\frac{3}{s^2+9}\right]\right]$$

$$= \frac{-1}{s}\left[\frac{-6s}{(s^2+9)^2}\right]$$

$$\therefore L\left[\int_0^t tsin3tdt\right] = \frac{6}{(s^2+9)^2}$$

(iv) $L\left[\int_0^t t\cos t\,dt\right] = \frac{-d}{ds} L\left[\int_0^t \cos t\,dt\right]$

$= \frac{-d}{ds}\left[\frac{1}{s}\left(\frac{s}{s^2+1}\right)\right]$

$= -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$

$= -\left[\frac{-2s}{(s^2+1)^2}\right]$

$\therefore L\left[\int_0^t t\sin 3t\,dt\right] = \frac{2s}{(s^2+1)^2}$

Example: Find the Laplace transform for $e^{-t}\int_0^t t\cos 4t\,dt$

Solution:

$L\left[e^{-t}\int_0^t t\cos 4t\,dt\right] = L\left[\int_0^t t\cos 4t\,dt\right]_{s\to s+1} = \left[\frac{-1}{s}\frac{d}{ds}L(\cos 4t)\right]_{s\to s+1}$

$= -\left(\frac{1}{s}\frac{d}{ds}\frac{s}{s^2+16}\right)_{s\to s+1}$

$= \left[\frac{-1}{s}\frac{(s^2+16)1-s(2s)}{(s^2+16)^2}\right]_{s\to s+1}$

$= \left[\frac{-1}{s}\frac{(s^2+16-2s^2)}{(s^2+16)^2}\right]_{s\to s+1}$

$= \left[\frac{-1}{s}\frac{(-s^2+16)}{(s^2+16)^2}\right]_{s\to s+1}$

$= \left[\frac{1}{s}\frac{(s^2-16)}{(s^2+16)^2}\right]_{s\to s+1}$

$\therefore L\left[e^{-t}\int_0^t t\cos 4t\,dt\right] = \frac{1}{s+1}\left[\frac{(s+1)^2-16}{((s+1)^2+16)^2}\right]$

Example: Find the Laplace transform of $e^{-t}\int_0^t \frac{\sin t}{t}dt$

Solution:

$L\left[e^{-t}\int_0^t \frac{\sin t}{t}dt\right] = L\left[\int_0^t \frac{\sin t}{t}dt\right]_{s\to s+1}$

$= \left[\frac{1}{s}L\left(\frac{\sin t}{t}\right)\right]_{s\to s+1}$

$= \left[\frac{1}{s}\int_s^\infty L(\sin t)ds\right]_{s\to s+1}$

$= \left[\frac{1}{s}\int_s^\infty \frac{1}{s^2+1}\right]_{s\to s+1}$

$= \left[\frac{1}{s}[\tan^{-1} s]_s^\infty\right]_{s\to s+1}$

$= \left[\frac{1}{s}(\tan^{-1}\infty - \tan^{-1} s)\right]_{s\to s+1}$

$= \left[\frac{1}{s}\left(\frac{\pi}{2} - \tan^{-1} s\right)\right]_{s\to s+1}$

$$= \left[\frac{1}{s}\cot^{-1}s\right]_{s\to s+1}$$

$$\therefore L\left[e^{-t}\int_0^t \frac{\sin t}{t}dt\right] = \frac{1}{s+1}\cot^{-1}(s+1)$$

Evaluation of integrals using Laplace transform

Note: (i) $\int_0^\infty f(t)e^{-st}dt = L[f(t)]$

(ii) $\int_0^\infty f(t)e^{-at}dt = [L[f(t)]]_{s=a}$

(iii) $\int_0^\infty f(t)dt = [L[f(t)]]_{s=0}$

Example: Find the values of the following integrals using Laplace transforms:

(i) $\int_0^\infty te^{-2t}\cos 2t\,dt$ **(ii)** $\int_0^\infty t^2e^{-t}\sin t\,dt$ **(iii)** $\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right)dt$

(iv) $\int_0^\infty e^{-t}\left(\frac{1-\cos t}{t}\right)dt$ **(v)** $\int_0^\infty \left(\frac{e^{-at}-\cos bt}{t}\right)dt$

Solution:

(i) $\int_0^\infty te^{-2t}\cos 2t\,dt = L[t\cos 2t]_{s=2} = \left[\frac{-d}{ds}L(\cos 2t)\right]_{s=2}$

$$= \frac{-d}{ds}\left(\frac{s}{s^2+4}\right)_{s=2}$$

$$= -\left[\frac{(s^2+4)1-s(2s)}{(s^2+4)^2}\right]_{s=2}$$

$$= -\left[\frac{(4-s^2)}{(s^2+4)^2}\right]_{s=2}$$

$$= -\frac{(4-4)}{(4+4)^2} = 0$$

(ii) $\int_0^\infty t^2e^{-t}\sin t\,dt = L[t^2\sin t]_{s=1} = \frac{d^2}{ds^2}L[\sin t]_{s=1}$

$$= \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right)_{s=1}$$

$$= \frac{d}{ds}\left[\frac{-1(2s)}{(s^2+1)^2}\right]_{s=1}$$

$$= -2\frac{d}{ds}\left[\frac{s}{(s^2+1)^2}\right]_{s=1}$$

$$= -2\left[\frac{[(s^2+1)^2(1)-s.2(s^2+1)(2s))]}{(s^2+1)^4}\right]_{s=1}$$

$$= -2\left[\frac{[(s^2+1)[(s^2+1)-4s^2)]]}{(s^2+1)^4}\right]_{s=1}$$

$$= -2\left[\frac{(1-3s^2)}{(s^2+1)^3}\right]_{s=1}$$

$$= \left[\frac{6s^3-2}{(s^2+1)^3}\right]_{s=1} = \frac{4}{8} = \frac{1}{2}$$

(iii) $\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right) dt = L\left[\frac{e^{-t}-e^{-2t}}{t}\right]_{s=0} = \int_s^\infty [L[e^{-t} - e^{-2t}]ds]_{s=0}$

$= \int_s^\infty [[L(e^{-t}) - L(e^{-2t})]\square s]_{s=0}$

$= \int_s^\infty \left[\left(\frac{1}{s+1} - \frac{1}{s+2}\right) ds\right]_{s=0}$

$= \{[\log(s+1) - \log(s+2)]_s^\infty\}_{s=0}$

$= \left\{\left[\log\frac{s+1}{s+2}\right]_s^\infty\right\}_{s=0}$

$= \left\{\log\frac{s(1+\frac{1}{s})}{s(1+\frac{2}{s})}\right\}_s^\infty{}_{s=0}$

$= \left[0 - \log\frac{s+1}{s+2}\right]_{s=0}$ $\quad\because \log 1 = 0$

$= \left[\log\frac{s+2}{s+1}\right]_{s=0} = \log 2$

(iv) $\int_0^\infty \left(\frac{1-\cos t}{t}\right) e^{-t}dt$

$\int_0^\infty \left(\frac{1-\cos t}{t}\right) e^{-t}dt = L\left[\frac{1-\cos t}{t}\right]_{S=1} = \int_s^\infty [L[(1-\cos t)]ds]_{S=1}$

$= \int_s^\infty [[L(1) - L(\cos t)]ds]_{S=1}$

$= \int_s^\infty \left[\left(\frac{1}{s} - \frac{s}{s^2+1}\right) ds\right]_{S=1}$

$= \left\{\left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty\right\}_{S=1}$

$= \left\{\left[\log s - \log\sqrt{s^2+1}\right]_s^\infty\right\}_{S=1}$

$= \left\{\left[\log\frac{s}{\sqrt{s^2+1}}\right]_s^\infty\right\}_{S=1}$

$= \left[0 - \log\frac{s}{\sqrt{s^2+1}}\right]_{s=1}$

$= \left[\log\frac{\sqrt{s^2+1}}{s}\right]_{s=1}$

$= \log\sqrt{2}$

(v) $\int_0^\infty \left(\frac{e^{-at}-\cos bt}{t}\right) dt$

$\int_0^\infty \left(\frac{e^{-at}-\cos bt}{t}\right) dt = L\left[\frac{e^{-at}-\cos bt}{t}\right]_{S=0} = \int_s^\infty [L[(e^{-at} - \cos bt)]ds]_{S=0}$

$= \int_s^\infty [[L(e^{-at}) - L(\cos bt)]ds]_{S=0}$

$= \int_s^\infty \left[\left(\frac{1}{s+a} - \frac{s}{s^2+b^2}\right) ds\right]_{S=0}$

$= \left\{\left[\log(s+a) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty\right\}_{S=0}$

$$= \{[\log(s+a) - \log\sqrt{s^2+b^2}]_s^{\infty}\}_{S=0}$$

$$= \{[log \frac{s+a}{\sqrt{s^2+b^2}}]_s^{\infty}\}_{S=0}$$

$$= [0 - log \frac{s+a}{\sqrt{s^2+b^2}}]_{s=0}$$

$$= [log \frac{\sqrt{s^2+b^2}}{s+a}]_{s=0}$$

$$= log \frac{\sqrt{b^2}}{a}$$

$$= log \frac{b}{a}$$

INITIAL AND FINAL VALUE THEOREMS

Initial value theorem

Statement: If $L[f(t)] = F(s)$, then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof:

We know that $L[f'(t)] = s\,L[f(t)] - f(0)$

$= sF(s) - f(0)$

$\therefore sF(s) = L[f'(t)] + f(0)$

$= \int_0^\infty e^{-st} f'(t)dt + f(0)$

Taking limit as $s \to \infty$ on both sides, we have

$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} [\int_0^\infty e^{-st} f'(t)dt + f(0)]$

$= \lim_{s\to\infty} [\int_0^\infty e^{-st} f'(t)dt] + f(0)$

$= \int_0^\infty \lim_{s\to\infty} [e^{-st} f'(t)]dt + f(0)$

$= 0 + f(0)$ $\quad\because e^{-\infty} = 0$

$= f(0)$

$= \lim_{t\to 0} f(t)$

$\therefore \lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t)$

Final value theorem

Statement: If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$,then

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Proof:

We know that $L[f'(t)] = s\,L[f(t)] - f(0)$

$= sF(s) - f(0)$

$\therefore sF(s) = L[f'(t)] + f(0)$

$= \int_0^\infty e^{-st} f'(t)dt + f(0)$

Taking limit as $s \to 0$ on both sides, we have

$\lim_{s\to 0} sF(s) = \lim_{s\to 0} [\int_0^\infty e^{-st} f'(t)dt + f(0)]$

$= \lim_{s\to 0} [\int_0^\infty e^{-st} f'(t)dt] + f(0)$

$= \int_0^\infty \lim_{s\to 0} [e^{-st} f'(t)]dt + f(0)$

$= \int_0^\infty f'(t)dt + f(0)$

$= [f(t)]_0^\infty + f(0)$

$= f(\infty) - f(0) + f(0)$

$= f(\infty)$

$= \lim_{t\to\infty} f(t)$

$\therefore \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

Example: Verify the initial value theorem for the function $f(t) = ae^{-bt}$

Solution:

Given $f(t) = ae^{-bt}$

$F(s) = L[f(t)]$

$= L[ae^{-bt}]$

$= a\ \frac{1}{s+b}$

$sF(s) = \frac{as}{s+b}$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$\lim_{t\to 0} f(t) = \lim_{t\to 0} ae^{-bt}$

$= a \cdots\cdots\cdots (1)$

$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} [\frac{as}{s+b}]$

$= \lim_{s\to\infty} [\frac{as}{s(1+\frac{b}{s})}] == \lim_{s\to\infty} [\frac{a}{(1+\frac{b}{s})}]$

$= a \cdots\cdots\cdots (2)$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$\therefore$ Initial value theorem is verified

Example: Verify the initial value theorem and Final value theorem for the function $f(t) = 1 + e^{-t}[sint + cost]$.

Solution:

Given $f(t) = 1 + e^{-t}[sint + cost]$

$F(s) = L[f(t)]$

$= L[1 + e^{-t}[sint + cost]]$

$= L[1] + L[e^{-t}[sint + cost]]$

$= L[1] + L[sint + cost]_{s\to s+1}$

$= \frac{1}{s} + [\frac{1}{s^2+1} + \frac{s}{s^2+1}]_{s\to s+1}$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$F(s) = \frac{1}{s} + \frac{1}{s^2+2s+2} + \frac{s+1}{s^2+2s+2}$$

$$sF(s) = 1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}$$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}[1 + e^{-t}[sint + cost]]$$

$$= 1 + 0 + 1 = 2 \cdots\cdots\cdots (1)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} [1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}]$$

$$= 1 + \lim_{s\to\infty} [\frac{1}{s(1+\frac{2}{s}+\frac{2}{s^2})} + \frac{(1+\frac{1}{s})}{(1+\frac{2}{s}+\frac{2}{s^2})}]$$

$$= 1 + 0 + 1 = 2 \cdots\cdots\cdots (2)$$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}(1 + e^{-t}[sint + cost])$$

$$= 1 + 0 = 1 \cdots\cdots\cdots (3)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} [1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}]$$

$$= 1 + 0 + 0 = 1 \cdots\cdots\cdots (4)$$

From (3) and (4), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

∴ Final value theorem is verified.

Example: Verify the initial value theorem and Final value theorem for the function

$$\boldsymbol{f(t) = L^{-1}[\frac{1}{s(s+2)^2}]}$$

Solution:

Given $f(t) = L^{-1}[\frac{1}{s(s+2)^2}] \cdots (1)$

$$= \int_0^t L^{-1}[\frac{1}{(s+2)^2}]\, dt = \int_0^t e^{-2t} L^{-1}[\frac{1}{s^2}]\, dt$$

$$= \int_0^t e^{-2t} t\, dt$$

$$= \int_0^t te^{-2t}\, dt$$

$$= [t(\frac{e^{-2t}}{-2}) - \frac{(1)e^{-2t}}{(-2)^2}]_0^t$$

$$= -t\frac{e^{-2t}}{2} - \frac{e^{-2t}}{4} - 0 + \frac{1}{4}$$

$$\therefore f(t) = \frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}$$

From (1), $F(s) = \frac{1}{s(s+2)^2}$

$$sF(s) = \frac{1}{(s+2)^2}$$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} [\frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}]$$

$$= \frac{1}{4} - 0 - \frac{1}{4} = 0$$

$$\therefore \lim_{t\to 0} f(t) = 0 \cdots (2)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{1}{(s+2)^2} = 0$$

$$\therefore \lim_{s\to\infty} sF(s) = 0 \cdots (3)$$

From (2) and (3), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$\therefore$ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} [\frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}]$$

$$= \frac{1}{4} - 0 - 0 = \frac{1}{4} \cdots (4)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0} [\frac{1}{(s+2)^2}]$$

$$= \frac{1}{4} \cdots (5)$$

From (4) and (5), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$\therefore$ Final value theorem is verified

Example: Verify the initial value theorem and Final value theorem for the function

$\boldsymbol{f(t) = e^{-t}(t+2)^2}$

Solution:

Given $f(t) = e^{-t}(t+2)^2$

$$= e^{-t}(t^2 + 4t + 4)$$

$$F(s) = L[f(t)]$$

$$= L[e^{-t}(t^2 + 4t + 4)]$$

$$= L[t^2 + 4t + 4]_{s\to s+1}$$

$$= [L(t^2) + 4L(t) + 4L(1)]_{s\to s+1}$$

$$= \left[\frac{2!}{s^3} + 4\frac{1}{s^2} + 4\frac{1}{s}\right]_{s \to s+1}$$

$$= \frac{2}{(s+1)^3} + 4\frac{1}{(s+1)^2} + 4\frac{1}{s+1}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}$$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}[e^{-t}(t^2 + 4t + 4)]$$

$$= 4 \cdots (1)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{2s}{(s+1)^3} + \frac{4}{(s+1)^2} + \frac{4s}{s+1}\right]$$

$$= \lim_{s\to\infty}\left[\frac{2s}{s^3(1+\frac{1}{s})^3} + \frac{4}{s^2(1+\frac{1}{s})^2} + \frac{4s}{s(1+\frac{1}{s})}\right]$$

$$= \lim_{s\to\infty}\left[\frac{2}{s^2(1+\frac{1}{s})^3} + \frac{4}{s(1+\frac{1}{s})^2} + \frac{4}{(1+\frac{1}{s})}\right]$$

$$= 0 + 0 + 4$$

$$= 4 \cdots (2)$$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}[e^{-t}(t^2 + 4t + 4)]$$

$$= 0 \cdots (3)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{2s}{(s+1)^3} + \frac{4}{(s+1)^2} + \frac{4s}{s+1}\right]$$

$$= 0 \cdots (4)$$

From (3) and (4), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

∴ Final value theorem is verified.

INVERSE LAPLACE TRANSFORM

Inverse Laplace transform of elementary functions

$L[f(t)] = F(s)$	$L^{-1}[F(s)] = f(t)$
$L[1] = \frac{1}{s}$	$L^{-1}\left[\frac{1}{s}\right] = 1$
$L[t] = \frac{1}{s^2}$	$L^{-1}\left[\frac{1}{s^2}\right] = t$
$L[t^n] = \frac{n!}{s^{n+1}}$ if n is an integer	$L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$ $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$
$L[e^{at}] = \frac{1}{s-a}$	$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
$L[e^{-at}] = \frac{1}{s+a}$	$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
$L[sinat] = \frac{a}{s^2+a^2}$	$L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{sinat}{a}$
$L[cosat] = \frac{s}{s^2+a^2}$	$L^{-1}\left[\frac{s}{s^2+a^2}\right] = cosat$

$L[sinhat] = \frac{a}{s^2 - a^2}$	$L^{-1}\left[\frac{1}{s^2 - a^2}\right] = \frac{sinhat}{a}$

$L[cosat] = \frac{s}{s^2 - a^2}$	$L^{-1}\left[\frac{s}{s^2 - a^2}\right] = coshat$

Result on inverse Laplace transform

Result: 1 Linear property

$L[f(t)] = F(s)$ and $L[g(t)] = G(s)$,then $L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$

Where a and b are constants.

Proof:

We know that $L[aF(s) \pm bG(s)] = aL[F(s)] \pm bL[G(s)]$

$$= a\,F(s) \pm b\,G(s)$$

$(i.e.)\, a\,F(s) \pm b\,G(s) = L[af(t) \pm bg(t)]$

Operating L^{-1} on both sides, we get

$L^{-1}[aF(s) \pm bG(s)] = af(t) \pm bg(t)$

$\because f(t) = L^{-1}[F(s)]$

$\because g(t) = L^{-1}[G(s)]$

$L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$

Result: 2 First shifting property

(i) $L^{-1}[F(s+a) = e^{-at}L^{-1}[F(s)]]$

(ii) $L^{-1}[F(s-a) = e^{at}L^{-1}[F(s)]]$

Proof:

Let $L[e^{-at}f(t)] = F[s+a]$

Operating L^{-1} on both sides, we get

$e^{-at}f(t) = L^{-1}[F[s+a]]$

$L^{-1}[F[s+a]] = e^{-at}L^{-1}[F(s)]$

Result: 3 Multiplication by s.

If $L^{-1}[F(s)] = f(t)$ and $f(0) = 0$, then $L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]$

Proof:

We know that $L[f'(t)] = sL[f(t)] - f(0) = sF(s)$

Operating L^{-1} on both sides, we get

$f'(t) = L^{-1}[sF(s)]$

$\frac{d}{dt}f(t) = L^{-1}[sF(s)]$

$\frac{d}{dt} L^{-1}[F(s)] = L^{-1}[sF(s)]$

$\therefore L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)]$

Result: 4 Division by s.

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]dt$$

Proof:

We know that $L\left[\int_0^t f(t)dt\right] = \frac{1}{s} L[f(t)] = \frac{1}{s} F(s)$

Operating L^{-1} on both sides ,we get

$\int_0^t f(t)dt = L^{-1}\left[\frac{1}{s} F(s)\right]$

$\int_0^t L^{-1}[F(s)]\, dt = L^{-1}\left[\frac{1}{s} F(s)\right]$

$\therefore L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\, dt$

Result: 5 Inverse Laplace transform of derivative

$$L^{-1}[F(s)] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds} F(s)\right]$$

Proof:

We know that $L[tf(t)] = \frac{-d}{ds} L[f(t)] = \frac{-d}{ds} F(s)$

Operating L^{-1} on both sides ,we get

$tf(t) = -L^{-1}\left[\frac{d}{ds} F(s)\right]$

$L^{-1}[F(s)] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds} F(s)\right]$

$f(t) = \frac{-1}{t} L^{-1}\left[\frac{d}{ds} F(s)\right]$

$L^{-1}[F(s)] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds} F(s)\right]$

Result: 6 Inverse Laplace transform of integral

$$L^{-1}[F(s)] = tL^{-1}\left[\int_s^\infty F(s)ds\right]$$

Proof:

We know that $L\left[\frac{f(t)}{t}\right] = \int_s^\infty L(f(t))\, ds$

$= \int_s^\infty F(s)\, ds$

Operating L^{-1} on both sides, we get

$\frac{f(t)}{t} = L^{-1}\left[\int_s^\infty F(s)\, ds\right]$

$f(t) = tL^{-1}\left[\int_s^\infty F(s)\, ds\right]$

$L^{-1}[F(s)] = tL^{-1}[\int_s^\infty F(s)\,ds]$

Problems under inverse Laplace transform of elementary functions

Example: Find the inverse Laplace for the following

(i) $\frac{1}{2s+3}$ **(ii)** $\frac{1}{4s^2+9}$ **(iii)** $\frac{s^3-3s^2+7}{s^4}$ **(iv)** $\frac{3s+5}{s^2+36}$

Solution:

(i) $L^{-1}\left[\frac{1}{2s+3}\right] = L^{-1}\left[\frac{1}{2[s+\frac{3}{2}]}\right]$

$= \frac{1}{2}e^{-\frac{3t}{2}}$

(ii) $L^{-1}\left[\frac{1}{4s^2+9}\right] = L^{-1}\left[\frac{1}{4[s^2+\frac{9}{4}]}\right]$

$= \frac{1}{4}L^{-1}\left[\frac{1}{[s^2+\frac{9}{4}]}\right]$

$= \frac{1}{4}\frac{1}{3/2}\sin\frac{3}{2}t$

$= \frac{1}{6}\sin\frac{3}{2}t$

(iii) $L^{-1}\left[\frac{s^3-3s^2+7}{s^4}\right] = L^{-1}\left[\frac{s^3}{s^4} - \frac{3s^2}{s^4} + \frac{7}{s^4}\right]$

$= L^{-1}\left[\frac{1}{s}\right] - 3L^{-1}\left[\frac{1}{s^2}\right] + 7L^{-1}\left[\frac{1}{s^4}\right]$

$L^{-1}\left[\frac{s^3-3s^2+7}{s^4}\right] = 1 - 3t + \frac{7t^3}{3!}$

(iv) $L^{-1}\left[\frac{3s+5}{s^2+36}\right] = 3L^{-1}\left[\frac{s}{s^2+36}\right] + 5L^{-1}\left[\frac{1}{s^2+36}\right]$

$L^{-1}\left[\frac{3s+5}{s^2+36}\right] = 3\cos 6t + \frac{5\sin 6t}{6}$

Inverse Laplace transform using First shifting theorem

$$L^{-1}[F(s+a)] = e^{-at}L^{-1}[F(s)]$$

Example: 5.40 Find the inverse Laplace transform for the following:

(i) $\frac{1}{(s+2)^2}$ **(ii)** $\frac{1}{(s-3)^4}$ **(iii)** $\frac{1}{(s+3)^2+9}$ **(iv)** $\frac{1}{s^2-2s+2}$

Solution:

(i) $L^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t}L^{-1}\left[\frac{1}{s^2}\right] = e^{-2t}t$

(ii) $L^{-1}\left[\frac{1}{(s-3)^4}\right] = e^{3t}L^{-1}\left[\frac{1}{s^4}\right] = e^{-2t}\frac{t^3}{3!}$

(iii) $L^{-1}\left[\frac{1}{(s+3)^2+9}\right] = e^{-3t}L^{-1}\left[\frac{1}{s^2+9}\right] = e^{-3t}\frac{\sin 3t}{3}$

(iv) $L^{-1}\left[\frac{1}{s^2-2s+2}\right] = L^{-1}\left[\frac{1}{(s-1)^2+1}\right] = e^t L^{-1}\left[\frac{1}{s^2+1}\right] = e^t sint$

Inverse using the formula

$$L^{-1}[F(s)] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}F(s)\right]$$

Note: This formula is used when $F(s)$ is $\cot^{-1}\emptyset(s)$ or $\tan^{-1}\emptyset(s)$ or $log\emptyset(s)$

Example: 5.41 Find the inverse Laplace transform for the following

(i) $cot^{-1}\left(\frac{s}{a}\right)$ **(ii)** $tan^{-1}\left(\frac{a}{s}\right)$ **(iii)** $cot^{-1}as$

Solution:

(i) $L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left(cot^{-1}\left(\frac{s}{a}\right)\right)\right]$

$= \frac{-1}{t} L^{-1}\left[\frac{-1}{1+\frac{s^2}{a^2}}\left(\frac{1}{a}\right)\right] = \frac{1}{t} L^{-1}\left[\frac{-1}{\frac{a^2+s^2}{a^2}}\left(\frac{1}{a}\right)\right]$

$= \frac{1}{t} L^{-1}\left[\frac{a}{s^2+a^2}\right]$

$L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{1}{t} sinat$

(ii) $L^{-1}\left[tan^{-1}\left(\frac{a}{s}\right)\right] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\left(tan^{-1}\left(\frac{a}{s}\right)\right)\right]$

$= \frac{-1}{t} L^{-1}\left[\frac{1}{1+\left(\frac{a}{s}\right)^2}\left(\frac{-a}{s^2}\right)\right] = \frac{-1}{t} L^{-1}\left[\frac{1}{\frac{s^2+a^2}{s^2}}\left(\frac{-a}{s^2}\right)\right]$

$= \frac{1}{t} L^{-1}\left[\frac{a}{s^2+a^2}\right]$

$L^{-1}\left[tan^{-1}\left(\frac{a}{s}\right)\right] = \frac{1}{t} sinat$

(iii) $L^{-1}[cot^{-1}as] = \frac{-1}{t} L^{-1}\left[\frac{d}{ds}(cot^{-1}(as))\right]$

$= \frac{-1}{t} L^{-1}\left[\frac{-1}{1+a^2s^2}(a)\right] = \frac{1}{t} L^{-1}\left[\frac{a}{a^2(s^2+\frac{1}{a^2})}\right]$

$= \frac{1}{at} L^{-1}\left[\frac{1}{s^2+\frac{1}{a^2}}\right] = \frac{1}{at}\left[\frac{sin\frac{1}{a}t}{\frac{1}{a}}\right]$

$L^{-1}[cot^{-1}as] = \frac{1}{t} sin\frac{t}{a}$

Inverse using the formula

$$L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)]$$

Example: Find $L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$

Solution:

$L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{d}{dt} L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] \cdots (1)$

$$L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = L^{-1}\frac{d}{ds}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(\log(s^2+a^2)-\log(s^2+b^2)\right)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{s^2+a^2}2s-\frac{1}{s^2+b^2}2s\right]$$

$$= \frac{-2}{t}L^{-1}\left[\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right]$$

$$= \frac{-2}{t}[cosat-cosbt]$$

$$= \frac{2}{t}[cosbt-cosat]$$

Substituting in (1), we get

$$L^{-1}\left[s\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{d}{dt}\left[\frac{2}{t}[cosbt-cosat]\right]$$

$$= 2\left[\frac{t(-bsinbt+asinat)-(cosbt-cosat)}{t^2}\right]$$

$$L^{-1}\left[s\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = 2\left[\frac{t(-bsinbt+asinat)-(cosbt-cosat)}{t^2}\right]$$

Inverse using the formula

$$\boldsymbol{L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]dt}$$

This formula is used when $F(s) = \frac{one\ term}{s(another\ term)}$

Example: Find $\boldsymbol{L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]}$

Solution:

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s^2+a^2)}\right]dt$$

$$= \int_0^t \left[\frac{sinat}{a}\right]dt$$

$$= \frac{1}{a}\left[\frac{-cosat}{a}\right]_0^t$$

$$= \frac{-1}{a^2}[cosat]_0^t$$

$$= \frac{-1}{a^2}(cosat-cos0) = \frac{-1}{a^2}(cosat-1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1-cosat}{a^2}$$

Example: Find $\boldsymbol{L^{-1}\left[\frac{1}{s(s^2-a^2)}\right]}$

Solution:

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s^2-a^2)}\right]dt$$

$$= \int_0^t \left[\frac{sinhat}{a}\right]dt$$

$$= \frac{1}{a}\left[\frac{\cosh at}{a}\right]_0^t$$

$$= \frac{1}{a^2}[\cosh at]_0^t$$

$$= \frac{1}{a^2}(\cosh at - \cosh 0) = \frac{1}{a^2}(\cosh at - 1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2-a^2)}\right] = \frac{\cosh at - 1}{a^2}$$

Example: Find $L^{-1}\left[\frac{1}{s(s+a)}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s+a)}\right] dt$$

$$= \int_0^t e^{-at} dt$$

$$= \left[\frac{e^{-at}}{-a}\right]_0^t$$

$$= \frac{-1}{a}(e^{-at} - 1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s+a)}\right] = \frac{1 - e^{-at}}{a}$$

CONVOLUTION THEOREM

Definition: Convolution of two functions

The convolution of two functions $f(t)$ and $g(t)$ is denoted by $f(t) * g(t)$ and defined by

$f(t) * g(t) = \int_0^t f(u)g(t-u)du.$

State and prove Convolution theorem

Statement: If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$, then $L[f(t)] * L[g(t)] = F(s)G(s)$

Proof:

We have $f(t) * g(t) = \int_0^t f(u)g(t-u)du$

$$L[f(t) * g(t)] = \int_0^\infty [f(t) * g(t)]\, e^{-st}dt$$

$$= \int_0^\infty \int_0^t f(u)g(t-u)du e^{-st}dt$$

$$= \int_0^\infty \int_0^t f(u)g(t-u)e^{-st}dudt \cdots (1)$$

Now we have no change the order of integration.

$u = 0, u = t; t = 0, t = \infty$

Change of order is . Draw horizontal strip PQ

At P, $t = u$, At A $u = \infty$

$$L[f(t) * g(t)] = \int_0^\infty \int_u^\infty f(u)g(t-u)e^{-st}dtdu$$

$$= \int_0^\infty f(u)\, [\int_u^\infty g(t-u)e^{-st}dt]du \cdots (2)$$

Put $t - u = x \cdots (3)$

$t = u + x \Rightarrow dt = dx$

When $t = u; (3) \Rightarrow x = 0$

When $t = \infty; (3) \Rightarrow x = \infty$

$$(2) \Rightarrow L[f(t) * g(t)] = \int_0^\infty f(u)\, [\int_0^\infty g(x)e^{-s(u+x)}dx]du$$

$$= \int_0^\infty f(u)\, [\int_0^\infty g(x)e^{-su}e^{-sx}dx]du$$

$$= \int_0^\infty f(u)e^{-su}du \int_0^\infty g(x)e^{-sx}dx$$

$$= L[f(u)]L[g(x)]$$

$\therefore L[f(t) * g(t)] = F(s)G(s)$

Note: Convolution theorem is very useful to compute inverse Laplace transform of product of two terms

Convolution theorem is $L[f(t) * g(t)] = F(s)G(s)$

$$L^{-1}[F(s)G(s)] = f(t) * g(t)$$

$$L^{-1}[F(s)G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

Example: Find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$ using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{(s+a)}\right] * L^{-1}\left[\frac{1}{(s+b)}\right]$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au} e^{-b(t-u)}\, du$$

$$= e^{-bt} \int_0^t e^{-au} e^{bu} du$$

$$= e^{-bt} \int_0^t e^{(b-a)u}\, du$$

$$= e^{-bt}\left[\frac{e^{(b-a)u}}{b-a}\right]_0^t$$

$$= \frac{e^{-bt}}{b-a}[e^{(b-a)t} - 1]$$

$$= \frac{e^{-bt}}{b-a}[e^{bt-at} - 1]$$

$$= \frac{1}{b-a}[e^{-bt+bt-at} - e^{-bt}]$$

$$\therefore L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = \frac{1}{b-a}[e^{-at} - e^{-bt}]$$

Example: Find the inverse Laplace transform $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}\,\frac{s}{(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{s}{(s^2+b^2)}\right]$$

$$= cosat * cosbt$$

$$= \int_0^t cosau\, cosb(t-u)du$$

$$= \int_0^t \frac{\cos(au+bt-bu)+\cos(au-bt+bu)}{2} du$$

$$= \frac{1}{2}\int_0^t (\cos(au + bt - bu) + \cos(au - bt + bu))\, du$$

$$= \frac{1}{2}\int_0^t [\cos(a - b)\, u + bt + \cos(a + b)\, u - bt]du$$

$$= \frac{1}{2}\left[\frac{sin[(a-b)u+bt]}{a-b} + \frac{sin[(a+b)u+bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{a-b} + \frac{\sin(at-bt+bt)}{a+b} - \frac{sinbt}{a-b} + \frac{sinbt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{(a+b)\sin at+(a-b)\sin at-(a+b)\sin bt+(a-b)\sin bt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at-2b\sin bt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2(a\sin at-b\sin bt)}{a^2-b^2}\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{a\sin at-b\sin bt}{a^2-b^2}$$

Example: Find the inverse Laplace transform $\frac{1}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\frac{1}{(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{(s^2+b^2)}\right]$$

$$= \frac{1}{a}\sin at * \frac{1}{b}\sin bt$$

$$= \frac{1}{ab}\int_0^t \sin au \sin b(t-u)du$$

$$= \frac{1}{ab}\int_0^t \frac{\cos(au-bt+bu)-\cos(au+bt-bu)}{2}du$$

$$= \frac{1}{2ab}\int_0^t (\cos(au-bt+bu)-\cos(au+bt-bu))\,du$$

$$= \frac{1}{2}\int_0^t [\cos[(a+b)u-bt]-\cos[(a-b)u+bt]]du$$

$$= \frac{1}{2ab}\left[\frac{\sin[(a+b)u-bt]}{a+b} - \frac{\sin[(a-b)u+bt]}{a-b}\right]_0^t$$

$$= \frac{1}{2ab}\left[\frac{\sin(at+bt-bt)}{a+b} - \frac{\sin(at-bt+bt)}{a-b} + \frac{\sin bt}{a+b} + \frac{\sin bt}{a-b}\right]$$

$$= \frac{1}{2ab}\left[\frac{\sin at}{a+b} - \frac{\sin at}{a-b} - \frac{\sin bt}{a+b} + \frac{\sin bt}{a-b}\right]$$

$$= \frac{1}{2ab}\left[\frac{(a-b)\sin at-(a+b)\sin at+(a-b)\sin bt+(a+b)\sin bt}{a^2-b^2}\right]$$

$$= \frac{1}{2ab}\left[\frac{-2b\sin at+2a\sin bt}{a^2-b^2}\right]$$

$$= \frac{1}{2ab}\left[\frac{2(a\sin bt-b\sin at)}{a^2-b^2}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = \frac{a\sin bt-b\sin at}{ab(a^2-b^2)}$$

Example: Find the inverse Laplace transform $\frac{s}{(s^2+4)(s^2+9)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = L^{-1}\left[\frac{1}{(s^2+4)}\frac{s}{(s^2+9)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+4)}\right] * L^{-1}\left[\frac{s}{(s^2+9)}\right]$$

$$= \frac{1}{2} sin2t * cos3t$$

$$= \frac{1}{2}\int_0^t sin2u\ cos3(t-u)du$$

$$= \frac{1}{2}\int_0^t \frac{\sin(2u+3t-3u)+\sin(2u-3t+3u)}{2}du$$

$$= \frac{1}{4}\int_0^t [\sin(3t-u)+\sin(5u-3t)]du$$

$$= \frac{1}{4}\left[\frac{-\cos(3t-u)}{-1} - \frac{\cos(5u-3t)}{5}\right]_0^t$$

$$= \frac{1}{4}\left[\frac{\cos(3t-t)}{1} - \frac{\cos(5t-3t)}{5} - \frac{cos3t}{1} + \frac{cos3t}{5}\right]$$

$$= \frac{1}{4}\left[cos2t - \frac{cos2t}{5} - cos3t + \frac{cos3t}{5}\right]$$

$$= \frac{1}{4}\left[\frac{5cos2t - cos2t - 5cos3t + cos3t}{5}\right]$$

$$= \frac{1}{20}[4cos2t - 4cos3t]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = \frac{cos2t - cos3t}{5}$$

Example: Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\ \frac{s}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+a^2)}\right] * L^{-1}\left[\frac{s}{(s^2+a^2)}\right]$$

$$= \frac{1}{a} sinat * cosat$$

$$= \frac{1}{a}\int_0^t sinau\ cosa(t-u)du$$

$$= \frac{1}{a}\int_0^t \frac{\sin(au+at-au)+\sin(au-at+au)}{2}du$$

$$= \frac{1}{2a}\int_0^t [sinat + \sin(2au-at)]du$$

$$= \frac{1}{2a}\left[\int_0^t sinatdu + \int_0^t \sin(2au-at)\,du\right]$$

$$= \frac{1}{2a}\left[sinat \int_0^t du + \int_0^t \sin(2au-at)\,du\right]$$

$$= \frac{1}{2a}\left[sinat(u)_0^t - \left(\frac{\cos(2au-at)}{2a}\right)_0^t\right]$$

$$= \frac{1}{2a}\left[tsinat - \frac{\cos(2at-at)}{2a} + \frac{cosat}{2a}\right]$$

$$= \frac{1}{2a}\left[tsinat - \frac{\cos at}{2a} + \frac{cosat}{2a}\right]$$

$$= \frac{1}{2a}\ tsinat$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{tsinat}{2a}$$

TRANSFORM OF PERIODIC FUNCTIONS

Definition: A function $f(t)$ is said to be periodic if $f(t+T)=f(t)$ for all values of t and for certain values of T. The smallest value of T for which $f(t+T)=f(t)$ for all t is called periodic function.

Example:

$sint = \sin(t+2\pi) = \sin(t+4\pi)\cdots$

$\therefore$ $sint$ is periodic function with period 2π.

Let $f(t)$ be a periodic function with period T. Then

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

Example: Find the Laplace transform of $f(t) = \begin{cases} \sin\omega t; & 0 < t < \frac{\pi}{\omega} \\ 0; & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$ $f\left(t+\frac{2\pi}{\omega}\right) = f(t)$

Solution:

The given function is a periodic function with period $T = \frac{2\pi}{\omega}$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} \sin\omega t e^{-st}dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st}(0)dt\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} \sin\omega t e^{-st}dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{-st}}{(-s)^2+\omega^2}(-s\sin\omega t - \omega\cos\omega t)\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left\{\frac{e^{\frac{-s\pi}{\omega}}}{s^2+\omega^2}[-s\sin\pi - \omega\cos\pi] + \frac{\omega}{s^2+\omega^2}\right\}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{\frac{-s\pi}{\omega}}\omega+\omega}{s^2+\omega^2}\right]$$

$$= \frac{1}{1^2-\left(e^{\frac{-\pi s}{\omega}}\right)^2}\left[\frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{s^2+\omega^2}\right]$$

$$= \frac{1}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(1+e^{\frac{-\pi s}{\omega}}\right)}\left[\frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{s^2+\omega^2}\right]$$

$$\therefore L[f(t)] = \frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)(s^2+\omega^2)}$$

Example: Find the Laplace transform of $f(t) = \begin{cases} E; 0 \le t \le a \\ -E; a \le t \le 2a \end{cases}$ **given that** $f(t+2a) = f(t)$.

Solution:

The given function is a periodic function with period $T = 2a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a Ee^{-st}dt + \int_a^{2a} -Ee^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[E\int_0^a e^{-st}dt - E\int_a^{2a} e^{-st}dt\right]$$

$$= \frac{E}{1-e^{-2as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^a - \left[\frac{e^{-st}}{-s}\right]_a^{2a}\right]$$

$$= \frac{E}{1-e^{-2as}}\left[\frac{e^{-as}}{-s} + \frac{1}{s} - \frac{e^{-2as}}{s} - \frac{e^{-as}}{s}\right]$$

$$= \frac{E}{1-e^{-2as}}\left[\frac{1-2e^{-as}+e^{-2as}}{s}\right]$$

$$= \frac{E}{1^2-(e^{-as})^2}\left[\frac{(1-e^{-as})^2}{s}\right]$$

$$= \frac{E}{(1-e^{-as})(1+e^{-as})}\left[\frac{(1-e^{-as})^2}{s}\right]$$

$$= \frac{E}{s}\frac{(1-e^{-as})}{(1+e^{-as})}$$

$$\therefore L[f(t)] = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$$

Example: Find the Laplace transform of $f(t) = \begin{cases} 1; 0 \le t \le \frac{a}{2} \\ -1; \frac{a}{2} \le t \le a \end{cases}$ **given that** $f(t+a) = f(t)$.

Solution:

The given function is a periodic function with period $T = a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-as}}\int_0^a e^{-st}f(t)dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}}(1)e^{-st}dt + \int_{\frac{a}{2}}^{a}(-1)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}} e^{-st}dt - \int_{\frac{a}{2}}^{a} e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^{\frac{a}{2}} - \left[\frac{e^{-st}}{-s}\right]_{\frac{a}{2}}^{a}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{e^{\frac{-sa}{2}}}{-s} + \frac{1}{s} + \frac{e^{-as}}{s} - \frac{e^{\frac{-sa}{2}}}{s}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\frac{1-2e^{\frac{-sa}{2}}+e^{-as}}{s}\right]$$

$$= \frac{1}{1^2-\left(e^{\frac{-sa}{2}}\right)^2}\left[\frac{\left(1-e^{\frac{-sa}{2}}\right)^2}{s}\right]$$

$$= \frac{1}{\left(1-e^{\frac{-sa}{2}}\right)\left(1+e^{\frac{-sa}{2}}\right)}\left[\frac{\left(1-e^{\frac{-sa}{2}}\right)^2}{s}\right]$$

$$= \frac{1}{s}\frac{\left(1-e^{\frac{-sa}{2}}\right)}{\left(1+e^{\frac{-sa}{2}}\right)} \qquad \left[\because\ tanhx = \frac{(1-e^{-2x})}{(1+e^{-2x})}\right]$$

$$\therefore L[f(t)] = \frac{1}{s}\tanh\left(\frac{as}{4}\right)$$

Example: Find the Laplace transform of $f(t) = \begin{cases} t; 0 \le t \le a \\ 2a - t; a \le t \le 2a \end{cases}$ **given that** $f(t+2a) = f(t)$.

Solution:

The given function is a periodic function with period $T = 2a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a te^{-st}dt + \int_a^{2a}(2a - t)e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[t\left(\frac{e^{-st}}{-s}\right) - \left(\frac{e^{-st}}{(-s)^2}\right)\right]_0^a - \left[(2a-t)\left(\frac{e^{-st}}{-s}\right) - (-1)\left(\frac{e^{-st}}{(-s)^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2} + \frac{e^{-2as}}{s^2} + \frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1-2e^{-as}+e^{-2as}}{s^2}\right]$$

$$= \frac{1}{1^2-(e^{-as})^2}\left[\frac{(1-e^{-as})^2}{s^2}\right]$$

$$= \frac{1}{(1-e^{-as})(1+e^{-as})}\left[\frac{(1-e^{-as})^2}{s^2}\right]$$

$$= \frac{1}{s^2}\frac{(1-e^{-as})}{(1+e^{-as})}$$

$$= \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

SOLUTION OF DIFFERENTIAL EQUATION BY LAPLACE TRANSFORM TECHNIQUE

$$L[y'(t)] = sL[y(t)] - y(0)$$

$$L[y''(t)] = s^2L[y(t)] - sy(0) - y'(0)$$

Example: Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2$, given $x = 0$ and $\frac{dx}{dt} = 5$ for $t = 0$ using Laplace transform method.

Solution:

Given $x'' - 3x' + 2x = 2; x(0) = 0; x'(0) = 5$

Taking Laplace transform on both sides, we get,

$$L[x''(t)] - 3L[x'(t)] + 2L[x(t)] = 2L(1)$$

$$[s^2L[x(t)] - sx(0) - x'(0)] - 3[sL[x(t)] - x(0)] + 2L[x(t)] = \frac{2}{s}$$

Substituting $x(0) = 0; x'(0) = 5$

$$[s^2L[x(t)] - 0 - 5] - 3[sL[x(t)] - 0] + 2L[x(t)] = \frac{2}{s}$$

$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

$$s^2L[x(t)] - 3sL[x(t)] + 2L[x(t)] = \frac{2}{s} + 5$$

Put $L[x(t)] = \bar{x}$

$$s^2\bar{x} - 3s\bar{x} + 2\bar{x} = \frac{2}{s} + 5$$

$$[s^2 - 3s + 2]\bar{x} = \frac{2}{s} + 5$$

$$(s-1)(s-2)\bar{x} = \frac{2}{s} + 5$$

$$\bar{x} = \frac{2+5s}{s(s-1)(s-2)}$$

Consider $\frac{2+5s}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{A(s-1)(s-2)+Bs(s-2)+Cs(s-1)}{s(s-1)(s-2)}$$

$$A(s-1)(s-2) + Bs(s-2) + Cs(s-1) = 2 + 5s \cdots (1)$$

Put $s = 0$ in (1)	Put $s = 1$ in (1)	Put $s = 2$ in (1)
$A(-1)(-2) = 2$	$B(1)(-1) = 7$	$C(2)(1) = 2 + 10$
$A = 1$	$B = -7$	$C = 6$

$$\frac{2+5s}{s(s-1)(s-2)} = \frac{1}{s} - \frac{7}{s-1} + \frac{6}{s-2}$$

$$\therefore \bar{x} = \frac{1}{s} - 7\frac{1}{s-1} + 6\frac{1}{s-2}$$

$$x(t) = L^{-1}\left[\frac{1}{s}\right] - 7L^{-1}\left[\frac{1}{s-1}\right] + 6L^{-1}\left[\frac{1}{s-2}\right]$$

$$x(t) = 1 - 7e^{t} + 6e^{2t}$$

Example: Using Laplace transform solve the differential equation $y'' - 3y' - 4y = 2e^{-t}$, with $y(0) = 1 = y'(0)$.

Solution:

Given $y'' - 3y' - 4y = 2e^{-t}$; with $y(0) = 1 = y'(0)$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 3L[y'(t)] - 4L[y(t)] = 2L(e^{-t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] - 4L[y(t)] = 2\,\frac{1}{s+1}$$

Substituting $y(0) = 1 = y'(0)$.

$$[s^2L[y(t)] - s - 1] - 3[sL[y(t)] - 1] - 4L[y(t)] = \frac{2}{s+1}$$

$$s^2L[y(t)] - s - 1 - 3sL[y(t)] + 3 - 4L[y(t)] = \frac{2}{s+1}$$

$$s^2L[y(t)] - 3sL[y(t)] - 4L[y(t)] = \frac{2}{s+1} + s - 2$$

Put $L[y(t)] = \bar{y}$

$$s^2\bar{y} - 3s\bar{y} - 4\bar{y} = \frac{2}{s+1} + s - 2$$

$$[s^2 - 3s - 4]\bar{y} = \frac{2}{s+1} + s - 2$$

$$[s^2 - 3s - 4]\bar{y} = \frac{2 + s(s+1) - 2(s+1)}{s+1}$$

$$= \frac{2 + s^2 + s - 2s - 2}{s+1}$$

$$(s+1)(s-4)\bar{y} = \frac{s^2 - s}{s+1}$$

$$\bar{y} = \frac{s^2 - s}{(s+1)(s+1)(s-4)}$$

$$\bar{y} = \frac{s^2 - s}{(s+1)^2(s-4)}$$

Consider $\frac{s^2 - s}{(s+1)^2(s-4)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-4}$

$$\frac{s^2 - s}{(s+1)^2(s-4)} = \frac{A(s+1)(s-4) + B(s-4) + C(s+1)^2}{(s+1)^2(s-4)}$$

$$A(s+1)(s-4) + B(s-4) + C(s+1)^2 = s^2 - s \cdots (1)$$

Put $s = -1$ in (1)	Put $s = 4$ in (1)	equating the coefficients of s^2, we get
$-5B = 1 + 1$	$25C = 16 - 4$	$A + C = 1 \Rightarrow A = 1 - C \Rightarrow 1 - \frac{12}{25}$
$B = \frac{-2}{5}$	$C = \frac{12}{25}$	$A = \frac{13}{25}$

$$\frac{s^2-s}{(s+1)^2(s-4)} = \frac{25}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$$

$$\therefore \bar{y} = \frac{13}{25(s+1)} - \frac{2}{5(s+1)^2} + \frac{12}{25(s-4)}$$

$$y(t) = \frac{13}{25}L^{-1}\left[\frac{1}{(s+1)}\right] - \frac{2}{5}L^{-1}\left[\frac{1}{(s+1)^2}\right] + \frac{12}{25}L^{-1}\left[\frac{1}{s-4}\right]$$

$$y(t) = \frac{13}{25}e^{-t} - \frac{2}{5}te^{-t} + \frac{12}{25}e^{4t}$$

Example: Solve the differential equation $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{-t}$, with $y(0) = 1$ and $y'(0) = 0$ using Laplace transform.

Solution:

Given $y'' - 3y' + 2y = e^{-t}$; with $y(0) = 1$ and $y'(0) = 1$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = L(e^{-t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = \frac{1}{s+1}$$

Substituting $y(0) = 1$ and $y'(0) = 0$.

$$[s^2L[y(t)] - s - 0] - 3[sL[y(t)] - 1] + 2L[y(t)] = \frac{1}{s+1}$$

$$s^2L[y(t)] - s - 3sL[y(t)] + 3 + 2L[y(t)] = \frac{1}{s+1}$$

$$s^2L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{1}{s+1} + s - 3$$

Put $L[y(t)] = \bar{y}$

$$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{1}{s+1} + s - 3$$

$$[s^2 - 3s + 2]\bar{y} = \frac{1}{s+1} + s - 3$$

$$[s^2 - 3s + 2]\bar{y} = \frac{1 + s(s+1) - 3(s+1)}{s+1}$$

$$= \frac{1 + s^2 + s - 3s - 3}{s+1}$$

$$(s-1)(s-2)\bar{y} = \frac{s^2 - 2s - 2}{s+1}$$

$$\bar{y} = \frac{s^2 - 2s - 2}{(s+1)(s-1)(s-2)}$$

Consider $\frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{s-2}$

$$\frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1)}{(s+1)(s-1)(s-2)}$$

$$A(s-1)(s-2) + B(s+1)(s-2) + C(s+1)(s-1) = s^2 - 2s - 2 \cdots (1)$$

Put $s = -1$ in (1)	put $s = 1$ in (1)	put $s = 2$ in (1)
$6A = 1 + 2 - 2$	$-2B = 1 - 4$	$3C = 4 - 4 - 2$

$A = \frac{1}{6}$ $\quad B = \frac{3}{2}$ $\quad C = \frac{-2}{3}$

$$\therefore \frac{s^2-2s-2}{(s+1)(s-1)(s-2)} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$$

$$\bar{y} = \frac{1}{6(s+1)} + \frac{3}{2(s-1)} - \frac{2}{3(s-2)}$$

$$y(t) = \frac{1}{6}L^{-1}\left[\frac{1}{(s+1)}\right] + \frac{3}{2}L^{-1}\left[\frac{1}{s-1}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{s-2}\right]$$

$$y(t) = \frac{1}{6}e^{-t} + \frac{3}{2}e^{t} - \frac{2}{3}e^{2t}$$

Example: Using Laplace transform solve the differential equation $y'' + 2y' - 3y = \sin t$, with $y(0) = y'(0) = 0$.

Solution:

Given $y'' + 2y' - 3y = \sin t$ with $y(0) = 0 = y'(0)$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] + 2L[y'(t)] - 3L[y(t)] = L(\sin t)$$

$$[s^2L[y(t)] - sy(0) - y'(0)] + 2[sL[y(t)] - y(0)] - 3L[y(t)] = \frac{1}{s^2+1}$$

Substituting $y(0) = 0 = y'(0)$.

$$[s^2L[y(t)] - 0 - 0] + 2[sL[y(t)] - 0] - 3L[y(t)] = \frac{1}{s^2+1}$$

$$s^2L[y(t)] + 2sL[y(t)] - 3L[y(t)] = \frac{1}{s^2+1}$$

$$s^2L[y(t)] + 2sL[y(t)] - 3L[y(t)] = \frac{1}{s^2+1}$$

Put $L[y(t)] = \bar{y}$

$$s^2\bar{y} + 2s\bar{y} - 3\bar{y} = \frac{1}{s^2+1}$$

$$[s^2 + 2s - 3]\bar{y} = \frac{1}{s^2+1}$$

$$(s-1)(s+3)\bar{y} = \frac{1}{s^2+1}$$

$$\bar{y} = \frac{1}{(s-1)(s+3)(s^2+1)}$$

Consider $\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A(s^2+1)(s+3)+B(s-1)(s^2+1)+(Cs+D)(s-1)(s+3)}{(s-1)(s+3)(s^2+1)}$$

$$A(s^2+1)(s+3) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3) = 1 \cdots (1)$$

Put $s = 1$ in (1)

$8A = 0 + 1$

Put $s = -3$ in (1)

$B(-4)(10) = 1$

equating the coefficients of s^2, we get

$A + B + C = 0 \Rightarrow C = -A - B = \frac{-1}{8} + \frac{1}{40}$

$A = \frac{1}{8}$ $\quad B = \frac{-1}{40}$ $\quad C = \frac{-1}{10}$

Put $s = 0$ in (1), we get

$3A - B - 3D = 1 \Rightarrow \frac{3}{8} + \frac{1}{40} - 3D = 1$

$3D = \frac{3}{8} + \frac{1}{40} - 1$

$3D = \frac{15+1-40}{40} \Rightarrow D = \frac{-24}{40\times 3} \Rightarrow D = \frac{-1}{5}$

$$\frac{1}{(s-1)(s+3)(s^2+1)} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} + \frac{\left(\frac{-1}{10}\right)s - \frac{1}{5}}{s^2+1}$$

$$\therefore \bar{y} = \frac{1}{8(s-1)} - \frac{1}{40(s+3)} - \frac{s}{10(s^2+1)} - \frac{1}{5(s^2+1)}$$

$$y(t) = \frac{1}{8}L^{-1}\left[\frac{1}{(s-1)}\right] - \frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}(cost - 2sint)$$

Example: Using Laplace transform solve the differential equation $y'' - 3y' + 2y = 4e^{2t}$, with $y(0) = -3$ and $y'(0) = 5$.

Solution:

Given $y'' - 3y' + 2y = 4e^{2t}$; with $y(0) = -3$ and $y'(0) = 5$.

Taking Laplace transform on both sides, we get,

$L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = 4L(e^{2t})$

$[s^2L[y(t)] - sy(0) - y'(0)] - 3[sL[y(t)] - y(0)] + 2L[y(t)] = 4\frac{1}{s-2}$

Substituting $y(0) = -3$ and $y'(0) = 5$.

$[s^2L[y(t)] + 3s - 5] - 3[sL[y(t)] + 3] + 2L[y(t)] = \frac{4}{s-2}$

$s^2L[y(t)] + 3s - 5 - 3sL[y(t)] - 9 + 2L[y(t)] = \frac{4}{s-2}$

$s^2L[y(t)] - 3sL[y(t)] + 2L[y(t)] = \frac{4}{s-2} - 3s + 14$

Put $L[y(t)] = \bar{y}$

$s^2\bar{y} - 3s\bar{y} + 2\bar{y} = \frac{4}{s-2} - 3s + 14$

$[s^2 - 3s + 2]\bar{y} = \frac{4}{s-2} + 14 - 3s$

$[s^2 - 3s + 2]\bar{y} = \frac{4+(14-3s)(s-2)}{s-2}$

$(s-1)(s-2)\bar{y} = \frac{4+(14-3s)(s-2)}{s-2}$

$\bar{y} = \frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2}$

Consider $\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$

$$\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{A(s-2)^2+B(s-1)(s-2)+C(s-1)}{(s-1)(s-2)^2}$$

$$A(s-2)^2 + B(s-1)(s-2) + C(s-1) = 4 + (14-3s)(s-2) \cdots (1)$$

Put $s = 1$ in (1)

$A = 4 - 11$

$A = -7$

Put $s = 2$ in (1)

$C = 4 + 0$

$C = 4$

equating the coefficients of s^2, we get

$A + B = -3 \Rightarrow -7 + B = -3$

$B = 4$

$$\frac{4+(14-3s)(s-2)}{(s-1)(s-2)^2} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$\therefore \bar{y} = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$$

$$y(t) = -7L^{-1}\left[\frac{1}{(s-1)}\right] + 4L^{-1}\left[\frac{1}{s-2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$$= -7e^t + 4e^{2t} + 4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$y(t) = -7e^t + 4e^{2t} + 4e^{2t}t$$

Example: Using Laplace transform solve the differential equation $y'' - 4y' + 8y = e^{2t}$, with $y(0) = 2$ and $y'(0) = -2$.

Solution:

Given $y'' - 4y' + 8y = e^{2t}$; with $y(0) = 2$ and $y'(0) = -2$.

Taking Laplace transform on both sides, we get,

$$L[y''(t)] - 4L[y'(t)] + 8L[y(t)] = L(e^{2t})$$

$$[s^2L[y(t)] - sy(0) - y'(0)] - 4[sL[y(t)] - y(0)] + 8L[y(t)] = \frac{1}{s-2}$$

Substituting $y(0) = 2$ and $y'(0) = -2$.

$$[s^2L[y(t)] - 2s + 2] - 4[sL[y(t)] - 2] + 8L[y(t)] = \frac{1}{s-2}$$

$$s^2L[y(t)] - 2s + 2 - 4sL[y(t)] + 8 + 8L[y(t)] = \frac{1}{s-2}$$

$$s^2L[y(t)] - 4sL[y(t)] + 8L[y(t)] = \frac{1}{s-2} + 2s - 10$$

Put $L[y(t)] = \bar{y}$

$$s^2\bar{y} - 4s\bar{y} + 8\bar{y} = \frac{1}{s-2} + 2s - 10$$

$$[s^2 - 4s + 8]\bar{y} = \frac{1}{s-2} + 2s - 10$$

$$[s^2 - 4s + 8]\bar{y} = \frac{1+(2s-10)(s-2)}{s-2}$$

$$\bar{y} = \frac{1+(2s-10)(s-2)}{(s-2)(s^2-4s+8)}$$

$$= \frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]}$$

Consider $\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{A}{s-2} + \frac{B(s-2)+C}{(s-2)^2+4}$

$$= \frac{A[(s-2)^2+4]+B[(s-2)+C](s-2)}{[s-2][(s-2)^2+4]}$$

$$A[(s-2)^2+4]+B[(s-2)+C](s-2) = 1+(2s-10)(s-2) \cdots (1)$$

Put $s=2$ in (1)	Put $s = 0\ in\ (1)$	equating the coefficients ofs^2, we get
$4A = 1+0$	$8A+4B-2C = 21$	$A+B = 2 \Rightarrow \frac{1}{4}+B = 2$
$A = \frac{1}{4}$	$C = -6$	$B = \frac{7}{4}$

$$\frac{1+(2s-10)(s-2)}{(s-2)[(s-2)^2+4]} = \frac{\frac{1}{4}}{s-2} + \frac{\frac{7}{4}(s-2)-6}{(s-2)^2+4}$$

$$\therefore \bar{y} = \frac{1}{4(s-2)} + \frac{7}{4}\frac{(s-2)}{(s-2)^2+4} - 6\frac{1}{(s-2)^2+4}$$

$$y(t) = \frac{1}{4}L^{-1}\left[\frac{1}{(s-2)}\right] + \frac{7}{4}L^{-1}\left[\frac{(s-2)}{(s-2)^2+4}\right] - 6L^{-1}\left[\frac{1}{(s-2)^2+4}\right]$$

$$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}L^{-1}\left[\frac{s}{s^2+4}\right] - 6e^{2t}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}cos2t - 6e^{2t}\frac{sin2t}{2}$$

$$y(t) = \frac{1}{4}e^{2t} + \frac{7}{4}e^{2t}cos2t - 3e^{2t}sin2t$$

$L[f(t)] = F(s)$	$L^{-1}[F(s)] = f(t)$
$L[1] = \frac{1}{s}$	$L^{-1}[\frac{1}{s}] = 1$
$L[t] = \frac{1}{s^2}$	$L^{-1}[\frac{1}{s^2}] = t$
$L[t^n] = \frac{n!}{s^{n+1}}$ $if\ n\ is\ an\ integer$	$L^{-1}[\frac{n!}{s^{n+1}}] = t^n$ $L^{-1}[\frac{1}{s^{n+1}}] = \frac{t^n}{n!}$
$L[e^{at}] = \frac{1}{s-a}$	$L^{-1}[\frac{1}{s-a}] = e^{at}$
$L[e^{-at}] = \frac{1}{s+a}$	$L^{-1}[\frac{1}{s+a}] = e^{-at}$

$L[sinat] = \frac{a}{s^2 + a^2}$	$L^{-1}\left[\frac{1}{s^2 + a^2}\right] = \frac{sinat}{a}$
$L[cosat] = \frac{s}{s^2 + a^2}$	$L^{-1}\left[\frac{s}{s^2 + a^2}\right] = cosat$
$L[sinhat] = \frac{a}{s^2 - a^2}$	$L^{-1}\left[\frac{1}{s^2 - a^2}\right] = \frac{sinhat}{a}$