Catalog

LINE INTEGRAL –CAUCHY INTEGRAL THEOREM

If $f(z)$ is a continuous function of the complex variable $z = x + iy$ and C is any continuous curve connecting two points A and B on the z – plane then the complex line integral of $f(z)$ along C from A to B is denoted by $\int_c f(z)dz$

When C is simple closed curve, then the complex integral is also called as a contour integral and is denoted as $\oint_c f(z)dz$. The curve C is always take in the anticlockwise direction.

Note: If the direction of C is reversed (clockwise), the integral changes its sign

$$(ie)\oint_c f(z)dz = -\oint f(z)dz$$

Standard theorems:

1. **Cauchy's Integral theorem (or) Cauchy's Theorem (or) Cauchy's Fundamental Theorem**

   Statement: If $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a

   simple closed curve C then $\oint_c f(z)\, dz = 0$

2. **Extension of Cauchy's integral theorem (or) Cauchy's theorem for multiply connected Region Statement:** If $f(z)$ is analytic at all points inside and on a multiply connected region whose outer boundary is C and inner boundaries are $C_1, C_2, \dots, C_n$ then

$$\int_c f(z)dz = \int_{c} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{c_n} f(z)dz$$

Example: Evaluate $\int_0^{3+i} z^2dz$ along the line joining the points (0, 0) and (3, 1)

Solution:

Given $\int_0^{3+i} z^2dz$

Let $z = x + iy$

Here $z = 0$ corresponds to (0, 0) and $z = 3 + i$ corresponds to (3, 1)

The equation of the line joining (0, 0) and (3, 1) is

$$y = \frac{x}{3} \Rightarrow x = 3y$$

Now $z^2dz = (x + iy)^2(dx + idy)$

$= [x^2 - y^2 + i2xy][dx + idy]$

$= [(x^2 - y^2) + i2xy][dx + idy]$

$$= [(x^2 - y^2)dx - 2xydy] + i[2xydx + (x^2 - y^2)dy]$$

Since $x = 3y \Rightarrow dx = 3dy$

$$\therefore z^2dz = [8y^2(3dy) - 6y^2dy] + i[18y^2dy + 8y^2dy]$$

$$= 18y^2dy + i26y^2dy$$

$$\therefore \int_0^{3+i} z^2dz = \int_0^{'}[18y^2 + i26y^2]dy$$

$$= [18\frac{y^2}{3} + i\,26\,\frac{y^3}{3}]_0^{'}$$

$$= 6 + i\frac{26}{3}$$

Example: Evaluate $\int_0^{2+i}(x^2 - iy)dz$

Solution:

Let $z = x + iy$

Here $z = 0$ corresponds to (0, 0) and $z = 2 + i$ corresponds to (2, 1)

Now $(x^2 - iy)dz = (x^2 - iy)(dx + idy)$

$$= x^2dx + y\,dy) + i\,(x^2dy - y\,dx)$$

Along the path $y = x^2 \Rightarrow dy = 2xdx$

$$\therefore (x^2 - iy)dz = (x^2dx + 2x^3dx) + i(2x^3dx - x^2dx)$$

$$\int_0^{2+i}(x^2 - iy)dz = \int_0^2(x^2 + 2x^3)dx + i(2x^3 - x^2)dx$$

$$= [\frac{x^3}{3} + \frac{2x^4}{4}]_0^2 + i\,[\frac{2x^4}{4} = \frac{x^3}{3}]_0^2$$

$$= (\frac{8}{3} + \frac{16}{2}) + i\,(\frac{16}{2} - \frac{8}{3})$$

$$= \frac{32}{3} + i\,\frac{16}{3}$$

Example: Evaluate $\int_c e^{\frac{1}{z}}dz$, where C is $|z| = 2$

Solution:

Let $f(z) = e^{\frac{1}{z}}$ clearly $f(z)$ is analytic inside and on C.

Hence, by Cauchy's integral theorem we get $\int_c e^{\frac{1}{z}}dz = 0$

Example: Evaluate $\int_c z^2e^{\frac{1}{z}}dz$, where C is $|z| = 1$

Solution:

Given $\int_c z^2\,e^{1/z}dz$

$$= \int_c \frac{z^2}{e^{-1/z}}dz$$

$Dr = 0 \Rightarrow z = 0$, We get $e^{\frac{-1}{0}} = e^{-\infty} = 0$

$z = 0$ lies inside $|z| = 1$.

Cauchy's Integral formula is

$$\int_c z^2\, e^{1/z} dz = 2\pi i f(0) = 0$$

Example: Evaluate $\int_c \frac{1}{2z-3} dz$ where C is $|z| = 1$

Solution:

Given $\int_c \frac{1}{2z-3} dz$

$Dr = 0 \Longrightarrow 2z - 3 = 0, \Longrightarrow z = \frac{3}{2}$

Given C is $|z| = 1$

$$\Rightarrow |z| = |\tfrac{3}{2}| = \tfrac{3}{2} > 1$$

$\therefore z = \frac{3}{2}$ lies outside C

$\therefore$ By Cauchy's Integral theorem, $\int_c \frac{1}{2z-3} dz = 0$

Example: Evaluate $\int_c \frac{dz}{z+4}$ where C is $|z| = 2$

Solution:

Given $\int_c \frac{dz}{z+4}$

$Dr = 0 \Longrightarrow z + 4 = 0 \Longrightarrow z = -4$

Given C is $|z| = 2$

$$\Rightarrow |z| = |-4| = 4 > 2$$

$\therefore z = -4$ lies outside C.

$\therefore$ By Cauchy's Integral Theorem, $\int_c \frac{dz}{z+4} = 0$

Cauchy's integral formula

Statement: If $f(z)$ is analytic inside and on a simple closed curve C of a simply connected region R

and if 'a' is any point interior to C, then

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,dz$$

(OR)

$$\int_c \frac{f(x)}{z-a}\,dz = 2\pi i\, f(a),$$

the integration around C being taken in the positive direction.

Cauchy's Integral formula for derivatives

Statement: If $f(z)$ is analytic inside and on a simple closed curve C of a simply connected Region R

and if 'a' is any point interior to C, then

$$\int_c \frac{f(z)}{(z-a)^2}\,dz = 2\pi i f'(a)$$

$$\int_c \frac{f(z)}{(z-a)^3}\,dz = 2\pi i f''(a)$$

In general, $\int_c \frac{f(z)}{(z-a)^n}\,dz = 2\pi i f^{(n-1)}(a)$

Example: Evaluate $\int_c \frac{e^{2z}}{z^2+1}\,dz, \text{ where } C \text{ is } |z| = \frac{1}{2}$

Solution:

Given $\int_c \frac{e^{2z}}{z^2+1}\,dz$

$Dr = 0 \Longrightarrow z^2 + 1 = 0 \;\Longrightarrow z = \pm i$

Given $C\ is\ |z| = \frac{1}{2}$

$\Rightarrow |z| = |\pm i| = 1 > \frac{1}{2}$

∴Clearly both the points $z = \pm i$ lies outside C.

∴By Cauchy's Integral Theorem, $\int_c \frac{e^{2z}}{z^2+1}\,dz = 0$

Example: Using Cauchy's integral formula Evaluate $\int_c \frac{z+1}{(z-3)(z-1)}\,dz, \text{ where } C \text{ is } |z| =$

Solution:

Given $\int_c \frac{z+1}{(z-3)(z-1)} dz$

$Dr = 0 \Longrightarrow z = 3, 1$

Given $C\ is\ |z| = 2$

∴Clearly $z = 1$ lies inside C and $z = 3$ lies outside C

$$\int_c \frac{z+1}{(z-3)(z-1)} dz = \int_c \frac{(z+1)/(z-3}{(z-1)} dz$$

∴ By Cauchy's Integral Theorem

$\int_c \frac{(z+1)/(z-3)}{(z-1)} dz = 2\pi i f(1)$ Where $f(z) = \frac{z+1}{z-3} \Rightarrow f(1) = \frac{2}{-2}$

$$= 2\pi i(-1) = -2\pi i$$

Example: Using Cauchy's integral formula, evaluate $\int_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)(z-3)} dz$ where C is the circle

$|z| = 4.$

Solution:

Given $\int_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)(z-3)} dz$

$Dr = 0 \ \Longrightarrow z = 2, 3$

Given $C\ is\ |z| = 4$

∴Clearly $z = 2\ and\ 3$ lies inside C.

Consider, $\frac{1}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3}$

$$\Rightarrow 1 = A(z-3) + B(Z = 2)$$

Put $z = -3 \Rightarrow 1 = B$

Put $z = 2 \Rightarrow \ -1 = A$

$\therefore \frac{1}{(z-2)(z-3)} = -\frac{1}{z-2} + \frac{1}{z-3}$

$\int_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)(z-3)} dz = -\int \frac{\sin\pi z^2+\cos\pi z^2}{z-2} dz + \int \frac{\sin\pi z^2+\cos\pi z^2}{z-3} dz$

$= -2\pi i f(2) + 2\pi i\ f(3)$ Where $f(z) = \sin(\pi z^2) + \cos\pi z^2$

$= -2\pi i(1) + 2\pi i(-1)$ $f(2) = \sin 4\pi + \cos 4\pi = 1$

$= -4\pi i$ $f(3) = \sin 9\pi + \cos 9\pi - 1 =$
-1

Example: Evaluate $\int_c \frac{z+4}{z^2+2z+5}$ Where C is the circle $(i)|z+1+i| = 2$ $(ii)|z+1-i| =$
2

(*iii*) $|z| = 1$

Solution:

Given $\int_c \frac{z+4}{z^2+2z+5} dz$

$Dr = 0 \Rightarrow z^2 + 2z + 5 = 0$

$\Rightarrow z = \frac{-2\pm\sqrt{4-20}}{2}$

$\Rightarrow z = -1 \pm 2i$

$\therefore \int_c \frac{z+4}{z^2+2z+5} dz = \int_c \frac{(z+4)\ dz}{[z-(-1+2i)[z-(-1-2i)]}$

(i) $|z + 1 + i| = 2$ *is the circle*

When $z = -1 + 2i, |-1 + 2i + 1 + i| = |3i| > 2$ lies outside C.

When $z = -1 - 2i, |-1 - 2i + 1 + i| = |-i| < 2$ lies inside C.

∴ By Cauchy's Integral formula

$\int_c \frac{[(z+1)/(z-(-1+2i)]}{[z-(-1-2i)]} dz = 2\pi i f(-1 - 2i)$ Where $f(z) = \frac{z+4}{[z-(-1+2i)]}$

$= 2\pi i\ [\frac{3-2i}{-4i}]$ $f(-1 - 2i) = \frac{-1-2i+4}{-1-2i+1-2i} = \frac{3-2i}{-4i}$

$= \frac{\pi}{2}(2i - 3)$

(ii) $|z + 1 - i| = 2$ is the circle

When $z = -1 + 2i, |-1 + 2i + 1 - i| = |i| < 2$ lies inside C

When $z = -1 - 2i, |-1 - 2i + 1 - i| = |-3i| > 2$ lies outside C

∴ By Cauchy's Integral formula

$\int_c \frac{(z+1)/[z-(-1-2i)]}{[z-(-1+2i)]} dz = 2\pi i\ f(-1 + 2i)$ Where $f(z) = \frac{z+4}{z-(-1-2i)}$

$= 2\pi i \frac{[3+2i]}{4i}$ $f(-1 + 2i) = \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}$

$= \frac{\pi}{2}(3 + 2i)$

(iii)$|z| = 1$ is the circle

When $z = -1 + 2i, 1 - 1 + 2i| = \sqrt{5} > 1$ lies outside C

When $z = -1 - 2i, 1 - 1 - 2i| = \sqrt{5} > 1$ lies outside C

∴ By Cauchy's Integral theorem

$$\int_c \frac{z+4}{z^2+2z+5} dz = 0$$

Example: Using Cauchy's integral formula, evaluate $\int_c \frac{z+1}{z^2+2z+4} dz$ where C is the circle $|z+1+i| = 2$

Solution:

Given $\int_c \frac{z+1}{z^2+2z+4} dz$

$Dr = 0 \Rightarrow z^2 + 2z + 4 = 0$

$\Rightarrow z = \frac{-2 \pm \sqrt{4-16}}{2}$

$\Rightarrow z = -1 \pm i\sqrt{3}$

$\therefore \int_c \frac{z+1}{z^2+2z+4} dz = \int_c \frac{(z+1)\, dz}{[z-(-1+i\sqrt{3})][z-(-1-i\sqrt{3}]}$

Given $C\ is\ |z+1+i| = 2$

When $z = -1 - i\sqrt{3}, |-1 - i\sqrt{3} + 1 + i| = |(1 - \sqrt{3}i)| < 2$ lies inside C.

When $z = -1 + i\sqrt{3}, |-1 + i\sqrt{3} + 1 + i| = |i + \sqrt{3}i| > 2$ lies outside C.

$\therefore$ By Cauchy's Integral Formula

$\int_c \frac{(z+1)/[z-(-1+i\sqrt{3})]}{[z-(-1-i\sqrt{3})]} dz = 2\pi i\, f(-1 - i\sqrt{3})$ Where $f(z) = \frac{z+1}{z-(-1+i\sqrt{3})}$

$= 2\pi i\ (\frac{1}{2}) = \pi i$ $f(-1 - i\sqrt{3}) = \frac{-1-i\sqrt{3}+1}{-1-i\sqrt{3}+1-i\sqrt{3}} = \frac{\sqrt{3}i}{-2i\sqrt{3}} = \frac{1}{2}$

$\therefore \int_c \frac{z+1}{z^2+2z+4} dz = \pi i$

Example: Evaluate $\int_c \frac{z^2+1}{z^2-1} dz$ where C is the circle $(i)|z-1| = 1\ (ii)|z+1| = 1 (iii)|z-i| = 1$

Solution:

Given $\int_c \frac{z^2+1}{z^2-1} dz = \int_c \frac{z^2+1}{(z+1)(z-1)} dz$

$Dr = 0 \Longrightarrow z = 1, -1$

(i) $(z-1) = 1$ is the circle

When $z = 1$, $|1-1| = 0 < 1$ lies inside C

When $z = -1, |-1-1| = 2 > 1$ lies outside C

$\therefore$ By Cauchy's Integral formula

$\int_c \frac{z^2+1}{(z+1)(z-1)} dz = \int_c \frac{(z^2+1)/z+1}{(z-1)} dz$

$= 2\pi i f(1)$ where $f(z) = \frac{z^2+1}{z+1} \Rightarrow f(1) = 1$

$$= 2\pi i(1)$$
$$= 2\pi i$$

(ii) $|z+1| = 1$ is the circle

When $z = 1, |1+1| = 2 > 1$ lies outside C

When $z = -1, |-1+1| = 0 < 1$ lies inside C

∴ By Cauchy's Integral formula

$$\int_c \frac{(z^2+1)/(z-1)}{z+1} dz = 2\pi i\, f(-1) \qquad \text{where } f(z) = \frac{z^2+1}{z-1} \Rightarrow$$

$f(-1) = -1$

$$= 2\pi i(-1) = -2\pi i$$

(iii) $|z-i| = 1$ is the circle

When $z = 1, |1-i| = \sqrt{2} > 1$ lies outside C

When $z = -1, |-1-i| = \sqrt{2} > 1$ lies outside C

∴ By Cauchy's Integral Formula

$$\int_c \frac{(z^2+1)}{(z+1)(z-1)} dz = 0$$

Example: Using Cauchy's Integral formula evaluate $\int_c \frac{zdz}{(z-1)(z-2)^2}$ where C is the circle $|z-2| = \frac{1}{2}$

Solution:

Given $\int_c \frac{zdz}{(z-1)(z-2)^2}$

$Dr = 0 \Longrightarrow z = 1$ is a pole of order 1, $z = 2$ is a pole of order 2.

Given C is $|z-2| = \frac{1}{2}$

When $z = 1, |1-2| = 1 > \frac{1}{2}$ lies outside C.

When $z = 2, |2-2| = 0 < \frac{1}{2}$ lies inside C.

∴ By Cauchy's Integral formula

$$\int_c \frac{z/z-1}{(z-2)^2} dz = 2\pi i\, f'(2) \qquad \text{Where } f(z) = \frac{z}{z-1}$$

$$= 2\pi i(-1) \qquad f'(z) = \frac{(z-1)1 - z(1)}{(z-1)^2} \Rightarrow f'(2) =$$

-1

$$= -2\pi i$$

Example: Evaluate $\int_c \frac{\sin^2 z}{\left(z-\frac{\pi}{6}\right)^3} dz$ where C is the circle $|z| = 1$

Solution:

Given $\int_c \frac{sin^2 z}{(z-\frac{\pi}{6})^3} dz$

$Dr = 0 \Rightarrow z = \frac{\pi}{6}$ is a pole of order 3.

Give C is $|z| = 1$.

Clearly $z = \frac{\pi}{6}$ lies inside the circle $|z| = 1$

∴ By Cauchy's Integral formula

$\int_c \frac{sin^2 z}{(z-\frac{\pi}{6})^3} dz = \frac{2\pi i}{2!} f''(\pi/6)$ Where $f(z) = sin^2 z$

$= \frac{2\pi i}{2!}(1)$ $f'(z) = 2\sin z \cos z = \sin 2z$

$= \pi i$ $f''(z) = \cos 2z(2) \Rightarrow f''(\frac{\pi}{6}) =$

$2\cos(\frac{2\pi}{6})$

$= 2\cos\frac{\pi}{3} = 2(\frac{1}{2}) = 1$

Example: Evaluate $\int_c \frac{z}{(z-1)^3} dz$ where C is the circle $|z| = 2$, using Cauchy's Integral formula

Solution:

Given $\int_c \frac{z}{(z-1)^3} dz$

$Dr = 0 \Rightarrow z = 1$ is a pole of order 3.

Given C is $|z| = 2$.

Clearly $z = 1$ lies inside the circle C

∴ By Cauchy's Integral formula

$\int_c \frac{sin^2 z}{(z-1)^3} dz = \frac{2\pi i}{2!} f''(1)$ Where $f(z) = z \Rightarrow f'(z) = 1$

$= \frac{2\pi i}{2!}(0)$ $\Rightarrow f''(z) = 0 \Rightarrow f''(1) = 0$

$= 0$

Example: Evaluate $\int_c \frac{z^2}{(2z-1)^2} dz$ where C is the circle $|z| = 1$

Solution:

Given $\int_c \frac{z^2}{(2z-1)^2} dz$

$Dr = 0 \Rightarrow 2z = 0 \Rightarrow z = \frac{1}{2}$ is a pole of order 2.

Given C is $|z| = 1$.

Clearly $z = \frac{1}{2}$ lies inside the circle C

∴ By Cauchy's Integral formula

$$\int_c \frac{z^2}{2^2\left(z-\frac{1}{2}\right)^2} dz = \frac{1}{4}\int_c \frac{z^2}{\left(z-\frac{1}{2}\right)^2} dz \qquad \text{Where } f(z) = z^2 \Rightarrow f'(z) = 2z$$

$$= \frac{1}{4}\left(2\pi i f'\left(\frac{1}{2}\right)\right) \qquad \Rightarrow f'\left(\frac{1}{2}\right) = 1$$

$$= \frac{1}{2}\pi i(1)$$

$$= \frac{\pi i}{2}$$

TAYLORS AND LAURENTS SERIES

Taylor's Series

If $f(z)$ is analytic inside and on a circle C with centre at point 'a' and radius 'R' then at each point Z inside C,

$$f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2\frac{f''(a)}{2!} + \cdots$$

(OR)

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^n(a)$$

This is known as Taylor's series of $f(z)$ about $z = a$.

Note: 1 Putting $a = 0$ in the Taylor's series we get

$f(z) = f(0) + (z-0)\frac{f'(0)}{1!} + (z-0)^2\frac{f''(0)}{1!} + \cdots$ this series is called Maclaurin's Series.

Note: 2 The Maclaurin's for some elementary functions are

1) $(1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots$, when $|z| < 1$
2) $(1+z)^{-1} = 1 - z + z^2 - z^3 + \cdots$, when $|z| < 1$
3) $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$, when $|z| < 1$
4) $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots$, when $|z| < 1$
5) $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$ when $|z| < \infty$
6) $e^z = 1 - \frac{z}{1!} + \frac{z^2}{2!} + \cdots$ when $|z| < \infty$
7) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$ when $|z| < \infty$
8) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$ when $|z| < \infty$

LAURENTS SERIES

If c_1 *and* c_2 are two concentric circles with centre at $z = a$ and radii r_1 *and* r_2 $(r_1 < r_2)$ and if $f(z)$ is analytic inside on the circles and within the annulus between c_1 *and* c_2 then for any z in the annulus, we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n} \quad \ldots (1)$$

Where $a_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(z)}{(z-a)^{n+1}}\, dz$ and $b_n = \frac{1}{2\pi i}\int_{c_2} \frac{f(z)}{(z-a)^{1-n}}\, dz$ and the integration being taken in positive direction. This series (1) is called Laurent series of $f(z)$ about the point $z = a$

Example: Expand $f(z) = \cos z$ as a Taylor's series about $z = \frac{\pi}{4}$

Solution:

Function	Value of function at $z = \frac{\pi}{4}$
$f(z) = cosz$	$f\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$
$f'(z) = -sinz$	$f'\left(\frac{\pi}{4}\right) = -sin\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
$f''(z) = -cosz$	$f''\left(\frac{\pi}{4}\right) = -cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$
$f'''(z) = sinz$	$f''\left(\frac{\pi}{4}\right) = sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

The Taylor series of $f(z)$ about $z = \frac{\pi}{4}$ is $f(z) = f\left(\frac{\pi}{4}\right) + \left(z - \frac{\pi}{4}\right)\frac{f'\left(\frac{\pi}{4}\right)}{1!} + \left(z - \frac{\pi}{4}\right)^2\frac{f''\left(\frac{\pi}{4}\right)}{2!} + \cdots$

$$cosz = \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\frac{-\frac{1}{\sqrt{2}}}{1!} + \left(z - \frac{\pi}{4}\right)^2\frac{-\frac{1}{\sqrt{2}}}{2!} + \cdots$$

Example: Expand $f(z) = log(1 + z)$ as a Taylor's series about $z = 0$.

Solution:

Function	Value of function at $z = 0$
$f(z) = log(1 + z)$	$f(0) = log(1 + 0) = 0$
$f'(z) = \frac{1}{1 + z}$	$f'(0) = \frac{1}{1 + 0} = 1$
$f''(z) = \frac{-1}{(1 + z)^2}$	$f''(0) = \frac{-1}{(1 + 0)^2} = -1$
$f'''(z) = \frac{2}{(1 + z)^3}$	$f'''(0) = \frac{2}{(1 + 0)^3} = 2$

The Taylor series of $f(z)$ about $z = 0$ is

$$f(z) = f(0) + (z - 0)\frac{f'(0)}{1!} + (z - 0)^2\frac{f''(0)}{2!} + \cdots$$

$$log(1 + z) = 0 + (z)\frac{1}{1!} + (z)^2\frac{-1}{2!} + \cdots$$

$$log(1 + z) = (z)\frac{1}{1!} - (z)^2\frac{1}{2!} + \cdots$$

Example: Expand $f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$ as a Laurent's series if (i) $|z| < 2$ (ii) $|z| > 3$

(iii) $2 < |z| < 3$

Solution:

Given $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ is an improper fraction. Since degree of numerator and degree of denominator of $f(z)$ are same

$\therefore$ Apply division process

$$\begin{array}{r|l} & 1 \\ z^2+5z+6 & z^2-1 \\ & z^2+5z+6 \\ \hline & -5z-7 \end{array}$$

$\therefore \frac{z^2-1}{(z+2)(z+3)} = 1 - \frac{5z+7}{(z+2)(z+3)} \quad \ldots (1)$

Consider $\frac{5z+7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$

$\Rightarrow 5z+7 = A(z+3) + B(z+2)$

Put $z = -2$, we get $-10+7 = A\,(1)$

$\Rightarrow A = -3$

Put $z = -3$, we get $-15+7 = B(-1)$

$\Rightarrow B = 8$

$\therefore \frac{5z+7}{(z+2)(z+3)} = \frac{-3}{z+2} + \frac{8}{z+3}$

$\therefore (1) \Rightarrow 1 - \frac{3}{z+2} - \frac{8}{z+3}$

(i) Given $|z| < 2$

$$\therefore f(z) = 1 + \frac{3}{2(1+z/2)} - \frac{8}{3(1+z/3)}$$

$$= 1 + \frac{3}{2}(1+z/2)^{-1} - \frac{8}{3}(1+z/3)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \left[\frac{z}{2}\right]^2 + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left[\frac{z}{3}\right]^2 + \cdots\right]$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left[\frac{z}{2}\right]^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left[\frac{z}{3}\right]^n$$

(ii) Given $|z| > 3$

$$\therefore f(z) = 1 + \frac{3}{z(1+2/z)} - \frac{8}{z(1+3/z)}$$

$$= 1 + \frac{3}{z}(1+2/z)^{-1} - \frac{8}{z}(1+3/z)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left[\frac{2}{z}\right]^2 + \cdots\right] - \frac{8}{z}\left[1 - \frac{3}{z} + \left[\frac{3}{z}\right]^2 \ldots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left[\frac{2}{z}\right]^n - \frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left[\frac{3}{z}\right]^n$$

(iii) Given $2 < |z| < 3$

$$\therefore\ f(z) = 1 + \frac{3}{z(1+2/z)} - \frac{8}{3(1+z/3)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left[\frac{2}{z}\right]^2 + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left[\frac{z}{3}\right]^2 \ldots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left[\frac{2}{z}\right]^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left[\frac{z}{3}\right]^n$$

Example: Find the Laurent's series expansion of $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ in $1 < |z+1| < 3$.

Also find the residue of $f(a)$ at $z = -1$

Solution:

Given $f(z) = \frac{7z-2}{z(z-2)(z+1)}$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$7z - 2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

Put $z = 2$, we get $14 - 2 = B\,(2)\,(2+1)$

$$\Rightarrow 12 = 6B$$

$$\Rightarrow B = 2$$

Put $z = -1$, we get $-7 - 2 = C(-1)(-1-2)$

$$\Rightarrow -9 = 3C$$

$$\Rightarrow C = -3$$

Put $z = 0$ we get $-2 = A(-2)$

$$\Rightarrow A = 1$$

$$\therefore\ f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

Given region is $1 < |z+1| < 3$

Let $u = z + 1 \Rightarrow z = u - 1$

$(i.e)\ 1 < |u| < 3$

Now $f(z) = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$

$$= \frac{1}{u(1-1/u)} + \frac{2}{-3(1-u/3)} - \frac{3}{u}$$

$$= \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1} - \frac{3}{u}$$

$$= \frac{1}{u}\left[1 + \frac{1}{u} + \left[\frac{1}{u}\right]^2 + \cdots\right] - \frac{2}{3}\left[1 + \frac{u}{3} + \left[\frac{u}{3}\right]^2 + \cdots\right] - \frac{3}{u}$$

$$= \frac{1}{z+1}\left[1 + \frac{1}{z+1} + \left[\frac{1}{z+1}\right]^2 + \cdots\right] - \frac{2}{3}\left[1 + \frac{z+1}{3} + \left[\frac{z+1}{3}\right]^2 + \cdots\right] - \frac{3}{z+1}$$

$$=\frac{1}{z+1}\sum_{n=0}^{\infty}\left[\frac{1}{z+1}\right]^n-\frac{2}{3}\sum_{n=0}^{\infty}\left[\frac{1}{\frac{z+1}{3}}\right]^n-\frac{3}{z+1}$$

Also $Res[f(z), z=-1]$ =coefficient of $\frac{1}{z+1} = -2$

Example: Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in a Laurent's series valid in the region

(i)$|z-1|>1$(ii) $0<|z-2|<1$ (iii) $|z|>2$ (iv) $0<|z-1|<1$

Solution:

Given $f(z) = \frac{1}{(z-1)(z-2)}$

Consider $\frac{1}{(z-1)(z-2)} = \frac{A}{z-1}+\frac{B}{z-2}$

$$\Rightarrow 1 = A(z-2)+B(z-1)$$

Put $z = 2$, we get $1 = B(1)$

$$\Rightarrow B = 1$$

Put $z = 1$we get $1 = A(1-2)$

$$\Rightarrow A = -1$$

$$\therefore f(z) = \frac{-1}{z-1}+\frac{1}{z-2}$$

(i) Given region is $|z-1|>1$

Let $u = z-1 \Rightarrow z = u+1$

$(i.e)\ |u|>1$

Now $f(z) = -\frac{1}{u}+\frac{1}{u-1}$

$$= \frac{-1}{u}+\frac{1}{u(1-{}^{1}/_{u})}$$

$$= \frac{-1}{u}+\frac{1}{u}(1-{}^{1}/_{u})^{-1}$$

$$= \frac{-1}{u}+\frac{1}{u}\left[1+\frac{1}{u}+\left[\frac{1}{u}\right]^2+\cdots\right]$$

$$= \frac{-1}{z+1}+\frac{1}{z+1}\left[1+\frac{1}{z+1}+\left[\frac{1}{z+1}\right]^2+\cdots\right]$$

$$= \frac{-1}{z+1}+\frac{1}{z+1}\sum_{n=0}^{\infty}\left[\frac{1}{z+1}\right]^n$$

(ii) Given $0<|z-2|<1$

Let $u = z-2 \Rightarrow z = u+2$

$(i.e)\ 0<|u|<1$

Now $f(z) = -\frac{1}{u+1}+\frac{1}{u}$

$$= -(1+u)^{-1}+\frac{1}{u}$$

$$= -[1 - u + [u]^2 + \cdots] + \frac{1}{u}$$

$$= -[1 - (z-2) + [z-2]^2 + \cdots] + \frac{1}{z-2}$$

$$= -\sum_{n=0}^{\infty} [-1]^n [z-2]^n + \frac{1}{z-2}$$

(iii) Given $|z| > 2$

Now $f(z) = -\frac{1}{z(1-\frac{1}{z})} + \frac{1}{z(1-\frac{2}{z})}$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{z}\left[1 + \frac{1}{z} + \left[\frac{1}{z}\right]^2 + \cdots\right] + \frac{1}{z}\left[1 + \frac{2}{z} + \left[\frac{2}{z}\right]^2 + \cdots\right]$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left[\frac{1}{z}\right]^n + \frac{1}{z}\sum_{n=0}^{\to\infty}\left[\frac{2}{z}\right]^n$$

(iv) Given $0 < |z-1| < 1$

Let $u = z - 1 \Rightarrow z = u + 1$

$(i.e)\ 0 < |u| < 1$

Now $f(z) = -\frac{1}{u} + \frac{1}{u-1}$

$$= -\frac{1}{u} + \frac{1}{-1\,[1-u]}$$

$$= -\frac{1}{u} - (1-u)^{-1}$$

$$= -\frac{1}{u} - [1 + u + [u]^2 + \cdots]$$

$$= -\frac{1}{z-1} - [1 + z - 1 + [z-1]^2 + \cdots]$$

$$= -\frac{1}{z-1} - \sum_{n=0}^{\infty} [z-1]^n$$

Example: Expand $f(z) = \frac{z}{(z+1)(z-2)}$ in a Laurent's series about (i) $z = -1$ (ii) $z = 2$

Solution:

Consider $\frac{z}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}$

$$\Rightarrow z = A(z-2) + B(z+1)$$

Put $z = 2$, we get $2 = B(3)$

$$\Rightarrow B = \frac{2}{3}$$

Put $z = -1$we get $-1 = A(-3)$

$$\Rightarrow A = \frac{1}{3}$$

$$\therefore f(z) = \frac{1}{3(z+1)} + \frac{2}{3(z-2)}$$

(i) To expand $f(z)$ about $z = -1$

(or) $|z-1|<1$

Put $z+1=u \Rightarrow z=u-1$

$\Rightarrow |z-1|<1 \Rightarrow |u|<1$

Now $f(z)=\frac{1}{3u}+\frac{2}{3(u-3)}$

$=\frac{1}{3u}+\frac{2}{3((-3)(1-u/3))}$

$=\frac{1}{3u}-\frac{2}{9}(1-u/3)^{-1}$

$=\frac{1}{3u}-\frac{2}{9}[1+\frac{u}{3}+[\frac{u}{3}]^2+\cdots]$

$=\frac{1}{3(z+1)}-\frac{2}{9}[1+\frac{(z+1)}{3}+[\frac{(z+1)}{3}]^2+\cdots]$

$=\frac{1}{3(z+1)}-\frac{2}{9}\sum_{n=0}^{\infty}[\frac{(z+1)}{3}]^n$

(ii) To expand $f(z)$ about $z=2$

(or) $|z-2|<1$

Put $z-2=u \Rightarrow z=u+2$

$\Rightarrow |z-2|<1 \Rightarrow |u|<1$

Now $f(z)=\frac{1}{3(u+3)}+\frac{2}{3(u)}$

$=\frac{1}{3((3)(1+u/3))}+\frac{2}{3(u)}$

$=\frac{1}{9}(1+u/3)^{-1}+\frac{2}{3(u)}$

$=\frac{1}{9}[1-\frac{u}{3}+[\frac{u}{3}]^2+\cdots]+\frac{2}{3(u)}$

$=\frac{1}{9}[1-\frac{(z-2)}{3}+[\frac{(z-2)}{3}]^2+\cdots]+\frac{2}{3(z-2)}$

$=\frac{1}{9}\sum_{n=0}^{\infty}(-1)^n[\frac{(z-2)}{3}]^n+\frac{2}{3(z-2)}$

Example: Expand the Laurent's series about for $f(z)=\frac{6z+5}{z(z-2)(z+1)}$ in the region $1<|z+1|<3$

Solution:

Consider $\frac{6z+5}{z(z-2)(z+1)}=\frac{A}{z}+\frac{B}{z-2}+\frac{c}{z+1}$

$\Rightarrow 6z+5=A(z-2)(z+1)+Bz(z+1)+Cz(z-2)$

Put $z=0$, we get $5=A(-2)(1)$

$$\Rightarrow A = \frac{-5}{2}$$

Put $z = -1$we get $-11 = C(-1)(-3)$

$$\Rightarrow C = -\frac{11}{3}$$

Put $z = 2$we get $17 = B(2)(3)$

$$\Rightarrow B = \frac{17}{6}$$

$$\therefore f(z) = \frac{-5}{2z} + \frac{17}{6(z-2)} - \frac{11}{3(z+1)}$$

Given region $1 < |z+1| < 3$

Put $z + 1 = u \Rightarrow z = u - 1$

$(i.e)\ 1 < |u| < 3$

Now $f(z) = \frac{-5}{2(u-1)} + \frac{17}{6(u-3)} - \frac{11}{3u}$

$$= \frac{-5}{2u(1-\frac{1}{u})} + \frac{17}{6(-3)(1-\frac{u}{3})} - \frac{11}{3u}$$

$$= \frac{-5}{2u}\left[1-\frac{1}{u}\right]^{-1} - \frac{17}{18}\left[1-\frac{u}{3}\right]^{-1} - \frac{11}{3u}$$

$$= \frac{-5}{2u}\left[1+\frac{1}{u}+\left[\frac{1}{u}\right]^2+\cdots\right] - \frac{17}{18}\left[1+\frac{u}{3}+\left[\frac{u}{3}\right]^2+\cdots\right] - \frac{11}{3u}$$

$$= \frac{-5}{2(z+1)}\left[1+\frac{1}{(z+1)}+\left[\frac{1}{(z+1)}\right]^2+\cdots\right] - \frac{17}{18}\left[1+\frac{(z+1)}{3}+\left[\frac{(z+1)}{3}\right]^2+\cdots\right] - \frac{11}{3(z+1)}$$

$$= \frac{-5}{2(z+1)}\sum_{n=0}^{\infty}\left[\frac{1}{(z+1)}\right]^n - \frac{17}{18}\sum_{n=0}^{\infty}\left[\frac{(z+1)}{3}\right]^n - \frac{11}{3(z+1)}$$

SINGULARITIES

Zeros of an analytic function

If a function $f(z)$ is analytic in a region R, is zero at a point $z = z_0$ in R, then z_0 is called a zero of $f(z)$.

Simple zero

If $f(z_0) = 0$ and $f'(z_0) \neq 0$, then $z = z_0$ is called a simple zero of $f(z)$ or a zero of the first order.

Zero of order n

If $f(z_0) = f'(z_0) = \cdots = f^{n-1}(z_0) = 0$ and $f^n(z_0) \neq 0$, then z_0 is called zero of order.

Example: Find the zeros of $f(z) = \frac{z^2+1}{1-z^2}$

Solution:

The zeros of $f(z)$are given by $f(z) = 0$

$$(i.e.)f(z) = \frac{z^2+1}{1-z^2} = \frac{(z+i)(z-i)}{1-z^2} = 0$$

$$\Rightarrow (z+i)(z-i) = 0$$

$$\Rightarrow z = i \text{ and } -i \text{ are simple zero.}$$

Example: Find the zeros of $f(z) = \sin\frac{1}{z-a}$

Solution:

The zeros are given by $f(z) = 0$

$$(i.e.) \sin\frac{1}{z-a} = 0$$

$$\Rightarrow \frac{1}{z-a} = n\pi, n = \pm 1, \pm 2, \ldots$$

$$\Rightarrow (z-a)n\pi = 1$$

$\therefore$ The zeros are $z = a + \frac{1}{n\pi}, n = \pm 1, \pm 2, ,,$

Example: Find the zeros of $f(z) = \frac{\sin z - z}{z^3}$

Solution:

The zeros are given by $f(z) = 0$

$$(i.e.)\frac{\sin z - z}{z^3} = 0$$

$$\Rightarrow \frac{[z - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots]}{z^3} - z = 0$$

$$\Rightarrow \frac{-\frac{z^3}{3!}+\frac{z^5}{5!}}{z^3} \ldots . = 0$$

$$\Rightarrow -\frac{1}{3!} + \frac{z^2}{5!} \ldots = 0$$

But $\lim_{z \to 0} \frac{\sin z - z}{z^3} = -\frac{1}{3!} + 0$

$\therefore f(z)$ has no zeros.

Singular points

A point $z = z_0$ at which a function $f(z)$ fails to be analytic is called a singular point or singularity of $f(z)$.

Example: Consider $f(z) = \frac{1}{z-5}$

Here, $z = 5$, is a singular point of $f(z)$

Types of singularity

A point $z = z_0$ is said to be isolated singularity of $f(z)$ if

(i) $f(z)$ is not analytic at $z = z_0$

(ii) There exists a neighbourhood of $z = z_0$ containing no other singularity

Example: $f(z) = \frac{z}{z^2-1}$

This function is analytic everywhere except at $z = 1, -1$

$\therefore z = 1, -1$ are two isolated singular points.

When $z = z_0$ is an isolated singular point of $f(z)$, it can expand $f(z)$ as a Laurent's series about $z = z_0$

Thus

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} b_n (z - z_0)^{-n}$$

Note: If $z = z_0$ is an isolated singular point of a function $f(z)$, then the singularity is called

(i) a removable singularity (or)

(ii) a pole (or)

(iii) an essential singularity

According as the Laurent's series about $z = z_0$ of $f(z)$ has

(i) no negative powers (or)

(ii) a finite number of negative powers (or)

(iii) an infinite number of negative powers

Removable singularity

If the principal part of $f(z)$ in Laurent's series expansion contains no term $(i.e.)b_n = 0$ for all n, then the singularity $z = z_0$ is known as the removable singularity of $f(z)$

$$\therefore f(z) = \sum_{n=0}^{\infty} a_n(z-z_o)^n$$

(OR)

A singular point $z = z_0$ is called a removable singularity of $f(z)$, if $\lim_{z\to z_0} f(z)$ ezists finitely

Example: $f(z) = \frac{\sin z}{z}$

$$\frac{\sin z}{z} = \frac{1}{z}[z - \frac{z^3}{3!} + \frac{z^5}{5!} \dots]$$
$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!}$$

There is no negative powers of z.

$\therefore z = 0$ is a removable singularity of $f(z)$.

Poles

If we can find the positive integer n such that $\lim_{z\to z_0} (z - z_0)^n f(z) \neq 0$, then $z = z_0$ is called a pole of order n for $f(z)$.

(or)

If $\lim_{z\to z_0} f(z) = \infty, then\ z = z_0$ is a pole of $f(z)$

Simple pole

A pole of order one is called a simple pole.

Example: $f(z) = \frac{1}{(z-1)^2(z+2)}$

Here $z = 1$ is a pole of order 2

$z = 2$ is a pole of order 1.

Essential singularity

If the principal part of $f(z)$ in Laurent's series expansion contains an infinite number of non zero terms, then $z = z_0$ is known as an essential singularity.

Example: $f(z) = e^{1/z} = 1 + \frac{\frac{1}{z}}{1!} + \frac{(\frac{1}{z})^2}{2!} + \cdots$ has $z = 0$ as an essential singularity since, $f(z)$ is an infinite series of negative powers of z.

$f(z) = e^{\frac{1}{z}-4}$ has $z = 4$ an essential singularity

Note: The removable singularity and the poles are isolated singularities. But, the essential singularity is either an isolated or non-isolated singularity.

Entire function (or) Integral function

A function $f(z)$ which is analytic everywhere in the finite plane (except at infinity) is called an entire function or an integral function.

Example: $e^z, \sin z, \cos z$ are all entire functions.

Example: What is the nature of the singularity $z = 0$ of the function $f(z) = \frac{\sin z - z}{z^3}$

Solution:

Given $f(z) = \frac{\sin z - z}{z^3}$

The function $f(z)$ is not defined at $z = 0$

By L' Hospital's rule.

$$\lim_{z \to 0} \frac{\sin z - z}{z^3} = \lim_{z \to 0} \frac{\cos z - 1}{3z^2}$$

$$= \lim_{z \to 0} \frac{-\sin z}{6z}$$

$$= \lim_{z \to 0} -\frac{\cos z}{6z} = \frac{-1}{6}$$

Since, the limit exists and is finite, the singularity at $z = 0$ is a removable singularity.

Example: Classify the singularities for the function $f(z) = \frac{z - \sin z}{z}$

Solution:

Given $f(z) = \frac{z - \sin z}{z}$

The function $f(z)$ is not defined at $z = 0$

But by L' Hospital's rule.

$$\lim_{z \to 0} \frac{z - \sin z}{z} = \lim_{z \to 0} 1 - \cos z = 1 - 1 = 0$$

Since, the limit exists and is finite, the singularity at $z = 0$ is a removable singularity.

Example: Find the singularity of $f(z) = \frac{e^{1/z}}{(z-a)^2}$

Solution:

Given $f(z) = \frac{e^{1/z}}{(z-a)^2}$

Poles of $f(z)$ are obtained by equating the denominator to zero.

$$(i.e.)(z - a)^2 = 0$$

$\Rightarrow z = a$ is a pole of order 2.

Now, Zeros of $f(z)$

$$\lim_{z \to 0} \frac{e^{1/z}}{(z-a)^2} = \frac{\infty}{a^2} = \infty \neq 0$$

$\Rightarrow z = 0$ is a removable singularity.

$\therefore f(z)$ has no zeros.

Example: Find the kind of singularity of the function $f(z) = \frac{cot\pi z}{(z-a)^2}$

Solution:

Given $f(z) = \frac{cot\pi z}{(z-a)^2}$

$= \frac{\cos \pi z}{sin\pi z(z-a)^2}$

Singular points are poles, are given by

$$\Rightarrow sin\pi z(z-a)^2 = 0$$

$(i.e.) sin\pi z = 0, (z-a)^2 = 0$

$\pi z = n\pi$, where $n = 0, \pm 1, \pm 2, ...$

$(i.e.) z = n$

$z = a$ is a pole of order 2

Since $z = n, n = 0, \pm 1, \pm 2, ...$

$z = \infty$ is a limit of these poles.

$\therefore z = \infty$ is non- isolated singularity.

Example: Find the singular point of the function $f(z) = sinz \frac{1}{z-a}$. State nature of singularity.

Solution:

Given $f(z) = sinz \frac{1}{z-a}$

$z = a$ is the only singular point in the finite plane.

$$sinz \frac{1}{z-a} = \frac{1}{z-a} - \frac{1}{3!(z-a)^3} + \frac{1}{5!(z-a)^5} - \cdots$$

$z = a$ is an essential singularity

It is an isolated singularity.

Example: Identify the type of singularity of the function $f(z) = sin\left(\frac{1}{1-z}\right)$.

Solution:

$z = 1$ is the only singular point in the finite plane.

$z = 1$ is an essential singularity

It is an isolated singularity.

Example: Find the singular points of the function $f(z) = \left(\frac{1}{sin\frac{1}{z-a}}\right)$, state their nature.

Solution:

$f(z)$has an infinite number of poles which are given by

$$\frac{1}{z-a} = n\pi, n = \pm 1, \pm 2, \ldots$$

$(i.e.) z - a = \frac{1}{n\pi}\ ; z = a + \frac{1}{n\pi}$

But $z = a$ is also a singular point.

It is an essential singularity.

It is a limit point of the poles.

So, It is an non - isolated singularity.

Example: Classify the singularity of $f(z) = \frac{tanz}{z}$.

Solution:

Given $f(z) = \frac{tanz}{z}$

$$= \frac{z + \frac{z^3}{3} + \frac{2z^5}{15} + \ldots}{z}$$

$$= 1 + \frac{z^2}{3} + \frac{2z^4}{15} + \ldots$$

$$\lim_{z \to 0} \frac{tanz}{z} = 1 \neq 0$$

$\Rightarrow z = 0$is a removable singularity of $f(z)$.

RESIDUES

The residue of $f(z)$ at $z = z_0$ is the coefficient of $\frac{1}{z-z_0}$ in the Laurent series of $f(z)$ about $z = z_0$

Evaluation of Residues

(i) If $z = z_0$ is a pole of order one (simple pole) for $f(z)$, then

$$[Res\ f(z), z = z_0] = \lim_{z\to z_0} (z - z_0)\ f(z).$$

(ii) If $z = z_0$ is a pole of order n for $f(z)$, then

$$[Res\ f(z), z = z_0] = \lim_{z\to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n\ f(z)$$

Example: Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its pole.

Solution:

Given $f(z) = \frac{e^{2z}}{(z+1)^2}$ Here, $z = -1$ is a pole of order 2.

We know that,

$$[Res\ f(z), z = z_0] = \lim_{z\to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m\ f(z)$$

Here, $m = 2$

$$[Res\ f(z), z = -1] = \lim_{z\to -1} \frac{1}{1!} \frac{d}{dz} (z + 1)^2 \frac{e^{2z}}{(z+1)^2}$$

$$= \lim_{z\to -1} \frac{d}{dz} [e^{2z}] \quad = \lim_{z\to -1} 2[e^{2z}] = 2\ e^{-2}$$

Example: Find the residues at $z = 0$ of the function (i) $f(z) = e^{1/z}$ (ii) $f(z) = \frac{sinz}{z^4}$ (iii) $f(z) = zcos\frac{1}{z}$

Solution:

The residues are the coefficients of $\frac{1}{z}$ in the Laurent's expansions of $f(z)$ about $z = 0$

(i) $e^{1/z} = 1 + \frac{(\frac{1}{z})}{1!} + \frac{(\frac{1}{z})^2}{2!} + \cdots$

$$= 1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \frac{1}{3!}\left(\frac{1}{z}\right)^3 + \cdots$$

$[Res\ f(z), 0\ ] =$ coefficient of $\frac{1}{z}$ in Laurent's expansion.

$[Res\ f(z), 0\ ] = \frac{1}{1!} = 1$ by definition of residue.

(ii) $f(z) = \frac{sinz}{z^4} = \frac{1}{z^4}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right] = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{z^5}{5!} - \ \ldots$

$[Res\, f(z), 0\,]$ = coefficient of $\frac{1}{z}$ in Laurent's expansion.

$[Res\, f(z), 0\,] = -\frac{1}{3!} = -\frac{1}{6}$ by definition of residue.

(iii) $f(z) = z\cos\frac{1}{z} = z\,[1 - \frac{1}{2!}\frac{1}{z^2} + \frac{1}{4!}\frac{1}{z^4} - \cdots]$

$= z - \frac{1}{2!}\frac{1}{z} + \frac{1}{4!}\frac{1}{z^3} - \cdots$

$[Res\, f(z), 0\,]$ = coefficient of $\frac{1}{z}$ in Laurent's expansion.

$[Res\, f(z), 0\,] = -\frac{1}{2!} = -\frac{1}{2}$

Example: Find the residue of $z^2 \sin\left(\frac{1}{z}\right)$ at $z = 0$

Solution:

Let $f(z) = z^2 \sin\left(\frac{1}{z}\right) = z^2\left[\frac{\left(\frac{1}{z}\right)}{1!} - \frac{\left(\frac{1}{z}\right)^3}{3!} + \cdots\right] = \frac{z}{1!} - \frac{1}{6z} + \cdots$

$[Res\, f(z), 0\,]$ = coefficient of $\frac{1}{z}$ in Laurent's expansion.

$= -\frac{1}{6}$

Example: Find the residue of the function $f(z) = \frac{4}{z^3(z-2)}$ at a simple pole.

Solution:

Here, $z = 2$ is a simple pole.

$$[Res\, f(z), z = 2] = \lim_{z\to 2}(z-2)\frac{4}{z^3(z-2)}$$

$$= \lim_{z\to 2}\frac{4}{z^3} = \frac{4}{8} = \frac{1}{2}$$

Example: Find the residue of $\frac{1-e^{-z}}{z^3}$ at $z = 0$

Solution:

$$\text{Given } f(z) = \frac{1-e^{-z}}{z^3} = \frac{1-\left[1-\frac{z}{1!}+\frac{(z)^2}{2!}-\frac{(z)^3}{3!}+\frac{(z)^4}{4!}-\ldots\right]}{z^3}$$

$$= \frac{\left[1-\frac{z}{2!}+\frac{z^2}{3!}-\frac{z^3}{4!}+\ldots\right]}{z^2}$$

Here, $z = 0$ is a pole of order 2.

$$[Res\, f(z), z = 0] = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}[(z)^2 f(z)]$$

$$= \lim_{z\to 0}\frac{d}{dz}\left[\left[1 - \frac{z}{2!} + \frac{z^2}{3!} - \frac{z^3}{4!} + \ldots\right]\right]$$

$$= \lim_{z\to 0}\left[\frac{-1}{2!} + \frac{2z}{3!} - \frac{3z^2}{4!} + \ldots\right]$$

$$= \frac{-1}{2!} = -\frac{1}{2}$$

CAUCHY RESIDUE THEOREM

Statement:

If $f(z)$ is analytic inside and on a simple closed curve C, except at a finite number of singular points $a_1, a_2, \ldots a_n$ inside C, then

$$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues\ of\ f(z) at\ a_1, a_2, \ldots a_n]$$

Note: Formulae for evaluation of residues

(i) If $z = a$ is a simple pole of $f(z)$ then

$$[Res\ f(z), z = a] = \lim_{z \to a} (z - a) f(z)$$

(ii) If $z = a$ is a pole of order n of $f(z)$, then

$$[[Res\ f(z)], z = a] = \frac{1}{(n-1)!} \lim_{z \to a} \{\frac{d^{n-1}}{dz^{n-1}}[(z - a)^n f(z)]\}$$

Example: Evaluate using Cauchy's residue theorem, $\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz$, where C is $|z| = 3$

Solution:

Let $f(z) = \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}$

The poles are given by $(z - 1)(z - 2) = 0$

$\Rightarrow z = 1, 2$ are poles of order 1.

Given C is $|z| = 3$

$\therefore$ Clearly $z = 1$and $z = 2$ lies inside $|z| = 3$

To find the residues:

(i) When $z = 1$

$$[Res\ f(z)]_{z=1} = \lim_{z \to 1} (z - 1) f(z)$$

$$= \lim_{z \to 1} (z - 1)\frac{\cos\pi z^2 + \sin\pi z^2}{(z-1)(z-2)}$$

$$= \lim_{z \to 1} \frac{\cos\pi z^2 + \sin\pi z^2}{(z-2)}$$

$$= \frac{\cos\pi + \sin\pi}{-1}$$

$$= \frac{-1+0}{-1} = 1$$

(ii) When $z = 2$

$$[Res\ f(z)]_{z=2} = \lim_{z \to 2} (z - 2) f(z)$$

$$= \lim_{z\to 2}(z-2)\frac{\cos\pi z^2+\sin\pi z^2}{(z-1)(z-2)}$$

$$= \lim_{z\to 2}\frac{\cos\pi z^2+\sin\pi z^2}{(z-1)}$$

$$= \frac{\cos 4\pi+\sin 4\pi}{1}$$

$$= \frac{1+0}{1} = 1$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\ (sum\ of\ residues)$$

$$= 2\pi i\ (1+1) = 4\pi i$$

Example: Evaluate $\int_c \frac{z^2}{z^2+1}\,dz$ where C is $|z| = 2$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{z^2}{z^2+1}$

The poles are given by $z^2 + 1 = 0$

$\Rightarrow z = \pm i$ are poles of order 1.

Given C is $|z| = 2$

∴ Clearly $z = i, -i$ lies inside $|z| = 2$

To find the residue:

(i) When $z = i$

$$[Res\ f(z)]_{z=i} = \lim_{z\to i}(z-i)\frac{z^2}{(z+i)(z-i)}$$

$$= \lim_{z\to i}\frac{z^2}{(z+i)} = \frac{-1}{2i}$$

(ii) When $z = -i$

$$[Res\ f(z)]_{z=-i} = \lim_{z\to -i}(z+i)\frac{z^2}{(z+i)(z-i)}$$

$$= \lim_{z\to -i}\frac{z^2}{(z-i)}$$

$$= \frac{-1}{-2i} = \frac{1}{2i}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\ (sum\ of\ residues)$$

$$= 2\pi i\left(\frac{-1}{2i}+\frac{1}{2i}\right) = 0$$

$$\therefore \int_c \frac{z^2}{z^2+1}\,dz = 0$$

Example: Evaluate $\int_c \frac{(z-1)}{(z+1)^2(z-2)} dz$ where C is the circle $|z-i| = 2$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{(z-1)}{(z+1)^2(z-2)}$

The poles are given by $(z+1)^2(z-2) = 0$

$\Rightarrow z+1 = 0; z-2 = 0$

$\Rightarrow z = -1$ *is a pole of order* 2 *and*

$\Rightarrow z = 2$ is a pole of order 1.

Given C is $|z-i| = 2$

When $z = -1, |z-i| = |-1-i| = \sqrt{2} < 2$

$\therefore z = -1$ lies inside C

When $z = 2, |z-i| = |2-i| = \sqrt{5} > 2$

$\therefore z = -1$ lies inside C

To find the residue for the inside pole:

$$[Res\, f(z)]_{z=-1} = \lim_{z\to -1} \frac{d}{dz}[(z+1)^2 f(z)]$$

$$= \lim_{z\to -1} \frac{d}{dz}[(z+1)^2 \frac{(z-1)}{(z+1)^2(z-2)}]$$

$$= \lim_{z\to -1} \frac{d}{dz}(\frac{z-1}{z-2})$$

$$= \lim_{z\to -1} [\frac{(z-2)(1)-(z-1)(1)}{(z-2)}] = -\frac{1}{9}$$

$\therefore$ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\,(sum\ of\ residues)$$

$$= 2\pi i\,(-\frac{1}{9})$$

$$\therefore \int_c \frac{(z-1)}{(z+1)^2(z-2)} dz = -\,2\pi i\,(\frac{1}{9})$$

Example: Evaluate $\int_c \frac{dz}{(z^2+4)^2}$ where C is the circle $|z-i| = 2$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{1}{(z^2+4)^2}$

The poles are given by $(z^2+4)^2$

$\Rightarrow z^2 + 4 = 0$

$\Rightarrow z = \pm 2i$ are poles of order 2

Given C is $|z - i| = 2$

When $z = 2i, |z - i| = |2i - i| = 1 < 2$

$\therefore z = 2i$ lies inside C

When $z = -2i, |z - i| = |-2i - i| = 3 > 2$

$\therefore z = -2i$ lies outside C

To find the residue for the inside pole

$$[Res\, f(z)]_{z=2i} = \lim_{z\to 2i} \frac{d}{dz}[(z-2i)^2 f(z)]$$

$$= \lim_{z\to 2i} \frac{d}{dz}[(z-2i)^2 \frac{1}{(z-2i)^2((z+2i)^2)}]$$

$$= \lim_{z\to 2i} \frac{d}{dz}(\frac{1}{z+2i})^2$$

$$= \lim_{z\to 2i} [\frac{-2)}{(z+2i)^3)}]$$

$$= -\frac{2}{(4i)^3} = -\frac{2}{-64i} = \frac{1}{32i}$$

$\therefore$ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\,(sum\ of\ residues)$$

$$= 2\pi i\,(\frac{1}{32i})$$

$$\therefore \frac{dz}{(z^2+4)^2} = \frac{\pi}{16}$$

Example: Evaluate $\int_c \frac{e^z dz}{(z^2+\pi^2)^2}$ where C is the circle $|z| = 4$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{e^z}{(z^2+\pi^2)^2}$

The poles are given by $(z^2 + \pi^2)^2 = 0$

$\Rightarrow z^2 + \pi^2 = 0$

$\Rightarrow z = \pm \pi i$ are poles of order 2

Given C is $|z| = 4$

Clearly $z = +\pi i, z = \pi i$ lies inside $|z| = 4$

To find the residue

(i) When $z = +\pi i$

$$[Res\, f(z)]_{z=\pi i} = \lim_{z\to\pi i} \frac{d}{dz}[(z-\pi i)^2 f(z)]$$

$$= \lim_{z\to\pi i} \frac{d}{dz}[(z-\pi i)^2 \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}]$$

$$= \lim_{z\to\pi i} \frac{d}{dz}(\frac{e^z}{(z+\pi i)^2})$$

$$= \lim_{z\to\pi i} [\frac{(z+\pi i)^2 e^z - 2(z+\pi i)e^z)}{(z+\pi i)^4)}]$$

$$= \lim_{z\to\pi i} [\frac{(z+\pi i)e^z[z+\pi i-2]}{(z+\pi i)^4}]$$

$$= \frac{e^{\pi i}(2\pi i-2)}{(2\pi i)^3}$$

$$= \frac{e^{\pi i}(\pi i-1)}{-4\pi^3 i}$$

$$= \frac{(cos\pi + isin\pi)(1-\pi i)}{4\pi^3 i}$$

$$= \frac{(-1+0)(1-\pi i)}{4\pi^3 i}$$

$$= \frac{(\pi i-1)}{4\pi^3 i}$$

(ii) When $z = -\pi i$

$$[Res\, f(z)]_{z=-\pi i} = \lim_{z\to-\pi i} \frac{d}{dz}[(z+\pi i)^2 f(z)]$$

$$= \lim_{z\to-\pi i} \frac{d}{dz}[(z+\pi i)^2 \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}]$$

$$= \lim_{z\to-\pi i} \frac{d}{dz}(\frac{e^z}{(z-\pi i)^2})$$

$$= \lim_{z\to-\pi i} [\frac{(z-\pi i)^2 e^z - 2(z-\pi i)e^z)}{(z-\pi i)^4}]$$

$$= \lim_{z\to-\pi i} [\frac{(z-\pi i)e^z[z-\pi i-2]}{(z-\pi i)^4}]$$

$$= \frac{e^{-\pi i}(-2\pi i-2)}{(-2\pi i)^3}$$

$$= \frac{(-2)(cos\pi - isin\pi)(\pi i+1)}{8\pi^3 i}$$

$$= \frac{-(-1-0)(\pi i+1)}{4\pi^3 i}$$

$$= \frac{(1+\pi i)}{4\pi^3 i}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\,(sum\ of\ residues)$$

$$= 2\pi i\,[\frac{(\pi i-1)}{4\pi^3 i} + \frac{(\pi i+1)}{4\pi^3 i}]$$

$$= \frac{2\pi i}{4\pi^3 i}[2\pi i\,] = \frac{i}{\pi}$$

$$\therefore \int_c \frac{e^z dz}{(z^2+\pi^2)^2} = \frac{i}{\pi}$$

Example: Evaluate $\int_c \frac{dz}{zsinz}$ where C is the circle $|z| = 1$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{1}{zsinz}$

The poles are given by $zsinz = 0$

$$\Rightarrow z\,[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots] = 0$$

$$\Rightarrow z^2\,[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots] = 0$$

$\Rightarrow z = 0$ is a pole of order 2

Given C is $|z| = 1$

$\therefore z = 0$ lies inside C

To find the residue for the inside pole

$$[Res\, f(z)]_{z=0} = \lim_{z\to 0} \frac{d}{dz}[(z-0)^2 f(z)]$$

$$= \lim_{z\to 0} \frac{d}{dz}[(z)^2 \frac{1}{zsinz}]$$

$$= \lim_{z\to 0} \frac{d}{dz}(\frac{z}{sinz})$$

$$= \lim_{z\to 0} [\frac{sinz(1) - z(cosz)}{(sinz)^2}]$$

$$= \frac{0-0}{0} = [\frac{0}{0}]\text{form}$$

$$= \lim_{z\to 0} \frac{cosz - [z\,(-sinz) + cosz(1)]}{2sinzcosz} \text{ (by L' Hospital rule)}$$

$$= \lim_{z\to 0} \frac{cosz + zsinz - cosz}{2sinzcosz}$$

$$= \lim_{z\to 0} \frac{zsinz}{2sinzcosz}$$

$$= \lim_{z\to 0} \frac{z}{2cosz}$$

$$= \frac{0}{2} = 0$$

$\therefore$ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\,(sum\ of\ residues)$$

$$= 2\pi i\,[0]$$

$$\therefore \int_c \frac{dz}{zsinz} = 0$$

Example: Evaluate $\int_c \frac{dz}{z^2 sinhz}$ where C is the circle $|z-1| = 2$ using Cauchy's residue theorem.

Solution:

Let $f(z) = \frac{1}{z^2 sinhz}$

The poles are given by $z^2 sinhz = 0$

$\Rightarrow z^2 = 0\ (or) sinhz = 0$

$\Rightarrow z = 0\ or)\ z = sinh^{-1}(0) = 0$is a pole of order 1.

Given C is $|z-1| = 2$

∴Clearly $z = 0$ lies inside C.

To find residue for the inside pole at $z = 0$

$$f(z) = \frac{1}{z^2 sinhz}$$

$$= \frac{1}{z^2[z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots]}$$

$$= \frac{1}{z^3[1+\frac{z^2}{6}+\frac{z^4}{120}+\cdots]}$$

$$= \frac{1}{z^3}[1+u]^{-1} \quad \text{where } u = 1+\frac{z^2}{6}+\cdots$$

$$= \frac{1}{z^3}[1-u+u^2-u^3 \ldots]$$

$$= \frac{1}{z^3}[1-(\frac{z^2}{6}+\frac{z^4}{120}+\cdots)+(\frac{z^2}{6}+\frac{z^4}{120}+\cdots)^2 \ldots]$$

$$= \frac{1}{z^3} - \frac{1}{6z} - \frac{z}{120} + \cdots$$

$[Res\, f(z)]_{z=0}$ = Coefficient of $\frac{1}{z}$ in the Laurent's expansion of $f(z)$

$$\therefore [Res\, f(z)]_{z=0} = -\frac{1}{6}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\ (sum\ of\ residues)$$

$$= 2\pi i\ [-\frac{1}{6}]$$

$$\therefore \int_c \frac{dz}{z^2 sinhz} = \frac{-\pi i}{3}$$

Example: Evaluate $\int_c \frac{z}{cosz} dz$ where C is the circle $|z - \frac{\pi}{2}| = \frac{\pi}{2}$

Solution:

Let $f(z) = \frac{z}{cosz}$

The poles are given by $cosz = 0$

$\Rightarrow z = (2n+1)\frac{\pi}{2}, n = 0, \pm 1, \pm 2, \ldots$ are poles of order 1

Given C is $|z - \frac{\pi}{2}| = \frac{\pi}{2}$

Here $z = \frac{\pi}{2}$ lies inside the circle and others lies outside.

$$[Res\, f(z)]_{z=\frac{\pi}{2}} = \lim_{z\to\frac{\pi}{2}} (z - \tfrac{\pi}{2})\, f(z)$$

$$[Res\, f(z)]_{z=\frac{\pi}{2}} = \lim_{z\to\frac{\pi}{2}} (z - \tfrac{\pi}{2})\, \frac{z}{cosz}$$

$$= \frac{0}{0} \text{ (form)}$$

Using L ‘ Hospital’s rule

$$= \lim_{z\to\frac{\pi}{2}} \frac{(z-\frac{\pi}{2})(1)+z(1)}{-sinz}$$

$$= \lim_{z\to\frac{\pi}{2}} \frac{(z-\frac{\pi}{2})+z}{-sinz}$$

$$= -\frac{\pi}{2}$$

∴ By Cauchy’s Residue theorem

$$\int_c f(z)dz = 2\pi i\,(sum\ of\ residues)$$

$$= 2\pi i\,[-\tfrac{\pi}{2}]$$

$$\therefore \int_c \frac{z}{cosz}\, dz = -\pi^2 i$$

Example: Evaluate $\int_c z^2 e^{1/z}\, dz$ where C is the unit circle using Cauchy’s residue theorem.

Solution:

Let $f(z) = z^2 e^{1/z}$

Here $z = 0$ is the only singular point.

Given C is $|z| = 1$

∴Clearly $z = 0$ lies inside C.

To find residue of $f(z)$ at $z = 0$

We find the Laurent’s series of $f(z)$ $about\ z = 0$

$$\Rightarrow f(z) = z^2 e^{1/z}$$

$$\Rightarrow z^2\left[1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots\right]$$

$[Res\, f(z)]_{z=0}$ = C0efficient of $\frac{1}{z}$ in the Laurent’s expansion of $f(z)$

$$\therefore [Res\, f(z)]_{z=0} = \frac{1}{6}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\ (sum\ of\ residues)$$

$$= 2\pi i\ \left[\frac{1}{6}\right]$$

$$\therefore \int_c z^2 e^{1/z}\, dz = \frac{\pi i}{3}$$

Contour Integration

Evaluation of Real Integrals

The evaluation of certain types of real definite integrals of complex functions over suitable closed paths or contours and applying Cauchy's Residue theorem is known as Contour Integration.

Type 1: Integration round the unit circle

Integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ where f is a rational function in $cos\theta$ and $\sin\theta$

To evaluate this type of integrals

We take the unit circle $|z| = 1$ as the contour C.

On $|z| = 1, let\ z = e^{i\theta}$

$$\Rightarrow \frac{dz}{d\theta} = ie^{i\theta} = iz$$

$$\therefore d\theta = \frac{dz}{iz}$$

Also, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$

and, $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z^2-1}{2iz}$

$|z| = 1 \Rightarrow \theta$ varies from $0\ to\ 2\pi$

$$\therefore \int_0^{2\pi} f\ (\cos\theta, \sin\theta)d\theta = \int_c f\ \left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz}$$

Example: Evaluate $\int_o^{2\pi} \frac{d\theta}{5+4\sin\theta}$ using Contour integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz}\ and \sin\theta = \frac{z^2-1}{2iz}$$

$$\therefore \int_o^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_c \frac{\frac{dz}{iz}}{5+4\left(\frac{z^2-1}{2iz}\right)}\ where\ c\ is\ |z| = 1$$

$$= \int_c \frac{\frac{dz}{iz}}{\frac{5iz+2z^2-2}{iz}}$$

$$= \int_c \frac{dz}{2z^2+5iz-2}$$

$$= \int_c f(z)\ dz \quad \ldots(1)$$

Where, $f(z) = \frac{1}{2z^2+5iz-2}$

To Evaluate, $\int_c f(z)\,dz$

To find poles of $f(z)$, put $2z^2 + 5iz - 2 = 0$

$$z = \frac{-5i \pm \sqrt{-25+16}}{4} = \frac{-5i \pm 3i}{4}$$

$$z = -\frac{i}{2}, -2i \text{ are poles of order one}$$

Given $C\ is\ |z| = 1$

Consider $z = -\frac{i}{2}$

$\Rightarrow |z| = |\frac{-i}{2}| = \frac{1}{2} < 1$

$\therefore z = -\frac{i}{2}$ lies inside C

Consider $z = -2i$

$\Rightarrow |z| = |-2i| = 2 > 1$

$\therefore z = -2i$ lies outside C.

Find the residue for inside pole $z = -\frac{i}{2}$

$$[Res\ f(z)]_{z=-\frac{i}{2}} = \lim_{z\to -\frac{i}{2}} (z + \frac{i}{2}) f(z)$$

$$= \lim_{z\to -\frac{i}{2}} (z + \frac{i}{2}) \frac{1}{2(z+\frac{i}{2})(z+2i)}$$

$$= \frac{1}{2(-\frac{i}{2}+2i)} = \frac{1}{3i}$$

$\therefore$ By Cauchy's residue theorem

$$\int_c f(z)dz = 2\pi i \quad [Sum\ of\ residues]$$

$$= 2\pi i\left(\frac{1}{3i}\right) = \frac{2\pi}{3}$$

$$(1) \Longrightarrow \int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}$$

Example: Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ using Contour Integration.

Solution:

Replacement Let $z = e^{i\theta}$

$\Rightarrow d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{z^2-1}{2iz}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = \int_c \frac{dz/iz}{13+5\left(\frac{z^2-1}{2iz}\right)} \quad where\ C is\ |z| = 1$$

$$= \int_c \frac{dz/iz}{\frac{26iz+5z^2-5}{2iz}}$$

$$= 2\int_c \frac{dz}{5z^2+26iz-5}$$

$$= 2\int_c f(z)dz \quad \dots (1)$$

Where, $f(z) = \frac{1}{5z^2+26iz-5}$

To evaluate $\int_c f(z)dz$

To find poles of $f(z)$, put $5z^2 + 26iz - 5 = 0$

$$z = \frac{-26i\pm\sqrt{-676+100}}{10} = \frac{-26i\pm 24i}{10}$$

$$\Rightarrow z = -\frac{i}{5}, -5i \text{ are poles of order one.}$$

Given C is $|z| = 1$

Consider $z = -\frac{i}{5}$

$$\Rightarrow |z| = |-\frac{i}{5}| = \frac{1}{5} < 1$$

$\therefore z = -\frac{i}{5}$ lies inside C

Consider $z = -5i$

$$\Rightarrow |z| = |-5i| = 5 > 1$$

$\therefore z = -5i$ lies outside C.

Find the residue for inside pole $z = -\frac{i}{5}$

$$[Res\, f(z)]_{z=-\frac{i}{5}} = \lim_{z\to-\frac{i}{5}} (z + \frac{i}{5}) f(z)$$

$$= \lim_{z\to-\frac{i}{5}} (z + \frac{i}{5}) \frac{1}{5z^2+26iz-5}$$

$$= \lim_{z\to-\frac{i}{5}} (z + \frac{i}{5}) \frac{1}{(5z+i)(z+5i)}$$

$$= \lim_{z\to-\frac{i}{5}} (z + \frac{i}{5}) \frac{1}{(5(z+\frac{i}{5})(z+5i)}$$

$$= \frac{1}{5(-\frac{i}{5}+5i)} = \frac{1}{24i}$$

$\therefore$ By Cauchy's residue theorem

$$\int_c f(z)dz = 2\pi i[Sum\ of\ residues]$$

$$= 2\pi i\,(\frac{1}{24i}) = \frac{\pi}{12}$$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = 2\,(\frac{\pi}{12}) = \frac{\pi}{6}$$

Example: Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}, a > b > 0$ byusing contour integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz} \text{ and } \cos\theta = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_c \frac{dz/iz}{a+b\left(\frac{z^2+1}{2z}\right)} \text{ where } c \text{ is } |z| = 1$$

$$= \int_c \frac{dz/iz}{\frac{2az+bz^2+b}{2z}}$$

$$= \frac{2}{i}\int_c \frac{dz}{bz^2+2az+b}$$

$$= \frac{2}{i}\int_c f(z)dz \quad \ldots (1)$$

Where, $f(z) = \frac{1}{bz^2+2az+b}$

To evaluate $\int_c f(z)dz$

To find poles of $f(z)$, put $bz^2 + 2az + b$

$$z = \frac{-2a \pm \sqrt{4(a^2-b^2)}}{2b} = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$$z = \frac{-a+\sqrt{a^2-b^2}}{b}, \frac{-a+\sqrt{a^2-b^2}}{b} \text{ are poles of order one.}$$

Clearly, $z = \frac{-a+\sqrt{a^2-b^2}}{b} = \alpha$ lies inside c

$and\ z = \frac{-a-\sqrt{a^2-b^2}}{b} = \beta$ lies outside c

Since a > b, we can write $bz^2 + 2az + b = b(z-\alpha)(z-\beta)$

Find the residue for inside pole $z = \alpha$

$$Res\ f(z)]_{z=\alpha} = \lim_{z\to\alpha}(z-\alpha)\frac{1}{b(z-\alpha)(z-\beta)}$$

$$= \frac{1}{b(\alpha-\beta)}$$

$$= \frac{1}{2\sqrt{a^2-b^2}}$$

∴ By Cauchy residue theorem

$$\int_c f(z)dz = 2\pi i\ [sum\ of\ residues]$$

$$= 2\pi i\left[\frac{1}{2\sqrt{a^2-b^2}}\right]$$

$$= \frac{\pi i}{\sqrt{a^2-b^2}}$$

$$(1) \Rightarrow \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2}{i}\left[\frac{\pi i}{\sqrt{a^2-b^2}}\right]$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

Example: Evaluate $\int_0^{2\pi} \frac{\cos 3\theta\, d\theta}{5-4\cos\theta}$ using contour integration.

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz} \text{ and } \cos\theta = \frac{z^2+1}{2z}$$

$\cos 3\theta = \text{Real part of } e^{i3\theta} = R.P\,(z^3)$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta\, d\theta}{5-4\cos\theta} = \int_0^{2\pi} \frac{R.P\,\frac{(z^3)dz}{iz}}{5-4\left(\frac{z^2+1}{2z}\right)} \text{ where } C \text{ is } |z| = 1$$

$$= R.P \int_c \frac{z^3 dz/iz}{\frac{5z-(2z^2+2)}{z}}$$

$$= R.P\,\left(-\frac{1}{i}\right) \int_c f(z)\,dz \qquad \ldots(1)$$

Where, $f(z) = \frac{z^3}{2z^2-5z+2}$

To evaluate $\int_c f(z)dz$

To find poles of $f(z)$, put $2z^2 - 5z + 2 = 0$

$$z = \frac{5+\sqrt{25-16}}{4} = \frac{5\pm 3}{4}$$

$\Rightarrow z = 2, \frac{1}{2}$ are poles of order one.

Given C is $|z| = 1$

Consider $z = \frac{1}{2}$

$$\Rightarrow |z| = \left|\frac{1}{2}\right| = \frac{1}{2} < 1$$

$\therefore z = \frac{1}{2}$ lies inside C

Consider $z = 2$

$$\Rightarrow |z| = |2| = 2 > 1$$

$\therefore z = 2$ lies outside C

Find the residue for inside pole $z = \frac{1}{2}$

$$[Res\, f(z)]_{z=\frac{1}{2}} = \lim_{z\to\frac{1}{2}} \left(z - \frac{1}{2}\right) f(z)$$

$$= \lim_{z\to\frac{1}{2}} \left(z - \frac{1}{2}\right) f(z)$$

$$= \lim_{z\to\frac{1}{2}} \left(z - \frac{1}{2}\right) \frac{z^3}{(2z-1)(z-2)}$$

$$= \lim_{z\to\frac{1}{2}} \left(z - \frac{1}{2}\right) \frac{z^3}{2\left(z-\frac{1}{2}\right)(z-2)}$$

$$= \frac{\left(\frac{1}{2}\right)^3}{2\left(\frac{1}{2}-2\right)} = -\frac{1}{24}$$

∴ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$= 2\pi i\left(-\frac{1}{24}\right) = -\frac{\pi i}{12}$$

$$(1) \Rightarrow \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}d\theta = R.P\ \left(-\frac{1}{i}\right)\left(-\frac{\pi i}{12}\right) = \frac{\pi}{12}$$

Example: Evaluate $\int_0^{2\pi} \frac{\sin^2\theta d\theta}{5-3\cos\theta} = \int_0^{2\pi} \frac{\cos 2\theta d\theta}{10-6\cos\theta}$

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz}\ and \cos\theta = \frac{z^2+1}{2z}$$

$$\cos 2\theta = Real\ part\ of\ e^{i2\theta} = R.P\ (Z^2)$$

$$\therefore \int_0^{2\pi} \frac{\sin^2\theta}{5-3\cos\theta}d\theta = \int_0 \frac{1-R.P\frac{(z^2)dz}{iz}}{10-6\left(\frac{z^2+1}{2z}\right)}$$

$where\ C\ is\ |z| = 1$

$$= R.P\int_C \frac{(1-z^2)dz/iz}{\frac{10z-3z^2-3}{z}}$$

$$= R.P\ \left(-\frac{1}{i}\right)\int_C \frac{(1-z^2)dz}{3z^2-10Z+3}$$

$$= R.P\ \left(-\frac{1}{i}\right)\int_C f(z)dz \qquad \text{.... (1)}$$

Where, $f(z) = \frac{1-z^2}{3z^2-10z+3}$

To evaluate $\int_c f(z)dz$

To find poles of $f(z), put\ 3z^2 - 10z + 3 = 0$

$$z = \frac{10\pm\sqrt{100-36}}{6} = \frac{10\pm 8}{6}$$

$\therefore z = 3, \frac{1}{3}$ are poles of order one.

Given $C\ is\ |z| = 1$

Consider $z = \frac{1}{3}$

$$\Rightarrow |z| = \left|\frac{1}{3}\right| = \frac{1}{3} < 1$$

$\therefore z = \frac{1}{3}$ lies inside C

Consider $z = 3$

$$\Rightarrow |z| = |3| = 3 < 1$$

$\therefore z = 3$ lies outside C

Find the residue for inside pole $z = \frac{1}{3}$

$$[Res\, f(z)]_{z=\frac{1}{3}} = \lim_{z\to\frac{1}{3}} (z - \tfrac{1}{3}) f(z)$$

$$= \lim_{z\to\frac{1}{3}} (z - \tfrac{1}{3}) \frac{(1-Z^2)}{3(Z-\frac{1}{3})(Z-3)}$$

$$= \frac{1-(\frac{1}{3})^2}{3\,(\frac{1}{3}-3)} = -\frac{1}{9}$$

∴ By Cauchy's Residue theorem

$$\int_C f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$= 2\pi i\,(-\tfrac{1}{9})$$

$$(1) \Rightarrow \int_0^{2\pi} \frac{sin^2\theta\, d\theta}{5-3\cos\theta} = R.P\,(-\tfrac{1}{i})\,(-\tfrac{2\pi i}{9}) = \frac{2\pi}{9}$$

Example: Using Contour Integration, evaluate the real integral $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta$

Solution:

Replacement Let $z = e^{i\theta}$

$$\Rightarrow d\theta = \frac{dz}{iz}\ and\ \cos\theta = \frac{Z^2+1}{2Z}$$

Now, $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta$

$$[\because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx, if\ f(2a-x) = f(x)]$$

$$\therefore \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = \frac{1}{2}\int_C \frac{1+[1+2\,(\frac{z^2+1)}{2z})]\frac{dz}{iz}}{5+4\,(\frac{z^2+1)}{2z})}$$

$$= \frac{1}{2i}\int_C \frac{(z^2+z+1)}{z(2z^2+5z+2}$$

$$= \frac{1}{2i}\int_C f(z)dz \qquad \ldots (1)$$

Where, $f(z) = \frac{z^2+z+1}{Z(2z^2+5z+2)}$

To evaluate $\int_C f(z)dz$

To find poles of $f(z), put\ z(2z^2 + 5z + 2) = 0$

$$\Rightarrow z = 0;\ 2z^2 + 5z + 2 = 0$$

$\Rightarrow z = 0; z = -2, z = -\frac{1}{2}$ are poles of order one.

Given $C\ is\ |z| = 1$

Consider $z = 0$

$$\Rightarrow |z| = |0| = 0 < 1$$

$\therefore z = 0$ lies inside C

Consider $z = -2$

$$\Rightarrow |z| = |-2| = 2 > 1$$

$\therefore z = -2$ lies outside C

Consider $z = -\frac{1}{2}$

$$\Rightarrow |z| = |-\frac{1}{2}| = \frac{1}{2} < 1$$

$\therefore z = -\frac{1}{2}$ lies inside C

Find the residue for the inside pole

(i) When $z = 0$

$$[Res\, f(z)]_{z=0} = \underset{z\to 0}{\text{Lt}} (z-0) f(z)$$

$$\underset{z\to 0}{\text{Lt}}\, z \frac{(z^2+z+1)}{Z(2Z^2+5z+2)} = \frac{1}{2}$$

(ii) When $z = -\frac{1}{2}$

$$[Res\, f(z)_{z=-\frac{1}{2}} = \lim_{z\to -\frac{1}{2}} (z+\frac{1}{2}) f(z)$$

$$= \lim_{z\to -\frac{1}{2}} (z+\frac{1}{2}) \frac{z^2+z+1}{Z(2z+1)(z+2)}$$

$$= \lim_{z\to -\frac{1}{2}} (z+\frac{1}{2}) \frac{z^2+z+1}{Z\, 2(z+\frac{1}{2})(z+2)}$$

$$= \frac{\frac{1}{4}-\frac{1}{2}+1}{2(-\frac{1}{2})(-\frac{1}{2}+2)}$$

$$= \frac{\frac{3}{4}}{-\frac{3}{2}} = -\frac{1}{2}$$

$\therefore$ By Cauchy's Residue Theorem

$$\int_C f(z)dz = 2\pi i\, [Sum\ of\ residues]$$

$$= 2\pi i\, [\frac{1}{2} - \frac{1}{2}] = 0$$

$$(1) \Rightarrow \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = \frac{1}{2i}[0] = 0$$

Type II: Integration around semi – circular contour

Integrals of the form $\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx,$

where $f(x)$ and $g(x)$ are polynomials in x, such that the degree of $f(x)$ is less than that of $g(x)$ atleast by two and $g(x)$ does not vanish for any value of x.

Let C be a closed contour of real axis from $-R$ to R and semicircle $'S'$ of radius R above real axis.

Thus,

$$\int_C \frac{f(z)}{g(z)} = \int_{-R}^{R} \frac{f(x)}{g(x)} dx + \int_S \frac{f(z)}{g(z)} dz$$

As $R \to \infty, \int_C \frac{f(z)}{g(z)} dz \to 0$ by Cauchy's lemma

$$= \int_C \frac{f(z)}{g(z)} dz = \int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx$$

Example: Evaluate $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)}$ where $a > b > 0$

Solution:

Replacement $put\ x = z \Rightarrow dx = dz$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \int_C \frac{z^2 dz)}{(z^2+a^2)(z^2+b^2)} \quad where$$

Where C is the upper semi circle

$$= \int_C f(z) dz \qquad \dots (1)$$

Where, $f(z) = \frac{z^2}{(z^2+a^2)(z^2+b^2)}$

To find the poles, put $(z^2 + a^2)(z^2 + b^2) = 0$

$\Rightarrow z = \pm ai, z = \pm bi$, are poles of order one.

Here $z = ai, bi$ lies in upper, half of the z –plane.

Find the residue for the inside pole

(i) When $z = ai$

$$[Res\ f(z)]_{z \to ai} = \lim_{z \to ai} (z - ai) \frac{z^2}{(z+ai)(z-ai)(z^2+b^2)}$$

$$= \frac{-a^2}{2ai\ (b^2-a^2)}$$

$$= \frac{a}{2i\ (a^2-b^2)}$$

(ii) When $z = bi$

$$[Res\ f(z)]_{z=bi} = \lim_{z \to bi} (z - bi) f(z)$$

$$= \lim_{z \to bi} (z - bi) \frac{z^2}{(z^2+a^2)(z+bi)(z-bi)}$$

$$= -\frac{b^2}{(a^2-b^2)2bi}$$

$$= -\frac{b}{2i\ (a^2-b^2)}$$

$\therefore$ By Cauchy's Residue theorem

$$\int_c f(z) dz = 2\pi i\ [sum\ of\ residues]$$

$$= 2\pi i\ \left[\frac{a}{2i(a^2-b^2)} - \frac{b}{2i\ (a^2-b^2)}\right]$$

$$= \frac{2\pi i}{2i}\left[\frac{a-b}{(a-b)(a+b)}\right]$$

$$= \frac{\pi}{a+b}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a+b}$$

Example: Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}, a > 0, b > 0$

Solution:

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(z^2+a^2)(x^2+b^2)}$$

Replacement put $x = z$

$$\Rightarrow dx = dz$$

$$\therefore \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

Where C is the upper semi circle

$$= \frac{1}{2}\int_c f(z)dz \ldots (1)$$

Where, $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$

To find the poles, put $(z^2 + a^2)(z^2 + b^2) = 0$

$\Rightarrow z = \pm ai, \pm bi$ are poles of order one.

Here $z = ai, bi$ lies in the upper half of the z- plane.

Find the residue for the inside pole

(i) When $z = ai$

$$[Res\, f(z)]_{z=ai} = \lim_{z\to ai}(z - ai)f(z)$$

$$= \lim_{z\to ai}(z - ai)\frac{1}{(z+ai)(z-ai)(z^2+b^2)}$$

$$= \frac{1}{2ai(b^2-a^2)} = -\frac{1}{2ai\ (a^2-b^2)}$$

(ii) When $z = bi$

$$[Res\, f(z)]_{z=bi} = \lim_{z\to ai}(z - bi)f(z)$$

$$= \lim_{z\to bi}(z - bi) = \frac{1}{(z^2+a^2)(z+bi)(z-bi)}$$

$$= \frac{1}{(a^2-b^2)2bi}$$

$\therefore$ By Cauchy's Residue theorem

$$\int_c f(z)dz = 2\pi i\left[-\frac{1}{2ai\ (a^2-b^2)} + \frac{1}{2bi\ (a^2-b^2)}\right]$$

$$= \frac{2\pi i}{2i(a^2-b^2)}\left[-\frac{1}{a} + \frac{1}{b}\right]$$

$$= \frac{\pi}{(a+b)(a-b)} \left(\frac{a-b}{ab}\right)$$

$$= \frac{\pi}{ab\,(a+b)}$$

$$(1) \Rightarrow \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2}\frac{\pi}{ab\,(a+b)}$$

$$= \frac{\pi}{2ab\,(a+b)}$$

Example: Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$

Solution:

Replacement $put\ x = z \Rightarrow dx = dz$

Now, $\int_{o}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)}$

$= \frac{1}{2}\int_{c} \frac{dz}{(z^2+a^2)^2}$ where C is the upper semi circle

$= \frac{1}{2}\int_{c} f(z)dz$ ... (1)

Where, $f(z) = \frac{1}{(z^2+a^2)^2}$

To find the poles, put $(z^2 + a^2)^2 = 0$

$\Rightarrow z = \pm ai$ are poles of order 2 here $z = ai$ lies in the upper half of z – plane. Find the residue of the inside pole.

(i) When $z = ai$

$$[Res\ f(z)]_{z=ai} = \lim_{z\to ai} \frac{d}{dz}(z-ai)^2 f(z)$$

$$= \lim_{z\to ai} \frac{d}{dz}\left[(z-ai)^2 \frac{1}{(z-ai)^2(z+ai)^2}\right]$$

$$= \lim_{z\to ai} \frac{d}{dz}\left[\frac{1}{(z+ai)^2}\right]$$

$$= \lim_{z\to ai}\left[\frac{-2}{(z+ai)^3}\right]$$

$$= -\frac{2}{(2ai)^3} = -\frac{2}{-8a^3 i} = \frac{1}{4ia^3}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$= 2\pi i \left(\frac{1}{4ia^3}\right)$$

$$= \frac{\pi}{2a^3}$$

Example: Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{(x^4+10x^2+9)} dx$

Solution:

Replacement Put $x = z \Rightarrow dx = dz$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2-x+2}{(x^4+10x^2+9)} dx = \int_C \frac{z^2-z+2}{(z^4+10z^2+9)} dz,$$ where C is the upper semi circle.

$$= \int_C f(z)dz \qquad \dots (1)$$

Where, $f(z) = \frac{z^2-z+2}{(z^4+10z^2+9)}$

To find the poles, put $z^4 + 10z^2 + 9 = 0$

$\Rightarrow (z^2 + 1)(z^2 + 9) = 0$

$\Rightarrow z = \pm i, \pm 3i$ are poles of order one.

Here $z = i, 3i$ lies in the inside pole

Find the residue of the inside pole.

(i) When $z = i$

$$[Res\ f(z)]_{z=i} = \lim_{z\to i}(z-i)\,f(z)$$

$$= \lim_{z\to i}\left[(z-i)\frac{(z^2-z+2)}{(z+i)(z-i)(z^2+9)}\right]$$

$$= \lim_{z\to i}\left[\frac{(z^2-z+2)}{(z+i)(z^2+9)}\right]$$

$$= \frac{-1-i+2}{(2i)(8)} = \frac{1-i}{16i}$$

(ii) When $z = 3i$

$$[Res\ f(z)]_{z=3i} = \lim_{z\to 3i} \frac{d}{dz}(z-3i)\,f(z)$$

$$= \lim_{z\to 3i}\left[(z-3i)\frac{(z^2-z+2)}{(z^2+1)(z+3i)(z-3i)}\right]$$

$$= \lim_{z\to 3i}\frac{(z^2-z+2)}{(z^2+1)(z+3i)}$$

$$= \frac{-9-3i+2}{(-8)(6i)} = \frac{-7-3i}{-48i}$$

$$= \frac{7+3i}{48i}$$

$\therefore$By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$= 2\pi i\left(\frac{1-i}{16i} + \frac{7+3i}{48i}\right)$$

$$= 2\pi i\left(\frac{3-3i+7+3i}{48i}\right)$$

$$= 2\pi i\left(\frac{10}{48i}\right) = \frac{5\pi}{12}$$

$$(1) \implies \int_{-\infty}^{\infty} \frac{x^2-x+2}{(x^4+10x^2+9)}\,dx = \frac{5\pi}{12}$$

Example: Evaluate $\int_0^\infty \frac{dx}{x^4+a^4}$

Solution:

$$\int_0^\infty \frac{dx}{x^4+a^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4}$$

Replacement Put $x = z \Rightarrow dx = dz$

$$\therefore \frac{1}{2}\int_{-\infty}^{\infty} \frac{dz}{x^4+a^4} = \frac{1}{2}\int_C \frac{dz}{z^4+a^4}$$ where C is the upper semi circle.

$$= \frac{1}{2}\int_c f(z)dz \qquad \dots (1)$$

Where, $f(z) = \frac{1}{z^4+a^4}$

To find the poles, put $z^4 + a^4 = 0$

$$\Rightarrow z^4 = -a^4$$

$$\Rightarrow z = (-a^4)^{\frac{1}{4}}$$

$$\Rightarrow z = (-1)^{1/4}a$$

$$= (\cos\pi + i\sin\pi)^{\frac{1}{4}}a$$

$$= [\cos(\pi+2k\pi) + i\sin(\pi+2k\pi)]^{\frac{1}{4}}a$$

$$= [\cos\left(\frac{\pi+2k\pi}{4}\right) + i\sin\left(\frac{\pi+2k\pi}{4}\right)]\,a$$

$$= ae^{+i\left(\frac{\pi+2k\pi}{4}\right)}\ ;\ k = 0, 1, 2, 3 \dots ....$$

When $k = 0, z = ae^{\frac{i\pi}{4}}$

When $k = 1, z = ae^{\frac{3i\pi}{4}}$

When $k = 2, z = ae^{\frac{5i\pi}{4}}$

When $k = 3, z = ae^{\frac{7i\pi}{4}}$ are all poles of order one.

Here $z = ae^{\frac{i\pi}{4}}$ and , $z = ae^{\frac{3i\pi}{4}}$ lies in the upper half of the z plane.

Find the residue for the inside pole

(i)When $z = ae^{\frac{i\pi}{4}}$

$$[Res\, f(z)]_{z\to ae^{\frac{i\pi}{4}}} (z - ae^{\frac{i\pi}{4}})\ f(z)$$

$$= \lim_{z\to ae^{\frac{i\pi}{4}}} [(z - ae^{\frac{i\pi}{4}}) \frac{1}{(z^4+a^4)}]$$

$$= \frac{0}{0}\ [Apply\ L'Hospital\ rule]$$

$$= \lim_{z \to ae^{\frac{i\pi}{4}}} \frac{1}{4z^3}$$

$$= \frac{1}{4a^3 e^{\frac{3i\pi}{4}}}$$

(ii) When $z = ae^{\frac{3i\pi}{4}}$

$$[Res\, f(z)]_{z=ae^{\frac{3i\pi}{4}}} = \lim_{z \to ae^{\frac{3i\pi}{4}}} (z - ae^{\frac{3i\pi}{4}}) f(z)$$

$$= \lim_{z \to ae^{\frac{3i\pi}{4}}} (z - ae^{\frac{3i\pi}{4}}) \frac{1}{z^4+a^4}$$

$$= \frac{0}{0} \; [Apply\; L'Hospital\; rule]$$

$$= \lim_{z \to ae^{\frac{3i\pi}{4}}} \frac{1}{4z^3}$$

$$= \frac{1}{4a^3 e^{\frac{9i\pi}{4}}}$$

∴ By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\, [sum\; of\; residues]$$

$$= 2\pi i \left(\frac{1}{4a^3 e^{\frac{3i\pi}{4}}} + \frac{1}{4a^3 e^{\frac{9i\pi}{4}}} \right)$$

$$= \frac{2\pi i}{4a^3} \left(e^{\frac{-i3\pi}{4}} + e^{\frac{-i9\pi}{4}} \right)$$

$$= \frac{\pi i}{2a^3} \left(e^{-\pi i} e^{\frac{i\pi}{4}} + e^{-i2\pi} e^{-\frac{i\pi}{4}} \right) \qquad [\because e^{-\pi i} = -1]$$

$$= \frac{\pi i}{2a^3} \left((-1) e^{\frac{i\pi}{4}} + (-1) e^{-\frac{i\pi}{4}} \right) \qquad [\because e^{-2\pi i} = -1]$$

$$= \frac{-\pi i}{a^3} \left(\frac{e^{\frac{i\pi}{4}} - e^{-\frac{i\pi}{4}}}{2} \right) \qquad [\because \frac{e^{ix} - e^{-ix}}{2} = i \sin x]$$

$$= \frac{-\pi i}{a^3} \left(i \sin \frac{\pi}{4} \right)$$

$$= \frac{\pi}{a^3} \left(\frac{1}{\sqrt{2}} \right)$$

$$(1) \implies \int_o^{\infty} \frac{dx}{(x^4+a^4)} = \frac{1}{2} \left(\frac{\pi}{\sqrt{2}a^3} \right)$$

Example: Evaluate $\int_{-\infty} \frac{dx}{(x^2+1)^3}$

Solution:

Replacement Put $x = z \Rightarrow dx = dz$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^4+1)^3} = \int_C \frac{dz}{(z^4+1)^3}$$ where C is the upper semi circle.

$$= \int_c f(z)dz \qquad \dots (1)$$

Where, $f(z) = \frac{1}{(z^4+1)^3}$

To find the poles, put $(z^2 + 1)^3 = 0$

$\Rightarrow z^2 + 1 = 0$

$\Rightarrow z = \pm i$ are poles of order 3.

Here $z = i$ lies in the upper half of z – plane.

Find the residue for the inside pole $z = i$

$$[Res\, f(z)]_{z=i} = \frac{1}{2} \underset{z\to i}{\text{Lim}} \frac{d^2}{dz^2} (z-i)^3 f(z)$$

$$= \frac{1}{2!} \underset{z\to i}{\text{Lim}} \frac{d^2}{dz^2} [(z-i)^3 \frac{1}{(z+i)^3(z-i)^3}]$$

$$= \frac{1}{2!} \underset{z\to i}{\text{Lim}} \frac{d^2}{dz^2} [\frac{1}{(z+i)^3}]$$

$$= \frac{1}{2} \underset{z\to i}{\lim} \frac{d}{dz} [\frac{-3}{(z+i)^4}]$$

$$= \frac{1}{2} \frac{12}{(2i)^5} = \frac{6}{32i} = \frac{3}{16i}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\, [sum\ of\ residues]$$

$$= 2\pi i \left(\frac{3}{16i}\right) = \frac{3\pi}{8}$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x^4+1)^3} = \frac{3\pi}{8}$$

Type III

Integrals of the form

$$\int_{-\infty}^{\infty} \frac{f(x)}{g(z)} \sin(nx)dx\ (or)\ \int_{-\infty}^{\infty} \frac{f(x)}{g(x)} \cos(nx)dx$$

To evaluate this integral, write $\sin(nx)$ and $\cos(nx)$ in terms of e^{inx} thus,

$$\int_C \frac{f(z)}{g(z)} e^{inz}dz = \int_{-\infty}^{\infty} \frac{d(z)}{g(x)} e^{inx}dx$$

Where C is the closed curve as in type II and finally equate imaginary part or real part accordingly to get the required integral.

Example: Evaluate $\int_o^{\infty} \frac{\cos mx}{x^2+a^2} dx, a > 0, m > 0$

Solution:

Replacement put $x = z \Rightarrow dx = dz$ and $\cos mn = R.P\ e^{imn}$

Now, $\int_o^{\infty} \frac{\cos mx}{x^2+a^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{RP\ e^{imx}}{x^2+a^2}dx$

$= \frac{1}{2} \int_C \frac{RP\ e^{imz}}{z^2+a^2}dz$ where C is the upper semi circle.

$$= \frac{R.P}{2} \int_c f(z)dz \qquad \dots (1)$$

Where $f(z) = \frac{e^{imz}}{z^2+a^2}$

To find the poles, put $z^2 + a^2 = 0$

$\Rightarrow z = \pm ai$ are poles of order one.

Here $z = ai$ lies in the upper half of z – plane.

Find the residue for the inside pole $z = ai$

$$[Res\, f(z)]_{z=ai} = \lim_{z\to ai} (z - ai)\, f(z)$$

$$= \lim_{z\to ai} (z - ai) \frac{e^{imz}}{(z+ai)(z-ai)}$$

$$= \lim_{z\to ai} \frac{e^{imz}}{(z+ai)}$$

$$= \frac{e^{-ma}}{2ai}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\, [sum\ of\ residues]$$

$$= 2\pi i \left(\frac{e^{-ma}}{2ai}\right)$$

$$= \frac{\pi\, e^{-ma}}{a}$$

$$(1) \Longrightarrow \int_o^\infty \frac{\cos mx}{x^2+a^2} dx = \frac{R.P}{2} \left(\pi \frac{e^{-ma}}{a}\right) = \frac{\pi}{2a} e^{-ma}$$

Example: Evaluate $\int_o^\infty \frac{x \sin mx}{x^2+a^2} dx$ where $a > 0, m > 0$

Solution:

Replacement $put\ x = z \Longrightarrow dx = dz\ and\ \sin(mx) = IP\ e^{imx}$

Now, $\int_0^\infty \frac{x \sin mx}{x^2+a^2} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{xIP\, e^{imx}}{x^2+a^2} dx$

$$= \frac{I.P}{2} \int_c \frac{z\, e^{imz}}{z^2+a^2} dz \text{ where C is the upper semi circle.}$$

$$= \frac{I.P}{2} \int_c f(z)dz \qquad \dots (1)$$

Where, $f(z) = \frac{z\, e^{imz}}{z^2+a^2}$

To find the poles, put $f(z)$, put $z^2 + a^2 = 0$

$\Rightarrow z = \pm ai$ are poles of order one.

Here $z = ai$ lies in the upper half of z – plane.

Find the residue for the inside pole $z = ai$

$$[Res\, f(z)]_{z=ai} = \lim_{z\to ai} (z - ai)\, f(z)$$

$$= \lim_{z \to ai} (z - ai) \frac{ze^{imz}}{(z+ai)(z-ai)}$$

$$= \frac{(ai)e^{-ma}}{2ai} = \frac{e^{-ma}}{2}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\,[sum\ of\ residues]$$

$$= 2\pi i\,\left(\frac{e^{-ma}}{2}\right) = \pi\, e^{-ma}$$

$$(1) \Longrightarrow \int_o^{\infty} \frac{x \sin mx}{x^2+a^2} dx = \frac{I.P}{2} (\pi i\, e^{-ma}) = \frac{\pi}{2} e^{-ma}$$

Example: Evaluate $\int_{-\infty}^{\infty} \frac{\cos x \ dx}{(x^2+a^2)(x^2+b^2)}, a > b > 0$

Solution:

Replacement $put\ n = z \Rightarrow dz = dz \cos x = R.P\ \ e^{ix}$

Now, $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+a^2} \frac{dx}{(x^2+b^2)} = \int_c \frac{RP\ e^{iz}\ dz}{(z^2+a^2)(z^2+b^2)}$

where C is the upper semi circle.

$$= \frac{R.P}{2} \int_c f(z)dz$$

Where, $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$

To find the poles, put $f(z)$, put $(z^2 + a^2)(z^2 + b^2) = 0$

$\Rightarrow z = \pm ai, \pm bi$ are poles of order one here $z = ai, bi$ lies in the upper half of z – plane.

To find the residue for the inside pole

(i) when $z = ai$

$$[Res\ f(z)]_{z=ai} = \lim_{z \to ai} (z - ai)\ f(z)$$

$$= \lim_{z \to ai} (z - ai) \frac{e^{iz}}{(z+ai)(z-ai)(z^2+b^2)}$$

$$= \lim_{z \to ai} \frac{e^{iz}}{(z+ai)(z^2+b^2)}$$

$$= \frac{e^{-a}}{(2ai)(b^2-a^2)} = \frac{-e^{-a}}{(2ai)(a^2-b^2)}$$

(ii) when $z = bi$

$$[Res\ f(z)]_{z=bi} = \lim_{z \to bi} (z - bi)\ f(z)$$

$$= \lim_{z \to bi} (z - bi) \frac{e^{iz}}{(z^2+a^2)(z+bi)(z-bi)}$$

$$= \lim_{z \to bi} \frac{e^{iz}}{(z^2+a^2)(z+bi)}$$

$$= \frac{e^{-b}}{2bi(a^2-b^2)}$$

∴By Cauchy's Residue theorem,

$$\int_C f(z)dz = 2\pi i\ [sum\ of\ residues]$$

$$= 2\pi i\ \left(\frac{e^{-a}}{(2ai)(a^2-b^2)} + \frac{e^{-b}}{(2bi)(a^2-b^2)}\right)$$

$$= \frac{2\pi i}{(2i)(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$= \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)}\ R.P.\ \frac{\pi}{a^2-b^2}\left(\frac{ae^{-b}-be^{-a}}{ab}\right) = \frac{\pi}{ab\ (a^2-b^2)}$$