Catalog

ANALYTIC FUNCTIONS – NECESSARY AND SUFFICIENT CONDITIONS FOR ANALYTICITY IN CARTESIAN AND POLAR CO-ORDINATES

Analytic [or] Holomorphic [or] Regular function

A function is said to be analytic at a point if its derivative exists not only at that point but also in some neighbourhood of that point.

Entire Function: [Integral function]

A function which is analytic everywhere in the finite plane is called an entire function.

An entire function is analytic everywhere except at $z = \infty$.

Example: $e^z, \sin z, \cos z, sinhz, \cosh z$

Example: Show that $f(z) = log\ z$ analytic everywhere except at the origin and find its derivatives.

Solution:

Let $z = re^{i\theta}$

$$f(z) = log\ z$$

$$= \log(re^{i\theta}) = logr + \log(e^{i\theta}) = logr + i\theta$$

But, at the origin, $r = 0$. Thus, at the origin,

$$f(z) = log0 + i\theta = -\infty + i\theta$$

So, $f(z)$is not defined at the origin and hence is not differentiable there.

Note : $e^{-\infty} = 0$
$\log e^{-\infty} = \log 0\,; -\infty = log0$

At points other than the origin, we have

$u(r,\theta) = log\ r$	$v(r,\theta) = \theta$
$u_r = \frac{1}{r}$ $u_\theta = 0$	$v_r = 0$ $v_\theta = 1$

So, $logz$ satisfies the C−R equations.

Further $\frac{1}{r}$ is not continuous at $z = 0$.

So, $u_r, u_\theta, v_r, v_\theta$ are continuous everywhere except at $z = 0$. Thus log z satisfies all the sufficient conditions for the existence of the derivative except at the origin. The derivative is

$$f'(z) = \frac{u_r + iv_r}{e^{i\theta}} = \frac{\left(\frac{1}{r}\right) + i(0)}{e^{i\theta}} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

Note: $f(z) = u + iv \Rightarrow f(re^{i\theta}) = u + iv$

Differentiate w.r.to 'r', we get

$$(i.e.)\ e^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

Example: Check whether $w = \bar{z}$ is analytics everywhere.

Solution:

Let $w = f(z) = \bar{z}$

$u + iv = x - iy$

$u = x$	$v = -y$
$u_x = 1$ $u_y = 0$	$v_x = 0$ $v_y = -1$

$u_x \neq v_y$ at any point p(x,y)

Hence, C−R equations are not satisfied.

∴The function $f(z)$ is nowhere analytic.

Example: Test the analyticity of the function $w = \sin z$.

Solution:

Let $w = f(z) = \sin z$

$u + iv = \sin(x + iy)$

$u + iv = \sin x \cos iy + \cos x \sin iy$

$u + iv = \sin x \cosh y + i \cos x \sin hy$

Equating real and imaginary parts, we get

$u = \sin x \cosh y$	$v = \cos x \sinh y$
$u_x = \cos x \cosh y$ $u_y = \sin x \sinh y$	$v_x = -\sin x \sinh y$ $v_y = \cos x \cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$

C −R equations are satisfied.

Also the four partial derivatives are continuous.

Hence, the function is analytic.

Example: Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not.

Solution:

Let $f(z) = 2xy + i(x^2 - y^2)$

(i.e.)

$u = 2xy$	$v = x^2 - y^2$
$\frac{\partial u}{\partial x} = 2y$ $\frac{\partial u}{\partial y} = 2x$	$\frac{\partial v}{\partial x} = 2x$ $\frac{\partial v}{\partial y} = -2y$

$u_x \neq v_y$ and $u_y \neq -v_x$

C−R equations are not satisfied.

Hence, $f(z)$ is not an analytic function.

Example: Prove that $f(z) = \cosh z$ is an analytic function and find its derivative.

Solution:

Given $f(z) = \cosh z = \cos(iz) = \cos[i(x + iy)]$

$= \cos(ix - y) = \cos ix \cos y + \sin(ix) \sin y$

$u + iv = \cosh x \cos y + i \sinh x \sin y$

$u = \cosh x \cos y$	$v = \sinh x \sin y$
$u_x = \sinh x \cos y$ $u_y = -\cosh x \sin y$	$v_x = \cosh x \sin y$ $v_y = \sinh x \cos y$

$\therefore u_x, u_y, v_x$ and v_y exist and are continuous.

$u_x = v_y$ and $u_y = -v_x$

C−R equations are satisfied.

$\therefore f(z)$ is analytic everywhere.

Now, $f'(z) = u_x + iv_x$

$= \sinh x \cos y + i \cosh x \sin y$

$= \sinh(x + iy) = \sinh z$

Example: If $w = f(z)$ is analytic, prove that $\frac{dw}{dz} = \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$ where $z = x + iy$, and prove that $\frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$.

Solution:

Let $w = u(x,y) + iv(x,y)$

As $f(z)$ is analytic, we have $u_x = v_y, u_y = -v_x$

Now, $\frac{dw}{dz} = f'(z) = u_x + iv_x = v_y - iu_y = i(u_y + iv_y)$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$= \frac{\partial}{\partial x}(u+iv) = -i\frac{\partial}{\partial y}(u+iv)$$

$$= \frac{\partial w}{\partial x} = -i\frac{\partial w}{\partial y}$$

We know that, $\frac{\partial w}{\partial z} = 0$

$$\therefore \frac{\partial^2 w}{\partial z \partial \bar{z}} = 0$$

Also $\frac{\partial^2 w}{\partial \bar{z} \partial z} = 0$

Example: Prove that every analytic function $w = u(x,y) + iv(x,y)$ can be expressed as a function of z alone.

Proof:

Let $z = x + iy$ and $\bar{z} = x - iy$

$x = \frac{z+\bar{z}}{2}$ and $y = \frac{z+\bar{z}}{2i}$

Hence, u and v and also w may be considered as a function of z and $\bar{z}$

Consider $\frac{\partial w}{\partial \bar{z}} = \frac{\partial u}{\partial \bar{z}} + i\frac{\partial v}{\partial \bar{z}}$

$$= \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) + \left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \bar{z}}\right)$$

$$= \left(\frac{1}{2}u_x - \frac{1}{2i}u_y\right) + i\left(\frac{1}{2}v_x - \frac{1}{2i}v_y\right)$$

$$= \frac{1}{2}(u_x - v_y) + \frac{i}{2}(u_y + v_x)$$

$= 0$ by C−R equations as w is analytic.

This means that w is independent of $\bar{z}$

$(i.e.)$ w is a function of z alone.

This means that if $w = u(x,y) + iv(x,y)$ is analytic, it can be rewritten as a function of $(x + iy)$.

Equivalently a function of $\bar{z}$ cannot be an analytic function of z.

Example: Find the constants a, b, c if $f(z) = (x + ay) + i(bx + cy)$ is analytic.

Solution:

$f(z) = u(x,y) + iv(x,y)$

$= (x + ay) + i(bx + cy)$

$u = x + ay$	$v = bx + cy$
$u_x = 1$ $u_y = a$	$v_x = b$ $v_y = c$

Given $f(z)$ is analytic

$\Rightarrow u_x = v_y$ and $u_y = -v_x$

$1 = c$ and $a = -b$

Example: Examine whether the following function is analytic or not f(z) = e⁻ˣ(cos y – i sin y).

Solution:

Given $f(z) = e^{-x}(\cos y - i \sin y)$

$\Rightarrow u + iv = e^{-x} \cos y - ie^{-x} \sin y$

$u = e^{-x} \cos y$	$v = -e^{-x} \sin y$
$u_x = -e^{-x} \cos y$ $u_y = -e^{-x} \sin y$	$v_x = e^{-x} \sin y$ $v_y = -e^{-x} \cos y$

Here, $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ C–R equations are satisfied

$\Rightarrow f(z)$ is analytic.

Example: Test whether the function $f(z) = \frac{1}{2}\log(x^2 + y^2 + \tan^{-1}\left(\frac{y}{x}\right)$ is analytic or not.

Solution:

Given $f(z) = \frac{1}{2}\log(x^2 + y^2 + i \tan^{-1}\left(\frac{y}{x}\right)$

$(i.e.)u + iv = \frac{1}{2}\log(x^2 + y^2 + i \tan^{-1}\left(\frac{y}{x}\right)$

$u = \frac{1}{2}\log(x^2 + y^2)$	$v = \tan^{-1}\left(\frac{y}{x}\right)$

$u_x = \frac{1}{2}\frac{1}{x^2+y^2}(2x)$ $= \frac{x}{x^2+y^2}$ $u_y = \frac{1}{2}\frac{1}{x^2+y^2}(2y)$ $= \frac{y}{x^2+y^2}$	$v_x = \frac{1}{1+\frac{y^2}{x^2}}[-\frac{y}{x^2}]$ $= \frac{-y}{x^2+y^2}$ $v_y = \frac{1}{1+\frac{y^2}{x^2}}[\frac{1}{x}]$ $= \frac{x}{x^2+y^2}$

Here, $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ C−R equations are satisfied

$\Rightarrow f(z)$ is analytic.

Example: Find where each of the following functions ceases to be analytic.

(i) $\frac{z}{(z^2-1)}$ **(ii)** $\frac{z+i}{(z-i)^2}$

Solution:

(i) Let $f(z) = \frac{z}{(z^2-1)}$

$$f'(z) = \frac{(z^2-1)(1)-z(2z)}{(z^2-1)^2} = \frac{-(z^2+1)}{(z^2-1)^2}$$

$f(z)$ is not analytic, where $f'(z)$ does not exist.

$(i.e.)\ f'(z) \to \infty$

$(i.e.)(z^2-1)^2 = 0$

$(i.e.)\quad z^2 - 1 = 0$

$z = 1$

$z = \pm 1$

$\therefore f(z)$ is not analytic at the points $z = \pm 1$

(ii) Let $f(z) = \frac{z+i}{(z-i)^2}$

$$f'(z) = \frac{(z-i)^2(1)(z+i)[2(z-i)]}{(z-i)^4} = \frac{(z+3i)}{(z-i)^3}$$

$f'(z) \to \infty, at\ z = i$

$\therefore f(z)$ is not analytic at $z = i$.

PROPERTIES – HARMONIC CONJUGATES

Laplace equation

$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$ is known as Laplace equation in two dimensions.

Properties of Analytic Functions

Property: 1 Prove that the real and imaginary parts of an analytic function are harmonic functions.

Proof:

Let $f(z) = u + iv$ be an analytic function

$u_x = v_y \ldots (1)$ and $u_y = -v_x \quad \ldots (2)$ by C−R

Differentiate (1) & (2) p.w.r. to x, we get

$u_{xx} = v_{xy} \ldots (3)$ and $u_{xy} = -v_{xx} \quad \ldots (4)$

Differentiate (1) & (2) p.w.r. to x, we get

$u_{yx} = v_{yy} \ldots (5)$ and $u_{yy} = -v_{yx} \quad \ldots (6)$

(3) +(6) ⇒ $u_{xx} + u_{yy} = 0 \quad [\because v_{xy} = v_{yx}]$

(5) – (4) ⇒ $v_{xx} + v_{yy} = 0 \quad [\because u_{xy} = u_{yx}]$

∴ u and v satisfy the Laplace equation.

Harmonic function (or) [Potential function]

A real function of two real variables x and y that possesses continuous second order partial derivatives and that satisfies Laplace equation is called a harmonic function.

Note: A harmonic function is also known as a potential function.

Conjugate harmonic function

If u and v are harmonic functions such that $u + iv$ is analytic, then each is called the conjugate harmonic function of the other.

Property: 2 If w = $u(x, y) + iv(x, y)$ is an analytic function the curves of the family $u(x, y) = c_1$ and the curves of the family $v(x, y) = c_2$ cut orthogonally, where c_1 and c_2 are varying constants.

Proof:

Let $f(z) = u + iv$ be an analytic function

$\Rightarrow u_x = v_y \ldots (1)$ and $u_y = -v_x \quad \ldots (2)$ by C−R

Given $u = c_1$ and $v = c_2$

Differentiate p.w.r. to x, we get

$$u_x + u_y \frac{dy}{dx} = 0 \text{ and } \quad v_x + v_y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-u_x}{u_y} \quad \text{and } \frac{dy}{dx} = \frac{-v_x}{v_y}$$

$$\Rightarrow m_1 = \frac{-u_x}{u_y} \qquad \Rightarrow m_2 = \frac{-v_x}{v_y}$$

$$m_1 . m_2 = \left(\frac{-u_x}{u_y}\right)\left(\frac{-v_x}{v_y}\right) = \left(\frac{u_x}{u_y}\right)\left(\frac{u_y}{u_x}\right) = -1 \quad \text{by (1) and (2)}$$

Hence, the family of curves form an orthogonal system.

Property: 3 An analytic function with constant modulus is constant.

Proof:

Let $f(z) = u + iv$ be an analytic function.

$\Rightarrow u_x = v_y \ldots(1)$ and $u_y = -v_x \quad \ldots(2)$ by C−R

Given $|f(z)| = \sqrt{u^2 + v^2} = c \neq 0$

$\Rightarrow |f(z)| = u^2 + v^2 = c^2$ (say)

$(i.e)\ u^2 + v^2 = c^2 \quad \ldots(3)$

Differentiate (3) p.w.r. to x and y; we get

$2uu_x + 2vv_x = 0 \Rightarrow uu_x + vv_x = 0 \quad \ldots(4)$

$2uu_y + 2vv_y = 0 \Rightarrow uu_y + vv_y = 0 \quad \ldots(5)$

$(4) \times u \quad \Rightarrow u^2u_x + uv\, v_x = 0 \quad \ldots(6)$

$(5) \times v \quad \Rightarrow uv\, u_y + v^2 v_y = 0 \quad \ldots(7)$

$(6)+(7) \quad \Rightarrow u^2u_x + v^2v_y + uv\,[v_x + u_y] = 0$

$\Rightarrow u^2u_x + v^2 u_x + uv\,[-u_y + u_y] = 0$ by (1) & (2)

$\Rightarrow (u^2+v^2)\, u_x = 0$

$\Rightarrow u_x = 0$

Similarly, we get $v_x = 0$

We know that $f'(z) = u_x + v_x = 0 + i0 = 0$

Integrating w.r.to z, we get, $f(z) = c$ [Constant]

Property: 4 An analytic function whose real part is constant must itself be a constant.

Proof :

Let $f(z) = u + iv$ be an analytic function.

$\Rightarrow u_x = v_y \ldots(1)$ and $u_y = -v_x \quad \ldots(2)$ by C−R

Given $u = c$ [Constant]

$\Rightarrow u_x = 0, \quad u_y = 0$

$\Rightarrow u_x = 0, \quad v_x = 0 \quad$ by (2)

We know that $f'(z) = u_x + iv_x = 0 + i0 = 0$

Integrating w.r.to z, we get $f(z) = c$ [Constant]

Property: 5 Prove that an analytic function with constant imaginary part is constant.

Proof:

Let $f(z) = u + iv$ be an analytic function.

$\Rightarrow u_x = v_y \ldots(1)$ and $u_y = -v_x$ $\ldots(2)$ by C−R

Given $v = c$ [Constant]

$\Rightarrow v_x = 0, \quad v_y = 0$

We know that $f'(z) = u_x + iv_x$

$= v_y + iv_x$ by (1) $= 0 + i0$

$\Rightarrow f'(z) = 0$

Integrating w.r.to z, we get $f(z) = c$ [Constant]

Property: 6 If $f(z)$ and $\overline{f(\bar{z})}$ are analytic in a region D, then show that $f(z)$ is constant in that region D.

Proof:

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function.

$\overline{f(\bar{z})} = u(x, y) - iv(x, y) = u(x, y) + i[-v(x, y)]$

Since, $f(z)$ is analytic in D, we get $u_x = v_y$ and $u_y = -v_x$

Since, $\overline{f(\bar{z})}$ is analytic in D, we have $u_x = -v_y$ and $u_y = v_x$

Adding, we get $u_x = 0$ and $u_y = 0$ and hence, $v_x = v_y = 0$

$\therefore f(z) = u_x + iv_x = 0 + i0 = 0$

$\therefore f(z)$ is constant in D.

Theorem: 1 If $f(z) = u + iv$ is a regular function of z in a domain D, then $\nabla^2|f(z)|^2 = 4|f'(z)|^2$

Solution:

Given $f(z) = u + iv$

$\Rightarrow |f(z)| = \sqrt{u^2 + v^2}$

$\Rightarrow |f(z)|^2 = u^2 + v^2$

$\Rightarrow \nabla^2|f(z)|^2 = \nabla^2(u^2 + v^2)$

$= \nabla^2(u^2) + \nabla^2(v^2) \quad \ldots(1)$

$\nabla^2(u^2) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u^2 += \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} \quad \ldots(2)$

$$\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right] = 2\left[u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2$

$$(2) \Rightarrow \quad \nabla^2(u^2) = 2u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 0 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \qquad [\therefore u \text{ is harmonic}]$$

$$\nabla^2(u^2) = 2u_x^2 + 2u_y^2$$

Similarly, $\nabla^2(v^2) = 2v_x^2 + 2v_y^2$

$$(1) \Rightarrow \nabla^2|f(z)|^2 = 2[u_x^2 + u_y^2 + v_x^2 + v_y^2]$$

$$= 2[u_x^2 + (-v_x)^2 + v_x^2 + u_x^2] \qquad [u_x = v_y ; u_y = -v_x]$$

$$= 4[u_x^2 + v_x^2]$$

$$(i.e.)\nabla^2|f(z)|^2 = 4|f'(z)|^2$$

Note : $f(z) = u + iv; f'(z) = u_x + iv_x$;

(or) $f'(z) = v_y + iu_y$; $|f'(z)| = \sqrt{u_x^2 + v_x^2}$; $|f'(z)|^2 = u_x^2 + v_x^2$

Theorem: 2 If f(z) = u + iv is a regular function of z in a domain D, then ∇^2 log |f(z)| = 0 if f(z) f′(z) ≠ 0 in D. i.e., log |f(z)| is harmonic in D.

Solution:

Given $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2 + v^2}$$

$$\log|f(z)| = \frac{1}{2}\log(u^2 + v^2)$$

$$\nabla^2\log|f(z)| = \frac{1}{2}\nabla^2\log(u^2 + v^2) = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log(u^2 + v^2)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x^2}[\log(u^2 + v^2)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[\log(u^2 + v^2)] \qquad \ldots(1)$$

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}[\log(u^2 + v^2)] = \frac{1}{2}\frac{\partial}{\partial x}\left[\frac{1}{u^2+v^2}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right)\right] = \frac{\partial}{\partial x}\left[\frac{uu_x+vv_x}{u^2+v^2}\right]$$

$$= \frac{(u^2+v^2)[uu_{xx}+u_xu_x+vv_{xx}+v_xv_x]-(uu_x+vv_x)(2uu_x+2vv_x)}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[uu_{xx}+vv_{xx}+u_x^2+v_x^2]-2(uu_x+vv_x)^2}{(u^2+v^2)^2}$$

Similarly, $\frac{1}{2}\frac{\partial^2}{\partial y^2}[\log(u^2 + v^2)] = \frac{(u^2+v^2)[uu_{yy}+vv_{yy}+u_y^2+v_y^2]-2(uu_y+vv_y)^2}{(u^2+v^2)^2}$

$$(1) \Rightarrow \nabla^2\log|f(z)| = \frac{(u^2+v^2)[u(u_{xx}+u_{yy})+v(v_{xx}+v_{yy})+(u_x^2+u_y^2)+(v_x^2+v_y^2)]-2[uu_x+vv_x]^2-2[uu_y+vv_y]^2}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[u(0)+(u_x^2+v_x^2)+u_y^2+v_y^2)--2[u^2u_x^2+v^2v_x^2+2uv\,u_x v_x+u^2 u_y^2+v^2v_y^2+2uv\,u_y v_y]}{(u^2+v^2)^2}$$

$[\because u_{xx}+u_{yy}=0, v_{xx}+vv_{yy}=0]$

$$=\frac{(u^2+v^2)[|f'(z)^2|+|f'(z)^2|-2[u^2(u_x^2+u_y^2)+v^2(v_x^2+v_y^2)+2uv(u_x v_x+u_y v_y)]}{(u^2+v^2)^2}$$

$[\because f'(z)=u+iv, |f'(z)|=u_x+iv_x$ (or) $f'(z)=v_y-iu_y, |f'(z)|^2=u_x^2+v_x^2$ (or) $|f'(z)|^2=u_y^2+v_y^2$

$$=\frac{2(u^2+v^2)[|f'(z)^2|-2[u^2|f'(z)|^2+v^2|f'(z)^2|+2uv(0)]}{(u^2+v^2)^2}$$

$[\because u_x=v_y,\ u_y=-v_x]$

$\Rightarrow u_x v_x+u_y v_y=0$

$\Rightarrow u_x^2+u_y^2=u_x^2+v_x^2=|f'(z)|^2$

$\Rightarrow v_x^2+v_y^2=u_y^2+v_y^2=|f'(z)|^2$

$$=\frac{2(u^2+v^2)|f(z)|^2-2(u^2+v^2)|f'(z)|^2}{(u^2+v^2)^2}$$

$(i.e.)\ \nabla^2\log|f(z)|=0$

Theorem: 3 If f(z) $= u+iv$ is a regular function of z in a domain D, then $\nabla^2(u^p)=p(p-1)\,u^{p-2}|f'(z)|^2$

Solution:

$$\nabla^2(u^p)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u^p)$$

$$=\frac{\partial^2}{\partial x^2}(u^p)+\frac{\partial^2}{\partial y^2}(u^p)$$

$$\frac{\partial^2}{\partial x^2}(u^p)=\frac{\partial}{\partial x}\left[pu^{p-1}\frac{\partial u}{\partial x}\right]=pu^{p-1}u_{xx}+p(p-1)u^{p-2}(u_x)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^p)=pu^{p-1}u_{yy}+p(p-1)u^{p-2}(u_y)^2$

$(1)\Rightarrow \nabla^2(u^p)=pu^{p-1}(u_{xx}+u_{yy})+p(p-1)u^{p-2}[u_x^2+u_y^2]$

$$=pu^{p-1}(0)+p(p-1)u^{p-2}|f'(z)|^2$$

$[\because u_{xx}+u_{yy}=0, f(z)=u+iv, f'(z)=u_x+iv_x,\ |f'(z)|^2=u_x^2+u_y^2$

$$\therefore\ \nabla^2(u^p)=p(p-1)\,u^{p-2}|f'(z)|^2$$

Theorem: 4 If f(z) $= u+iv$ is a regular function of z, then $\nabla^2|f(z)|^p=p^2|f(z)|^{p-2}|f'(z)|^2$.

Solution:

Let $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2 + v^2} \qquad \ldots(a)$$

$$|f(z)|^p = (u^2 + v^2)^{p/2} \qquad \ldots(b)$$

$$\nabla^2 |f(z)|^p = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2 + v^2)^{p/2}$$

$$= \frac{\partial^2}{\partial x^2}(u^2 + v^2)^{p/2} + \frac{\partial^2}{\partial y^2}(u^2 + v^2)^{p/2}$$

$$\frac{\partial^2}{\partial x^2}(u^2 + v^2)^{p/2} = \frac{\partial}{\partial x}\left[\frac{p}{2}(u^2 + v^2)^{\frac{p}{2}-1}\left[2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right]\right]$$

$$= p(u^2 + v^2)^{\frac{p}{2}-1}[uu_{xx} + u_x u_x + vv_{xx} + v_x v_x]$$

$$+p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}(uu_x + vv_x)(2uu_x + 2vv_x)$$

$$= p(u^2 + v^2)^{\frac{p}{2}-1}[uu_{xx} + u_x^2 + vv_{xx} + v_x^2]$$

$$+2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}(uu_x + vv_x)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2 + v^2)^{p/2} = p(u^2 + v^2)^{\frac{p}{2}-1}[uu_{yy} + u_y^2 + vv_{yy} + v_y^2]$

$$+2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}(uu_y + vv_y)^2$$

$\Rightarrow \nabla^2 |f(z)|^p = p(u^2 + v^2)^{\frac{p}{2}-1}[u(u_{xx} + u_{yy}) + v(v_{xx} + v_{yy}) + u_x^2 + u_y^2 + v_x^2 + v_y^2] +$

$2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}[u^2 u_x^2 + v^2 v_x^2 + 2uv\, u_x v_x + u^2 u_y^2 + v^2 v_y^2 + 2uv\, u_y v_y]$

$$= p(u^2 + v^2)^{\frac{p}{2}-1}[u(0) + v(0) + 2(u_x^2 + u_y^2)]$$

$+2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}[u^2(u_x^2 + u_y^2) + v^2(v_x^2 + v_y^2) + 2uv(u_x v_x + u_y v_y)]$

$= 2p(u^2 + v^2)^{\frac{p}{2}-1}|f'(z)|^2 + 2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}[u^2|f'(z)|^2 + v^2|f'(z)|^2 + 2uv(0)]$

$$= 2p(u^2 + v^2)^{\frac{p}{2}-1}|f'(z)|^2 + 2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-2}(u^2 + v^2)|f'(z)|^2$$

$$= 2p(u^2 + v^2)^{\frac{p}{2}-1}|f'(z)|^2 + 2p\left(\frac{p}{2} - 1\right)(u^2 + v^2)^{\frac{p}{2}-1}|f'(z)|^2$$

$$= 2p(u^2 + v^2)^{\frac{p}{2}-1}|f'(z)|^2\left[1 + \frac{p}{2} - 1\right]$$

$$= 2p(u^2 + v^2)^{\frac{p}{2}-1}|f'(z)|^2 = p^2(u^2 + v^2)^{\frac{p-2}{2}}|f'(z)|^2$$

$$= p^2\left(\sqrt{u^2 + v^2}\right)^{p-2}|f'(z)|^2$$

$= p^2|f(z)|^{p-2}|f'(z)|^2$ by (a) & (b)

Theorem: 5 If $f(z) = u + iv$ is a regular function of z, in a domain D, then $\left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = |f'(z)|^2$

Solution:

Given $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2 + v^2}$$

$$\frac{\partial}{\partial x}|f(z)| = \frac{\partial}{\partial x}\left[\sqrt{u^2 + v^2}\right]$$

$$= \frac{1}{2\sqrt{u^2+v^2}}[2uu_x + 2vv_x] = \frac{uu_x+vv_x}{\sqrt{u^2+v^2}}$$

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2 = \frac{(uu_x+vv_x)^2}{u^2+v^2} = \frac{u^2u_x^2+v^2v_x^2+2uv\,u_xv_x}{u^2+v^2}$$

Similarly, $\left[\frac{\partial}{\partial y}|f(z)|\right]^2 = \frac{u^2u_y^2+v^2v_y^2+2uv\,u_yv_y}{u^2+v^2}$

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = \frac{u^2[u_x^2+u_y^2]+v^2[v_x^2+v_y^2]+2uv\,[u_xv_x+u_yv_y]}{u^2+v^2}$$

$$= \frac{u^2|f'(z)|^2+v^2|f'(z)|^2+2u\,v\,(0)}{u^2+v^2}\left[\because u_x = v_y\,;\, u_y = -v_x\right]$$

$$= \frac{(u^2+v^2)|f(z)|^2}{u^2+v^2} = |f'(z)|^2\left[\because u_xv_x + u_yv_y = 0\right]$$

Theorem: 6 If $f(z) = u + iv$ is a regular function of z, then $\nabla^2|\text{Re } f(z)|^2 = 2|f'(z)|^2$

Solution:

Let $f(z) = u + iv$

$\text{Re} f(z) = u$

$|\text{Re } f'(z)|^2 = u^2$

$\nabla^2|\text{Re } f'(z)|^2 = \nabla^2 u^2$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2)$$

$$= \left(\frac{\partial^2}{\partial x^2}\right)(u^2) + \left(\frac{\partial^2}{\partial y^2}\right)(u^2)$$

$$= 2[u_x^2 + u_y^2]$$

$$= 2\,|f'(z)|^2$$

Theorem: 7 If $f(z) = u + iv$ is a regular function of z, then prove that $\nabla^2|\text{Im } f(z)|^2 = 2|f'(z)|^2$

Proof:

Let $f(z) = u + iv$

$\text{I m} f(z) = v$

$$|Im\, f(z)|^2 = v^2$$

$$\frac{\partial}{\partial x}(v^2) = 2vv_x$$

$$\frac{\partial^2}{\partial x^2}(v^2) = 2[vv_{xx} + v_x v_x] = 2[vv_{xx} + v_x^2]$$

Similarly, $\frac{\partial^2}{\partial y^2}(v^2) = 2[vv_{yy} + v_y^2]$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Im\, f(z)|^2 = 2[v(v_{xx} + v_{yy}) + v_x^2 + v_y^2]$$

$$= 2[v(0) + u_x^2 + v_x^2] \quad \text{by C−R equation}$$

$$= 2|f'(z)|^2$$

Theorem: 8 Show that $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$ **(or) S T** $\nabla^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

Proof:

Let x & y are functions of z and $\bar{z}$

that is $x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left[\frac{1}{2i}\right] = \frac{1}{2}\left[\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right]$$

$$2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y} \quad \ldots(1)$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial}{\partial y}\left[\frac{-1}{2i}\right] = \frac{1}{2}\left[\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right]$$

$$2\frac{\partial}{\partial \bar{z}} = \left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) \quad \ldots(2)$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) \; [\because (a+b)(a-b) = a^2 - b^2]$$

$$= \left(2\frac{\partial}{\partial z}\right)\left(2\frac{\partial}{\partial \bar{z}}\right) \text{ by (1) \& (2)}$$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

Theorem: 9 If $f(z)$ is analytic, show that $\nabla^2 |f(z)|^2 = 4|f'(z)|^2$

Solution:

We know that, $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$

$$|f(z)|^2 = f(z)\overline{f(z)}$$

$$\nabla^2 |f(z)|^2 = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}[f(z)\overline{f(z)}]$$

$$= 4\left[\frac{\partial}{\partial z} f(z)\right]\left[\frac{\partial}{\partial \bar{z}} \bar{f}(\bar{z})\right]$$

[$\because f(z)$ is independent of $\bar{z}$ and $\overline{f(z)}$ is independent of z]

$$\therefore \nabla^2 |f(z)|^2 = 4 f'(z) \left[\frac{\partial}{\partial \bar{z}} \overline{f(z)}\right] = 4 f'(z) \overline{f'(z)}$$

$$= 4|f'(z)|^2 \quad [\because z\bar{z} = |z|^2]$$

Example: Find the value of m if $u = 2x^2 - my^2 + 3x$ is harmonic.

Solution:

Given $u = 2x^2 - my^2 + 3x$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ [$\because$ u is harmonic]} \qquad \ldots(1)$$

$\frac{\partial u}{\partial x} = 4x + 3$	$\frac{\partial u}{\partial y} = -2my$
$\frac{\partial^2 u}{\partial x^2} = 4$	$\frac{\partial^2 u}{\partial y^2} = -2m$

$$\therefore (1) \Rightarrow (4) + (-2m) = 0$$

$$\Rightarrow m = 2$$

CONSTRUCTION OF ANALYTIC FUNCTION

Method: [Milne – Thomson method]

(i) To find $f(z)$ when u is given

Let $f(z) = u + iv$

$f'(z) = u_x + iv_x$

$= u_x - iv_y$ [by C−R condition]

$\therefore f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

(ii) To find $f(z)$ when v is given

Let $f(z) = u + iv$

$f'(z) = u_x + iv_x$

$= v_y + iv_x$ [by C−R condition]

$\therefore f(z) = \int v_y(z,0)dz + i\int v_x(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

Example: Construct the analytic function $f(z)$ for which the real part is $e^x \cos y$.

Solution:

Given $u = e^x \cos y$

$\Rightarrow u_x = e^x \cos y$ [$\therefore \cos 0 = 1$]

$\Rightarrow u_x(z,0) = e^x$

$\Rightarrow u_y = e^x \cos y$ [$\therefore \sin 0 = 0$]

$\Rightarrow u_y(z,0) = 0$

$\therefore f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$\therefore f(z) = \int e^z\, dz - i\int 0\, dz + C$

$= e^z + C$

Example: Determine the analytic function $w = u + iv$ if $u = e^{2x}(x\cos 2y - y\sin 2y)$

Solution:

Given $u = e^{2x}(x\cos 2y - y\sin 2y)$

$u_x = e^{2x}[\cos 2y] + (x\cos 2y - y\sin 2y)[2e^{2x}]$

$u_x(z,0) = e^{2z}[1] + [z(1) - 0][2e^{2z}]$

$$= e^{2z} + 2ze^{2z}$$

$$= (1 + 2z)e^{2x}$$

$$u_y = e^{2x}[-2x \sin 2y - (y2cos2y + \sin 2y)]$$

$$u_y(z, 0) = e^{2z}[-0 - (0 + 0)] = 0$$

$\therefore f(z) = \int u_x(z, 0)dz - i \int u_y(z, 0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$$f(z) = \int (1 + 2z)e^{2z}dz - i \int 0 + dz + C$$

$$= \int (1 + 2z)e^{2z}dz + C$$

$$= (1 + 2z)\frac{e^{2z}}{2} - 2\frac{e^{2z}}{4} + C \quad [\because \int uv\, dz = uv_1 - u'v_2 + u''v_3 - \cdots]$$

$$= \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2} + C$$

$$= ze^{2z} + C$$

Example: Determine the analytic function where real part is

$\boldsymbol{u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1}$.

Solution:

Given $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$u_x = 3x^2 - 3y^2 + 6x$$

$$\Rightarrow u_x(z, 0) = 3z^2 - 0 + 6z$$

$$u_y = 0 - 6xy + 0 - 6y$$

$$\Rightarrow u_y(z, 0) = 0$$

$f(z) = \int u_x(z, 0)dz - i \int u_y(z, 0)dz + C$ [by Milne−Thomson rule],

Where, C is a complex constant.

$$f(z) = \int (3z^2 + 6z)dz - i \int 0 + dz + C$$

$$= 3\frac{z^2}{3} + 6\frac{z^2}{2} + C$$

$$= z^3 + 3z^2 + C$$

Example: Determine the analytic function whose real part in $\frac{\sin 2x}{\cosh 2y - \cos 2x}$

Solution:

Given $u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$

$$u_x = \frac{(\cosh 2y - \cos 2x)[2 \cos 2x] - \sin 2x[2 \sin 2x]}{[\cosh 2y - \cos 2x]^2}$$

$$u_x(z, 0) = \frac{(1 - \cos 2z)(2 \cos 2z) - 2\sin^2 2z}{[\cosh 0 - \cos 2z]^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2[\cos^2 2z + \sin^2 2z]}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)^2}$$

$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1-\cos 2z)}$$

$$= \frac{-2}{2sin^2 2}$$

$$= -cosec^2 z$$

$$u_y = \frac{(\cosh 2y - \cos 2x)\,(0) - \sin 2x[2\sin 2y]}{[\cosh 2y - \cos 2x]^2}$$

$$\Rightarrow u_y(z,0) = 0$$

$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne–Thomson rule],

where C is a complex constant.

$$f(z) = \int(-cosec^2 z)dz - i\int 0\,dz + C$$

$$= cot\, z + C$$

Example: Show that the function $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic and determine its conjugate. Also find $f(z)$

Solution:

Given $u = \frac{1}{2}\log(x^2 + y^2)$

$$u_x = \frac{1}{2}\frac{1}{(x^2+y^2)}(2x) = \frac{x}{x^2+y^2},$$

$$\Rightarrow u_x(z,0) = \frac{z}{z^2} = \frac{1}{z}$$

$$u_{xx} = \frac{(x^2+y^2)[1] - x[2x]}{[x^2+y^2]^2} = \frac{x^2+y^2-2x^2}{[x^2+y^2]^2} = \frac{y^2-x^2}{[x^2+y^2]^2} \quad \ldots(1)$$

$$u_y = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$$

$$\Rightarrow u_y(z,0) = 0$$

$$u_{yy} = \frac{(x^2+y^2)[1] - y[2y]}{[x^2+y^2]^2} = \frac{x^2-y^2}{[x^2+y^2]^2} \quad \ldots(2)$$

To prove u is harmonic:

$$\therefore u_{xx} + u_{yy} = \frac{(y^2-x^2)+(x^2-y^2)}{[x^2+y^2]^2} = 0 \qquad by\ (1)\&(2)$$

$\Rightarrow u$ is harmonic.

To find $f(z)$:

$$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C \text{ [by Milne–Thomson rule],}$$

Where, C is a complex constant.

$$f(z) = \int \frac{1}{z} dz - i\int 0\, dz + C$$

$$= \log z + C$$

To find v :

$$f(z) = \log(re^{i\theta}) \qquad [\because z = re^{i\theta}]$$

$$u + iv = \log r + \log e^{i\theta} = \log r + i\theta$$

$$\Rightarrow u = \log r, v = \theta$$

Note: $z = x + iy$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\log r = \frac{1}{2}\log(x^2 + y^2)$$

$$\tan\theta = \frac{y}{x}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad i.e., v = \tan^{-1}\left(\frac{y}{x}\right)$$

Example: Construct an analytic function $f(z) = u + iv$, given that $u = e^{x^2-y^2}\cos 2xy$. Hence find v.

Solution:

Given $u = e^{x^2-y^2}\cos 2xy = e^{x^2}e^{-y^2}\cos 2xy$

$$u_x = e^{-y^2}[e^{x^2}(-2y\sin 2xy) + \cos 2xy\, e^{x^2}2x]$$

$$u_x(z,0) = 1[e^{z^2}(0) + 2ze^{z^2}] = 2ze^{z^2}$$

$$u_y = e^{x^2}[e^{-y^2}(-2x\sin 2xy) + \cos 2xy e^{-y^2}(-2y)]$$

$$u_y(z,0) = e^{z^2}[0+0] = 0$$

$$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C \text{ [by Milne–Thomson rule]}$$

$$= \int 2z\, e^{z^2} dz + C$$

$$= 2\int z\, e^{z^2} dz + C$$

put $t = z^2$, dt= $2z\,dz$

$$= \int e^t\, dt + C$$

$$= e^t + C$$

$$f(z) = e^{z^2} + C$$

To find v :

$$u + iv = e^{(x+iy)^2} = e^{x^2-y^2+i\,2xy} = e^{x^2-y^2}e^{i2\,xy}$$

$$= e^{x^2-y^2}[\cos(2xy) + i\sin(2xy)]$$

$$v = e^{x^2-y^2}\sin 2xy \quad [\because \text{equating the imaginary parts}]$$

Example: Find the regular function whose imaginary part is $e^{-x}(x\cos y + y\sin y)$.

Solution:

Given $v = e^{-x}(x\cos y + y\sin y)$

$$v_x = e^{-x}[\cos y] + (x\cos y + y\sin y)[-e^{-x}]$$

$$v_x(z,0) = e^{-z} + (z)(-e^{-z}) = (1-z)e^{-z}$$

$$v_y = e^{-x}[-x\sin y + (y\cos y + \sin y\,(1))]$$

$$v_x(z,0) = e^{-z}[0+0+0] = 0$$

$$\therefore f(z) = \int v_y(z,0)dz + i\int v_x(z,0)dz + C \quad [\text{by Milne–Thomson rule}]$$

Where, C is a complex constant.

$$f(z) = \int 0dz + i\int(1-z)e^{-z}dz + C$$

$$= i\int(1-z)e^{-z}dz + C$$

$$= i\left[(1-z)\left[\frac{e^{-z}}{-1}\right] - (-1)\left[\frac{e^{-z}}{(-1)^2}\right]\right] + C$$

$$= i[-(1-z)e^{-z} + e^{-z}] + C$$

$$= ize^{-z} + C$$

Example: In a two dimensional flow, the stream function is $\psi = tan^{-1}\left(\frac{y}{x}\right)$. Find the velocity potential φ.

Solution:

Given $\psi = tan^{-1}(y/x)$

We should denote, ϕ by u and ψ by v

$$\therefore v = tan^{-1}(y/x)$$

$$v_x = \frac{1}{1+(y/x)^2}\left[\frac{-y}{x^2}\right] = \frac{-y}{x^2+y^2}, \qquad v_x(z,0) = 0$$

$$v_y = \frac{1}{1+(y/x)^2}\left[\frac{1}{x}\right] = \frac{x}{x^2+y^2} \qquad v_x(z,0) = \frac{z}{z^2} = \frac{1}{z}$$

$$\therefore f(z) = \int v_y(z,0)dz + i\int v_x(z,0)dz + C$$

$$f(z) = \int \frac{1}{z}dz + i\int 0\,dz + C = \log z + C$$

To find φ:

$$f(z) = log\,(re^{i\theta}) \qquad [\because z = re^{i\theta}]$$

$$u + iv = \log r + \log e^{i\theta}$$

$$u + iv = \log r + i\theta$$

$$\Rightarrow u = \log r$$

$$\Rightarrow u = log\sqrt{x^2 + y^2}$$

$$= \frac{1}{2}\log(x^2 + y^2)$$

$z = x + iy, \lvert z\rvert = \sqrt{x^2 + y^2}$

So, the velocity potential φ is

$$\varphi = \frac{1}{2}\log(x^2 + y^2)$$

Example: If $f(z) = u + iv$ is an analytic function and $u - v = e^x(\cos y - \sin y)$, find $f(z)$ interms of z.

Solution:

Given $u - v = e^x(\cos y - \sin y)$, $\quad\dots(A)$

Differentiate (A) p.w.r. to x, we get

$$u_x - v_x = e^x(\cos y - \sin y),$$

$$u_x(z,0) - v_x(z,0) = e^z \quad\dots(1)$$

Differentiate (A) p.w.r. to y, we get

$$u_y - v_y = e^x(-\sin y - \cos y)$$

$$u_y(z,0) - v_y(z,0) = e^z[-1]$$

$$\text{i.e., } u_y(z,0) - v_y(z,0) = -e^z$$

$$-v_x(z,0) - u_x(z,0) = -e^z \quad\dots(2) \text{ [by C−R conditions]}$$

$$(1) + (2) \Rightarrow -2v_x(z,0) = 0$$

$$\Rightarrow v_x(z,0) = 0$$

$$(1) \Rightarrow u_x(z,0) = e^z$$

$$f(z) = \int u_x(z,0)dz + i\int v_x(z,0)dz + C \quad \text{[by Milne−Thomson rule]}$$

$$f(z) = \int e^z dz + i0 + C$$

$$= e^z + C$$

Example: Find the analytic functions $f(z) = u + iv$ given that

(i) $\quad$ $\mathbf{2u + v = e^x(\cos y - \sin y)}$

(ii) $\quad$ $\mathbf{u - 2v = e^x(\cos y - \sin y)}$

Solution:

Given (i) $2u + v = e^x(\cos y - \sin y)$ $\quad\dots(A)$

Differentiate (A) p.w.r. to x, we get

$$2u_x + v_x = e^x(\cos y - \sin y)$$

$$2u_x - u_y = e^x(\cos y - \sin y) \quad \text{[by C−R condition]}$$

$$2u_x(z,0) - u_y(z,0) = e^z \qquad \ldots(1)$$

Differentiate (A) p.w.r. to y, we get

$$2u_y + v_y = e^x[-\sin y - \cos y]$$

$$2u_y + u_x = e^x\,[-\sin y - \cos y] \quad \text{[by C−R condition]}$$

$$2u_y(z,0) + u_x(z,0) = e^z(-1) = -e^z \qquad \ldots(2)$$

$$(1)\times(2) \quad \Rightarrow 4u_x(z,0) - 2u_y(z,0) = 2e^z \qquad \ldots(3)$$

$$(2)+(3) \quad \Rightarrow 5u_x(z,0) = e^z$$

$$\Rightarrow u_x(z,0) = \frac{1}{5}e^z$$

$$(1) \Rightarrow u_y(z,0) = \frac{2}{5}e^z - e^z = -\frac{3}{5}e^z$$

$$\Rightarrow u_y(z,0) = -\frac{3}{5}e^z$$

$$f(z) = \int u_x(z,0)dz - i\int u_y(z,0)dz + C \quad \text{[by Milne−Thomson rule]}$$

Where, C is a complex constant.

$$f(z) = \int \frac{1}{5}e^z dz - i\int -\frac{3}{5}e^z dz + C$$

$$= \frac{2}{5}e^z + \frac{3}{5}ie^z + C$$

$$= \frac{1+3i}{5}e^z + C$$

(ii) $u - 2v = e^x(\cos y - \sin y) \qquad \ldots(B)$

Differrentiate (B) p.w.r. to x, we get

$$u_x - 2v_x = e^x(\cos y - \sin y)$$

$$u_x + 2u_y = e^x(\cos y - \sin y) \quad \text{[by C−R condition]}$$

$$u_x(z,0) + 2u_y(z,0) = e^z \qquad \ldots(1)$$

Differentiate (B) p.w.r. to y, we get

$$u_y - 2v_y = e^x[-\sin y - \cos y]$$

$$u_y - 2u_x = e^x\,[-\sin y - \cos y] \quad \text{[by C−R condition]}$$

$$u_y(z,0) - 2u_x(z,0) = -e^z \qquad \ldots(2)$$

$$(1)\times(2) \quad \Rightarrow 2u_x(z,0) + 4u_y(z,0) = 2e^z \qquad \ldots(3)$$

$$(2)+(3) \quad \Rightarrow 5u_y(z,0) = e^z$$

$$\Rightarrow u_y(z,0) = \frac{1}{5}e^z$$

$$(1) \quad \Rightarrow u_x(z,0) = -\frac{2}{5}e^z + e^z$$

$$= \frac{3}{5}e^z$$

$f(z) = \int u_x\,(z,0)dz - i\int u_y(z,0)dz + C$ [by Milne−Thomson rule]

Where, C is a complex constant.

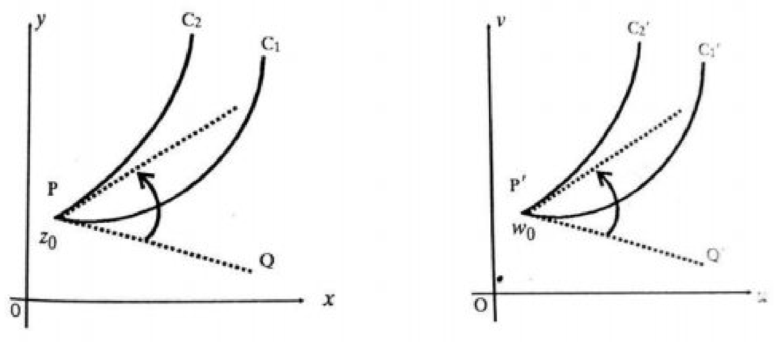
$f(z) = \int \frac{3}{5}\, e^z dz - i\int \frac{1}{5}\, e^z dz + C$

$= \frac{3}{5}e^z - i\,\frac{1}{5}e^z + C = \frac{3-i}{5}e^z + C$

CONFORMAL MAPPING-MAPPING BY FUNCTIONS

Definition: Conformal Mapping

A transformation that preserves angels between every pair of curves through a point, both in magnitude and sense, is said to be conformal at that point.



Some standard transformations

Translation:

The transformation $w = C + z$, where C is a complex constant, represents a translation.

Let $z = x + iy$

$w = u + iv$ and $C = a + ib$

Given $w = z + C$,

$(i.e.)\ u + iv = x + iy + a + ib$

$\Rightarrow u + iv = (x + a) + i(y + b)$

Equating the real and imaginary parts, we get $u = x + a, v = y + b$

Hence the image of any point $p(x, y)$in the z−plane is mapped onto the point $p'(x + a, y + b)$ in the w-plane. Similarly every point in the z−plane is mapped onto the w plane.

If we assume that the w−plane is super imposed on the z-plane, we observe that the point $(x,\ y)$ and hence any figure is shifted by a distance $|C| = \sqrt{a^2 + b^2}$ in the direction of C i.e., translated by the vector representing C.

Hence this transformation transforms a circle into an equal circle. Also the corresponding regions in the z and w planes will have the same shape, size and orientation.

Example: What is the region of the w plane into which the rectangular region in the Z plane bounded by the lines $x = 0,\ y = 0,\ x = 1\ and\ y = 2$ is mapped under the transformation $\mathbf{w = z + (2 - i)}$

Solution:

Given $w = z + (2 - i)$

$(i.e.)\ u + iv = x + iy + (2 - i) = (x + 2) + i(y - 1)$

Equating the real and imaginary parts

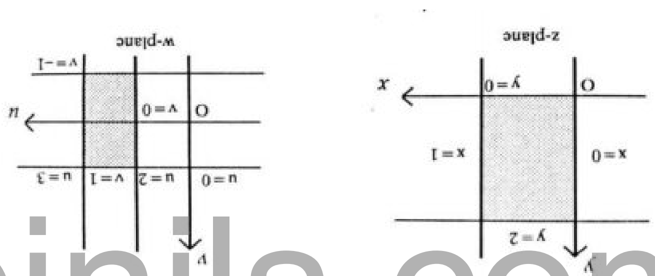
$$u = x + 2, v = y - 1$$

Given boundary lines are	transformed boundary lines are
$x = 0$	$u = 0 + 2 = 2$
$y = 0$	$v = 0 - 1 = -1$
$x = 1$	$u = 1 + 2 = 3$
$y = 2$	$v = 2 - 1 = 1$

Hence, the lines $x = 0, y = 0, x = 1,$ and $y = 2$ are mapped into the lines $u = 2, v = -1,$ $u = 3, and\ v = 1$ respectively which form a rectangle in the w plane.

Example: Find the image of the circle $|z| = 1$ by the transformation $w = z + 2 + 4i$

Solution:



Given $w = z + 2 + 4i$

$$(i.e.)\ u + iv = x + iy + 2 + 4i$$
$$= (x + 2) + i(y + 4)$$

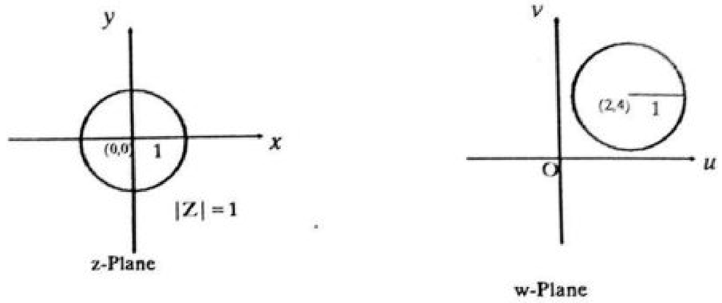
Equating the real and imaginary parts, we get

$$u = x + 2, v = y + 4,$$
$$x = u - 2, y = v - 4,$$

Given $|z| = 1$

$$(i.e.)\ x^2 + y^2 = 1$$
$$(u - 2)^2 + (v - 4)^2 = 1$$

Hence, the circle $x^2 + y^2 = 1$is mapped into $(u - 2)^2 + (v - 4)^2 = 1$ in w plane which is also a circle with centre $(2, 4)$and radius 1.



2. Magnification and Rotation

The transformation $w = cz$, where c is a complex constant, represents both magnification and rotation.

This means that the magnitude of the vector representing z is magnified by $a = |c|$ and its direction is rotated through angle $\alpha = amp\ (c)$. Hence the transformation consists of a magnification and a rotation.

Example: Determine the region 'D' of the w-plane into which the triangular region D enclosed by the lines $x = 0, y = 0, x + y = 1$ is transformed under the transformation $w = 2z$.

Solution:

Let $w = u + iv$

$z = x + iy$

Given $w = 2z$

$u + iv = 2(x + iy)$

$u + iv = 2x + i2y$

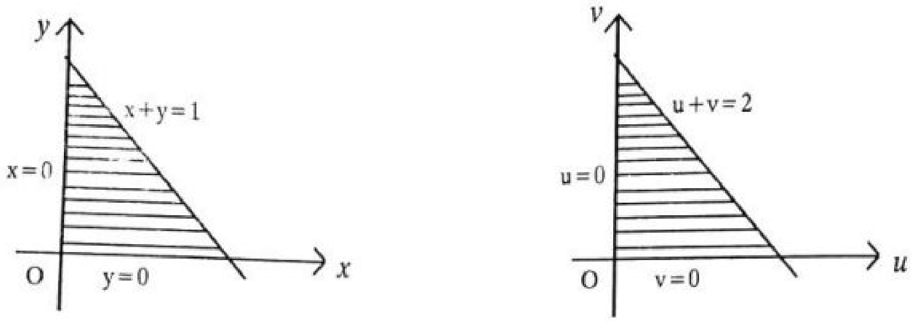
$u = 2x \Rightarrow x = \frac{u}{2}, v = 2y \Rightarrow y = \frac{v}{2}$

Given region (D) whose boundary lines are		Transformed region D' whose boundary lines are
$x = 0$	$\Rightarrow$	$u = 0$
$y = 0$	$\Rightarrow$	$v = 0$
$x + y = 1$	$\Rightarrow$	$\frac{u}{2} + \frac{v}{2} = 1[\because x = \frac{u}{2}, y = \frac{v}{2}]$ $(i.e.)\ u + v = 2$

In the z plane the line $x = 0$ is transformed into $u = 0$in the w plane.

In the z plane the line $y = 0$ is transformed into $v = 0$in the w plane.

In the z plane the line $x + y =$ is transformed into $u + v = 2$ in the w plane.



z-plane w-plane

Example: Find the image of the circle $|z| = \lambda$ under the transformation $w = 5z$.

Solution:

Given $w = 5z$

$|w| = 5|z|$

i.e., $|w| = 5\lambda \quad [\because |z| = \lambda]$

Hence, the image of $|z| = \lambda$ in the z plane is transformed into $|w| = 5\lambda$ in the w plane under the transformation $w = 5z$.

Example: Find the image of the circle $|z| = 3$ under the transformation $w = 2z$

Solution:

Given $w = 2z, \; |z| = 3$

$|w| = (2)|z|$

$= (2)(3), \quad$ Since $|z| = 3$

$= 6$

Hence, the image of $|z| = 3$ in the z plane is transformed into $|w| = 6$ w plane under the transformation $w = 2z$.

Example: Find the image of the region $y > 1$ under the transformation $w = (1 - i)z$.

Solution:

Given $w = (1 - i)z$.

$u + v = (1 - i)(x + iy)$

$= x + iy - ix + y$

$= (x + y) + i(y - x)$

i.e., $u = x + y, \quad v = y - x$

$u + v = 2y \qquad u - v = 2x$

$$y = \frac{u+v}{2} \qquad x = \frac{u-v}{2}$$

Hence, image region $y > 1$ *is* $\frac{u+v}{2} > 1$ i.e., $u + v > 2$ in the w plane.

3. Inversion and Reflection

The transformation $w = \frac{1}{z}$ represents inversion w.r.to the unit circle $|z| = 1$, followed by reflection in the real axis.

$$\Rightarrow w = \frac{1}{z}$$

$$\Rightarrow z = \frac{1}{w}$$

$$\Rightarrow x + iy = \frac{1}{u+iv}$$

$$\Rightarrow x + iy = \frac{1}{u^2+v^2}$$

$$\Rightarrow x = \frac{1}{u^2+v^2} \qquad \dots(1)$$

$$\Rightarrow y = \frac{-v}{u^2+v^2} \qquad \dots(2)$$

We know that, the general equation of circle in z plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \dots(3)$$

Substitute, (1) and (2) in (3)we get

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + 2g\left(\frac{u}{u^2+v^2}\right) + 2f\left(\frac{-v}{u^2+v^2}\right) + c = 0$$

$$\Rightarrow c(u^2 + v^2) + 2gu - 2fv + 1 = 0 \qquad \dots(4)$$

which is the equation of the circle in w plane

Hence, under the transformation $w = \frac{1}{z}$ a circle in z plane transforms to another circle in the w plane. When the circle passes through the origin we have $c = 0$ in (3). When $c = 0$, equation (4) gives a straight line.

Example: Find the image of $|z - 2i| = 2$ under the transformation $w = \frac{1}{z}$

Solution:

Given $|z - 2i| = 2$......(1) is a circle.

Centre $= (0,2)$

radius $= 2$

Given $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

(1) $\Rightarrow \left|\frac{1}{w} - 2i\right| = 2$

$$\Rightarrow |1 - 2wi| = 2|w|$$

$$\Rightarrow |1 - 2(u + iv)i| = 2|u + iv|$$

$$\Rightarrow |1 - 2ui + 2v| = 2|u + iv|$$

$$\Rightarrow |1 + 2v - 2ui| = 2|u + iv|$$

$$\Rightarrow \sqrt{(1 + 2v)^2 + (-2u)^2} = 2\sqrt{u^2 + v^2}$$

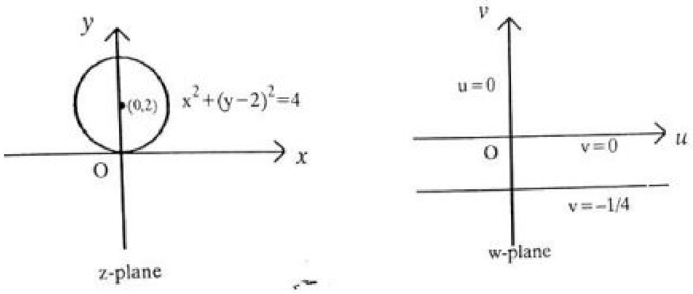
$$\Rightarrow (1 + 2v)^2 + 4u^2 = 4(u^2 + v^2)$$

$$\Rightarrow 1 + 4v^2 + 4v + 4u^2 = 4(u^2 + v^2)$$

$$\Rightarrow 1 + 4v = 0$$

$$\Rightarrow v = -\frac{1}{4}$$

Which is a straight line in w plane.



Example: Find the image of the circle $|z - 1| = 1$ in the complex plane under the mapping $w = \frac{1}{z}$

Solution:

Given $|z - 1| = 1$ ..... (1) is a circle.

Centre =(1,0)

radius = 1

Given $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

(1) $\Rightarrow |\frac{1}{w} - 1| = 1$

$$\Rightarrow |1 - w| = |w|$$

$$\Rightarrow |1 - (u + iv)| = |u + iv|$$

$$\Rightarrow |1 - u + iv| = |u + iv|$$

$$\Rightarrow \sqrt{(1 - u)^2 + (-v)^2} = \sqrt{u^2 + v^2}$$

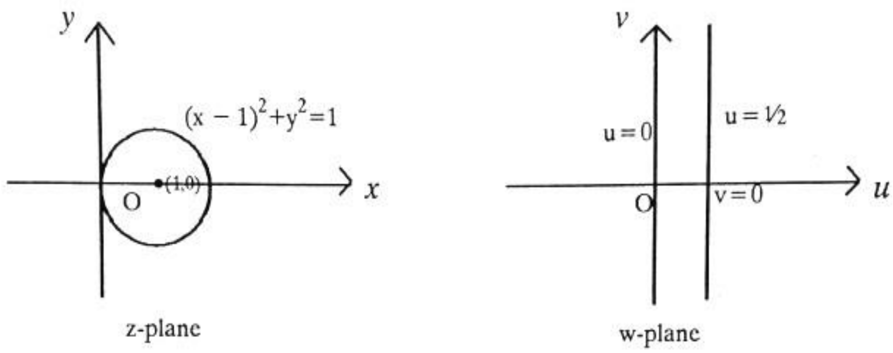
$$\Rightarrow (1 - u)^2 + v^2 = u^2 + v^2$$

$$\Rightarrow 1 + u^2 - 2v + v^2 = u^2 + v^2$$

$$\Rightarrow 2u = 1$$

$$\Rightarrow u = \frac{1}{2}$$

which is a straight line in the w- plane



Example: Find the image of the infinite strips

(i) $\frac{1}{4} < y < \frac{1}{2}$ **(ii)** $0 < y < \frac{1}{2}$ **under the transformation** $w = \frac{1}{z}$

Solution :

Given $w = \frac{1}{z}$ (given)

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \frac{u-iv}{u^2+v^2} = [\frac{u}{u^2+v^2}] + i\,[\frac{-v}{u^2+v^2}]$$

$$x = \frac{u}{u^2+v^2} \ldots.(1), y = \frac{-v}{u^2+v^2} \ldots.(2)$$

(i) Given strip is $\frac{1}{4} < y < \frac{1}{2}$

when $y = \frac{1}{4}$

$\frac{1}{4} = \frac{-v}{u^2+v^2}$ by (2)

$\Rightarrow u^2 + v^2 = -4v$

$\Rightarrow u^2 + v^2 + 4v = 0$

$\Rightarrow u^2 + (v+2)^2 = 4$

which is a circle whose centre is at $(0, -2)$ in the w plane and radius is 2k.

when $y = \frac{1}{2}$

$\frac{1}{2} = \frac{-v}{u^2+v^2}$ by (2)

$\Rightarrow u^2 + v^2 = -2v$

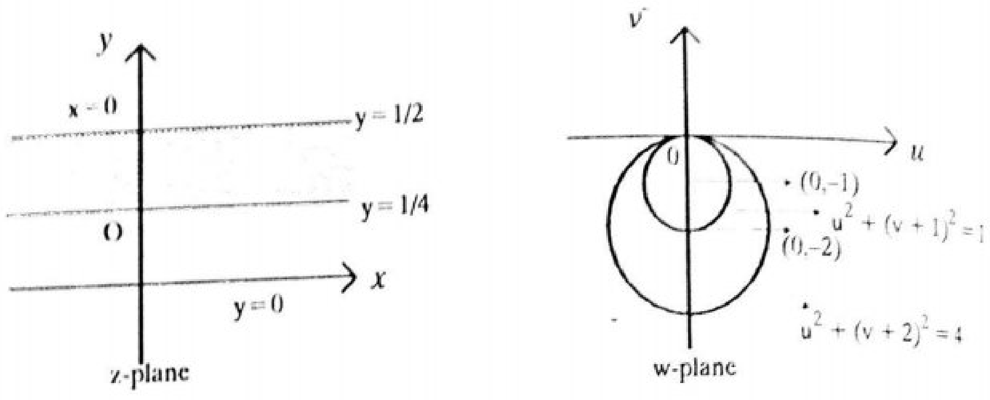
$\Rightarrow u^2 + v^2 + 2v = 0$

$\Rightarrow u^2 + (v+1)^2 = 0$

$$\Rightarrow u^2 + (v+1)^2 = 1 \qquad ......(3)$$

which is a circle whose centre is at $(0, -1)$ in the w plane and unit radius

Hence the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region in between circles $u^2 + (v+1)^2 = 1$ and $u^2 + (v+2)^2 = 4$ in the w plane.



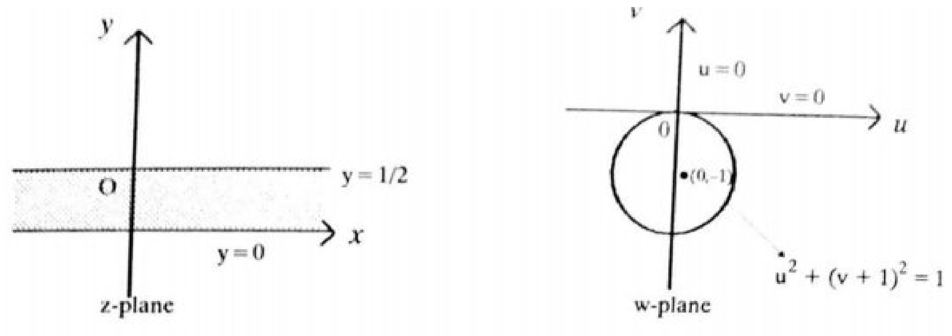
ii) Given strip is $0 < y < \frac{1}{2}$

when $y = 0$

$\Rightarrow v = 0$ by (2)

when $y = \frac{1}{2}$ we get $u^2 + (v+1)^2 = 1$ by (3)

Hence, the infinite strip $0 < y < \frac{1}{2}$ is mapped into the region outside the circle $u^2 + (v+1)^2 = 1$ in the lower half of the w plane.



Example: Find the image of $x = 2$ under the transformation $w = \frac{1}{z}$.

Solution:

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = [\frac{u}{u^2+v^2}] + i\,[\frac{-v}{u^2+v^2}]$$

$$\text{i.e.,} \quad x = \frac{u}{u^2+v^2} \;.... (1), y = \frac{-v}{u^2+v^2} \qquad ... (2)$$

Given $x = 2$ in the z plane.

$$\therefore 2 = \frac{u}{u^2+v^2} \qquad \text{by } (1)$$

$$2(u^2 + v^2) = u$$

$$u^2 + v^2 - \frac{1}{2}u = 0$$

which is a circle whose centre is $(\frac{1}{4}, 0)$ and radius $\frac{1}{4}$

$\therefore x = 2$ in the z plane is transformed into a circle in the w plane.

Example: What will be the image of a circle containing the origin(i.e., circle passing through the origin) in the XY plane under the transformation $w = \frac{1}{z}$?

Solution:

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = [\frac{u}{u^2+v^2}] + i\,[\frac{-v}{u^2+v^2}]$$

$$\text{i.e.,} \quad x = \frac{u}{u^2+v^2} \qquad ... (1),$$

$$y = \frac{-v}{u^2+v^2} \qquad ... (2)$$

Given region is circle $x^2 + y^2 = a^2$ in z plane.

Substitute, (1) and (2), we get

$$[\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}] = a^2$$

$$[\frac{u^2+v^2}{(u^2+v^2)^2}] = a^2$$

$$\frac{1}{(u^2+v^2)} = a^2$$

$$u^2 + v^2 = \frac{1}{a^2}$$

Therefore the image of circle passing through the origin in the XY $-$plane is a circle passing through the origin in the $w -$ plane.

Example: Determine the image of $1 < x < 2$ under the mapping $w = \frac{1}{z}$

Solution:

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = [\frac{u}{u^2+v^2}] + i\,[\frac{-v}{u^2+v^2}]$$

i.e., $x = \frac{u}{u^2+v^2}$ ....(1), $y = \frac{-v}{u^2+v^2}$ ....(2)

Given $1 < x < 2$

When $x = 1$

$\Rightarrow 1 = \frac{u}{u^2+v^2}$ by ....(1)

$\Rightarrow u^2 + v^2 = u$

$\Rightarrow u^2 + v^2 - u = 0$

which is a circle whose centre is $(\frac{1}{2}, 0)$ and is $\frac{1}{2}$

When $x = 2$

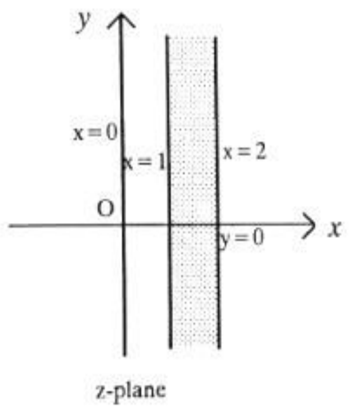
$\Rightarrow 2 = \frac{u}{u^2+v^2}$ by ....(1)

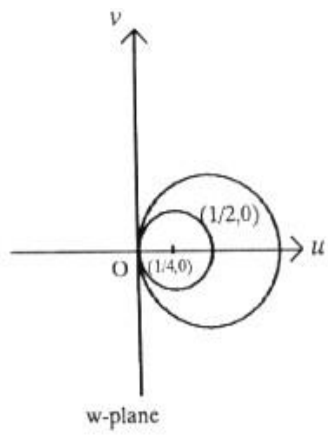
$\Rightarrow u^2 + v^2 = \frac{u}{2}$

$\Rightarrow u^2 + v^2 - \frac{u}{2} = 0$

which is a circle whose centre is $(\frac{1}{4}, 0)$ and is $\frac{1}{4}$

Hence, the infinite strip $1 < x < 2$ is transformed into the region in between the circles in the $w -$ plane.





4. Transformation $w = z^2$

Problems based on $w = z^2$

Example: Discuss the transformation $w = z^2$.

Solution:

Given $w = z^2$

$$u + iv = (x + iy)^2 = x^2 + (iy)^2 + i2xy = x^2 - y^2 + i2xy$$

$$i.e., u = x^2 - y^2 \quad ....(1), \quad v = 2xy \quad ....(2)$$

Elimination:

$$(2) \Rightarrow x = \frac{v}{2y}$$

$$(1) \Rightarrow u = \left(\frac{v}{2y}\right)^2 - y^2$$

$$\Rightarrow u = \frac{v^2}{4y^2} - y^2$$

$$\Rightarrow 4uy^2 = v^2 - 4y^4$$

$$\Rightarrow 4uy^2 + 4y^4 = v^2$$

$$\Rightarrow y^2[4u + 4y^2] = v^2$$

$$\Rightarrow 4y^2[u + y^2] = v^2$$

$$\Rightarrow v^2 = 4y^2(y^2 + u)$$

when $y = c\ (\neq 0)$, we get

$$v^2 = 4c^2(u+c^2)$$

which is a parabola whose vertex at $(-c^2, 0)$ and focus at $(0,0)$

Hence, the lines parallel to X-axis in the z plane is mapped into family of confocal parabolas in the w plane.

when $y = 0$, we get $v^2 = 0$ i.e., $v = 0$, $u = x^2$ i.e., $u > 0$

Hence, the line $y = 0$, in the z plane are mapped into $v = 0$, in the w plane.

Elimination:

$$(2) \Rightarrow y = \frac{v}{2x}$$

$$(1) \Rightarrow u = x^2 - \left(\frac{v}{2x}\right)^2$$

$$\Rightarrow u = x^2 - \frac{v^2}{4x^2}$$

$$\Rightarrow \frac{v^2}{4x^2} = x^2 - u$$

$$\Rightarrow v^2 = (4x^2)(x^2 - u)$$

when $x = c\ (\neq 0)$, we get $v^2 = 4c^2(c^2 - u) = -4c^2(u - c^2)$

which is a parabola whose vertex at $(c^2, 0)$ and focus at (0,0) and axis lies along the u −axis and which is open to the left.

Hence, the lines parallel to y axis in the z plane are mapped into confocal parabolas in the w plane when $x = 0$, we get $v^2 = 0$. i.e., $v = 0, u = -y^2$ i.e., $u < 0$

i.e., the map of the entire y axis in the negative part or the left half of the u −axis.

Example: Find the image of the hyperbola $x^2 - y^2 = 10$ under the transformation $w = z^2$ if

$w = u + iv$

Solution:

Given $w = z^2$

$$u + iv = (x + iy)^2$$

$$= x^2 - y^2 + i2xy$$

$$i.e., u = x^2 - y^2 \quad \ldots\ldots (1)$$

$$v = 2xy \quad \ldots\ldots (2)$$

Given $x^2 - y^2 = 10$

i.e., $u = 10$

Hence, the image of the hyperbola $x^2 - y^2 = 10$ in the z plane is mapped into $u = 10$ in the w plane which is a straight line.

Example: Find the critical points of the transformation $w^2 = (z - \alpha)\,(z - \beta)$.

Solution:

Given $w^2 = (z - \alpha)\,(z - \beta) \quad \ldots(1)$

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow 2w\frac{dw}{dz} = (z - \alpha) + (z - \beta)$$

$$= 2z - (\alpha + \beta)$$

$$\Rightarrow \frac{dw}{dz} = \frac{2z - (\alpha + \beta)}{2w} \quad \ldots (2)$$

Case (i) $\frac{dw}{dz} = 0$

$$\Rightarrow \frac{2z - (\alpha + \beta)}{2w} = 0$$

$$\Rightarrow 2z - (\alpha + \beta) = 0$$

$$\Rightarrow 2z = \alpha + \beta$$

$$\Rightarrow z = \frac{\alpha+\beta}{2}$$

Case (ii) $\frac{dz}{dw} = 0$

$$\Rightarrow \frac{2w}{2z-(\alpha+\beta)} = 0$$

$$\Rightarrow \frac{w}{z-\frac{\alpha+\beta}{2}} = 0$$

$$\Rightarrow w = 0 \Rightarrow (z-\alpha)\,(z-\beta) = 0$$

$$\Rightarrow z = \alpha, \beta$$

$\therefore$ The critical points are $\frac{\alpha+\beta}{2}$, α and β.

Example: Find the critical points of the transformation $w = z^2 + \frac{1}{z^2}$.

Solution:

Given $w = z^2 + \frac{1}{z^2}$ ... (1)

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow \frac{dw}{dz} = 2z - \frac{2}{z^3} = \frac{2z^4-2}{z^3}$$

Case (i) $\frac{dw}{dz} = 0$

$$\Rightarrow \frac{2z^4-2}{z^3} = 0 \Rightarrow 2z^4 - 2 = 0$$

$$\Rightarrow z^4 - 1 = 0$$

$$\Rightarrow z = \pm 1, \pm i$$

Case (ii) $\frac{dz}{dw} = 0$

$$\Rightarrow \frac{z^3}{2z^4-2} = 0 \Rightarrow z^3 = 0 \Rightarrow z = 0$$

$\therefore$ The critical points are ± 1 , $\pm i$, 0

Example: Prove that the transformation $w = \frac{z}{1-z}$ maps the upper half of the z plane into the upper half of the w plane. What is the image of the circle $|z| = 1$ under this transformation.

Solution:

Given $|z| = 1$ is a circle

Centre $= (0,0)$

Radius $= 1$

Given $w = \frac{z}{1-z}$

$$\Rightarrow z = \frac{w}{w+1}$$

$$\Rightarrow |z| = \left|\frac{w}{w+1}\right| = \frac{|w|}{|w+1|}$$

Given $|z| = 1$

$$\Rightarrow \frac{|w|}{|w+1|} = 1$$

$$\Rightarrow |w| = |w+1|$$

$$\Rightarrow |u+iv| = |u+iv+1|$$

$$\Rightarrow \sqrt{u^2+v^2} = \sqrt{(u+1)^2+v^2}$$

$$\Rightarrow u^2+v^2 = (u+1)^2+v^2$$

$$\Rightarrow u^2+v^2 = u^2+2u+1+v^2$$

$$\Rightarrow 0 = 2u+1$$

$$\Rightarrow u = \frac{-1}{2}$$

Further the region $|z| < 1$ transforms into $u > \frac{-1}{2}$

BILINEAR TRANSFORMATION

A bilinear transformation is also called a linear fractional transformation because $\frac{az+b}{cz+d}$ is a fraction formed by the linear fuctions $az - b$ and $cz + d$.

Theorem: 1 Under a bilinear transformation no two points in z plane go to the same point in w plane.

Proof:

Suppose z_1 and z_2 go to the same point in the w plane under the transformation $w = \frac{az+b}{cz+d}$.

Then $\frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d}$

$$\Rightarrow (az_1 + b)(cz_2 + d) = (az_2 + b)(cz_1 + d)$$

$$i.e., \quad (az_1 + b)(cz_2 + d) - (az_2 + b)(cz_1 + d) = 0$$

$$\Rightarrow acz_1 z_2 + adz_1 + bcz_2 + bd - acz_1 z_2 - adz_2 - bcz_1 - bd = 0$$

$$\Rightarrow (ad - bc)(z_1 - z_2) = 0$$

$$\text{or} \quad z_1 = z_2 \qquad [\because ad - bc \neq 0]$$

This implies that no two distinct points in the z plane go to the same point in w plane. So, each point in the z plane go to a unique point in the w plane.

Theorem: 2 The bilinear transformation which transforms z_1, z_2, z_3, into w_1, w_2, w_3 is $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$

Proof:

If the required transformation $w = \frac{az+b}{cz+d}$.

$$\Rightarrow w - w_1 = \frac{az+b}{cz+d} - \frac{az_1+b}{cz_1+d} = \frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}$$

$$\Rightarrow (cz + d)(cz_1 + d)(w - w_1) = (ad - bc)(z - z_1)$$

$$\Rightarrow (cz_2 + d)(cz_3 + d)(w_2 - w_3) = (ad - bc)(z_2 - z_3)$$

$$\Rightarrow (cz + d)(cz_3 + d)(w - w_3) = (ad - bc)(z - z_3)$$

$$\Rightarrow (cz_2 + d)(cz_1 + d)(w_2 - w_1) = (ad - bc)(z_2 - z_1)$$

$$\Rightarrow \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{\left[\frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}\right]\left[\frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}\right]}{\left[\frac{(ad-bc)(z-z_3)}{(cz+d)(cz_3+d)}\right]\left[\frac{(ad-bc)(z_2-z_1)}{(cz_2+d)(cz_1+d)}\right]}$$

$$= \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

Now, $\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$ ... (1)

Let : $A = \frac{w_2-w_3}{w_2-w_1}, B = \frac{z_2-z_3}{z_2-z_1}$

$(1) \Rightarrow \frac{w-w_1}{w-w_3} A = \frac{z-z_1}{z-z_3} B$

$\frac{wA-w_1A}{w-w_3} = \frac{zB-z_1B}{z-z_3}$

$\Rightarrow wAz - wAz_3 - w_1Az + w_1Az_3 = wBz - wz_1B - w_3zB + w_3z_1B$

$\Rightarrow w[(A-B)z + (Bz_1 - Az_3)] = (Aw_1 - Bw_3)z + (Bw_3z_1 - Aw_1z_3)$

$\Rightarrow w = \frac{(Aw_1-Bw_3)z+(Bw_3z_1-Aw_1z_3)}{(A-B)z+(Bz_1-Az_3)}$

$\frac{az+b}{cz+d}$, Hence $a = Aw_1 - Bw_3, b = Bw_3z_1 - Aw_1z_3, c = A - B, d = Bz_1 - Az_3$

Cross ratio

Definition:

Given four point z_1, z_2, z_3, z_4 in this order, the ratio $\frac{(z-z_1)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is called the cross ratio of the points.

Note: (1) $w = \frac{az+b}{cz+d}$ can be expressed as $cwz + dw - (az + b) = 0$

It is linear both in w and z that is why, it is called bilinear.

Note: (2) This transformation is conformal only when $\frac{dw}{dz} \neq 0$

$i.e., \frac{ad-bc}{(cz+d)^2} \neq 0$

$i.e., ad - bc \neq 0$

If $ad - bc \neq 0$, every point in the z plane is a critical point.

Note: (3) Now, the inverse of the transformation $w = \frac{az+b}{cz+d}$ is $z = \frac{-dw+b}{cw-a}$ which is also a bilinear transformation except $w = \frac{a}{c}$.

Note: (4) Each point in the plane except $z = \frac{-d}{c}$ corresponds to a unique point in the w plane.

The point $z = \frac{-d}{c}$ corresponds to the point at infinity in the w plane.

Note: (5) The cross ratio of four points

$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$ is invariant under bilinear transformation.

Note: (6) If one of the points is the point at infinity the quotient of those difference which involve this points is replaced by 1.

Suppose $z_1 = \infty$, then we replace $\frac{z-z1}{z_2-z_1}$ by 1 (or)Omit the factors involving ∞

Example: Find the fixed points of $w = \frac{2zi+5}{z-4i}$.

Solution:

The fixed points are given by replacing w by z

$$z = \frac{2zi+5}{z-4i}$$

$$z^2 - 4iz = 2zi + 5\ ;\ z^2 - 6iz - 5 = 0$$

$$z = \frac{6i\pm\sqrt{-36+20}}{2} \qquad \therefore z = 5i, i$$

Example: Find the invariant points of $w = \frac{1+z}{1-z}$

Solution:

The invariant points are given by replacing w by z

$$z = \frac{1+z}{1-z}$$

$$\Rightarrow z - z^2 = 1 + z$$

$$\Rightarrow z^2 = -1$$

$$\Rightarrow z = \pm i$$

Example: Obtain the invariant points of the transformation $w = 2 - \frac{2}{z}$

Solution:

The invariant points are given by

$$z = 2 - \frac{2}{z}\ ; \qquad z = \frac{2z-2}{z}$$

$$z^2 = 2z - 2; \qquad z^2 - 2z + 2 = 0$$

$$z = \frac{2\pm\sqrt{4-8}}{2} = \frac{2\pm\sqrt{-4}}{2} = \frac{2\pm 2i}{2} = 1 \pm i$$

Example: Find the fixed point of the transformation $w = \frac{6z-9}{z}$

Solution:

The fixed points are given by replacing $w = z$

i.e., $w = \frac{6z-9}{z} \Rightarrow z = \frac{6z-9}{z}$

$$\Rightarrow z^2 = 6z - 9$$

$$\Rightarrow z^2 - 6z + 9 = 0$$

$$\Rightarrow (z-3)^2 = 0$$

$$\Rightarrow z = 3,3$$

The fixed points are $3, 3$.

Example: Find the bilinear transformation that maps the points $z = 0, -1, i$ into the points $w = i, 0, \infty$ respectively.

Solution:

Given $z_1 = 0,\ z_2 = -1,\ z_3 = i,$

$w_1 = i,\ w_2 = 0,\ w_3 = \infty,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[omit the factors involving w_3, since $w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{w-i}{0-i} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\Rightarrow \frac{w-i}{-i} = \frac{z}{(z-i)}(1+i)$$

$$\Rightarrow w - i = \frac{z}{(z-i)}(-i+1)$$

$$\Rightarrow w = \frac{z}{(z-i)}(-i+1) + i = \frac{-iz+z+iz+1}{(z-i)} = \frac{z+1}{z-i}$$

Aliter: Given $z_1 = 0,\ z_2 = -1,\ z_3 = i,$

$w_1 = i,\ w_2 = 0,\ w_3 = \infty,$

Let the required transformation be

$$w = \frac{az+b}{cz+d} \ldots (1),\ ad - bc \neq 0$$

$$i = \frac{b}{d}$$

$$w_1 = \frac{az_1+b}{cz_1+d}$$

$$i = \frac{b}{d}$$

$$b = di$$

$$w_2 = \frac{az_2+b}{cz_2+d}$$

$$0 = \frac{-a+b}{-c+d}$$

$$\Rightarrow -a + b = 0$$

$$\Rightarrow a = b$$

$$w_3 = \frac{az_3+b}{cz_3+d}$$

$$\frac{1}{0} = \frac{ai+b}{ci+d}$$

$$\Rightarrow ci + d = 0$$

$$\Rightarrow d = -ci$$

$$\therefore a = b = di = c$$

$$\therefore (1) \Rightarrow w = \frac{az+a}{az+\frac{a}{i}} = \frac{z+1}{z+\frac{1}{i}} = \frac{z+1}{z-i}$$

Example: Find the bilinear transformation that maps the points $\infty, i, 0$ onto $0, i, \infty$ respectively.

Solution:

Given $z_1 = \infty,\ z_2 = i,\ z_3 = 0, w_1 = 0,\ w_2 = i,\ w_3 = \infty,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[omit the factors involving z_1, and w_3, since $z_1 = \infty, w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z_2-z_3)}{z-z_3}$$

$$\Rightarrow \frac{w-0}{i-0} = \frac{i-0}{z-0}$$

$$\Rightarrow w = \frac{-1}{z}$$

Example: Find the bilinear transformation which maps the points $1, i, -1$ onto the points $0, 1, \infty$, show that the transformation maps the interior of the unit circle of the $z -$ plane onto the upper half of the $w -$ plane

Solution:

Given $z_1 = 1,\ z_2 = i,\ z_3 = -1$

$w_1 = 0,\ w_2 = 1,\ w_3 = \infty,$

Let the transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

[Omit the factors involving w_3, since , $w_3 = \infty$]

$$\Rightarrow \frac{w-w_1}{w_2-w_1} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\Rightarrow \frac{w-0}{1-0} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$ $\because [(\frac{i+1}{i-1})(\frac{i+1}{i+1})] = [\frac{i^2+i+i+1}{i^2-i^2}]$

$=[\frac{2i}{-2}] = -i$

$$\Rightarrow w = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$= \frac{z-1}{z+1}[-i]$$

$$\Rightarrow w = \frac{(-i)z+i}{(1)z+1} [\because w = \frac{az+b}{cz+d}, ad - bc \neq 0 \text{ Form}]$$

To find z:

$$\Rightarrow wz + w = -iz + i$$

$$\Rightarrow wz + iz = -w + i$$

$$\Rightarrow z[w + i] = -w + i$$

$$\Rightarrow z = \frac{(w-i)}{w+i}$$

To prove: $|z| < 1$ maps $v > 0$

$$\Rightarrow |z| < 1$$

$$\Rightarrow |\frac{-(w-i)}{w+i}| < 1$$

$$\Rightarrow \left[\frac{w-i}{w+i}\right] < 1$$

$$\Rightarrow |w-i| < |w+i|$$

$$\Rightarrow |u+iv-i| < |u+iv+i|$$

$$\Rightarrow |u+i(v-1)| < |u+i(v+i)|$$

$$\Rightarrow u^2+(v-1)^2 < u^2+(v+1)^2$$

$$\Rightarrow (v-1)^2 < (v+1)^2$$

$$\Rightarrow v^2-2v+1 < v^2+2v+1$$

$$\Rightarrow -4v < 0$$

$$\Rightarrow v > 0$$

Example: Find the bilinear transformation which maps $z = 1, i, -1$ respectively onto $w = i, 0, -i$. Hence find the fixed points. [A.U, May 2001] [A.U April 2016 R-15 U.D]

Solution:

Given $z_1 = 1,\ z_2 = i,\ z_3 = -1,$

$w_1 = i,\ w_2 = 0,\ w_3 = -i,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$Let\quad A = \frac{w_2-w_3}{w_2-w_1} = \frac{0+i}{0-i} = -1$$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{i+1}{i-1} = -i$$

$$\Rightarrow a = Aw_1 - Bw_3 = (-1)(i)-(-i)(-i) = -i+1$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = (-i)(-i)(1)-(-1)(i)(-1) = -1-i$$

$$\Rightarrow c = A - B = (-1)-(-i) = -1+i$$

$$\Rightarrow d = Bz_1 - Az_3 = (-i)(1)-(-1)(-1) = -i-1$$

We know that, $w = \frac{az+b}{cz+d}, ad-bc \neq 0$

$$\therefore w = \frac{(-i+1)z+(-1-i)}{(-1+i)z+(-i-1)} = \frac{iz+1}{(-i)z+1}$$

Example: Find the bilinear transformation which maps $z = 0$ onto $w = -i$ and has -1 and 1 as the invariant points. Also show that under this transformation the upper half of the z plane maps onto the interior of the unit circle in the w plane.

Solution:

Given $z_1 = 0,\ z_2 = -1,\ z_3 = 1,$

$w_1 = -i,\ w_2 = -1,\ w_3 = 1,$

Let the required transformation be

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$Let \quad A = \frac{w_2-w_3}{w_2-w_1} = \frac{-1-1}{-1+i} = \frac{-2}{-1+i} = 1+i$$

$$B = \frac{z_2-z_3}{z_2-z_1} = \frac{-1-1}{-1-0} = 2$$

$$\Rightarrow a = Aw_1 - Bw_3 = (1+i)(-i) - 2(1) = -i + 1 - 2 = -i - 1$$

$$\Rightarrow b = Bw_3z_1 - Aw_1z_3 = (2)(1)(0) - (1+i)(-i)(1) = i - 1$$

$$\Rightarrow c = A - B = (1+i) - 2 = i - 1$$

$$\Rightarrow d = Bz_1 - Az_3 = (2)(0) - (1+i)(1) = -(1+i)$$

We know that, $w = \frac{az+b}{cz+d}, ad - bc \neq 0$

$$\therefore w = \frac{(-i+1)z+(i-1)}{(i-1)z+(-1-i)} = \frac{z+(-i)}{(-i)z+1}$$

We know that, $z = \frac{-dw+b}{cw-a} = \frac{-w-i}{-iw-1} = \frac{w+i}{1+wi}$

$$z = \frac{u+iv+i}{1(u+iv)i}$$

$$= \frac{u+iv+i}{1+iu-v} = \frac{u+iv+i}{(1-v)+iu}$$

$$= \left[\frac{u+iv+i}{(1-v)+iu}\right]\left[\frac{1-v-iu}{(1-v)-iu}\right]$$

$$= \frac{u-uv-iu^2+iv-iv^2+uv+i-iv+u}{(1-v)^2+u^2}$$

$$x + iy = \frac{2u+i[-u^2-v^2+1]}{(1-v)^2+u^2}$$

$$\Rightarrow \quad y \quad = \frac{1-u^2-v^2}{(1-v)^2+u^2}$$

Upper half of the z –plane

$$\Rightarrow y \geq 0$$

$$\Rightarrow \frac{1-u^2-v^2}{(1-v)^2+u^2} \geq 0$$

$$\Rightarrow 1 - u^2 - v^2 \geq 0$$

$$\Rightarrow 1 \geq u^2 + v^2$$

$$\Rightarrow u^2 + v^2 \leq 1$$

Therefore the upper half of the z –plane maps onto the interior of the unit circles in the w-plane.