Catalog

EIGEN VALUES AND EIGEN VECTORS

Definition

The values of λ obtained from the characteristic equation $|A-\lambda I| = 0$ are called Eigenvalues of 'A'.[or Latent values of A or characteristic values of A]

Definition

Let A be square matrix of order 3 and λ be scaler. The column matrix $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ which satisfies $(A-\lambda I)X = 0$ is called Eigen vector or Latent vector or characteristic vector.

Example: Find the Eigen values for the matrix $\begin{pmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 = sum of the main diagonal element

$= 2 + 3 + 2 = 7$

s_2 = sum of the minors of the main diagonalelement

$= \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4 + 3 + 4 = 11$

$s_3 = |A| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 2(6-2) - 2(2-1) + 1(2-3)$

$= 8 - 2 - 1 = 5$

Characteristic equation is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$\Rightarrow \lambda = 1, \lambda^2 - 6\lambda + 5 = 0$

$\Rightarrow \lambda = 1, \ (\lambda - 1)(\lambda - 5) = 0$

The Eigen values are $\lambda = 1, 1, 5$

Example: Determine the Eigen values for the matrix $\begin{pmatrix} \mathbf{-2} & \mathbf{2} & \mathbf{-3} \\ \mathbf{2} & \mathbf{1} & \mathbf{-6} \\ \mathbf{-1} & \mathbf{-2} & \mathbf{0} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 = sum of the main diagonal element

$= -2 + 1 + 0 = -1$

s_2 = sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} = -12 - 3 - 6 = -21$$

$$s_3 = |A| = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix} = -2(0-12) - 2(0-6) - 3(-4+1)$$

$$= 24 + 12 + 9 = 45$$

Characteristic equation is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$$\Rightarrow \lambda = -3 \;, \qquad \lambda^2 - 2\lambda - 15 = 0$$

$$\Rightarrow \lambda = -3,\; (\lambda+3)(\lambda-5) = 0$$

The Eigen values are $\lambda = -3, -3, 5$

Eigen values and Eigen vectors for Non – Symmetric matrix

Example: Find the Eigen values and Eigen vectors for the matrix $\begin{pmatrix} \mathbf{8} & \mathbf{-6} & \mathbf{2} \\ \mathbf{-6} & \mathbf{7} & \mathbf{-4} \\ \mathbf{2} & \mathbf{-4} & \mathbf{3} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 = sum of the main diagonal element

$$= 8 + 7 + 3 = 18$$

s_2 = sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix} = 5 + 20 + 20 = 45$$

$$s_3 = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = 8(21-16) + 6(-18+8) + 2(24-14)$$

$$= 40 - 40 + 20 = 0$$

Characteristic equation is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$$\Rightarrow \lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\Rightarrow \lambda = 0,\; (\lambda - 15)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 0, 3, 15$$

To find the Eigen vectors:

Case(i) When $\lambda = 0$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$8x_1 - 6x_2 + 2x_3 = 0 \ldots (1)$$

$$-6x_1 + 7x_2 - 4x_3 = 0 \ldots (2)$$

$$2x_1 - 4x_2 + 3x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{24-14} = \frac{x_2}{-12+32} = \frac{x_3}{56-36}$$

$$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Case (ii) When $\lambda = 3$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$5x_1 - 6x_2 + 2x_3 = 0 \ldots (4)$$

$$-6x_1 + 4x_2 - 4x_3 = 0 \ldots (5)$$

$$2x_1 - 4x_2 + 0x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{24-8} = \frac{x_2}{-12+20} = \frac{x_3}{20-36}$$

$$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Case (iii) When $\lambda = 15$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$7x_1 - 6x_2 + 2x_3 = 0 \ldots (7)$$
$$-6x_1 - 8x_2 - 4x_3 = 0 \ldots (8)$$
$$2x_1 - 4x_2 - 12x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$
$$\frac{x_1}{40} = \frac{x_2}{-40} = \frac{x_3}{20}$$
$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$
$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$; $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$; $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Example: Determine the Eigen values and Eigen vectors of the matrix $\begin{pmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 = sum of the main diagonal element

$= 7 + 6 + 5 = 18$

s_2 = sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 6 & -2 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} 7 & 0 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 7 & -2 \\ -2 & 6 \end{vmatrix} = 26 + 35 + 38 = 99$$

$$s_3 = |A| = \begin{vmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{vmatrix} = 182 - 20 + 0 = 162$$

Characteristic equation is $\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$

$$\Rightarrow \lambda = 3, (\lambda^2 - 15\lambda + 54) = 0$$
$$\Rightarrow \lambda = 3, \ (\lambda - 9)(\lambda - 6) = 0$$
$$\Rightarrow \lambda = 3, 6, 9$$

To find the Eigen vectors:

Case (i) When $\lambda = 3$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 7-3 & -2 & 0 \\ -2 & 6-3 & -2 \\ 0 & -2 & 5-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$4x_1 - 2x_2 + 0x_3 = 0 \ldots (1)$$

$$-2x_1 + 3x_2 - 2x_3 = 0 \ldots (2)$$

$$0x_1 - 2x_2 + 2x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{4-0} = \frac{x_2}{0+8} = \frac{x_3}{12-4}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{8}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2}$$

$$X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Case (ii) When $\lambda = 6$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 7-6 & -2 & 0 \\ -2 & 6-6 & -2 \\ 0 & -2 & 5-6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 - 2x_2 + 0x_3 = 0 \ldots (4)$$

$$-2x_1 + 0x_2 - 2x_3 = 0 \ldots (5)$$

$$0x_1 - 2x_2 - x_3 = 0 \ldots (6)$$

From (4)and (5)

$$\frac{x_1}{4-0} = \frac{x_2}{0+2} = \frac{x_3}{0-4}$$

$$\frac{x_1}{4} = \frac{x_2}{2} = \frac{x_3}{-4}$$

$$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2}$$

$$X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

Case (iii) When $\lambda = 9$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 7-9 & -2 & 0 \\ -2 & 6-9 & -2 \\ 0 & -2 & 5-9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 2x_2 + 0x_3 = 0 \dots (7)$$

$$-2x_1 - 3x_2 - 2x_3 = 0 \dots (8)$$

$$0x_1 - 2x_2 - 4x_3 = 0 \dots (9)$$

From (7) and (8)

$$\frac{x_1}{4-0} = \frac{x_2}{0-4} = \frac{x_3}{6-4}$$

$$\frac{x_1}{4} = \frac{x_2}{-4} = \frac{x_3}{2}$$

$$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$; $X_2 = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$; $X_3 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

Example: Determine the Eigen values and Eigen vectors of the matrix $\begin{pmatrix} \mathbf{3} & \mathbf{1} & \mathbf{4} \\ \mathbf{0} & \mathbf{2} & \mathbf{6} \\ \mathbf{0} & \mathbf{0} & \mathbf{5} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 3 + 2 + 5 = 10$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 2 & 6 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 0 & 5 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 0 & 2 \end{vmatrix} = 10 + 15 + 6 = 31$$

$$s_3 = |A| = \begin{vmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{vmatrix} = 30$$

Characteristic equation is $\lambda^3 - 10\lambda^2 + 31\lambda - 30 = 0$

$$\Rightarrow \lambda = 2, (\lambda^2 - 8\lambda + 15) = 0$$

$$\Rightarrow \lambda = 2, \ (\lambda - 5)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 2,3,5$$

To find the Eigen vectors:

Case(i) When $\lambda = 2$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-2 & 1 & 4 \\ 0 & 2-2 & 6 \\ 0 & 0 & 5-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 + 4x_3 = 0 \ldots (1)$$

$$0x_1 + 0x_2 + 6x_3 = 0 \ldots (2)$$

$$0x_1 + 0x_2 + 3x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{6-0} = \frac{x_2}{0-6} = \frac{x_3}{0-0}$$

$$\frac{x_1}{6} = \frac{x_2}{-6} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Case (ii) When $\lambda = 3$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-3 & 1 & 4 \\ 0 & 2-3 & 6 \\ 0 & 0 & 5-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 + x_2 + 4x_3 = 0 \ldots (4)$$

$$0x_1 - x_2 + 6x_3 = 0 \ldots (5)$$

$$0x_1 + 0x_2 + 2x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{4+6} = \frac{x_2}{0-0} = \frac{x_3}{0-0}$$

$$\frac{x_1}{10} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Case (iii) When $\lambda = 5$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-5 & 1 & 4 \\ 0 & 2-5 & 6 \\ 0 & 0 & 5-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 + 4x_3 = 0 \ldots (7)$$

$$0x_1 - 3x_2 + 6x_3 = 0 \ldots (8)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{6+12} = \frac{x_2}{0+12} = \frac{x_3}{6-0}$$

$$\frac{x_1}{18} = \frac{x_2}{12} = \frac{x_3}{6}$$

$$\frac{x_1}{3} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$; $X_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $X_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

Problems on Symmetric matrices with repeated Eigen values

Example: Determine the Eigen values and Eigen vectors of the matrix $\begin{pmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 = sum of the main diagonal element

$= 6 + 3 + 3 = 12$

s_2 = sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = 8 + 14 + 14 = 36$$

$$s_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 32$$

Characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$$\Rightarrow \lambda = 2, (\lambda^2 - 10\lambda + 16) = 0$$

$$\Rightarrow \lambda = 2, \ (\lambda - 2)(\lambda - 8) = 0$$

$$\Rightarrow \lambda = 2,2,8$$

To find the Eigen vectors:

Case (i) When $\lambda = 8$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \ldots (1)$$

$$-2x_1 - 5x_2 - x_3 = 0 \ldots (2)$$

$$2x_1 - x_2 - 5x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 2$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$4x_1 - 2x_2 + 2x_3 = 0 \ldots (4)$$

$$-2x_1 + x_2 - x_3 = 0 \ldots (5)$$

$$2x_1 - x_2 + x_3 = 0 \ldots (6)$$

In (1) put $x_1 = 0 \quad \Rightarrow -2x_2 = -2x_3$

$$\Rightarrow \frac{x_2}{1} = \frac{x_3}{1} \Rightarrow X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

In (1) put $x_2 = 0 \quad \Rightarrow 4x_1 + 2x_3 = 0$

$$\Rightarrow 4x_1 = -2x_3$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_3}{2} \Rightarrow X_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$; $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

Example: Find the Eigen values and Eigen vectors of the matrix $\begin{pmatrix} \mathbf{2} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{2} & \mathbf{2} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 2 + 3 + 2 = 7$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4 + 3 + 4 = 1$$

$$s_3 = |A| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 5$$

Characteristic equation is $\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$

$$\Rightarrow \lambda = 1, (\lambda^2 - 6\lambda + 5) = 0$$

$$\Rightarrow \lambda = 1,\ (\lambda - 1)(\lambda - 5) = 0$$

$$\Rightarrow \lambda = 1, 1, 5$$

To find the Eigen vectors:

Case (i) When $\lambda = 5$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-5 & 2 & 1 \\ 1 & 3-5 & 1 \\ 1 & 2 & 2-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-3x_1 + 2x_2 + x_3 = 0 \ldots (1)$$

$$x_1 - 2x_2 + x_3 = 0 \ldots (2)$$

$$x_1 + 2x_2 - 3x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{2+2} = \frac{x_2}{1+3} = \frac{x_3}{6-2}$$

$$\frac{x_1}{4} = \frac{x_2}{4} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 1$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-1 & 2 & 1 \\ 1 & 3-1 & 1 \\ 1 & 2 & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + 2x_2 + x_3 = 0 \ldots (4)$$

$$x_1 + 2x_2 + x_3 = 0 \ldots (5)$$

$$x_1 + 2x_2 + x_3 = 0 \ldots (6)$$

In (1) put $x_1 = 0 \quad \Rightarrow 2x_2 = -x_3$

$$\Rightarrow \frac{x_2}{-1} = \frac{x_3}{2} \Rightarrow X_2 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$$

In (1) put $x_2 = 0 \quad => x_1 = -x_3$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_3}{1} \Rightarrow X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $X_2 = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$; $X_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

CHARACTERISTIC EQUATION

If A is any square matrix of order n, the matrix $A - \lambda I$ where I is the unit matrix and λ be scaler of order n can be formed as

$$|A - \lambda I| = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$ is called the characteristic equation of A.

Working Rule for Characteristic Equation

Type I: For 2×2 matrix

If $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, then the characteristic equation of A is $\lambda^2 - s_1 \lambda + s_2 = 0$

Where s_1 =Sum of the leading diagonal elements $= a_{11} + a_{22}$

$s_2 = |A|$ =Determinant of a matrix A.

Type II: For 3×3 matrix

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then the characteristic equation of A is $\lambda^3 - s_1\lambda^2 + s_2 \lambda - s_3 = 0$

Where s_1 =Sum of the leading diagonal elements $= a_{11} + a_{22} + a_{33}$

s_2 =Sum of minors of leading diagonal elements

$= \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$s_3 = |A|$ = Determinant of a matrix A.

Example: Find the characteristic equation of the matrix $\begin{pmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{2} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^2 - s_1\lambda + s_2 = 0$

s_1 = sum of the main diagonal element

$= 1 + 2 = 3$

$s_2 = |A| = \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} = 2$

Characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$

Example: Find the characteristic equation of the matrix $\begin{pmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{-5} & \mathbf{4} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^2 - s_1\lambda + s_2 = 0$

s_1 = sum of the main diagonal element

$= 1 + 4 = 5$

$s_2 = |A| = \begin{vmatrix} 1 & -2 \\ -5 & 4 \end{vmatrix} = 4 - 10 = -6$

Characteristic equation is $\lambda^2 - 5\lambda - 6 = 0$

Example: Find the characteristic equation of the matrix $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 = sum of the main diagonal element

$= 2 + 2 + 2 = 6$

s_2 = sum of the minors of the main diagonalelement

$= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 11$

$s_3 = |A| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2(4 - 0) - 0 + 1(0 - 2)$

$= 8 - 2 = 6$

Characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Example: Find the characteristic equation of the matrix $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 = sum of the main diagonal element

$= 2 + 2 + 1 = 5$

s_2 = sum of the minors of the main diagonalelement

$= \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 7$

$s_3 = |A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2(2 - 0) - 1(1 - 0) + 1(0 - 0)$

$= 4 - 1 = 3$

Characteristic equation is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

Example: Find the characteristic polynomial of the matrix $\begin{pmatrix} 0 & -2 & -2 \\ -1 & 1 & 2 \\ -1 & -1 & 2 \end{pmatrix}$

Solution:

The characteristic polynomial is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3$

$s_1 =$ sum of the main diagonal element

$$s_3 = |A| = \begin{vmatrix} -1 & 1 & 2 \\ -1 & -1 & 2 \end{vmatrix} = 0 + 2(-2+2) - 2(1+1)$$

$$= -4$$

PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS

Property: 1(a) The sum of the Eigen values of a matrix is equal to the sum of the elements of the principal (main) diagonal.

(or)

The sum of the Eigen values of a matrix is equal to the trace of the matrix.

1. (b) product of the Eigen values is equal to the determinant of the matrix.

Proof:

Let A be a square matrix of order n.

The characteristic equation of A is $|A - \lambda I| = 0$

$(i.e.)\lambda^n - S_1\lambda^{n-1} + S_2\lambda^{n-2} - \cdots + (-1)S_n = 0$ ... (1)

where $S_1 =$ Sum of the diagonal elements of A.

. . .

. . .

. . .

$S_n =$ determinant of A.

We know the roots of the characteristic equation are called Eigen values of the given matrix.

Solving (1) we get n roots.

Let the n be $\lambda_1, \lambda_2, \ldots \lambda_n$.

i.e., $\lambda_1, \lambda_2, \ldots \lambda_n$. are the Eignvalues of A.

We know already,

$\lambda^n -$ (Sum of the roots λ^{n-1} + [sum of the product of the roots taken two at a time] λ^{n-2} −

... $+ (-1)^n$(Product of the roots) $= 0$

... (2)

Sum of the roots $= S_1 by$ (1)&(2)

$(i.e.)\ \lambda_1 + \lambda_2 + \cdots + \lambda_n = S_1$

$(i.e.)\ \lambda_1 + \lambda_2 + \cdots + \lambda_n = a_{11} + a_{22} + \cdots + a_{nn}$

Sum of the Eigen values = Sum of the main diagonal elements

Product of the roots $= S_n$by (1)&(2)

$(i.e.)\lambda_1\lambda_2 \ldots \lambda_n = \det$ of A

Product of the Eigen values $= \|A\|$

Property: 2 A square matrix A and its transpose A^T have the same Eigenvalues.

(or)

A square matrix A and its transpose A^T have the same characteristic values.

Proof:

Let A be a square matrix of order n.

The characteristic equation of A and A^T are

$$|A - \lambda I| = 0 \quad \dots\dots(1)$$

and $$|A^T - \lambda I| = 0 \quad \dots\dots(2)$$

Since, the determinant value is unaltered by the interchange of rows and columns.

We know $|A| = |A^T|$

Hence, (1) and (2) are identical.

$\therefore$ The Eigenvalues of A and A^T are the same.

Property: 3 The characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

(or)

The Eigen values of a triangular matrix are just the diagonal elements of the matrix.

Proof: Let us consider the triangular matrix.

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Characteristic equation of is

$$|A - \lambda I| = 0$$

i.e., $$\begin{vmatrix} a_{11}-\lambda & 0 & 0 \\ a_{21} & a_{22}-\lambda & 0 \\ a_{31} & a_{32} & a_{33}-\lambda \end{vmatrix} = 0$$

On expansion it gives $(a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda) = 0$

i.e., $\lambda = a_{11}, a_{22}, a_{33}$

which are diagonal elements of the matrix A.

Property: 4 If λ is an Eigenvalue of a matrix A, then $\frac{1}{\lambda}$, $(\lambda \neq 0)$ is the Eignvalue of A^{-1}.

(or)

If λ is an Eigenvalue of a matrix A, what can you say about the Eigenvalue of matrix A^{-1}. Prove your statement.

Proof:

If X be the Eigenvector corresponding to λ,

then $AX = \lambda X$ ... (i)

Pre multiplying both sides by A^{-1}, we get

$$A^{-1}AX = A^{-1}\lambda X$$

$$(1) \Rightarrow X = \lambda A^{-1}X$$

$$X = \lambda A^{-1}X$$

$$\div \lambda \Rightarrow \quad \frac{1}{\lambda}X = A^{-1}X$$

$$(i.e.) \quad A^{-1}X = \frac{1}{\lambda}X$$

This being of the same form as (i), shows that $\frac{1}{\lambda}$ is an Eigenvalue of the inverse matrix A^{-1}.

Property: 5 If λ is an Eigenvalue of an orthogonal matrix, then $\frac{1}{\lambda}$ is an Eigenvalue.

Proof:

Definition: Orthogonal matrix.
A square matrix A is said to be orthogonal if $AA^T = A^TA = I$
i.e., $A^T = A^{-1}$

Let A be an orthogonal matrix.

Given λ is an Eignevalue of A.

$\Rightarrow \frac{1}{\lambda}$ is an Eigenvalue of A^{-1}

Since, $A^T = A^{-1}$

$\therefore \frac{1}{\lambda}$ is an Eigenvalue of A^T

But, the matrices A and A^T have the same Eigenvalues, since the determinants $|A - \lambda I|$and $|A^T - \lambda I|$ are the same.

Hence, $\frac{1}{\lambda}$ is also an Eigenvalue of A.

Property: 6 If $\lambda_1, \lambda_2, \dots \lambda_n$. are the Eignvalues of a matrix A, then A^m has the Eigenvalues $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$ (m being a positive integer)

Proof:

Let A_i be the Eigenvalue of A and X_i the corresponding Eigenvector.

Then $AX_i = \lambda_i X_i$ ... (1)

We have $A^2X_i = A(AX_i)$

$= A(\lambda_i X_i)$

$= \lambda_i A(X_i)$

$= \lambda_i(\lambda_i X_i)$

$= \lambda_i^2 X_i$

||| 1y $A^3X_i = \lambda_i^3 X_i$

In general, $A^m X_i = \lambda_i^m X_i$ ....(2)

Hence, λ_i^m is an Eigenvalue of A^m.

The corresponding Eigenvector is the same X_i.

Note: If λ is the Eigenvalue of the matrix A then λ^2 is the Eigenvalue of A^2

Property: 7 The Eigen values of a real symmetric matrix are real numbers.

Proof:

Let λ be an Eigenvalue (may be complex) of the real symmetric matrix A. Let the corresponding Eigenvector be X. Let A denote the transpose of A.

We have $AX = \lambda X$

Pre-multiplying this equation by $1 \times n$ matrix $\bar{X}$, where the bar denoted that all elements of $\bar{X}$ are the complex conjugate of those of X', we get

$X'AX = \lambda X'X$ ....(1)

Taking the conjugate complex of this we get $X'\,AX = \lambda X'X$ or

$X'A\,X = \lambda\,X'\,X$ since, $A = A$ for A is real.

Taking the transpose on both sides, we get

$(X'AX)' = (\lambda\,X'\,X)'(i.e.,)X'\,A'\,X = \lambda\,X'\,X$

$(i.e.)X'\,A'\,X = \lambda\,X'\,X$ since $A' = A$ for A is symmetric.

But, from (1), $X'\,A\,X = \lambda\,X'\,X$ Hence $\lambda\,X'\,X = \lambda\,X'\,X$

Since, $\overline{X'}\,X$ is an 1×1 matrix whose only element is a positive value, $\lambda = \lambda$ $(i.e.)$ λ is real).

Property: 8 The Eigen vectors corresponding to distinct Eigen values of a real symmetric matrix are orthogonal.

Proof:

For a real symmetric matrix A, the Eigen values are real.

Let X_1, X_2 be Eigenvectors corresponding to two distinct eigen values λ_1, λ_2 [λ_1, λ_2 are real]

$$AX_1 = \lambda_1 X_1 \qquad \dots (1)$$

$$AX_2 = \lambda_2 X_2 \qquad \dots (2)$$

Pre multiplying (1) by X_2', we get

$$X_2' A X_1 = X_2' \lambda_1 X_1$$

$$= \lambda_1 X_2' X_1$$

Pre-multiplying (2) by X_1', we get

$$X_1' A X_2 = \lambda_2 X_1' X_2 \qquad \dots (3)$$

$$\text{But}(X_2' A X_1)' = (\lambda_1 X_2' X_1)'$$

$$X_1' A\, X_2 = \lambda_1 X_1' X_2$$

$$(i.e) \quad X_1' A\, X_2 = \lambda_1 X_1' X_2 \qquad \dots\dots (4) \quad [\because A' = A]$$

From (3) and (4)

$$\lambda_1 X_1' X_2 = \lambda_2 X_1' X_2$$

$$\text{(i.e.,)}(\lambda_1 - \lambda_2) X_1' X_2 = 0$$

$$\lambda_1 \neq \lambda_2, X_1' X_2 = 0$$

$\therefore X_1, X_2$ are orthogonal.

Property: 9 The similar matrices have same Eigen values.

Proof:

Let A, B be two similar matrices.

Then, there exists an non-singular matrix P such that $B = P^{-1} AP$

$$B - \lambda I = P^{-1}AP - \lambda I$$

$$= P^{-1}AP - P^{-1} \lambda IP$$

$$= P^{-1}(A - \lambda I)P$$

$$|B - \lambda I| = |P^{-1}|\, |A - \lambda I|\, |P|$$

$$= |A - \lambda I|\, |P^{-1}P|$$

$$= |A - \lambda I|\, |I|$$

$$= |A - \lambda I|$$

Therefore, A, B have the same characteristic polynomial and hence characteristic roots.

$\therefore$ They have same Eigen values.

Property: 10 If a real symmetric matrix of order 2 has equal Eigen values, then the matrix is a scalar matrix.

Proof :

Rule 1 : A real symmetric matrix of order n can always be diagonalised.

Rule 2 : If any diagonalized matrix with their diagonal elements are equal, then the matrix is a scalar matrix.

Given A real symmetric matrix 'A' of order 2 has equal Eigen values.

By Rule: 1 A can always be diagonalized, let λ_1 and λ_2 be their Eigenvalues then we get the diagonlized matrix $= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

Given $\lambda_1 = \lambda_2$

Therefore, we get $= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

By Rule: 2 The given matrix is a scalar matrix.

Property: 11 The Eigen vector X of a matrix A is not unique.

Proof :

Let λ be the Eigenvalue of A, then the corresponding Eigenvector X such that $AX = \lambda X$.

Multiply both sides by non-zero K,

$K(AX) = K(\lambda X)$

$\Rightarrow A(KX) = \lambda(KX)$

$(i.e.)$ an Eigenvector is determined by a multiplicative scalar.

$(i.e.)$ Eigenvector is not unique.

Property: 12 $\lambda_1, \lambda_2, \ldots \lambda_n$ be distinct Eigenvalues of an $n \times n$ matrix, then the corresponding Eigenvectors $X_1, X_2, \ldots X_n$ form a linearly independent set.

Proof:

Let $\lambda_1, \lambda_2, \ldots \lambda_m (m \le n)$ be the distinct Eigen values of a square matrix A of order n.

Let $X_1, X_2, \ldots X_m$ be their corresponding Eigenvectors we have to prove $\sum_{i=1}^{m} \alpha_i X_i = 0$ implies each $\alpha_i = 0, i = 1,2, \ldots, m$

Multiplying $\sum_{i=1}^{m} \alpha_i X_i = 0$ by $(A - \lambda_1 I)$, we get

$$(A - \lambda_1 I)\alpha_1 X_1 = \alpha_1(AX_1 - \lambda_1 X_1) = \alpha_1(0) = 0$$

When $\sum_{i=1}^{m} \alpha_i X_i = 0$ Multiplied by

$$(A - \lambda_2 I)(A - \lambda_2 I) \ldots (A - \lambda_{i-1} I)(A - \lambda_i I)\ (A - \lambda_{i+1} I) \ldots (A - \lambda_m I)$$

We get, $\alpha_i(\lambda_i - \lambda_1)(\lambda_i - \lambda_2) \ldots (\lambda_i - \lambda_{i-1})(\lambda_i - \lambda_{i+1}) \ldots (\lambda_i - \lambda_m) = 0$

Since, λ's are distinct, $\alpha_i = 0$

Since, i is arbitrary, each $\alpha_i = 0, i = 1, 2, \dots, m$

$\sum_{i=1}^{m} \alpha_i X_i = 0$ implies each $\alpha_i = 0, i = 1,2, \dots, m$

Hence, $X_1, X_2, \dots X_m$ are linearly independent.

Property: 13 If two or more Eigen values are equal it may or may not be possible to get linearly independent Eigenvectors corresponding to the equal roots.

Property: 14 Two Eigenvectors X_1 and X_2 are called orthogonal vectors if $X_1^T X_2 = 0$

Property: 15 If A and B are $n \times n$ matrices and B is a non singular matrix, then A and B^{-1} AB have same eigenvalues.

Proof:

Characteristic polynomial of B^{-1} AB

$= |B^{-1} AB - \lambda I| = |B^{-1} AB - B^{-1}(\lambda I)B|$

$= |B^{-1} (A - \lambda I)B| = |B^{-1}||A - \lambda I||B|$

$= |B^{-1}|\, |B|\, |A - \lambda I| = |B^{-1}B||A - \lambda I|$

= Characterisstisc polynomial of A

Hence, A and B^{-1} AB have same Eigenvalues.

Example: Find the sum and product of the Eigen values of the matrix $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Solution:

Sum of the Eigen values =Sum of the main diagonal elements

$= (-2) + (1) + (0)$

$= -1$

Product of the Eigen values = $\begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$

$= -2(0 - 12) - 2(0 - 6) - 3(-4 + 1)$

$= 24 + 12 + 9 = 45$

Example: Find the sum and product of the Eigen values of the matrix A= $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Solution:

Sum of the Eigen values = Sum of its diagonal elements $= 1 + 2 + 1 = 4$

Product of Eigen values $= |C| = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$

$= 1(2-1) - 2(-1-1) + 3(-1-2)$

$= 1(1) - 2(-2) + 3(-3)$

$= 1 + 4 - 9 = -4$

Example: The product of two Eigen values of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. Find the third Eigenvalue.

Solution:

Let Eigen values of the matrix A be $\lambda_1, \lambda_2, \lambda_3$.

Given $\lambda_1\lambda_2 = 16$

We know that, $\lambda_1\lambda_2\lambda_3 = |A|$

[Product of the Eigen values is equal to the determinant of the matrix]

$\therefore \lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$

$= 6(9-1) + (-6+2) + 2(2-6)$

$= 6(8) + 2(-4) + 2(-4)$

$= 48 - 8 - 8$

$\Rightarrow \lambda_1\lambda_2\lambda_3 = 32$

$\Rightarrow 16\,\lambda_3 = 32$

$\Rightarrow \lambda_3 = \frac{32}{16} = 2$

Example: Two of the Eigen values of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ are 2 and 8. Find the third Eigen value.

Solution:

We know that, Sum of the Eigen values = Sum of its diagonal elements

$= 6 + 3 + 3 = 12$

Given $\lambda_1 = 2, \lambda_2 = 8, \lambda_3 = ?$

We get, $\lambda_1 + \lambda_2 + \lambda_3 = 12$

$$2 + 8 + \lambda_3 = 12$$

$$\lambda_3 = 12 - 10$$

$$\lambda_3 = 2$$

∴ The third Eigenvalue = 2

Example: If 3 and 15 are the two Eigen values of $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ find |A|, without expanding the determinant.

Solution:

Given $\lambda_1 = 3, \lambda_2 = 15, \lambda_3 = ?$

We know that, Sum of the Eigen values = Sum of the main diagonal elements

$$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 8 + 7 + 3$$

$$3 + 15 + \lambda_3 = 18$$

$$\Rightarrow \lambda_3 = 0$$

We know that, Product of the Eigen values = |A|

$$\Rightarrow (3)(15)(0) = |A|$$

$$\Rightarrow |A| = (3)(15)(0)$$

$$\Rightarrow |A| = 0$$

Example: If 2, 2, 3 are the Eigen values of $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$ find the Eigen values of A^T.

Solution:

By Property "A square matrix A and its transpose A^T have the same Eigen values".

Hence, Eigen values of A^T are 2, 2, 3

Example: If the Eigen values of the matrix $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ are $2, -2$ then find the Eigen values of A^T.

Solution:

Eigen values of A = Eigen values of A^T

∴ Eigen values of A^T are $2, -2$.

Example: Two of the Eigen values of $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ are 3 and 6. Find the Eigen values of A^{-1}.

Solution:

Sum of the Eigen values = Sum of the main diagonal elements

$$= 3 + 5 + 3 = 11$$

Let K be the third Eigen value

$$\therefore 3 + 6 + k = 11$$

$$\Rightarrow 9 + k = 11$$

$$\Rightarrow k = 2$$

$\therefore$ The Eigenvalues of A^{-1} are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$

Example: Two Eigen values of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to 1 each. Find the Eigenvalues of A^{-1}.

Solution:

Given $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Let the Eigen values of the matrix A be $\lambda_1, \lambda_2, \lambda_3$

Given condition is $\lambda_2 = \lambda_3 = 1$

We have, Sum of the Eigen values = Sum of the main diagonal elements

$$\Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 2 + 3 + 2$$

$$\Rightarrow \lambda_1 + 1 + 1 = 7$$

$$\Rightarrow \lambda_1 + 2 = 7$$

$$\Rightarrow \lambda_1 = 5$$

Hence, the Eigen values of A are 1, 1, 5

Eigen values of A^{-1} are $\frac{1}{1}, \frac{1}{1}, \frac{1}{5}$, i.e., $1, 1, \frac{1}{5}$

CAYLEY HAMILTON THEOREM

Cayley Hamilton Theorem

Statement: Every square matrix satisfies its own characteristic equation.

Uses of Cayley Hamilton Theorem:

To calculate (i) the positive integral power of A and

(ii) the inverse of a non-singular square matrix A.

Example: Show that the matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ satisfies its own characteristic equation.

Solution:

Let $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

The characteristic equation of the given matrix is $|A - \lambda I| = 0$

$$\lambda^2 - S_1\lambda + S_1 = 0$$

Where S_1 = sum of the main diagonal elements.

$= 1 + 1 = 2$

$S_2 = |A| = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 1 + 4 = 5$

$\therefore$ The characteristic equation is $\lambda^2 - 2\lambda + 5 = 0$

To prove: $A^2 - 2A + 5I = O$

$$A^2 = AA = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix}$$

$$A^2 - 2A + 5I = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} - 2\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix} + \begin{pmatrix} -2 & 4 \\ -4 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = O$$

Therefore, the given matrix satisfies its own characteristic equation.

Example: Verify Cayley – Hamilton theorem find A^4 and A^{-1} when $A = \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$

i.e., $\lambda^3 - S_1\lambda^2 + S_2\lambda = 0$ where

S_1 = sum of its leading diagonal elements $= 2 + 2 + 2 = 6$

S_2 = sum of the minors of its leading diagonal elements

$$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (4-1) + (4-2) + (4-1) = 3 + 2 + 3 = 8$$

$$S_3 = |A| = 2(4-1) + 1(-2+1) + 2(1-2)$$

$$= 2(3) + 1(-1) + 2(-1) = 6 - 1 - 2 = 3$$

∴ The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$i.e., \lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

By Cayley-Hamilton theorem

[Every square matrix satisfies its own characteristic equation]

$$(i.e.)\ A^3 - 6A^2 + 8A - 3I = 0 \qquad \ldots(1)$$

Verification:

$$A^2 = A \times A = \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$A^3 = A \times A^2 = \begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix}$$

$$\therefore\ A^3 - 6A^2 + 8A - 3I = \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - 6\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix}$$

$$+8\begin{bmatrix} -2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & -28 & 38 \\ -22 & 23 & -28 \\ 22 & -22 & 29 \end{bmatrix} - \begin{bmatrix} 42 & -36 & 54 \\ -30 & 36 & -36 \\ 30 & -30 & 42 \end{bmatrix} + \begin{bmatrix} 16 & -8 & 16 \\ -8 & 16 & -8 \\ 8 & -8 & 16 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

To find A^4:

$$(1) \Rightarrow A^3 - 6A^2 - 8A + 3I \quad \ldots(2)$$

Multiply A on both sides, we get

$$A^4 = 6A^3 - 8A^2 + 3A = 6[6A^2 - 8A + 3I] - 8A^2 + 3A \text{ by } (2)$$

$$= 36A^2 - 48A + 18I - 8A^2 + 3A$$

$$A^4 = 28A^2 - 45A + 18I \qquad \ldots(3)$$

$$(1) \Rightarrow \quad A^4 = 28\begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 45\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 196 & -168 & 252 \\ -140 & 168 & -168 \\ 140 & -140 & 196 \end{bmatrix} - \begin{bmatrix} 90 & -45 & 90 \\ -45 & 90 & -45 \\ 45 & -45 & 90 \end{bmatrix} + \begin{bmatrix} 18 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{bmatrix}$$

To find A^{-1}:

$(1) \times A^{-1} \Rightarrow A^2 - 6A + 8I - 3A^{-1} = 0$

$3A^{-1} = A^2 - 6A + 8I$

$$3A^{-1} = \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -6 & 9 \\ -5 & 6 & -6 \\ 5 & -5 & 7 \end{bmatrix} + \begin{bmatrix} -12 & 6 & -12 \\ 6 & -12 & 6 \\ -6 & 6 & -12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$3A^{-1} = \begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{bmatrix}$$

Example: Find A^{-1} if $A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$, using Cayley- Hamilton theorem.

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$

$(i.e.)\ \lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

S_1 = sum of its leading diagonal elements

$= 1 + 2 + (-1) = 2$

S_2 = sum of the minors of its leading diagonal elements

$$= \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$$

$= (-2 + 1) + (-1 - 8) + (2 + 3)$

$= (-1) + (-9) + 5 = -5$

$$S_3 = |A| = \begin{vmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix}$$

$= 1(-2 + 1) + 1(-3 + 2) + (3 - 4)$

$$= 1(-1) + 1(-1) + 4(-1)$$

$$= -1 - 1 - 4 = -6$$

$\therefore$ The Characteristic equation is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

By Cayley Hamilton Theorem we get

[Every square matrix satisfies its own characteristic equation]

$\therefore A^3 - 2A^2 - 5A + 6I = 0 \qquad \ldots (1)$

To find A^{-1}

$(1) \times A^{-1} \Rightarrow A^2 - 2A - 5I + 6A^{-1} = 0$

$$A^2 - 2A - 5I + 6A^{-1} = 0$$

$$6A^{-1} = -A^2 + 2A + 5I$$

$$A^{-1} = \frac{1}{6}[-A^2 + 2A + 5I] \qquad \ldots (2)$$

$$A^2 = A \times A$$

$$= \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3+8 & -1-2+4 & 4+1-4 \\ 3+6-2 & -3+4-1 & 12-2+1 \\ 2+3-2 & -2+2-1 & 8-1+1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{bmatrix}$$

$$-A^2 + 2A + 5I = \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + 2\begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{bmatrix} + \begin{bmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$$

From (2) $\Rightarrow$ $A^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{bmatrix}$

Example: Use Cayley – Hamilton theorem to find the value of the matrix given by

(i) $f(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

(ii) $A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$ **if the matrix** $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Solution:

The characteristic equation of A is $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \text{ where}$$

S_1 = sum of the main diagonal elements

$= 2 + 1 + 2 = 5$

S_2 = sum of the minors of main diagonal elements

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$= (2 - 0) + (4 - 1) + (2 - 0) = 2 + 3 + 2 = 7$

$= (-1) + (-9) + 5 = -5$

$$S_3 = |A| = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$= 2(2 - 0) - 1(0 - 0) + 1(0 - 1) = 4 - 1 = 3$

Therefore, the characteristic equation is $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By C – H theorem, we get

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \ldots (1)$$

Let i) $f(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

ii) $g(A) = A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$

(i) $A^5 + A$

$A^3 - 5A^2 + 7A - 3I$ $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$A^8 - 5A^7 + 7A^6 - 3A^5$

$(-)$ $A^4 - 5A^3 + 8A^2 - 2A$

$A^4 - 5A^3 + 7A^2 - 3A$

$(-)$ $A^2 + A + 1\,I$

$f(A) = (A^3 - 5A^2 + 7A - 3I)(A^2 + A) + A^2 + A + I$

$= 0 + A^2 + A + I$ by (1)

$= A^2 + A + I \quad \ldots (2)$

$$\text{Now, } A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$\therefore A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

(ii) $A^5 + 8A + 35I$

$A^3 - 5A^2 + 7A - 3I$) $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + 1\,I$

$A^8 - 5A^7 + 7A^6 - 3A^5$

$(-)$ $8A^4 - 5A^3 + 8A^2 - 2A$

$8A^4 - 40A^3 + 56A^2 - 24A$

$(-)$ $35A^3 - A^2 + 22A + 1\,I$

$35A^3 - 175\,A^2 + 245A - 105I$

$(-)$ $127\,A^2 - 223A + 106\,I$

$$g(A) = (A^3 - 5A^2 + 7A - 3I)(A^4 + 8A + 35I) + 127A^2 - 223A + 106I$$

$$= 0 + 127A^2 - 223A + 106\,I$$

$$= 127A^2 - 223A + 106\,I$$

$$= 127 \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 223 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 106 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$g(A) = \begin{bmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 285 & 285 & 295 \end{bmatrix}$$

Example: Using Cayley Hamilton theorem find A^{-1} when $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

Solution:

The Characteristic equation of A is $|A - \lambda I| = 0$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$ where

S_1 = sum of the main diagonal elements $= 1 + 1 + 1 = 3$

S_2 = Sum of the minors of the main diagonal elements.

$$= \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}$$

$$= (1-1) + (1-3) + (1-0)$$

$$= 0 - 2 + 1 = -1$$

$$S_3 = |A| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= 1(1-1) - 0(2+1) + 3(-2-1)$$

$$= 0 - 0 + 3(-3) = -9$$

∴ The characteristic equation A is $\lambda^3 - 3\lambda^2 - \lambda + 9 = 0$

By Cayley - Hamilton Theorem every square matrix satisfies its own Characteristic equation

$$\therefore A^3 - 3A^2 - A + 9\,I = O$$

$$A^{-1} = \frac{-1}{9}[A^2 - 3A - I] \qquad \dots (1)$$

$$A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+3 & 0+0-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0-1-1 & 3+1+1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$$

$$-3A = \begin{bmatrix} -3 & 0 & -9 \\ -6 & -3 & 3 \\ -3 & 3 & -3 \end{bmatrix}$$

$$(1) \Rightarrow A^{-1} = \frac{-1}{9}\left[\begin{pmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{pmatrix} + \begin{pmatrix} -3 & 0 & -9 \\ -6 & -3 & -3 \\ -3 & 3 & -3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{-1}{9}\begin{bmatrix} 0 & -3 & -3 \\ -3 & -2 & 7 \\ -3 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 0 & 3 & 3 \\ 3 & 2 & -7 \\ 3 & -1 & -1 \end{bmatrix}$$

Example: Verify Cayley- Hamilton for the matrix A= $\begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{7} \\ \mathbf{4} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{2} & \mathbf{1} \end{bmatrix}$

Solution :

$$\text{Given } A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

The characteristic equation A is $|A - \lambda I| = 0$

$$\lambda^3 - S_1\lambda^2 + S_2\lambda S_3 = 0 \cdots (1) \quad \text{where}$$

S_1 = Sum of the main diagonal elements

$= 1 + 2 + 1 = 4$

S_2 = Sum of the minors of its leading diagonal elements

$$= \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 7 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix}$$

$= (2 - 6) + (1 - 7) + (2 - 12)$

$= -4 - 6 - 10 = -20$

$$S_3 = |A| = \begin{vmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{vmatrix}$$

$= 1(2 - 6) - 3(4 - 3) + 7(8 - 2)$

$= -4 - 3(1) + 7(6)$

$= -4 - 3 + 42 = 35$

$\therefore$ (1) $\Rightarrow \lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$

By Cayley –Hamilton theorem

(2) $\Rightarrow$ $A^3 - 4A^2 - 20A - 351 = O$

To find A^2 *and* A^3:

$$A^2 = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 12 + 7 & 3 + 6 + 14 & 7 + 9 + 7 \\ 4 + 8 + 3 & 12 + 4 + 6 & 28 + 6 + 3 \\ 1 + 8 + 1 & 3 + 4 + 2 & 7 + 6 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 20 + 92 + 23 & 60 + 46 + 46 & 140 + 69 + 23 \\ 15 + 88 + 37 & 45 + 44 + 74 & 105 + 66 + 37 \\ 10 + 36 + 14 & 30 + 18 + 28 + & 70 + 27 + 14 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix}$$

$A^3 - 4A^2 - 20A - 35\,I$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} - 4\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - 20\begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} - 35\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 135 & 152 & 232 \\ 140 & 163 & 208 \\ 60 & 76 & 111 \end{bmatrix} + \begin{bmatrix} -80 & -92 & -92 \\ -60 & -88 & -148 \\ -40 & -36 & -56 \end{bmatrix} + \begin{bmatrix} -20 & -60 & -140 \\ -80 & -40 & -60 \\ -20 & -40 & -20 \end{bmatrix} +$$

$$\begin{bmatrix} -35 & 0 & 0 \\ 0 & -35 & 0 \\ 0 & 0 & -35 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ The given matrix A satisfies its own characteristic equation.

Hence, cayley Hamilton theorem is verified.

DIAGONALISATION OF A MATRIX BY ORTHOGONAL TRANSFORMATION

Orthogonal matrix

Definition

A matrix 'A' is said to be orthogonal if $AA^T = A^TA = I$

Example: Show that the following matrix is orthogonal $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Solution:

Let $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$$\Rightarrow A^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$AA^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$\therefore$ A is orthogonal.

Modal Matrix

Modal matrix is a matrix in which each column specifies the eigenvectors of a matrix .It is denoted by N.

A square matrix A with linearity independent Eigen vectors can be diagonalized by a similarly transformation, $D = N^{-1}AN$, where N is the modal matrix .The diagonal matrix D has as its diagonal elements, the Eigen values of A.

Normalized vector

Eigen vector X_r is said to be normalized if each element of X_r is being divided by the square root of the sum of the squares of all the elements of X_r .i.e.,the normalized vector is $\frac{X}{|X|}$

$$X_r = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{; Normalized vector of } X_r = \begin{bmatrix} x_1/\sqrt{x_1^2 + x_2^2 + x_3^2} \\ x_2/\sqrt{x_1^2 + x_2^2 + x_3^2} \\ x_3/\sqrt{x_1^2 + x_2^2 + x_3^2} \end{bmatrix}$$

Working rule for diagonalization of a square matrix A using orthogonal reduction:

i) Find all the Eigen values of the symmetric matrix A.

ii) Find the Eigen vectors corresponding to each Eigen value.

iii) Find the normalized modal matrix N having normalized Eigen vectors as its column vectors.

iv) Find the diagonal matrix $D = N^T AN$. The diagonal matrix D has Eigen values of A as its diagonal elements.

Example: Diagonalize the matrix $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 = sum of the main diagonal element

$= 2 + 1 + 1 = 4$

s_2 = sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -3 + 1 + 1 = -1$$

$$s_3 = |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix} = -4$$

Characteristic equation is $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$$\Rightarrow \lambda = 1, (\lambda^2 - 3\lambda - 4) = 0$$

$$\Rightarrow \lambda = 1, \ (\lambda + 1)(\lambda - 4) = 0$$

$$\Rightarrow \lambda = -1, 1, 4$$

To find the Eigen vectors:

Case (i) When $\lambda = 1$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-1 & 1 & -1 \\ 1 & 1-1 & -2 \\ -1 & -2 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 - x_3 = 0 \ldots (1)$$

$$x_1 + 0x_2 - 2x_3 = 0 \ldots (2)$$

$$-x_1 - 2x_2 + 0x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{-2-0} = \frac{x_2}{-1+2} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_1 = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

Case(ii) When $\lambda = -1$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2+1 & 1 & -1 \\ 1 & 1+1 & -2 \\ -1 & -2 & 1+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x_1 + x_2 - x_3 = 0 \ldots (4)$$

$$x_1 + 2x_2 - 2x_3 = 0 \ldots (5)$$

$$-x_1 - 2x_2 + 2x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{-2+2} = \frac{x_2}{-1+6} = \frac{x_3}{6-1}$$

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = 4$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-4 & 1 & -1 \\ 1 & 1-4 & -2 \\ -1 & -2 & 1-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 - x_3 = 0 \ldots (7)$$

$$x_1 - 3x_2 - 2x_3 = 0 \ldots (8)$$

$$-x_1 - 2x_2 - 3x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{-2-3} = \frac{x_2}{-1-4} = \frac{x_3}{6-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{5}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$; $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

To check X_1, X_2 & X_3 are orthogonal

$$X_1^T X_2 = \begin{pmatrix} -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 + 1 - 1 = 0$$

$$X_2^T X_3 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0$$

$$X_3^T X_1 = \begin{pmatrix} -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} = 2 - 1 - 1 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{-2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T A N$

$$= \begin{pmatrix} \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{-2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Example: Diagonalize the matrix $\begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & -3 & 5 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 10 + 2 + 5 = 17$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 2 & 3 \\ -3 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -5 \\ -5 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix} = 1 + 25 + 16 = 42$$

$$s_3 = |A| = \begin{vmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & -3 & 5 \end{vmatrix} = 0$$

Characteristic equation is $\lambda^3 - 17\lambda^2 + 42\lambda = 0$

$$\Rightarrow \lambda(\lambda^2 - 17\lambda + 42) = 0$$

$$\Rightarrow \lambda = 0, 3, 14$$

To find the Eigen vectors:

Case (i) When $\lambda = 0$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 10-0 & -2 & -5 \\ -2 & 2-0 & 3 \\ -5 & -3 & 5-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$10x_1 - 2x_2 - 5x_3 = 0 \ldots (1)$$

$$-2x_1 + 2x_2 + 3x_3 = 0 \ldots (2)$$

$$-5x_1 + 3x_2 + 5x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{-6+10} = \frac{x_2}{10-30} = \frac{x_3}{20-4}$$

$$\frac{x_1}{4} = \frac{x_2}{-20} = \frac{x_3}{16}$$

$$\frac{x_1}{1} = \frac{x_2}{-5} = \frac{x_3}{4}$$

$$X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$$

Case (ii) When $\lambda = 3$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 10-3 & -2 & -5 \\ -2 & 2-3 & 3 \\ -5 & -3 & 5-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$7x_1 - 2x_2 - 5x_3 = 0 \ldots (4)$$

$$-2x_1 - x_2 + 3x_3 = 0 \ldots (5)$$

$$-5x_1 + 3x_2 + 2x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{-6-5} = \frac{x_2}{10-21} = \frac{x_3}{-7-4}$$

$$\frac{x_1}{-11} = \frac{x_2}{-11} = \frac{x_3}{-11}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = 14$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 10-14 & -2 & -5 \\ -2 & 2-14 & 3 \\ -5 & -3 & 5-14 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-4x_1 - 2x_2 - 5x_3 = 0 \ldots (7)$$

$$-2x_1 - 12x_2 + 3x_3 = 0 \ldots (8)$$

$$-5x_1 + 3x_2 - 9x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{-6-60} = \frac{x_2}{10+12} = \frac{x_3}{48-4}$$

$$\frac{x_1}{-66} = \frac{x_2}{22} = \frac{x_3}{44}$$

$$\frac{x_1}{-6} = \frac{x_2}{2} = \frac{x_3}{4}$$

$$X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$; $X_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$

To check X_1, X_2 & X_3 are orthogonal

$$X_1^T X_2 = \begin{pmatrix} 1 & -5 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 - 5 + 4 = 0$$

$$X_2^T X_3 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -3 + 1 + 2 = 0$$

$$X_3^T X_1 = \begin{pmatrix} -3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} = -3 - 5 + 8 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{1}{\sqrt{42}} \\ \frac{-5}{\sqrt{42}} \\ \frac{4}{\sqrt{42}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T A N$

$$= \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \end{pmatrix} \begin{pmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & -3 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{-3}{\sqrt{14}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{14}} \\ \frac{4}{\sqrt{42}} & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{14}} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 14 \end{pmatrix}$$

Example: Diagonalize the matrix $\begin{pmatrix} \mathbf{6} & \mathbf{-2} & \mathbf{2} \\ \mathbf{-2} & \mathbf{3} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 6 + 3 + 3 = 12$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix} = 8 + 14 + 14 = 36$$

$$s_3 = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 32$$

Characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$$\Rightarrow \lambda = 2, (\lambda^2 - 10\lambda + 16) = 0$$

$$\Rightarrow \lambda = 2,2,8$$

To find the Eigen vectors:

Case (i) When $\lambda = 8$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \dots (1)$$

$$-2x_1 - 5x_2 - x_3 = 0 \dots (2)$$

$$2x_1 - x_2 - 5x_3 = 0 \dots (3)$$

From (1) and (2)

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 2$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$4x_1 - 2x_2 + 2x_3 = 0 \dots (4)$$

$$-2x_1 + x_2 - x_3 = 0 \ldots (5)$$

$$2x_1 - x_2 + x_3 = 0 \ldots (6)$$

Put $x_1 = 0 \quad \Rightarrow -2x_2 = -2x_3$

$$\frac{x_2}{1} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case (iii) Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a new vector orthogonal to both X_1 and X_2

$(i.e)\ X_1^T X_3 = 0 \ \&\ X_2^T X_3 = 0$

$$\begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \ \&\ \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$2a - b + c = 0 \ldots (7)$$

$$a + b + c = 0 \ldots (8)$$

From (7) and (8)

$$\frac{a}{-1-1} = \frac{b}{0-2} = \frac{c}{2}$$

$$\frac{a}{-2} = \frac{b}{-2} = \frac{c}{2}$$

$$\frac{a}{-1} = \frac{b}{-1} = \frac{c}{1}$$

$$X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$; $X_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T A N$

$$= \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Example: Diagonalize the matrix $\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

s_1 = sum of the main diagonal element

$= 3 + 3 + 3 = 9$

s_2 = sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8 + 8 + 8 = 24$$

$$s_3 = |A| = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 16$$

Characteristic equation is $\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$

$$\Rightarrow \lambda = 1, (\lambda^2 - 8\lambda + 16) = 0$$

$$\Rightarrow \lambda = 1, 4, 4$$

To find the Eigen vectors:

Case (i) When $\lambda = 1$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-1 & 1 & 1 \\ 1 & 3-1 & -1 \\ 1 & -1 & 3-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_1 + x_2 + x_3 = 0 \ldots (1)$$

$$x_1 + 2x_2 - x_3 = 0 \ldots (2)$$

$$x_1 - x_2 + 2x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 4$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 3-4 & 1 & 1 \\ 1 & 3-4 & -1 \\ 1 & -1 & 3-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 + x_2 + x_3 = 0 \ldots (4)$$

$$x_1 - x_2 - x_3 = 0 \ldots (5)$$

$$x_1 - x_2 - x_3 = 0 \ldots (6)$$

put $x_1 = 0 \Rightarrow x_2 = -x_3$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii) Let $X_3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ be a new vector orthogonal to both X_1 and X_2

$(i.e)$ $X_1^T X_3 = 0$ & $X_2^T X_3 = 0$

$$\begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \ \& \ \begin{pmatrix} 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$-a + b + c = 0 \ldots (7)$$

$$0a - b + c = 0 \ldots (8)$$

From (7) and (8)

$$\frac{a}{1+1} = \frac{b}{0+1} = \frac{c}{1+0}$$

$$\frac{a}{2} = \frac{b}{1} = \frac{c}{1}$$

$$X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$; $X_2 = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$; $X_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T A N$

$$= \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

REDUCTION OF QUADRATIC FORM TO CANON ICAL FORM BY ORTHOGONAL TRANSFORMATION

Quadratic Form

A homogeneous polynomial of second degree in any number of variables is called a quadratic form.

The general Quadratic form in three variables $\{x_1, x_2, x_3\}$ is given by

$$f(x_1, x_2, x_3) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_1x_2 + a_{22}x_2^2 + a_{23}x_2x_3 + a_{31}x_3x_1 + a_{32}x_2x_2 + a_{33}x_3^2$$

This Quadratic form can written as $f(x_1, x_2, x_3) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}x_iy_j$

$$f(x_1, x_2, x_3) = (x_1 \quad x_2 \quad x_3)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= X'AX$$

Where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and A is called the matrix of the Quadratic form.

Note: To write the matrix of a quadratic form as

$$A = \begin{pmatrix} \text{coeff. of} x^2 & 1/2\text{coeff. of} xy & 1/2\text{coeff. of} xz \\ 1/2\text{coeff. of} xy & \text{coeff. of} y^2 & 1/2\text{coeff. of} yz \\ 1/2\text{coeff. of } xz & 1/2\text{coeff. of} yz & \text{coeff. of } z^2 \end{pmatrix}$$

Example: Write down the Quadratic form in to matrix form

(i) $2x^2 + 3y^2 + 6xy$

Solution:

$$A = \begin{pmatrix} \text{coeff. of } x^2 & 1/2\text{coeff. of } xy \\ 1/2\text{coeff. of } xy & \text{coeff. of} y^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 \\ 3 & 3 \end{pmatrix}$$

(ii) $2x^2 + 5y^2 - 6z^2 - 2xy - yz + 8zx$

Solution:

$$A = \begin{pmatrix} \text{coeff. of} x^2 & 1/2\text{coeff. of} xy & 1/2\text{coeff. of} xz \\ 1/2\text{coeff. of} xy & \text{coeff. of} y^2 & 1/2\text{coeff. of} yz \\ 1/2\text{coeff. of } xz & 1/2\text{coeff. of} yz & \text{coeff. of } z^2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -1 & 4 \\ -1 & 5 & -1/2 \\ 4 & -1/2 & -6 \end{pmatrix}$$

Example: Write down the matrix form in to Quadratic form

(i) $\begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & -2 & 5 \end{pmatrix}$

Solution:

Quadratic form is $2x_1^2 - 2x_2^2 + 6x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$

(ii) $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 6 \end{pmatrix}$

Solution:

Quadratic form is $x_1^2 + 3x_2^2 + 6x_3^2 + 2x_1x_2 + 4x_1x_3 + 2x_2x_3$.

Example: Reduce the Quadratic form $x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 + 2x_2x_3 + 6x_2x_3$ to canonical form through an orthogonal transformation .Find the nature rank, index, signature and also find the non zero set of values which makes this Quadratic form as zero.

Solution:

Given $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 1 + 2 + 1 = 4$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} = 1 + 1 + 1 = 3$$

$$s_3 = |A| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

Characteristic equation is $\lambda^3 - 4\lambda^2 + 3\lambda = 0$

$$\Rightarrow \lambda = 0 \,;\, (\lambda^2 - 4\lambda + 3) = 0$$

$$\Rightarrow \lambda = 0,1,3$$

To find the Eigen vectors:

Case (i) When $\lambda = 0$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 - x_2 + 0x_3 = 0 \ldots (1)$$

$$-x_1 + 2x_2 + x_3 = 0 \ldots (2)$$

$$0x_1 + x_2 + x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{2-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$X_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 3$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-3 & -1 & 0 \\ -1 & 2-3 & 1 \\ 0 & 1 & 1-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 - x_2 + 0x_3 = 0 \ldots (4)$$

$$-x_1 - x_2 + x_3 = 0 \ldots (5)$$

$$0x_1 + x_2 - 2x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{2-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$X_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = 1$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-1 & -1 & 0 \\ -1 & 2-1 & 1 \\ 0 & 1 & 1-1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 - x_2 + 0x_3 = 0 \ldots (7)$$

$$-x_1 + x_2 + x_3 = 0 \ldots (8)$$

$$0x_1 + x_2 + 0x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{-1-0} = \frac{x_2}{0-0} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{1}$$

$$X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$; $X_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

To check $X_1, X_2 \& X_3$ are orthogonal

$$X_1^T X_2 = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = -1 + 2 - 1 = 0$$

$$X_2^T X_3 = \begin{pmatrix} -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = -1 + 0 + 1 = 0$$

$$X_3^T X_1 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 1 + 0 - 1 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{-1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^T AN$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Canonical form $= Y^T DY$ where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$Y^T DY = (y_1, y_2, y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 0y_1^2 + y_2^2 + 3y_3^2$$

Rank = 2

Index = 2

Signature = 2 – 0 = 2

Nature is positive semi definite.

To find non zero set of values:

Consider the transformation X = NY

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$x_1 = \frac{y_1}{\sqrt{3}} - \frac{y_2}{\sqrt{6}} + \frac{y_3}{\sqrt{2}}$$

$$x_2 = \frac{y_1}{\sqrt{3}} + \frac{2y_2}{\sqrt{6}} + 0y_3$$

$$x_3 = \frac{-y_1}{\sqrt{3}} + \frac{y_2}{\sqrt{6}} + \frac{y_3}{\sqrt{2}}$$

Put $y_2 = 0$ & $y_3 = 0$

$x_1 = \frac{y_1}{\sqrt{3}}$; $x_2 = \frac{y_1}{\sqrt{3}}$; $x_3 = \frac{-y_1}{\sqrt{3}}$

Put $y_1 = \sqrt{3}$

$x_1 = 1; x_2 = 1; x_3 = -1$ which makes the Quadratic equation zero.

Example: Reduce the Quadratic form $x_1^2 + x_2^2 + x_3^2 - 2x_1x_2$ to canonical form through an orthogonal transformation .Find the nature rank,index,signature and also find the non zero set of values which makes this Quadratic form as zero

Solution:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 1 + 1 + 1 = 3$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 1 + 1 + 0 = 2$$

$$s_3 = |A| = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

Characteristic equation is $\lambda^3 - 3\lambda^2 + 2\lambda = 0$

$\Rightarrow \lambda = 0 ; (\lambda^2 - 3\lambda + 2) = 0$

$\Rightarrow \lambda = 0,1,2$

To find the Eigen vectors:

Case (i) When $\lambda = 0$ the eigen vector is given by$(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 - x_2 + 0x_3 = 0 \ldots (1)$

$-x_1 + x_2 + 0x_3 = 0 \ldots (2)$

$0x_1 + 0x_2 + x_3 = 0 \ldots (3)$

From (1) and (2)

$$\frac{x_1}{1-0} = \frac{x_2}{0+1} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$

$$X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Case (ii) When $\lambda = 1$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-1 & -1 & 0 \\ -1 & 1-1 & 0 \\ 0 & 0 & 1-0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$0x_1 - x_2 + 0x_3 = 0 \ldots (4)$$

$$-x_1 + 0x_2 + 0x_3 = 0 \ldots (5)$$

$$0x_1 + 0x_2 + 0x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$X_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

Case (iii) When $\lambda = 2$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1-2 & -1 & 0 \\ -1 & 1-2 & 0 \\ 0 & 0 & 1-2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-x_1 - x_2 + 0x_3 = 0 \ldots (7)$$

$$-x_1 - x_2 + 0x_3 = 0 \ldots (8)$$

$$0x_1 + 0x_2 - x_3 = 0 \ldots (9)$$

From (7) and (8)

$$\frac{x_1}{1-0} = \frac{x_2}{0-1} = \frac{x_3}{0-0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$X_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$; $X_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$; $X_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

To check $X_1, X_2 \& X_3$ are orthogonal

$$X_1^T X_2 = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0 + 0 + 0 = 0$$

$$X_2^T X_3 = \begin{pmatrix} 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0 + 0 + 0 = 0$$

$$X_3^T X_1 = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 1 - 1 + 0 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & -1 & 0 \end{pmatrix}$$

$$N^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$$

Thus the diagonal matrix $D = N^T A N$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & -1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Canonical form $= Y^T D Y$ where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$Y^T D Y = (y_1, y_2, y_3)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= 0y_1^2 + y_2^2 + 2y_3^2$$

Rank = 2

Index = 2

Signature = 2 - 0 = 2

Nature is positive semi definite.

To find non zero set of values:

Consider the transformation X = NY

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$x_1 = \frac{y_1}{\sqrt{2}} + 0 + \frac{y_3}{\sqrt{2}}$$

$$x_2 = \frac{y_1}{\sqrt{2}} + 0 - \frac{y_3}{\sqrt{2}}$$

$$x_3 = 0 - y_2 - 0$$

Put $y_2 = 0$ & $y_3 = 0$

$x_1 = \frac{y_1}{\sqrt{2}}$; $x_2 = \frac{y_1}{\sqrt{2}}$; $x_3 = 0$

Put $y_1 = \sqrt{2}$

$x_1 = 1; x_2 = 1;\ x_3 = 0$ which makes the Quadratic equation zero.

Example: Reduce the Quadratic form $2x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3 - 4x_2x_3$ to canonical form through an orthogonal transformation .Find the nature rank, index, signature

Solution:

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix}$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 =$ sum of the main diagonal element

$= 2 + 1 + 1 = 4$

$s_2 =$ sum of the minors of the main diagonalelement

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -3 + 1 + 1 = -1$$

$$s_3 = |A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix} = -4$$

Characteristic equation is $\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$

$$\lambda = -1,1,4$$

To find the Eigen vectors:

Case (i) When $\lambda = -1$ the Eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2+1 & 1 & -1 \\ 1 & 1+1 & -2 \\ -1 & -2 & 1+1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$3x_1 + x_2 - x_3 = 0 \ldots (1)$$

$$x_1 + 2x_2 - 2x_3 = 0 \ldots (2)$$

$$-x_1 - 2x_2 + 2x_3 = 0 \ldots (3)$$

From (1) and (2)

$$\frac{x_1}{-2+2} = \frac{x_2}{-1+6} = \frac{x_3}{6-1}$$

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

$$X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Case (ii) When $\lambda = 1$ the Eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-1 & 1 & -1 \\ 1 & 1-1 & -2 \\ -1 & -2 & 1-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$x_1 + x_2 - x_3 = 0 \ldots (4)$$

$$x_1 + 0x_2 - 2x_3 = 0 \ldots (5)$$

$$-x_1 - 2x_2 + 0x_3 = 0 \ldots (6)$$

From (4) and (5)

$$\frac{x_1}{-2+0} = \frac{x_2}{-1+2} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

Case (iii) When $\lambda = 4$ the eigen vector is given by $(A - \lambda I)X = 0$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-4 & 1 & -1 \\ 1 & 1-4 & -2 \\ -1 & -2 & 1-4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 - x_3 = 0 \dots (7)$$

$$x_1 - 3x_2 - 2x_3 = 0 \dots (8)$$

$$-x_1 - 2x_2 - 3x_3 = 0 \dots (9)$$

From (7) and (8)

$$\frac{x_1}{-2-3} = \frac{x_2}{-1-4} = \frac{x_3}{6-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{5}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Hence the corresponding Eigen vectors are $X_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$; $X_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$; $X_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

To check $X_1, X_2 \& X_3$ are orthogonal

$$X_1^T X_2 = \begin{pmatrix} 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0$$

$$X_2^T X_3 = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2 - 1 - 1 = 0$$

$$X_3^T X_1 = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 + 1 - 1 = 0$$

Normalized Eigen vectors are

$$\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{3}} \end{pmatrix}$$

Normalized modal matrix

$$N = \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

$$N^T = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

Thus the diagonal matrix $D = N^TAN$

$$= \begin{pmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Canonical form $= Y^TDY$ where $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$Y^TDY = (y_1, y_2, y_3) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= -y_1^2 + y_2^2 + 4y_3^2$$

Rank = 3

Index = 2

Signature = 2 – 1 = 1

Nature is indefinite.

NATURE OF QUADRATIC FORM DETERMINED BY PRINCIPAL MINORS

Let A be a square matrix of order n say $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ddots & a_{2n} \\ \cdots & & & \cdots & \cdots \end{bmatrix}$

The principal sub determinants of A are defined as below.

$$s_1 = a_{11}$$

$$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$s_n = |A|$$

The quadratic form $Q = X^T AX$ is said to be

1. Positive definite: If $s_1, s_2, s_3 \ldots\ldots s_n > 0$
2. Positive semidefinite: If $s_1, s_2, s_3 \ldots\ldots s_n \geq 0$ and atleast one $s_i = 0$
3. Negative definite: If $s_1, s_3, s_5 \ldots\ldots < 0$ and $s_2, s_4, s_6 \ldots\ldots > 0$
4. Negative semidefinite: If $s_1, s_3, s_5 \ldots\ldots < 0$ and $s_2, s_4, s_6 \ldots\ldots > 0$ and atleast one $s_i = 0$
5. Indefinite: In all other cases

Example: Determine the nature of the Quadratic form $12x_1^2 + 3x_2^2 + 12x_3^2 + 2x_1x_2$

Solution:

$$A = \begin{pmatrix} 12 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

$s_1 = a_{11} = 12 > 0$

$$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 12 & 1 \\ 1 & 3 \end{vmatrix} = 35 > 0$$

$$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 12 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 12 \end{vmatrix} = 430 > 0$$, Postive definite

Example: Determine the nature of the Quadratic form $x_1^2 + 2x_2^2$

Solution:

$$\text{Let } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$s_1 = a_{11} = 1 > 0$

$$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 2 > 0$$

$$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 0 \text{ , Positive semidefinite}$$

Example: Determine the nature of the Quadratic form $x^2 - y^2 + 4z^2 + 4xy + 2yz + 6zx$

Solution:

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{pmatrix}$$

$s_1 = a_{11} = 1 > 0$

$$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -5 < 0$$

$$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 0 \text{ , Indefinite}$$

Example: Determine the nature of the Quadratic form $xy + yz + zx$

Solution:

$$\text{Let } A = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

$s_1 = a_{11} = 0$

$$s_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = -1/4 < 0$$

$$s_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{vmatrix} = \frac{1}{4} > 0 \text{ , Indefinite}$$

RANK, INDEX AND SIGNATURE OF A REAL QUADRATIC FORMS

Let $Q = X^T AX$ be quadratic form and the corresponding canonical form is $d_1y_1^2 + d_2y_2^2 + \cdots + d_ny_n^2$.

The **rank** of the matrix A is number of non –zero Eigen values of A. If the rank of A is ‘r’, the canonical form of Q will contain only “r” terms .Some terms in the canonical form may be positive or zero or negative.

The number of positive terms in the canonical form is called the **index**(p) of the quadratic form.

The excess of the number of positive terms over the number of negative terms in the canonical form .i.e,$p-(r-p)=2p-r$ is called the signature of the quadratic form and usually denoted by s. Thus $s=2p-r$.