Catalog

Higher order linear differential equations with constant coefficient

General form of a linear differential equation of the n[th] order with constant coefficient is $\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1}y}{dx^{n-1}} + k_2 \frac{d^{n-2}y}{dx^{n-2}} \ldots k_n y = x$ ... (1)

Where $k_1, k_2, \ldots, k_n$ are constants it will be convenient to denote the operation $\frac{d}{dx}$ by a single letter D.

$Dy = \frac{dy}{dx}$ similarly $D^2 y = \frac{d^2y}{dx^2}$, $D^3 y = \frac{d^3y}{dx^3}$ etc

The equation (1) above can be written as

$$(D^n + k_1 D^{n-1} + \cdots + k_n)y = x$$

$$ie\ f(D)y = x$$

Note:

1. $\frac{1}{D}x = \int x\,dx$
2. $\frac{1}{D-a}x = e^{ax}\int x e^{-ax}dx$
3. $\frac{1}{D+a}x = e^{-ax}\int x e^{ax}dx$

Result:

1. $\frac{1}{D-a}\varphi(x) = e^{ax}\int e^{-ax}\varphi(x)dx$
2. $\frac{1}{D+a}\varphi(x) = e^{-ax}\int e^{ax}\varphi(x)dx$

(i) **The general form of the linear differential equation of second order is**

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Where P and Q are constants and R is a function of x or constant

(ii) **Differential Operators**

The symbol D stands for the operation of differential

$$Dy = \frac{dy}{dx},\ D^2y = \frac{d^2y}{dx^2}$$

$\frac{1}{D}$ stands for the operation of integration

$\frac{1}{D^2}$ stands for the operation of integration twice

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ can be written in the form

$$(D^2 + PD + Q)y = R$$

(iii) Complete solution is y = complementary function + Particular Integral

(iv)To find the complementary function

	Roots of A.E	**C.F**
1.	Roots are real and different m_1, m_2 ($m_1 \neq m_2$)	$Ae^{m_1 x} + Be^{m_2 x}$
2.	Roots are real and equal $m_1 = m_2 = m$(say)	$(Ax + B)e^{mx}$ or $(A + Bx)e^{mx}$
3.	Roots are imaginary $\alpha \pm i\beta$	$e^{\alpha x}[A\cos\beta x + B\sin\beta x]$
4.	Roots are $\alpha \pm i\beta$ (twice)	$e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$

(V) To find the particular integral

$$\text{P.I.} = \frac{1}{f(D)}x$$

	x	P.I
1	e^{ax}	$\text{P.I.} = \frac{1}{f(D)} e^{ax} = e^{ax}\frac{1}{f(a)}; f(a) \neq 0$ $= xe^{ax}\frac{1}{f'(a)}, f(a) = 0; f'(a) \neq 0$ $= x^2 e^{ax}\frac{1}{f''(a)}; f(a) = 0, f'(a) = 0\ f''(a) \neq 0$
2	x^n	$\text{P.I.} = \frac{1}{f(D)}x^n$ $= [f(D)]^{-1}x^n$ Expand $[f(D)]^{-1}$ and then operate
3	$\sin ax$ (or) $\cos ax$	$\text{P.I.} = \frac{1}{f(D)}[\cos ax \text{ (or) } \sin ax]$ Replace D^2 by $-a^2$
4	$e^{ax}\varphi(x)$	$\text{P.I.} = \frac{1}{f(D)}e^{ax}\varphi(x)$ $= e^{ax}\frac{1}{f(D+a)}\varphi(x)$

Problem Based on R.H.S of the given differential equation is zero

Example:

Solve $(D^2 + 2D + 1)y = 0$

Solution:

Auxiliary Equation is $m^2 + 2m + 1 = 0$

$$m = -1, -1$$

$$y = C.F$$

$$= (Ax + B)e^{-x}$$

Example:

Solve $(D^2 + 1)y = 0$ given $y(0) = 0, y'(0 = 1$

Solution:

Auxiliary Equation is $m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$

$y = A\cos x + B\sin x$

$y(x) = A\cos x + B\sin x \ldots (1)$

$y'(x) = -A\sin x + B\cos x \ldots (2)$

Given $y(0) = 0$

$(1) \Rightarrow A = 0$

Given $y'(0) = 1$

$(2) \Rightarrow B = 1$

$(1) \Rightarrow y(x) = \sin x$

Type I : Problems Based on P.I = $\frac{1}{f(D)} e^{ax}$ ⟶ **Replace D by a**

Example :

Solve $(D^2 - D - 6)y = 3e^{4x} + 5$

Solution:

Auxiliary Equation is $m^2 - m - 6 = 0$

$(m - 3)(m + 2) = 0$

$m_1 = 3, \; m_2 = -2$

$$\text{C.F} = Ae^{3x} + Be^{-2x}$$

$$\text{P.I}_1 = \frac{1}{D^2-D-6}3e^{4x} \qquad \text{Replace D by 4}$$

$$= \frac{1}{16-4-6}3e^{4x}$$

$$= \frac{1}{6}3e^{4x} \qquad = \frac{1}{2}e^{4x}$$

$$\text{P.I}_2 = \frac{1}{D^2-D-6}5$$

$$= \frac{1}{D^2-D-6}5e^{0x} \quad \text{Replace D by 0}$$

$$= \frac{-1}{6}5 \qquad = \frac{-5}{6}$$

The general solution is y = C.F + P.I

$$y = Ae^{3x} + Be^{-2x} + \frac{1}{2}e^{4x} - \frac{5}{6}$$

Example:

Solve $(D^2 + 7D + 12)y = 14e^{-3x}$

Solution:

Auxiliary Equation is $m^2 + 7m + 12 = 0$

$$m = -3,\ m = -4$$

$$\text{C.F} = Ae^{-3x} + Be^{-4x}$$

$$\text{P.I} = \frac{1}{D^2+7D+12}14e^{-3x}$$

$$= 14\frac{1}{9-21+12}e^{-3x} \quad \text{Replace D by -3}$$

$$= \frac{1}{0}e^{-3x} \qquad \text{(fails)}$$

$$= 14x\frac{1}{2D+7}e^{-3x}$$

$$= 14x\frac{1}{-6+7}e^{-3x}$$

$$= 14xe^{-3x}$$

The general solution is y = C.F + P.I

$$y = Ae^{-3x} + Be^{-4x} + 14xe^{-3x}$$

Example:

Find the P.I of $(D^2 - 1)\,y = (e^x + 1)^2$

Solution:

Given $(D^2-1)y=(e^x+1)^2$

$=(e^x)^2+1+2e^x$

$=e^{2x}+e^0+2e^x$

$P.I_1 = \frac{1}{D^2-1}e^{2x}$ Replace D by 2

$=\frac{1}{4-1}e^{2x}$

$=\frac{1}{3}e^{2x}$

$P.I_2 = \frac{1}{D^2-1}e^{0x}$

$=\frac{1}{-1}e^{0x}$ Replace D by 0

$=-e^{0x}$

$=-1$

$P.I_3 = \frac{1}{D^2-1}2e^x$

$=2\frac{1}{D^2-1}e^x$ Replace D by 1

$=2\frac{1}{1-1}e^x$ (fails)

$=2x\frac{1}{2D}e^x$ Replace D by 1

$=2x\frac{1}{2}e^x$

$=xe^x$

$P.I = P.I_1+P.I_2+P.I_3$

$=\frac{1}{3}e^{2x}-1+xe^x$

Type II:

Problems Based on P.I = $\frac{1}{f(D)}\sin ax$ **(or)** $\frac{1}{f(D)}\cos ax$ → **Replace D^2 by $-a^2$**

Example :

Solve $(D^2-4D+4)y=e^{2x}+\sin^2 x$

Solution:

Auxiliary Equation is $m^2-4m+4=0$

$(m-2)(m-2)=0$

$m=2,2$

C.F = $(Ax+B)e^{2x}$

$$P.I = \frac{1}{D^2-4D+4}[e^{2x} + \sin^2 x]$$

$$= \frac{1}{D^2-4D+4}[e^{2x} + \frac{1-\cos 2x}{2}]$$

$$P.I_1 = \frac{1}{D^2-4D+4} e^{2x}$$ Replace D by 2

$$= \frac{1}{4-8+4} e^{2x}$$

$$= \frac{1}{0} e^{2x}$$ (fails)

$$= x.\frac{1}{2D-4} e^{2x}$$ Replace D by 2

$$= x.\frac{1}{4-4} e^{2x}$$

$$= x\frac{1}{0} e^{2x}$$ (fails)

$$= x^2 \frac{1}{2} e^{2x}$$

$$P.I_2 = \frac{1}{D^2-4D+4}(\frac{1}{2})\ e^{0x}$$ Replace D by 0

$$= \frac{1}{8}$$

$$P.I_3 = \frac{1}{D^2-4D+4}(\frac{-\cos 2x}{2})$$ Replace D^2 by -4

$$= \frac{1}{-4-4D+4}(\frac{-\cos 2x}{2})$$

$$= \frac{1}{-4D}(\frac{-\cos 2x}{2})$$

$$= \frac{1}{8D}\cos 2x$$

$$= \frac{1}{8}\int \cos 2x\ dx$$

$$= \frac{1}{8}(\frac{\sin 2x}{2})$$

$$= \frac{1}{16}\sin 2x$$

The general solution y = C.F + P.I$_1$ + P.I$_2$ + P.I$_3$

$$y = (Ax + B)e^{2x} + \frac{x^2}{2}e^{2x} + \frac{1}{8} + \frac{1}{16}\sin 2x$$

Example :

Find the P.I of (D^2+4) y = cos2x

Solution:

$$P.I = \frac{1}{D^2+4}\cos 2x$$ Replace D^2 by -4

$= \frac{1}{-4+4} \cos 2x$

$= \frac{1}{0} \cos 2x$ (fails)

$= x \frac{1}{2D} \cos 2x$

$= \frac{x}{2D} \cos 2x$

$= \frac{x}{2} \int \cos 2x \, dx$

$= \frac{x}{2} \frac{\sin 2x}{2} = \frac{x}{4} sin2x$

$P.I = \frac{x}{4} sin2x$

Example :

Find the P.I of $\frac{d^3y}{dx^3} + 4\frac{dx}{dx} = sin2x$

Solution:

$P.I = \frac{1}{D^3+4D} \sin 2x$

$= \frac{1}{D(D^2+4)} \sin 2x$ Replace D^2 by -4

$= \frac{1}{D(-4+4)} \sin 2x$ (fails)

$= x \frac{1}{3D^2+4} \sin 2x$ Replace D^2 by -4

$= x \frac{1}{3(-4)+4} \sin 2x$

$= x \frac{1}{-12+4} \sin 2x$

$= \frac{-x}{8} \sin 2x$

$P.I = \frac{-x}{8} \sin 2x$

Type III: Problems Based on R.H.S= $e^{ax} + \sin ax$ (or) $e^{ax} + \cos ax$

Example :

Solve $(D^2 - 3D + 2)y = 2\cos(2x + 3) + 2e^x$

Solution:

Auxiliary Equation is $m^2 - 3m + 2 = 0$

$m = 1, m = 2$

C.F = $Ae^x + Be^{2x}$

$$P.I_1 = \frac{1}{D^2-3D+2} 2e^x$$

$= 2 \frac{1}{1-3+2} e^x$ Replace D by 1

$= 2 \frac{1}{0} e^x$ (fails)

$= 2x \frac{1}{2D-3} e^x$

$= 2x \frac{1}{2-3} e^x$ Replace D by 1

$= -2xe^x$

$$P.I_2 = \frac{1}{D^2-3D+2} 2\cos(2x+3)$$

$= 2 \frac{1}{-4-3D+2} \cos(2x+3)$ Replace D^2 by -4

$= 2 \frac{1}{-3D-2} \cos(2x+3)$

$= 2 \frac{1}{-3D-2} \frac{-3D+2}{-3D+2} \cos(2x+3)$

$= 2 \frac{-3D+2}{9D^2-4} \cos(2x+3)$ Replace D^2 by -4

$= 2 \frac{-3D+2}{-40} \cos(2x+3)$

$= \frac{-3D+2}{-20} \cos(2x+3)$

$= 6\sin(2x+3) + 2\cos(2x+3) / -20$

$= -\frac{1}{10}\cos(2x+3) - \frac{3}{10}\sin(2x+3)$

The general solution is y = C.F + $P.I_1$ + $P.I_2$

$$y = Ae^x + Be^{2x} - 2xe^x - \frac{1}{10}\cos(2x+3) - \frac{3}{10}\sin(2x+3)$$

Type IV : Problems Based on R.H.S = Polynomial in x

Binomial expression

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots \ldots \ldots \ldots \ldots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots \ldots \ldots \ldots \ldots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots \ldots \ldots \ldots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \ldots \ldots \ldots ..$

Example:

Solve $y'' + 4y' + 5y = 3x - 2$

Solution:

Auxiliary Equation is $m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{16-20}}{2}$$

$$= \frac{-4 \pm \sqrt{-4}}{2}$$

$$= \frac{-4 \pm 2i}{2} = -2 \pm i$$

$\propto = -2, \ \beta = 1$

C.F $= e^{-2x}[A\cos x + B\sin x]$

P.I $= \frac{1}{D^2+4D+5}(3x-2)$

$$= \frac{1}{5[1+\frac{D^2+4D}{5}]}(3x-2)$$

$$= \frac{1}{5}[1 + \frac{D^2+4D}{5}]^{-1} (3x-2)$$

$$= \frac{1}{5}[1 - \frac{D^2+4D}{5}] (3x-2)$$

$$= \frac{1}{5}[1 - \frac{D^2}{5} - \frac{4D}{5}] (3x-2)$$

$$= \frac{1}{5}[(3x-2) - \frac{D^2}{5}(3x-2) - \frac{4D}{5}(3x-2)]$$

$$= \frac{1}{5}[3x - 2 - 0 - \frac{4}{5}(3)]$$

$$= \frac{1}{5}[\frac{15x-10-12}{5}]$$

$$= \frac{1}{25}[15x - 22]$$

The general solution is y = C.F +P.F

$$y = e^{-2x}[A\cos x + B\sin x] + \frac{1}{25}[15x - 22]$$

Example:

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2 + 3$

Solution:

$$(D^2 - 5D + 6)y = x^2 + 3$$

Auxiliary Equation is $m^2 - 5m + 6 = 0$

$$m = 3, 2$$

C.F = $Ae^{3x} + Be^{2x}$

P.I$_1$ = $\frac{1}{D^2-5D+6}x^2$

$$= \frac{1}{6[1+\frac{D^2-5D}{6}]} x^2$$

$$= \frac{1}{6}[1 + \frac{D^2-5D}{6}]^{-1} x^2$$

$$= \frac{1}{6}[1 - (\frac{D^2-5D}{6}) + (\frac{D^2-5D}{6})^2 \ldots.] x^2$$

$$= \frac{1}{6}[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{25}{36} D^2] x^2$$

$$= \frac{1}{6}[x^2 - \frac{D^2(x^2)}{6} + \frac{5D(x^2)}{6} + \frac{25}{36} D^2(x^2)]$$

$$= \frac{1}{6}[x^2 - \frac{2}{6} + \frac{5\times 2x}{6} + \frac{25}{36}(2)]$$

$$= \frac{1}{6}[x^2 + \frac{5}{3}x + \frac{19}{18}]$$

$$\text{P.I}_2 = \frac{1}{D^2-5D+6} 3e^{0x}$$

$$= \frac{1}{2}$$

The general solution is y =C.F + P.I$_1$ +P.I$_2$

$$y = Ae^{3x} + Be^{2x} + \frac{1}{6}[x^2 + \frac{5}{3}x + \frac{19}{18}] + \frac{1}{2}$$

Example:

Solve (D³ +8) $y = x^4 + 2x + 1$

Solution :

Auxiliary Equation is $m^3 + 8 = 0$

$$m = -2, m^2 - 2m + 4 = 0$$

$$m = \frac{1 \pm i\sqrt{3}}{2}$$

$$= \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\alpha = \frac{1}{2} \qquad \beta = \frac{\sqrt{3}}{2}$$

$$\text{C.F} = Ae^{-2x} + Be^{\frac{1}{2}x}[B\cos\frac{\sqrt{3}}{2}x + C\cos\frac{\sqrt{3}x}{2}]$$

$$\text{P.I} = \frac{1}{D^3+8}(x^4 + 2x + 1)$$

$$= \frac{1}{8[1+\frac{D^3}{8}]}(x^4 + 2x + 1)$$

$$= \frac{1}{8}[1 + \frac{D^3}{8}]^{-1} (x^4 + 2x + 1)$$

$$= \frac{1}{8}\left[1 - \frac{D^3}{8} + \left(\frac{D^3}{8}\right)^2 \ldots\right](x^4 + 2x + 1)$$

$$= \frac{1}{8}\left[1 - \frac{D^3}{8}\right](x^4 + 2x + 1)$$

$$= \frac{1}{8}\left[(x^4 + 2x + 1) - \frac{D^3}{8}(x^4 + 2x + 1)\right]$$

$$= \frac{1}{8}\left[x^4 + 2x + 1 - \frac{24x}{8}\right]$$

$$= \frac{1}{8}[x^4 + 2x + 1 - 3x]$$

$$= \frac{1}{8}[x^4 - x + 1]$$

The general solution is $y = C.F + P.I$

$$y = Ae^{-2x} + Be^{\frac{1}{2}x}\left[B\cos\frac{\sqrt{3}}{2}x + C\cos\frac{\sqrt{3}}{2}x\right] + \frac{1}{8}[x^4 - x + 1]$$

Type V: Problems based on R.H.S $e^{ax}F(x)$

P.I $= \frac{1}{f(D+a)} e^{ax}F(x)$ Replace x by D+a

$= e^{ax}\frac{1}{f(D+a)}F(x)$

Example :

Solve $(D^2 + 1)y = xsinhx$

Solution:

Auxiliary Equation is $m^2 + 1 = 0$

$m^2 = -1$

$m^2 = \pm i$

C.F $= Acosx + Bsinx$

P.I $= \frac{1}{D^2+1} xsinhx$

$= \frac{1}{D^2+1}\left[x\left(\frac{e^x - e^{-x}}{2}\right)\right]$

$= \frac{1}{2}\left[\frac{1}{D^2+1}xe^x - \frac{1}{D^2+1}xe^{-x}\right]$

Replace $by\ D + 1$; Replace D by $D - 1$

$= \frac{1}{2}\left[e^x\frac{1}{(D+1)^2+1}x - e^x\frac{1}{(D-1)^2+1}\right]$

$= \frac{1}{2}\left[e^x\frac{1}{D^2+2D+2}x - e^{-x}\frac{1}{D^2-2D+2}x\right]$

$$= \frac{1}{2}\left[e^{x} \frac{1}{2[1+(\frac{D^2+2D}{2})]} x - e^{-x} \frac{1}{2[1+(\frac{D^2-2D}{2})]} x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{2}\left[1+(\frac{D^2+2D}{2})\right]^{-1} x - \frac{e^{-x}}{2}\left[1+(\frac{D^2-2D}{2})\right]^{-1} x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{2}(1-\frac{D^2}{2}-\frac{2D}{2})\,x - \frac{e^{-x}}{2}(1-\frac{D^2}{2}+\frac{2D}{2})\,x\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{2}(x-1) - \frac{e^{-x}}{2}(x+1)\right]$$

$$= \frac{1}{2}\left[\frac{e^x x}{2} - \frac{e^x}{2} - \frac{xe^{-x}}{2} - \frac{e^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[x\left(\frac{e^x-e^{-x}}{2}\right) - \left(\frac{e^x+e^{-x}}{2}\right)\right]$$

$$= \frac{1}{2}[xsinhx - coshx]$$

The general solution is y = C.F + P.I

$$y = Acosx + Bsinx + \frac{1}{2}(xsinhx - coshx)$$

TYPE VI:

Problems based on f(x) = $x^n \sin ax$ or $x^n \cos ax$ $P.I = \frac{1}{f(x)} x^n \sin ax$ or $x^n \cos ax$

Example

Solve $(D^2 + 1)y = xsin\,x$

Solution:

Auxiliary Equation is $m^2 + 1 = 0$

$$m^2 + 1 = -1$$

$$m = \pm i$$

C.F $= Acos\,x + Bsin\,x$

P.I $= \frac{1}{D^2+1}\, xsin\,x$

$= \frac{1}{D^2+1}\, x\; I.P\ of\ e^{ix}$ Replace D $by D + i$

$= I.P\ of e^{ix}\ \frac{1}{(D+i)^2+1}\ x$

$= I.P\ of e^{ix}\ \frac{1}{D^2+2Di+i^2+1}\ x$

$= I.P\ of e^{ix}\ \frac{1}{D^2+2Di+i^2+1}\ x$

$= I.P\ of e^{ix}\ \frac{1}{D^2+2Di}\ x$

$= I.P\ of e^{ix}\ \frac{1}{2Di}\ (1+\frac{D}{2i})^{-1}\ x$

$= I.P\ of e^{ix}\ \frac{1}{2Di}\ (x-\frac{D(x)}{2i})$

$= I.P\ of (\cos x + i\sin x)\ (\frac{x^2}{4i}+\frac{x}{4})$

$= I.P\ of (\cos x + i\sin x)\ (\frac{-x^2 i}{4}+\frac{x}{4})$

$= I.P\ of\ (\frac{-ix^2}{4}\cos x+\frac{x\cos x}{4}-\frac{x^2 sinx}{4}+\frac{ix\ sinx}{4})$

$= \frac{-x^2}{4}\cos x+\frac{xsin\ x}{4}$

The general solution is y = C.F + P.I

$$y = Acosx + Bsinx - \frac{x^2}{4}cosx + \frac{xsin\ x}{4}$$

Example:

Solve $(D^2-4D+4)y = 3x^2e^{2x}sin\ 2x$

Solution:

Auxiliary Equation is $m^2-4m+4=0$

$m = 2, 2$

C.F = $(A+Bx)e^{2x}$

P.I $= \frac{1}{D^2-4D+4}\ 3x^2e^{2x}sin\ 2x$ Replace D by $D+2$

$= 3e^{2x}\ \frac{1}{(D+2)^2-4(D+2)+4}\ x^2sin2x$

$= 3e^{2x}\ \frac{1}{D^2}\ x^2sin2x$

$= 3e^{2x}\ \frac{1}{D^2}\ x^2 I.P\ of e^{i2x}$ Replace D by $D+2i$

$= 3e^{2x}I.P\ of e^{i2x}\ \frac{1}{(D+2i)^2}\ x^2$

$= 3e^{2x}I.P\ of e^{i2x}\ \frac{1}{-4[1-\frac{D^2+4Di}{4}]}\ x^2$

$= 3e^{2x}I.P\ of e^{i2x}\ \frac{1}{-4}[1-\frac{D^2+4Di}{4}]^{-1}\ x^2$

$$= 3e^{2x} I.P\ of\ e^{i2x}\frac{1}{-4}\left[1+\left(\frac{D^2+4Di}{4}\right)+\left(\frac{D^2+4Di}{4}\right)^2+\cdots\right]x^2$$

$$= 3e^{2x} I.P\ of\ e^{i2x}\frac{1}{-4}\left[1+\frac{D^2}{4}+Di+D^2 i\right](x^2)$$

$$= \frac{3e^{2x}}{-4} I.P\ of(\cos 2x + i\sin 2x)\left(x^2+\frac{1}{2}+i2x-2\right)$$

$$= \frac{-3}{4}e^{2x} I.P\ of(\cos 2x + i\sin 2x)\left(x^2+2xi-\frac{3}{2}\right)$$

$$= \frac{-3}{4}e^{2x} I.P\ of\left(x^2\cos 2x + i2x\cos 2x - \frac{3}{2}\cos 2x + i\,x^2\sin 2x - 2x\sin 2x - i\frac{3}{2}\sin 2x\right)$$

$$= \frac{-3e^{2x}}{4}\left[2x\cos 2x + x^2\sin 2x - \frac{3}{2}\sin 2x\right]$$

The general solution y = C.F + P.I

$$y = (A+Bx)e^{2x} - \frac{3}{4}e^{2x}\left[2x\cos 2x + x^2\sin 2x - \frac{3}{2}\sin 2x\right]$$

Method of variation of parameters

This method is very useful in finding the general solution of the second order equation

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = x$$

Solution is $y = Au + Bv$

Where A, B are constants

$$A = \int \frac{-Rv}{uv' - vu'}\, dx + C_1$$

$$B = \int \frac{Ru}{uv' - vu'}\, dx + C_2$$

Example:

Solve $(D^2 + 4)y = \tan 2x$

Solution:

Auxiliary Equation is $m^2 + 4 = 0$

$$m^2 = -4$$

$$m = \pm 2i$$

C.F = $A\cos ax + B\sin ax$

Let the solution be $y = Au + Bv$

$u = \cos 2x$	$v = \sin 2x$
$u' = -2\sin 2x$	$v' = 2\cos 2x$

$$uv' - vu' = 2\cos^2 2x + 2\sin^2 2x$$

$$= 2$$

$$A = \int \frac{-Rv}{uv' - vu'}\, dx + C_1 \qquad \text{R= tan 2x}$$

$$= \int \frac{-\tan 2x \sin 2x}{2}\, dx + C_1$$

$$= \frac{-1}{2}\int \frac{\sin 2x}{\cos 2x}\sin 2x\, dx + C_1$$

$$= \frac{-1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx + C_1$$

$$= \frac{-1}{2}\int \frac{(1-\cos^2 2x)}{\cos 2x} dx + C_1$$

$$= \frac{-1}{2}\int \left(\frac{1}{\cos 2x} - \frac{\cos^2 2x}{\cos 2x}\right) dx + C_1$$

$$= \frac{-1}{2}\int (\sec 2x - \cos 2x)\, dx + C_1$$

$$= \frac{-1}{2}\left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right] + C_1$$

$$B = \int \frac{-Ru}{uv' - vu'}\, dx + C_2$$

$$= \int \frac{\tan 2x \cos 2x}{2}\, dx + C_2$$

$$= \int \frac{1}{2} \sin 2x\, dx + C_2$$

$$= -\frac{1}{2}\frac{\cos 2x}{2} + C_2$$

$$= -\frac{\cos 2x}{4} + C_2$$

$$\therefore y = Au + Bv$$

$$y = \{[\frac{-1}{4}(\log \sec 2x + \tan 2x) - \frac{\sin 2x}{4}] + C_1\} \cos 2x + (-\frac{\cos 2x}{4} + C_2) \sin 2x$$

Example:

Solve the equation $\frac{d^2y}{dx^2} + y = cosec\ x$

Solution:

$$(D^2 + 1)y = cosec\ x$$

Auxiliary Equation is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i$$

$$\alpha = 0, \beta = 1$$

C.F = $A\cos x + B \sin x$

Let the solution be $y = Au + Bv$

Here $u = \cos x$ | $v = \sin x$

$u' = -\sin x$ | $v' = \cos x$

$$uv' - vu' = \cos^2 x + \sin^2 x$$

$= 1$; R = cosec x

$$A = \int \frac{-Rv}{uv' - vu'}\, dx + C_1$$

$$= \int \frac{-\sin x \operatorname{cosec} x}{1} dx + C_1$$

$$= \int -dx + C_1$$

$$= -x + C_1$$

$$B = \int \frac{Ru}{uv' - vu'}\, dx + C_2$$

$$= \int \frac{\operatorname{cosec} x \cos x}{1} dx + C_2$$

$$= \int \frac{\cos x}{\sin x} dx + C_2$$

$$= \int \cot x \, dx + C_2$$

$$= \log \sin x + C_2$$

$$\therefore \quad y = Au + Bv$$

$$y = (-x + C_1) \cos x + (log \sin x + C_2) \sin x$$

Example:

Solve $(D^2 + 4)y = cot\ 2x$

Solution:

Auxiliary Equation is $m^2 + 4 = 0$

$$m^2 = -4$$

$$m = \pm 2i$$

C.F = $Acos\ 2x + B \sin 2x$

Let the solution be $y = Au + Bv$

Here $u = cos\ 2x$ | $v = sin\ 2x$

$u' = -2\sin 2x$ | $v' = 2\cos 2x$

$$uv' - vu' = 2cos^2 2x + 2sin^2 2x$$

$= 2$; R = cot 2x

$$A = \int \frac{-Rv}{uv' - vu'} dx + C_1$$

$$= \int \frac{-\cot 2x \sin 2x}{2} dx + C_1$$

$$= -\frac{1}{2} \int \cos 2x \, dx + C_1$$

$$= -\frac{1}{2} \left(\frac{\sin 2x}{2}\right) + C_1$$

$$= \frac{-\sin 2x}{4} + C_1$$

$$B = \int \frac{Ru}{uv' - vu'} dx + C_2$$

$$= \int \frac{\cot 2x \cos 2x}{2} dx + C_2$$

$$= \frac{1}{2} \int \frac{\cos^2 2x}{\sin 2x} dx + C_2$$

$$= \frac{1}{2} \int \left(\frac{1 - sin^2 2x}{\sin 2x}\right) dx + C_2$$

$$= \frac{1}{2} \int (\text{cosec } 2x - \sin 2x) \, dx + C_2$$

$$= \frac{1}{2}\left[-\frac{1}{2}\log(cosec\ 2x + \cot 2x) + \frac{1}{2}\cos 2x\right] + C_2$$

$$= -\frac{1}{4}\log(cosec\ 2x + \cot 2x) + \frac{1}{4}\cos 2x + C_2$$

$$\therefore \quad y = Au + Bv$$

$$y = \left[-\frac{\sin 2x}{4} + C_1\right]\cos 2x + \left[-\frac{1}{4}\log(cosec\ 2x + \cot 2x) + \frac{1}{4}\cos 2x + C_2\right]\sin 2x$$

Example:

Solve $\frac{d^2y}{dx^2} + a^2y = \sec ax$

Solution:

$(D^2 + a^2)y = \sec ax$

Auxiliary Equation is $m^2 + a^2 = 0$

$m = \pm ia$

C.F = $(A\,cos\ ax + B\sin ax)$

Let the solution be $y = Au + Bv$

Here

u = cos ax	v= sin ax
$u' = -a\sin ax$	$v' = a\cos ax$

$uv' - vu' = a\,cos^2\,ax + a\,sin^2\,ax = a$

$$A = \int \frac{-Rv}{uv' - vu'}\,dx + C_1$$

$$= \int \frac{-\sec ax \sin ax}{a}\,dx + C_1$$

$$= \frac{-1}{a}\int \frac{1}{\cos ax}\sin ax\,dx + C_1$$

$$= \frac{-1}{a}\int \tan ax\ dx + C_1$$

$$= +\frac{\log\cos ax}{a^2} + C_1$$

$$B = \int \frac{Ru}{uv' - vu'}\,dx + C_2$$

$$= \int \frac{\sec ax \cos ax}{a}dx + C_2$$

$$= \frac{1}{a}\int \frac{1}{\cos ax}\cos ax\,dx + C_2$$

$$= \frac{1}{a}x + C_2$$

$$\therefore y = Au + Bv$$

$$y = \cos ax \left[\frac{\log(\cos ax)}{a^2} + C_1\right] + \sin ax \left[\frac{x}{a} + C_2\right]$$

$$y = C_1 \cos ax + \cos ax \frac{\log(\cos ax)}{a^2} + C_2 \sin ax + \frac{x}{a} \sin ax$$

Homogeneous equation of Euler's and Legendre's type

The general form of linear equation of second order is given by $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + Qy = R$

Where P,Q and R are functions of x only

Homogeneous equation of Euler type (Cauchy type)

An equation of the form

$$\frac{x^n d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots \ldots . + a_n y = f(x)$$

Where $a_1, a_2, \ldots, a_n$ are constants and $f(x)$ is a function of x

Equation (1) can be reduced to linear differential equation with constant coefficients by putting substitution

$x = e^t$ (or) t= $\log x$

Now $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$

$x\frac{dy}{dx} = \frac{dy}{dt}$ (or) $x\frac{dy}{dx} = Dy$ $where\ D = \frac{d}{dt}$

$\frac{d^2y}{dx^2} = D(D-1)y$

Similarly $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on

Example:

Solve $(x^2D^2 - xD + 4)y = \sin(\log x)$

Solution:

Put $t = log\ x \Rightarrow x = e^t$

$xD = D$

$x^2D^2 = D(D-1)$

$[D(D-1) - D + 4]y = \sin t$

$(D^2 - 2D + 4D)y = \sin t$

Auxiliary Equation is $m^2 - 2m + 4 = 0$

$m = \frac{2 \pm \sqrt{4-16}}{2}$

$= \frac{2 \pm \sqrt{-12}}{2}$

$= \frac{2 \pm 2i\sqrt{3}}{2}$ $= 1 \pm i\sqrt{3}$

$$\alpha = 1 \; \beta = \sqrt{3}$$

C.F $= e^t[A\cos\sqrt{3}\,t + B\sin\sqrt{3}\;t]$

$$P.I = \frac{1}{D^2-2D+4}\; sint \qquad \text{Replace } D^2 \; by -1$$

$$= \frac{1}{-1-2D+4}\; sint$$

$$= \frac{1}{3-2D}\; sint$$

$$= \frac{3+2D}{(3+2D)(3-2D)}\; sint$$

$$= \frac{(3+2D)}{9-4D^2}\; sint \qquad \text{Replace } D^2 \; by -1$$

$$= \frac{3\sin t + 2D(\sin t)}{9-16(-1)} sint$$

$$= \frac{3\sin t + 2\cos t}{25}$$

The general solution is $y =$ C.F +P.I

$$y = e^t[Acos\sqrt{3}t + Bsin\;\sqrt{3}t] + \frac{3\sin t + 2\cos t}{25}$$

$$= x[Acos(\sqrt{3}\log x) + Bsin\;(\sqrt{3}\log x)] + \frac{3\sin(\log x) + 2\cos(\log x)}{25}$$

Example:

Solve $(x^2D^2 - 3xD + 4)y = x^2\cos(\log x)$

Solution:

Put $log\,x = t \Rightarrow x = e^t$

$$xD = D$$

$$x^2D^2 = D(D-1)$$

$$[D(D-1) - 3D + 4]y = e^{2t}\sin t$$

$$(D^2 - 4D + 4)y = e^{2t}\sin t$$

Auxiliary Equation is $m^2 - 4m + 4 = 0$

$$m = 2, 2$$

C.F $= (At + B)e^{2t}$

$$P.I = \frac{1}{D^2-4D+4}\; e^{2t}sint$$

$$= \frac{1}{(D-2)^2}\; e^{2t}sint \qquad \text{Replace } D \; by \; D+2$$

$$= e^{2t}\frac{1}{(D+2-2)^2}\; sin\,t$$

$= e^{2t}\frac{1}{D^2}\sin t$ Replace D by -1

$= e^{2t}\frac{1}{-1}\sin t = -e^{2t}\sin t$

The general solution is $y =$ C.F + P.I

$$y = (At + B)e^{2t} - e^{2t}\cos t$$

$$= (A\log x + B)x^2 - x^2\cos\log x$$

Legendre's Linear Differential Equation

An equation of the form

$$(ax + b)^n\frac{d^n y}{dx^n} + k_j(ax + b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + x_n y = 0 \ldots (1)$$

Where $k's$ are constant and Q is a function of x is called such equations are reduced by using substitution

$$ax + b = e^t$$

$$t = \log(ax + b)$$

$$(ax + b)D = a\,D$$

$$(ax + b)^2 D^2 = a^2 D(D - 1) \text{ and so on.}$$

After making these substitution in (1) it reduces to a linear differential equation with constant coefficients

Example:

Solve $(3x + 2)^2 y'' + 3(3x + 2)y' - 36y = 3x^2 + 4x + 3$

Solution:

Put t = log (3x+2)

$3x + 2 = e^t;\ x = \frac{e^t - 2}{3}$

(3 x +2) D = 3D

$(3x + 2)^2 D^2 = 9D(D - 1)$

$$[9D(D - 1) + 9D - 36]y = 3\left(\frac{e^t - 2}{3}\right)^2 + 4\left(\frac{e^t - 2}{3}\right) + 1$$

$$[9D^2 - 9D + 9D - 36]y = 3\frac{(e^{2t} - 4e^t + 4)}{9} + 4\left(\frac{e^t - 2}{3}\right) + 1$$

$$[9D^2 - 36]y = \frac{(e^{2t} - 4e^t + 4e^t + 4 - 8 + 3)}{3}$$

$$= \frac{e^{2t} - 1}{3}$$

$$(D^2-4)y = \frac{e^{2t}-1}{27}$$

$$(D^2-4)y = \frac{1}{27}e^{2t} - \frac{1}{27}$$

Auxiliary Equation is $m^2-4=0$

$$m^2 = 4$$

$$m = \pm 2$$

$$\text{C.F} = Ae^{2t} + Be^{-2t}$$

$$P.I = \frac{1}{D^2-4}\frac{1}{27}e^{2t} \qquad \text{Replace D by 2}$$

$$= \frac{1}{0}\frac{1}{27}e^{2t}$$

$$= \frac{t}{2D}\frac{1}{27}e^{2t}$$

$$= \frac{t}{108}e^{2t}$$

$$P.I_2 = \frac{1}{D^2-4}\left(\frac{-1}{27}\right) \qquad \text{Replace } D \text{ by } 0$$

$$= \frac{1}{-4}\left(\frac{-1}{27}\right)$$

$$= \frac{1}{108}$$

The general solution is y = C.F + P.I

$$y = Ae^{2t} + Be^{-2t} + \frac{t}{108}e^{2t} + \frac{1}{108}$$

$$y = A(3x+2)^2 + B(3x+2)^{-2} + (3x+2)^2\frac{\log x}{108} + \frac{1}{108}$$

Example:

Solve $(\mathbf{1}+\boldsymbol{x})^2\frac{d^2y}{dx^2} + (\mathbf{1}+\boldsymbol{x})\frac{dy}{dx} + \boldsymbol{y} = \mathbf{4}\cos[\log(\mathbf{1}+\boldsymbol{x})]$

Solution:

Put t = log (1+x)

$$(1+x) = e^t$$

$$(1+x)^2D^2 = D(D-1)$$

$$[D(D-1)+D+1]y = 4\cos t$$

$$[D^2+1]y = 4\cos t$$

Auxiliary Equation is $m^2+1=0$

$$m^2 = -1$$

$$m = \pm i$$

$\alpha = 0, \beta = 1$

C.F = $A\cos t + B\sin t$

$P.I = \frac{1}{D^2+1} \, 4\cos t$ Replace D $by -1$

$= \frac{1}{-1+1} \, 4\cos t$

$= \frac{t}{2D} \, 4\cos t$

$= 2t \sin t$

The general solution is y = C.F+P.I

$y = A\cos t + B\sin t + 2t\sin t$

$y = A\cos\log(1+x) + B\sin\log(1+x) + 2\log(1+x)\sin\log(1+x)$

Example:

Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin[\log(1+x)]$

Solution:

Put $1 + x = e^t$

$t = \log(1+x)$

$(1+x)\frac{dy}{dx} = Dy$

$(1+x)^2\frac{d^2y}{dx^2} = D(D-1)y$

$[D(D-1) + D + 1]y = 2\sin t$

$[D^2 + 1]y = 2\sin t$

Auxiliary Equation is $m^2 + 1 = 0$

$m = \pm i$

C.F = $A\cos t + B\sin t$

$= A\cos[\log(1+x)] + B\sin[\log(1+x)]$

$P.I = \frac{1}{D^2+1} \, 2\sin t$ Replace $D^2 by -1$

$= 2\,\frac{1}{0}\sin t$

$= \frac{2t}{2D}\sin t$

$= \frac{t}{D}\sin t$

$$= -t\cos t$$

$$= -\log(1+x)\cos[\log(1+x)]$$

The general solution is $y = C.F + P.I$

$$y = A\cos[\log(1+x)] + B\sin[\log(1+x)] - \log(1+x)\cos[\log(1+x)]$$

System of Simultaneous Linear Differential equations with constant coefficients

Simultaneous Linear equations

Linear differential equations in which there are two or more dependent variables and a single independent variable such equations are known as simultaneous linear equations.

Consider the simultaneous equation in two dependent variables x and y and one independent variable t.

$$f_1(D)x + g_1(D)y = h_1(t) \ldots (1)$$

$$f_2(D)x + g_2(D)y = h_2(t) \ldots (2)$$

Where f_1, f_2, g_1 and g_2 are polynomials in the operator D

Example:

Solve $\boldsymbol{\frac{dy}{dx} + x = t^2; \quad \frac{dx}{dt} - y = t}$

Solution:

$$x + Dy = t^2 \ \ldots (1)$$

$$Dx - y = t \ \ldots (2)$$

Eliminate $'x'$

(1) x D $\Rightarrow$ $Dx + D^2y = D\,(t^2)$

$$Dx + D^2y = 2t \ldots (3)$$

(2) $\Rightarrow$ $Dx - y = t \ \ldots (4)$

(3) – (2) $D^2y + y = t$

$$(D^2 + 1)y = t$$

Auxiliary Equation is $m^2 + 1 = 0$

$$m^2 = -1$$

$$m = \pm i$$

$$\propto = 0, \beta = 1$$

$$C.F = e^{at}[A\cos\beta t + B\sin\beta t]$$

$$= A\cos t + B\sin t$$

$$\text{P.I} = \frac{1}{D^2+1}(t)$$

$$= \frac{1}{1+D^2}(t)$$

$= (1 + D^2)^{-1} t$

$= (1 - D^2) t$

$= t - D^2(t)$

$= t$

$y = Acost + Bsint + t$

$Dy = -Asint + Bcost + 1$

$(1) \Rightarrow \quad x = t^2 - Dy$

$= t^2 - [-Asint + Bcost + 1]$

$= t^2 + Asint - Bcost - 1$

$x = t^2 + Asint - Bcost - 1$

$y = Acost + Bsint + t$

Example:

Solve the simultaneous differential equations

$\frac{dx}{dt} + 2y = sin2t \quad ; \quad \frac{dy}{dt} - 2x = cos2t$

Solution:

$\frac{dx}{dt} + 2y = sin2t$

$Dx + 2y = sin2t \ldots (1)$

$\frac{dy}{dt} - 2x = cos2t$

$-2x + Dy = cos2t \ldots (2)$ Eliminate $'x'$

$(1) \times 2 \quad \Rightarrow \quad 2Dx + 4y = 2sin2t \ldots (3)$

$(2) \times D \quad \Rightarrow \quad -2Dx + D^2y = -2sin2t \ldots (4)$

$(3) + (4) \quad \Rightarrow \quad D^2y + 4y = 0$

$(D^2 + 4)y = 0$

Auxiliary Equation is $m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i$

$\propto = 0, \beta = 2$

$C.F = e^{\propto t}[Acos\beta t + Bsin\beta t]$

$y = Acos2t + Bsin2t$

To find x , $\quad (5) \quad \Rightarrow \quad Dy = -2Asin2t + 2Bcos2t$

(2) $\Rightarrow$ $2x = Dy - cos2t$

$$= -2Asin2t + 2Bcos2t - cos2t$$

$$x = -Asin2t + Bcos2t - \frac{cos2t}{2}$$

Example:

Solve $\frac{dx}{dt} + 2x + 3y = 2e^{2t}$; $\frac{dy}{dt} + 3x + 2y = 0$

Solution:-

Given $Dx + 3x + 2y = 0$

$$(D+2)x + 3y = 2e^{2t} \ldots (1)$$

$$Dy + 3x + 2y = 0$$

$$3x + (D+2)y = 0 \ldots (2)$$

Eliminate x

(2) x (D+2) $\Rightarrow$ $3(D+2)x + (D+2)^2 y = 0 \ \ldots (3)$

(1) x (3) $\Rightarrow$ $3(D+2)x + 9y = 6e^{2t} \ldots (4)$

(3) – (4) $\Rightarrow$ $[(D+2)^2 - 9]y = -6e^{2t}$

$$(D^2 + 4D - 5)y = -6e^{2t}$$

Auxiliary Equation is $m^2 + 4m - 5 = 0$

$$m = -5, 1$$

$$C.F = Ae^{-5t} + Be^{t}$$

P.I $= \frac{1}{D^2+4D-5}(-6e^{2t})$ Replace D by 2

$$= \frac{-6}{7} e^{2t}$$

$$y = Ae^{-5t} + Be^{t} - \frac{6}{7}e^{2t} \ldots (5)$$

To find x (5) $\Rightarrow$ $Dy = -5Ae^{-5t} + Be^{t} - \frac{12}{7}e^{2t}$

(2) $\Rightarrow$ $3x = -(D+2)y$

$$= -Dy - 2y$$

$$= 5Ae^{-5t} - Be^{t} + \frac{12}{7}e^{2t} - 2Ae^{-5t} - 2Be^{t} + \frac{12}{7}e^{2t}$$

$$3x = 3Ae^{-5t} - 3Be^{t} + \frac{24}{7}e^{2t}$$

$$x = Ae^{-5t} - Be^{t} + \frac{8}{7}e^{2t}$$

Example:

Solve $\frac{dx}{dt} + 2x - 3y = t$; $\frac{dy}{dt} - 3x + 2y = e^{2t}$

Solution:

Given $\frac{dx}{dt} + 2x - 3y = t$ ...(1)

$Dx + 2x - 3y = t$

$(D + 2)x - 3y = t$ ...(2)

Given $\frac{dy}{dt} - 3x + 2y = e^{2t}$ ...(3)

$Dx - 3x + 2y = e^{2t}$

$-3x + (D + 2)y = e^{2t}$ ...(4)

(2) x (3) $\Rightarrow$ $3(D + 2)x - 9y = 3t$

(4) x (D+2) $\Rightarrow$ $-3(D + 2)x + (D + 2)^2y = (D + 2)e^{2t}$

$-9y + (D + 2)^2y = 3t + (D + 2)e^{2t}$

$(-9 + D^2 + 4D + 4)y = 3t + 4e^{2t}$

$(D^2 + 4D - 5)y = 3t + 4e^{2t}$

Auxiliary Equation is $m^2 + 4m - 5 = 0$

$m = -5, 1$

$C.F = Ae^t + Be^{-5t}$

$$\text{P.I}_1 = \frac{1}{D^2+4D-5}(3t)$$

$$= \frac{3}{-5[1 - \frac{D^2+4D}{5}]}t$$

$$= \frac{3}{-5[1 - \frac{D^2+4D}{5}]}t$$

$$= \frac{-3}{5}\left[1 - \left(\frac{D^2+4D}{5}\right)\right]^{-1}(t)$$

$$= \frac{-3}{5}\left[1 + \left(\frac{D^2+4D}{5}\right) + \left(\frac{D^2+4D}{5}\right)^2 + - - -\right](t)$$

$$= \frac{-3}{5}\left[t + \frac{4}{5}\right]$$

$$= \frac{-3}{5}t - \frac{12}{25}$$

$$\text{P.I}_2 = \frac{1}{D^2+4D-5}4e^{2t}$$

$$= \frac{4}{4-8-5} e^{2t}$$ Replace D by 2

$$= \frac{4}{7} e^{2t}$$

$$y = C.F + P.I_1 + P.I_2$$

$$y = Ae^t + Be^{-5t} - \frac{3}{5}t - \frac{12}{25} + \frac{4}{7}e^{2t}$$

$$Dy = Ae^t - 5Be^{-5t} - \frac{3}{5} + \frac{8}{7}e^{2t}$$

(3) ⇒ $3x = \frac{dy}{dt} + 2y - e^{2t}$

$$= \left[Ae^t - 5Be^{-5t} - \frac{3}{5} + \frac{8}{7}e^{2t}\right] + 2\left[Ae^t + Be^{-5t} - \frac{3}{5}t - \frac{12}{25} + \frac{4}{7}e^{2t}\right] - e^{2t}$$

$$= Ae^t - 5Be^{-5t} - \frac{3}{5} + \frac{8}{7}e^{2t} + 2Ae^t + 2Be^{-5t} - \frac{6}{5}t + \frac{24}{25} + \frac{8}{7}e^{2t} - e^{2t}$$

$$3x = 3Ae^t - 3Be^{-5t} - \frac{6}{5}t - \frac{39}{25} + \frac{9}{7}e^{2t}$$

$$x = Ae^t - Be^{-5t} - \frac{2}{5}t - \frac{13}{25} + \frac{3}{7}e^{2t}$$

$$\therefore x = Ae^t - Be^{-5t} - \frac{2}{5}t - \frac{13}{25} + \frac{3}{7}e^{2t}$$

$$y = Ae^t + Be^{-5t} - \frac{3}{5}t + \frac{4}{7}e^{2t} - \frac{12}{25}$$

Method of Undetermined Coefficient

The given differential equation is $F(D)y = f(x)$

To find the particular integral (P.I) of the given equation, we have to assume a trial solution that contains unknown constants. This unknown constants are to be determined by substitution in the given equation and the trial solution depends on the given function $f(x)$.

Sl.No	Function $f(x)$	Choice of P.I
1	ke^{px}	Ce^{px}
2	$ksin(ax+b)$ (or) $kcos(ax+b)$	$C_1sin(ax+b)+C_2cos(ax+b)$
3	$ke^{px}sin(ax+b)$ (or) $ke^{px}cos(ax+b)$	$C_1e^{px}sin(ax+b)+C_2e^{px}cos(ax+b)$
4	$kx^m where\ m = 0,1,2, \dots \dots$	$C_0 + C_1x + C_2x^2 + \cdots \dots \dots C_mx^m$

Straight Case:

If the R.H.S function $f(x)$ is not a member of the solution set, then choose, P.I,(y_p) from the above table depending on the nature of $f(x)$

Sum Case:

When the R.H.S $f(x)$ is a Combination (sum) of the functions in column "2" of the table, then P.I is chosen as a Combination of the corresponding function in third column and proceed as in straight case.

Modified Case:

When any term of $f(x)$ is a member of the solution set S, then the method fails. If we choose y_p from the table. In such cases the choice from the table should be modified as follows.

a) If a term u of $f(x)$ is also a term of the C.F then the choice from the table corresponding to u should be multiplied by

* x if u corresponds to a simple root of C.F
* x^2 if u corresponds to a double root of C.F
* x^s if u corresponds to a s-fold root of C.F

b) Suppose $x^r u$ is a term $f(x)$ and u is a term of C.F corresponding to an S-fold root then the choice from the table corresponding to $x^r u$ should be multiplied by x^s.

Type I : Straight Case:

Example :

Solve $(D^2 - 3D + 2)y = 6e^{3x}$

Solution:

Given $y'' - 3y' + 2y = 6e^{3x}$ ... (1)

Auxiliary Equation is $m^2 - 3m + 2 = 0$

$$m = 1,2$$

$$C.F = Ae^x + Be^{2x} \text{ ... (2)}$$

Here the Solution Set $S = \{e^x, e^{2x}\}$

R.H.S of (1) is not a member of S

$$P.I \quad y_p = Ce^{3x} \text{ ... (3)}$$

$$y_p' = 3Ce^{3x}$$

$$y_p'' = 9Ce^{3x}$$

(1) $\Rightarrow$ $9Ce^{3x} - 9Ce^{3x} + 2Ce^{3x} = 6e^{3x}$

$$2Ce^{3x} = 6e^{3x}$$

$$2C = 6$$

$$C = 3$$

(3) $\Rightarrow y_p = 3e^{3x}$

The general solution is $y = y_c + y_p$

$$y = Ae^x + Be^{2x} + 3e^{3x}$$

Type II : Sum Case:

Example :

Solve $(D^2 + 2D + 5)y = 2x^2 + 3e^{-x}$

Solution:

Given $y'' + 2y' + 4y = 2x^2 + 3e^{-x}$ ... (1)

Auxiliary Equation is $m^2 + 2m + 4 = 0$

$$m = \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= -1 \pm \sqrt{3}i$$

$$C.F = e^{-x}[A\cos\sqrt{3}x + B\sin\sqrt{3}x] \ldots (2)$$

Here the Solution Set $S = \{e^{-x}\cos\sqrt{3}x, e^{-x}\sin\sqrt{3}x\}$

R.H.S of (1) is not a member of S

Choose $P.I \quad y_p = C_0 + C_1x + C_2x^2 + C_3e^{-x} \ldots (3)$

$$y_p' = C_1 + 2C_2x - C_3e^{-x}$$

$$y_p'' = 2C_2 + C_3e^{-x}$$

$$(1) \Rightarrow 2C_2 + C_3e^{-x} + 2C_1 + 4C_2x - 2C_3e^{-x} + 4C_0 + 4C_1x + 4C_2x^2 + 4C_3e^{-x}$$
$$= 2x^2 + 3e^{-x}$$

Equating the coefficients of

x^2 : $4C_2 = 2$

$$C_2 = \frac{2}{4} = \frac{1}{2}$$

e^{-x} : $4C_3 + C_3 - 2C_3 = 3$

$$3C_3 = 3$$

$$C_3 = 1$$

x : $4C_2 + 4C_1 = 0$

$$4\left(\frac{1}{2}\right) + 4C_1 = 0$$

$$4C_1 = \frac{-4}{2}$$

$$C_1 = \frac{-4}{8} = \frac{-1}{2}$$

Constant: $2C_2 + 2C_1 + 4C_0 = 0$

$$2\left(\frac{1}{2}\right) + 2\left(\frac{-1}{2}\right) + 4C_0 = 0$$

$$C_0 = 0$$

$$(3) \Rightarrow y_p = \frac{-1}{2}x + \frac{1}{2}x^2 + e^{-x}$$

$$= \frac{-x}{2} + \frac{x^2}{2} + e^{-x}$$

The general solution is $y = C.F + P.I$

$$y = e^{-x}[A\cos\sqrt{3}x + B\sin\sqrt{3}x] - \frac{x}{2} + \frac{x^2}{2} + e^{-x}$$

Example :

Solve $(D^2 + D - 2)y = x + sinx$

Solution:

Given $y'' + y' - 2y = x + sinx \ldots (1)$

Auxiliary Equation is $m^2 + m - 2 = 0$

$m = 1, -2$

$C.F = Ae^x + Be^{-2x} \ldots (2)$

Here the Solution Set $S = \{e^x, e^{-2x}\}$

R.H.S of (1) is not a member of S

choose $P.I \quad y_p = C_0 + C_1x + C_2sinx + C_3cosx \ldots (3)$

$y_p' = C_1 + C_2cosx - C_3sinx$

$y_p'' = -C_2sinx - C_3cosx$

$(1) \Rightarrow \quad -C_2sinx - C_3cosx + C_1 + C_2cosx - C_3Sinx$

$-2C_0 - 2C_1x - 2C_3Sinx - 2C_3cosx = x + sinx$

Equating the coefficients of

x: $-2C_1 = 1$

$C_1 = \frac{-1}{2}$

$Constant$: $C_1 - 2C_0 = 0$

$C_0 = \frac{-1}{4}$

$sinx$: $-C_2 - C_3 - 2C_2 = 1$

$-3C_2 - C_3 = 1 \ldots (4)$

$cosx$: $-C_3 + C_2 - 2C_3 = 0$

$C_2 - 3C_3 = 0 \ldots (5)$

Solving (4) & (5) we get $C_2 = \frac{-3}{10}$ $\quad C_3 = \frac{-1}{10}$

(3) $\Rightarrow \quad y_p = \frac{-1}{4} - \frac{1}{2}x - \frac{3}{10}sinx - \frac{1}{10}cosx$

The general solution is $y = C.F + P.I$

$y = Ae^x + Be^{-2x} - \frac{1}{4} - \frac{1}{2}x - \frac{3}{10}sinx \frac{-1}{10}cosx$

Type: III Modified Case:-

Example:

Solve $(D^2 + 9)y = cos3x$

Solution:

Given $y" + 9y = cos3x \dots (1)$

Auxiliary Equation is $m^2 + 9 = 0$

$m = \pm 3i$

$C.F = Acos3x + Bsin3x \dots (2)$

Here the solution set $S = \{cos3x, sin3x\}$

R.H.S of (1) is a member of S

Choose $P.I$ $y_p = C_1 sin3x + C_2 cos3x$

Corresponding terms should multiplied by x

$y_p = x[C_1 sin3x + C_2 cos3x] \dots (3)$

$y'_p = x[3C_1 cos3x - 3C_2 sin3x] + [C_1 cos3x + C_2 sin3x] +$

$y_p" = x[-9C_1 sin3x - 9C_2 cos3x] +$

$[3C_1 cos3x - 3C_2 sin3x] + 3C_1 cos3x - 3C_2 sin3x$

(1)⇒ $-9C_1 x sin3x - 9C_2 x cos3x +$

$6C_1 cos3x - 6C_2 sin3x + 9C_1 x sinx + 9C_2 x cos3x = cos3x$

Equating the coefficients of

$cos3x$: $6C_1 = 1$

$C_1 = \frac{1}{6}$

$sin3x$: $-6C_2 = 0$

$C_2 = 0$

(3) ⇒ $y_p = x\left[\frac{1}{6} sin3x\right]$

The general solution is $y = C.F + P.I$

$y = Acos3x + Bsin3x + \frac{x}{6} sin3x$

Example:

Solve $y" + y' - 6y = 10e^{2x} - 18e^{3x} - 6x - 11$

Solution:

$y" + y' - 6y = 10e^{2x} - 18e^{3x} - 6x - 11 \dots (1)$

Auxiliary Equation is $m^2 + m - 6 = 0$

$m = 2, -3$

$C.F = Ae^{2x} + Be^{-3x}$ ... (2)

Here the Solution Set $S = \{e^{2x}, e^{-3x}\}$

R.H.S of (1) is not a member of S

Choose $P.I$ $\quad y_p = C_1e^{2x} + C_2e^{3x} + C_3x + C_4$

Corresponding terms should multiplied by x

$y_p = x(C_1e^{2x}) + C_2e^{3x} + C_3x + C_4$ ... (3)

$y_p' = C_1x2e^{2x} + C_1e^{2x} + 3C_2e^{3x} + C_3$

$= (2C_1x + C_1)e^{2x} + 3C_2e^{2x} + C_3$

$y_p" = (2C_1x + C_1)2e^{2x} + e^{2x}(2C_1) + 9C_2e^{3x}$

$= 2e^{2x}[2C_1x + 2C_1] + 9C_2e^{3x}$

$= 4e^{2x}(x + 1)C_1 + 9C_2e^{3x}$

(1) $\Rightarrow 4(x + 1)C_1e^{3x} + 9C_2e^{3x} + (2x + 1)C_1e^{3x} + 3C_2e^{3x}$

$-6xC_1e^{3x} - 6xC_2e^{3x} - 6C_3x - 6C_4 = 10e^{2x} - 18e^{2x} - 6x - 11$

$(4x + 4 + 2x + 1 - 6x)C_1e^{2x} + 6C_2e^{3x} + C_3 - 6C_3x - 6C_4$

$= 10e^{2x} - 18e^{3x} - 6x - 11$

Equating the coefficients of

e^{2x} : $\quad 5C_1 = 10$

$C_1 = 2$

e^{3x} : $\quad 6C_2 = 18$

$C_2 = -3$

x : $\quad -6C_3 = -6$

$C_3 = 1$

Constant : $\quad C_3 - 6C_4 = -11$

$C_4 = 2$

(3) $\Rightarrow$ $\quad y_p = 2xe^{2x} - 3e^{3x} + x + 2$

The general solution is $y = C.F + P.I$

$y = Ae^{2x} + Be^{-3x} + 2xe^{2x} - 3e^{3x} + x + 2$