Catalog

GRADIENT – DIRECTIONAL DERIVATIVE

Vector differential operator

The vector differential operator ∇ (read as Del) is denoted by $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along the three rectangular axes OX, OY and OZ.

It is also called Hamiltonian operator and it is neither a vector nor a scalar, but it behaves like a vector.

The gradient of a scalar function

If $\varphi(x, y, z)$ is a scalar point function continuously differentiable in a given region of space, then the gradient of φ is defined as $\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

It is also denoted as Grad φ.

Note

(i) $\nabla\varphi$ is a vector quantity.

(ii) $\nabla\varphi = 0$ if φ is constant.

(iii) $\nabla(\varphi_1\varphi_2) = \varphi_1\nabla\varphi_2 + \varphi_2\nabla\varphi_1$

(iv) $\nabla\left(\frac{\varphi_1}{\varphi_2}\right) = \frac{\varphi_2\nabla\varphi_1 - \varphi_1\nabla\varphi_2}{\varphi_2^2}$ if $\varphi_2 \neq 0$

(v) $\nabla(\varphi \pm \chi) = \nabla\varphi \pm \nabla\chi$

Example: If $\varphi = \log(x^2 + y^2 + z^2)$ then find $\nabla\varphi$.

Solution:

Given $\varphi = \log(x^2 + y^2 + z^2)$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\left(\frac{2x}{x^2+y^2+z^2}\right) + \vec{j}\left(\frac{2y}{x^2+y^2+z^2}\right) + \vec{k}\left(\frac{2z}{x^2+y^2+z^2}\right)$$

$$= \frac{2}{x^2+y^2+z^2}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{2}{r^2}\vec{r}$$

Example: Find $\nabla(r), \nabla\left(\frac{1}{r}\right), \nabla(\log r)$ where $r = |\vec{r}|$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\Rightarrow |\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x, \quad 2r\frac{\partial r}{\partial y} = 2y, \quad 2r\frac{\partial r}{\partial z} = 2z$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

(i) $\nabla(r) = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$

$= \vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}$

$= \frac{1}{r}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{1}{r}\vec{r}$

(ii) $\nabla\left(\frac{1}{r}\right) = \vec{i}\frac{\partial(\frac{1}{r})}{\partial x} + \vec{j}\frac{\partial(\frac{1}{r})}{\partial y} + \vec{k}\frac{\partial(\frac{1}{r})}{\partial z}$

$= \vec{i}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + \vec{j}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + \vec{k}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}$

$= \left(-\frac{1}{r^2}\right)\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$

$= -\frac{1}{r^3}(x\vec{i} + y\vec{j} + z\vec{k}) = -\frac{1}{r^3}\vec{r}$

(iii) $\nabla(\log r) = \sum \vec{i}\frac{\partial(\log r)}{\partial x}$

$= \sum \vec{i}\frac{1}{r}\frac{\partial r}{\partial x}$

$= \sum \vec{i}\frac{1}{r}\frac{x}{r}$

$= \sum \vec{i}\frac{x}{r^2}$

$= \frac{1}{r^2}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{1}{r^2}\vec{r}$

Example: Prove that $\nabla(r^n) = nr^{n-2}\vec{r}$

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\nabla(r^n) = \vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z}$

$= \vec{i}\,nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}\,nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}\,nr^{n-1}\frac{\partial r}{\partial z}$

$= nr^{n-1}\left[\vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)\right]$

$= \frac{nr^{n-1}}{r}(x\vec{i} + y\vec{j} + z\vec{k}) = nr^{n-2}\vec{r}$

Example: Find $|\nabla\varphi|$ if $\varphi = 2xz^4 - x^2y$ at $(2, -2, -1)$

Solution:

Given $\varphi = 2xz^4 - x^2y$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Now $\frac{\partial\varphi}{\partial x} = 2z^4 - 2xy,\ \frac{\partial\varphi}{\partial y} = -x^2,\ \frac{\partial\varphi}{\partial z} = 8xz^3$

$$\therefore \nabla\varphi = \vec{i}(2z^4 - 2xy) + \vec{j}(-x^2) + \vec{k}(8xz^3)$$

$$\therefore (\nabla\varphi)_{(2,-2,-1)} = 10\vec{i} - 4\vec{j} - 16\vec{k}$$

$$|\nabla\varphi| = \sqrt{100+16+256} = \sqrt{372}$$

Directional Derivative (D.D) of a scalar point function

The derivative of a point function (scalar or vector) in a particular direction is called its directional derivative along the direction.

The directional derivative of a scalar function φ in a given direction $\vec{a}$ is the rate of change of φ in that direction. It is given by the component of $\nabla\varphi$ in the direction of $\vec{a}$.

The directional derivative of a scalar point function in the direction of $\vec{a}$ is given by

$$\textbf{D.D} = \frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$$

The maximum directional derivative is $|\nabla\varphi|$ or $|\text{grad }\varphi|$.

Example: Find the directional derivative of $\varphi = 4xz^2 + x^2yz$ at $(1,-2,1)$ in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution:

Given $\varphi = 4xz^2 + x^2yz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}(2xyz + 4z^2) + \vec{j}(x^2z) + \vec{k}(x^2y + 8xz)$$

$$\therefore (\nabla\varphi)_{(1,-2,-1)} = 8\vec{i} - \vec{j} - 10\vec{k}$$

Given $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$

$$|\vec{a}| = \sqrt{4+1+4} = 3$$

$$\text{D.D} = \frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$$

$$= (8\vec{i} - \vec{j} - 10\vec{k})\cdot\frac{(2\vec{i} - \vec{j} - 2\vec{k})}{3}$$

$$= \frac{1}{3}(16+1+20) = \frac{37}{3}$$

Example: Find the directional derivative of $\varphi(x,y,z) = xy^2 + yz^3$ at the point $P(2,-1,1)$ in the direction of PQ where Q is the point $(3,1,3)$

Solution:

Given $\varphi = xy^2 + yz^3$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y^2) + \vec{j}\,(2xy + z^3) + \vec{k}\,(3yz^2)$$

$$\therefore (\nabla\varphi)_{(2,-1,\ 1)} = \vec{i} - 3\vec{j} - 3\vec{k}$$

Given $\vec{a} = \vec{PQ} = \vec{OQ} - \vec{OP}$

$$= (3\vec{i} + \vec{j} + 3\vec{k}) - (2i - \vec{j} + \vec{k})$$

$$= \vec{i} + 2\vec{j} + 2\vec{k}$$

$$|\vec{a}| = \sqrt{1+4+4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

$$= \frac{(\vec{i} - 3\vec{y} - 3\vec{k}) \cdot (\vec{i} + 2\vec{y} + 2\vec{k})}{3}$$

$$= \frac{1}{3}(1 - 6 - 6) = -\frac{11}{3}$$

Example: In what direction from $(-1, 1, 2)$ is the directional derivative of $\varphi = xy^2 z^3$ a maximum? Find also the magnitude of this maximum.

Solution:

Given $\varphi = xy^2 z^3$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y^2z^3) + \vec{j}\,(2xy\,z^3) + \vec{k}\,(3xy^2z^2)$$

$$\therefore (\nabla\varphi)_{(-1,\ 1,\ 2)} = 8\vec{i} - 16\vec{j} - 12\vec{k}$$

The maximum directional derivative occurs in the direction of $\nabla\varphi = 8\vec{i} - 16\vec{j} - 12\vec{k}$.

$\therefore$ The magnitude of this maximum directional derivative

$$|\nabla\varphi| = \sqrt{64 + 256 + 144} = \sqrt{464}$$

Example: Find the directional derivative of the scalar function $\varphi = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point $(3, 1, 3)$.

Solution:

Given $\varphi = xyz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(yz) + \vec{j}\,(xz) + \vec{k}\,(xy)$$

$$\therefore (\nabla \varphi)_{(3,\ 1,\ 3)} = 3\vec{i} + 9\vec{j} + 3\vec{k}$$

Given surface is $z = xy \Rightarrow z - xy = 0$

$$\nabla\chi = \vec{i}\frac{\partial z}{\partial x} + \vec{j}\frac{\partial z}{\partial y} + \vec{k}\frac{\partial z}{\partial z}$$

$$= \vec{i}\,(-y) + \vec{j}\,(-x) + \vec{k}\,(1)$$

Let $\vec{a} = \nabla\chi_{(3,1\,,3)} = -\vec{i} - 3\vec{j} + \vec{k}$

$$\Rightarrow |\vec{a}| = \sqrt{1+9+1} = \sqrt{11}$$

$$\text{D. D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

$$= \frac{(3\vec{i}+9\vec{j}+3\vec{k}) \cdot (-\vec{i} - 3\vec{j} + \vec{k})}{\sqrt{11}}$$

$$= \frac{1}{\sqrt{11}}\,(-3 - 27 + 3) = -\frac{27}{\sqrt{11}}$$

DIVERGENCE AND CURL

Divergence of a vector function

If $\vec{F}(x,y,z)$ is a continuously differentiable vector point function in a given region of space, then the divergence of $\vec{F}$ is defined by

$$\nabla.\vec{F} = \text{div}\,\vec{F} = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).(F_1\vec{i}+F_2\vec{j}+F_3\vec{k})$$

$$\text{div}\vec{F} = \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z} \quad \text{where}\ \vec{F} = F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$$

Note: $\nabla.\vec{F}$ Is a scalar point function.

Curl of a vector function

If $\vec{F}(x,y,z)$ is a differentiable vector point function defines at each point (x,y,z) in some region of space, then the curl of $\vec{F}$ is defined by

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_1 & F_2 & F_3\end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y}-\frac{\partial F_2}{\partial z}\right)-\vec{j}\left(\frac{\partial F_3}{\partial x}-\frac{\partial F_1}{\partial z}\right)+\vec{k}\left(\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

Where $\vec{F} = F_1\vec{i}+F_2\vec{j}+F_3\vec{k}$

Note: $\nabla\times\vec{F}$ Is a vector point function.

Example: If $\vec{F} = xy^2\vec{i}+2x^2yz\vec{j}-3yz^2\vec{k}$ find $\nabla.\vec{F}$ and $\nabla\times\vec{F}$ at the point (1,-1, 1).

Solution:

Given $\vec{F} = xy^2\vec{i}+2x^2yz\vec{j}-3yz^2\vec{k}$

(i) $\nabla.\vec{F} = \frac{\partial}{\partial x}(xy^2)+\frac{\partial}{\partial y}(2x^2yz)+\frac{\partial}{\partial z}(-3yz^2)$

$= y^2+2x^2z-6yz$

$\nabla.\vec{F}_{(1,-1,1)} = 1+2+6 = 9$

(ii) $$\nabla\times\vec{F} = \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy^2 & 2x^2yz & 3yz^2\end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial(-3yz^2)}{\partial y}-\frac{\partial(2x^2yz)}{\partial z}\right]-\vec{j}\left[\frac{\partial(-3yz^2)}{\partial x}-\frac{\partial(xy^2)}{\partial z}\right]+\vec{k}\left[\frac{\partial(2x^2yz)}{\partial x}-\frac{\partial(xy^2)}{\partial y}\right]$$

$$= \vec{i}(-3z^2-2x^2y)-\vec{j}(0)+\vec{k}(4xyz-2xy)$$

$$\nabla \times \vec{F}_{(1,-1,1)} = \vec{i}(-3+2) + \vec{k}(-4+2)$$

$$= -\vec{i} - 2\vec{k}$$

Example: If $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$, then find $\nabla \cdot \vec{F}$, $\nabla(\nabla \cdot \vec{F})$, $\nabla \times \vec{F}$, $\nabla \cdot (\nabla \times \vec{F})$, and $\nabla \times (\nabla \times \vec{F})$ at the point (1,1,1).

Solution:

Given $\vec{F} = (x^2 - y^2 + 2xz)\vec{i} + (xz - xy + yz)\vec{j} + (z^2 + x^2)\vec{k}$

(i) $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - y^2 + 2xz) + \frac{\partial}{\partial y}(xz - xy + yz) + \frac{\partial}{\partial z}(z^2 + x^2)$

$$= (2x + 2z) + (-x + z) + 2z$$

$$= x + 5z$$

$\therefore \nabla \cdot \vec{F}_{(1,1,1)} = 6$

(ii) $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 + 2xz & xz - xy + yz & z^2 + x^2 \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial(z^2+x^2)}{\partial y} - \frac{\partial(xz-xy+yz)}{\partial z}\right] - \vec{j}\left[\frac{\partial(z^2+x^2)}{\partial x} - \frac{\partial(x^2-y^2+2xz)}{\partial z}\right] + \vec{k}\left[\frac{\partial(xz-xy+yz)}{\partial x} - \frac{\partial(x^2-y^2+2xz)}{\partial y}\right]$$

$$= -(x+y)\vec{i} - (2x - 2x)\vec{j} + (y+z)\vec{k}$$

$$\therefore \nabla \times \vec{F}_{(1,1,1)} = -2\vec{i} + 2\vec{k}$$

(iii) $\nabla(\nabla \cdot \vec{F}) = \vec{i}\frac{\partial}{\partial x}(x + 5z) + \vec{j}\frac{\partial}{\partial y}(x + 5z) + \vec{k}\frac{\partial}{\partial z}(x + 5z)$

$$= \vec{i} + 5\vec{k}$$

$\therefore \nabla(\nabla \cdot \vec{F})_{(1,1,1)} = \vec{i} + 5\vec{k}$

(iv) $\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}(-(x+y)) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(y+z)$

$$= -1 + 0 + 1$$

$\nabla \cdot (\nabla \times \vec{F})_{(1,1,1)} = 0$

(v) $\nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -(x+y) & 0 & y+z \end{vmatrix}$

$\therefore \nabla \times (\nabla \times \vec{F})_{(1,1,1)} = \vec{i} + \vec{k}$

Example: Find div $\vec{F}$ and curl $\vec{F}$, where $\vec{F} = \text{grad}(x^3 + y^3 + z^3 - 3xyz)$

Solution:

Given $\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

$$= \vec{i}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz) + \vec{j}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz) + \vec{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$\vec{F} = \vec{i}(3x^2-3yz) + \vec{j}(3y^2-3xz) + \vec{k}(3z^2-3xy)$$

Now $\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(3x^2-3yz) + \frac{\partial}{\partial y}(3y^2-3xz) + \frac{\partial}{\partial z}(3z^2-3xy)$

$$= 6x + 6y + 6z$$

$$= 6(x+y+z)$$

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$$

$$= \vec{i}[-3x+3x] - \vec{j}[-3y+3y] + \vec{k}[-3z+3z]$$

$$= \vec{0}$$

Example: Find div(grad φ) and curl(grad φ) at (1,1,1) for $\varphi = x^2y^3z^4$

Solution:

Given $\varphi = x^2y^3z^4$

$$\text{grad}\,\varphi = \nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}(2xy^3z^4) + \vec{j}(x^2 3y^2z^4) + \vec{k}(x^2y^3 4z^3)$$

$\text{Div}(\text{grad}\,\varphi) = \nabla\cdot(\text{grad}\,\varphi)$

$$= \frac{\partial}{\partial x}(2xy^3z^4) + \frac{\partial}{\partial y}(x^2 3y^2z^4) + \frac{\partial}{\partial z}(x^2y^3 4z^3)$$

$$= 2y^3z^4 + 6x^2yz^4 + 12x^2y^3z^4$$

$$\therefore \text{Div}(\text{grad}\,\varphi)_{(1,1,1)} = 2+6+12 = 20$$

$$\text{Curl}(\text{grad}\,\varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & x^2 3y^2z^4 & x^2y^3 4z^3 \end{vmatrix}$$

$$= \vec{i}(12x^2y^2z^3 - 12x^2y^2z^3) - \vec{j}(8xy^3z^3 - 8xy^3z^3) + \vec{k}(6xy^2z^4 - 6xy^2z^4)$$

$$= \vec{0}$$

$$\therefore \text{Curl grad}\varphi_{(1,1,1)} = \vec{0}$$

1) If φ is a scalar point function, $\vec{F}$ is a vector point function, then

$$\nabla\cdot(\varphi\vec{F}) = \varphi(\nabla\cdot\vec{F}) + \vec{F}\cdot\nabla\varphi$$

Proof:

$$\nabla\cdot(\varphi\vec{F}) = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\cdot(\varphi\vec{F})$$

$$= \sum\vec{i}\cdot\frac{\partial}{\partial x}(\varphi\vec{F})$$

$$= \sum\vec{i}\cdot\left(\varphi\frac{\partial\vec{F}}{\partial x}+\vec{F}\frac{\partial\varphi}{\partial x}\right)$$

$$= \varphi\left(\sum\vec{i}\cdot\frac{\partial\vec{F}}{\partial x}+\vec{F}\frac{\partial\varphi}{\partial x}\right)+\vec{F}\cdot\left(\sum\vec{i}\frac{\partial\varphi}{\partial x}\right)$$

$$\therefore\nabla\cdot(\varphi\vec{F}) = \varphi(\nabla\cdot\vec{F}) + \vec{F}\cdot\nabla\varphi$$

2) If φ is a scalar point fuction, $\vec{F}$ is a vector point function, then $\nabla\times(\varphi\vec{F}) = \varphi(\nabla\times\vec{F}) + (\nabla\varphi)\times\vec{F}$

Proof:

$$\nabla\times(\varphi\vec{F}) = \sum\vec{i}\times\frac{\partial}{\partial x}(\varphi\vec{F})$$

$$=\sum\vec{i}\times\left[\varphi\frac{\partial\vec{F}}{\partial x}+\vec{F}\frac{\partial\varphi}{\partial x}\right]$$

$$=\sum\vec{i}\times\left(\frac{\partial\varphi}{\partial x}\vec{F}+\varphi\frac{\partial\vec{F}}{\partial x}\right)$$

$$=\left(\sum\vec{i}\frac{\partial\varphi}{\partial x}\right)\times\vec{F}+\varphi\left[\sum\vec{i}\times\frac{\partial\vec{F}}{\partial x}\right]$$

$$\therefore\nabla\times(\varphi\vec{F}) = \nabla\varphi\times\vec{F}+\varphi(\nabla\times\vec{F})$$

3) If $\vec{A}$ and $\vec{B}$ are vector point functions, then $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

Proof:

$$\nabla.(\vec{A}\times\vec{B}) = \sum\vec{i}\cdot\frac{\partial}{\partial x}(\vec{A}\times\vec{B})$$

$$=\sum\vec{i}\cdot\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}+\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)$$

$$=\sum\vec{i}\cdot\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)+\sum\vec{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)$$

$$=-\left(\sum\vec{i}\times\frac{\partial\vec{B}}{\partial x}\right)\cdot\vec{A}+\left(\sum\vec{i}\times\frac{\partial\vec{A}}{\partial x}\right).\vec{B}$$

$$=-(\nabla\times\vec{B}).\vec{A}+(\nabla\times\vec{A})\cdot\vec{B}$$

$$\therefore\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) \qquad [\because(\nabla\times\vec{A})\cdot\vec{B} = \vec{B}\cdot(\nabla\times\vec{A})]$$

(4) If $\vec{F}$ is a vector point function, then $\nabla \cdot (\nabla \times \vec{F}) = 0$.

(or)

Prove that $div(curl\,\vec{F}) = 0$.

Solution:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\nabla \cdot (\nabla \times \vec{F}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left[\vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y}$$

$$= 0$$

(5) If is a vector point function, then $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$

(or)

Prove that $curl\,(curl\,\vec{F}) = grad\,(div\,\vec{F}) - \nabla^2\vec{F}$

Solution:

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$\nabla \times (\nabla \times \vec{F}) = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

And $\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

L.H.S $\quad \nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} & -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z} & \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{vmatrix}$

$$= \vec{i}\left[\frac{\partial^2 F_2}{\partial y \partial x} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_3}{\partial z \partial x} + \frac{\partial^2 F_1}{\partial z^2}\right] - \vec{j}\left[\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial z \partial y} + \frac{\partial^2 F_2}{\partial z^2}\right]$$

$$+\vec{k}\left[-\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y \partial z}\right]$$

R.H.S $\quad \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)$$

$$= \vec{i}\left[\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z}\right] + \vec{j}\left[\frac{\partial^2 F_1}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial y^2} + \frac{\partial^2 F_3}{\partial y \partial z}\right] + \vec{k}\left[\frac{\partial^2 F_1}{\partial z \partial x} + \frac{\partial^2 F_2}{\partial z \partial y} + \frac{\partial^2 F_3}{\partial z^2}\right]$$

$$- \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)$$

$$= \vec{i}\left[\frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial z^2}\right] - \vec{j}\left[\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x \partial y} - \frac{\partial^2 F_3}{\partial z \partial y} + \frac{\partial^2 F_2}{\partial z^2}\right] +$$

$$\vec{k}\left[-\frac{\partial^2 F_3}{\partial x^2} + \frac{\partial^2 F_1}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y^2} + \frac{\partial^2 F_2}{\partial y \partial z}\right]$$

L.H.S = R.H.S

$$\therefore \nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2 \vec{F}$$

(6) $\nabla \cdot (\nabla \varphi) = (\nabla \cdot \nabla)\varphi = \nabla^2 \varphi$

Proof:

$$\nabla \varphi = \vec{i}\frac{\partial \varphi}{\partial x} + \vec{j}\frac{\partial \varphi}{\partial y} + \vec{k}\frac{\partial \varphi}{\partial z}$$

$$\therefore \nabla \cdot (\nabla \varphi) = \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial z}\right)$$

$$= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla \cdot (\nabla \varphi) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = \nabla^2 \varphi$$

Example: Find (i) $\nabla \cdot \vec{r}$ (ii) $\nabla \times \vec{r}$

Solution:

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

(i) $\nabla \cdot \vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(x\vec{i} + y\vec{j} + z\vec{k}\right)$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$

(ii) $\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$

$$= \vec{i}(0) + \vec{j}(0) + \vec{k}(0) = \vec{0}$$

Example: Find $\nabla \cdot \left(\frac{1}{r}\vec{r}\right)$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Solution:

$$\nabla \cdot \left(\frac{1}{r}\vec{r}\right) = \nabla \cdot \left[\frac{1}{r}(x\vec{i} + y\vec{j} + z\vec{k})\right]$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{x}{r}\vec{i} + \frac{}{r}\vec{j} + \frac{z}{r}\vec{k}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \sum \left[\frac{1}{r}(1) + x\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x}\right]$$

$$= \sum \left[\frac{1}{r} - \frac{x}{r^2}\left(\frac{x}{r}\right)\right] \qquad \left(\because \frac{\partial r}{\partial x} = \frac{x}{r}\right)$$

`

$$= \sum \left[\frac{1}{r} - \frac{x^2}{r^3}\right]$$

$$= \frac{3}{r} - \frac{1}{r^3}(x^2 + y^2 + z^2)$$

$$= \frac{3}{r} - \frac{r^2}{r^3} \qquad \because r^2 = (x^2 + y^2 + z^2)$$

$$= \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$$= 2(a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) = 2\vec{a}$$

Example: Prove that $curl(f(r)\vec{r}) = \vec{0}$

Solution:

Let $f(r)\vec{r} = f(r)[x\vec{i} + y\vec{j} + z\vec{k}]$

$$= xf(r)\vec{i} + yf(r)\vec{j} + zf(r)\vec{k}$$

$$\nabla \times (f(r)\vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix}$$

$$= \sum \vec{i}\left[zf'(r)\frac{\partial r}{\partial y} - yf'(r)\frac{\partial r}{\partial z}\right]$$

$$= \sum \vec{i}\left[zf'(r)\left(\frac{y}{r}\right) - yf'(r)\left(\frac{z}{r}\right)\right]$$

$$= \sum \vec{i}\left[\frac{zy}{r}f'(r) - \frac{zy}{r}f'(r)\right]$$

$$= \sum \vec{i}(0)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$$

Example: Prove that $curl[\varphi\nabla\varphi] = \vec{0}$

(or)

Prove that $\nabla \times [\varphi \nabla\varphi] = \vec{0}$

Solution:

$$\varphi\nabla\varphi = \varphi\left[\vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}\right]$$

$$= \vec{i}\left(\varphi\frac{\partial\varphi}{\partial x}\right) + \vec{j}\left(\varphi\frac{\partial\varphi}{\partial y}\right) + \vec{k}\left(\varphi\frac{\partial\varphi}{\partial z}\right)$$

$$\nabla \times (\varphi\nabla\varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi\frac{\partial\varphi}{\partial x} & \varphi\frac{\partial\varphi}{\partial y} & \varphi\frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \sum \vec{i}\left[\frac{\partial}{\partial y}\left(\varphi\frac{\partial\varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\varphi\frac{\partial\varphi}{\partial y}\right)\right]$$

$$= \sum \vec{i}\left[\varphi\frac{\partial^2\varphi}{\partial y\partial z} + \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\varphi}{\partial z} - \varphi\frac{\partial^2\varphi}{\partial z\partial y} - \frac{\partial\varphi}{\partial y}\cdot\frac{\partial\varphi}{\partial z}\right]$$

$$= \sum \vec{i}\,(0)$$

$$= 0\,\vec{i} + 0\,\vec{j} + 0\,\vec{k} = \vec{0}$$

Example: If $\vec{\omega}$ is a constant vector and $\vec{v} = \vec{\omega} \times \vec{r}$, then prove that $\vec{\omega} = \frac{1}{2}(\nabla \times \vec{v})$.

Solution:

Let $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

$$\vec{\omega} = \omega_1\,\vec{i} + \omega_2\,\vec{j} + \omega_3\,\vec{k}$$

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(\omega_2 z - \omega_3 y) - \vec{j}(\omega_1 z - \omega_3 x) + \vec{k}(\omega_1 y - \omega_2 x)$$

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & -\omega_1 z + \omega_3 x & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \vec{i}(\omega_1 + \omega_1) - \vec{j}(-\omega_2 - \omega_2) + \vec{k}(\omega_3 + \omega_3)$$

$$= 2\omega_1\vec{i} + 2\omega_2\vec{j} + 2\omega_3\vec{k}$$

$$= 2(\omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k}) = 2\vec{\omega}$$

$$\vec{\omega} = \frac{1}{2}(\nabla \times \vec{v})$$

Irrotational and Solenoidal vector fields

Solenoidal vector

A vector $\vec{F}$ is said to be solenoidal if $div\,\vec{F} = 0$ (i.e) $\nabla.\vec{F} = 0$

Irrotational vector

A vector is said to be irrotational if Curl $\vec{F} = 0$ $(i.\ e)$ $\nabla \times \vec{F} = 0$

Example: Prove that the vector $\vec{F} = z\,\vec{i} + x\,\vec{j} + y\,\vec{k}$ is solenoidal.

Solution:

Given $\vec{F} = z\,\vec{i} + x\,\vec{j} + y\,\vec{k}$

To prove $\nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$

$$= 0$$

$\therefore$ $\vec{F}$ is solenoidal.

Example: If $\vec{F} = (x + 3y)\vec{i} + (y - 2z)\vec{j} + (x + \lambda z)\vec{k}$ is solenoidal, then find the value of λ.

Solution:

Given $\vec{F}$ is solenoidal.

$(ie)\nabla \cdot \vec{F} = 0$

$$\Rightarrow \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + \lambda z) = 0$$

$\Rightarrow 1 + 1 + \lambda = 0$

$\therefore \lambda = -2$

Example: Find a such that $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.

Solution:

Given $(3x - 2y + z)\vec{i} + (4x + ay - z)\vec{j} + (x - y + 2z)\vec{k}$ is solenoidal.

$(ie)\nabla \cdot \vec{F} = 0$

$$\Rightarrow \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + ay - z) + \frac{\partial}{\partial z}(x - y + 2z) = 0$$

$\Rightarrow 3 + a + 2 = 0$

$\therefore a = -5$

Example: Show that the vector $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$ is irrotational.

Solution:

Given $\vec{F} = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$

To prove $curl\ \vec{F} = 0$

$$(i.e) To\ prove\ \nabla \times \vec{F} = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - 3z^2 & 3xz^2 - y \end{vmatrix}$$

$$= \vec{i}(-1 + 1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x) = \vec{0}$$

$\therefore\ \vec{F}$ is irrotational.

Example: Find the constants a, b, c so that the vectors is irrotational

$$\vec{F} = (x + 2y + az)\vec{i} + (bx + 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}\,.$$

Solution:

Given $\vec{F} = (x + 2y + az)\vec{i} + (bx + 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ is irrotational.

$$(ie)\nabla \times \vec{F} = 0$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx + 3y - z & 4x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\Rightarrow \vec{i}(c + 1) - \vec{j}(4 - a) + \vec{k}(b - 2) = \vec{0}$$

$$\Rightarrow c + 1 = 0\ ; \quad 4 - a = 0\ ; \quad b - 2 = 0$$

$$\Rightarrow c = -1\ ; \quad 4 = a\ ; \quad b = 2$$

Line Integral over a plane curve

An integral which is evaluated along a curve then it is called line integral.

Let C be the curve in same region of space described by a vector valued function $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ of a point (x, y, z) and let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ be a continuous vector valued function defined along a curve C. Then the line integral $\vec{F}$ over C is denoted by

$\int_c \vec{F} \cdot d\vec{r}$.

Work done by a Force

If $\vec{F}(x, y, z)$ is a force acting on a particle which moves along a given curve C, then $\int_c \vec{F} \cdot d\vec{r}$ gives the total work done by the force $\vec{F}$ in the displacement along C.

Thus work done by force $\vec{F} = \int_c \vec{F} \cdot d\vec{r}$

Conservative force field

The line integral $\int_A^B \vec{F} \cdot d\vec{r}$ depends not only on the path C but also on the end points A and B.

If the integral depends only on the end points but not on the path C, then $\vec{F}$ is said to be conservative vector field.

If $\vec{F}$ is conservative force field, then it can be expressed as the gradient of some scalar function φ.

$(ie)\ \vec{F} = \nabla\varphi$

$$\vec{F} = \nabla\varphi = (\vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z})$$

$$\vec{F} \cdot d\vec{r} = (\vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}) \cdot (dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$

$$= \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = \partial\varphi$$

$$\int_c \vec{F} \cdot d\vec{r} = \int_A^B \partial\varphi$$

$$= [\varphi]_A^B$$

$$= \varphi[B] - \varphi[A]$$

$\therefore$ work done by $\vec{F} = \varphi[B] - \varphi[A]$

Note:

If $\vec{F}$ is conservative, then $\nabla \times \vec{F} = \nabla \times (\nabla\varphi) = \vec{0}$ and hence $\vec{F}$ is irrotational.

Example: If $\vec{F} = 3xy\vec{\imath} - y^2\vec{\jmath}$, evaluate $\int_c \vec{F}\cdot d\vec{r}$ where c is the curve $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Solution:

Given $\vec{F} = 3xy\vec{\imath} - y^2\vec{\jmath}$

$d\vec{r} = dx\vec{\imath} + dy\,\vec{\jmath}$

$\vec{F}\cdot d\vec{r} = 3xy\,dx - y^2dy$

Given C is $y = 2x^2$

$\therefore dy = 4xdx$

Along C, x varies from 0 to 1.

$\int_c \vec{F}\cdot d\vec{r} = \int_0^1 3x\,(2x^2)dx - 4x^4(4xdx)$

$= \int_0^1 6x^3 - 16x^5\,dx$

$= [6\frac{x^4}{4} - 16\frac{x^6}{6}]$

$= \frac{6}{4} - \frac{16}{6} = -\frac{7}{6}$ units.

Example: Find the work done, when a force $\vec{F} = (x^2 - y^2 + x)\vec{\imath} - (2xy + y)\vec{\jmath}$ moves a particle from the origin to the point $(1,1)$ along $y^2 = x$.

Solution:

Given $\vec{F} = (x^2 - y^2 + x)\vec{\imath} - (2xy + y)\vec{\jmath}$

$d\vec{r} = dx\vec{\imath} + dy\vec{\jmath}$

$\vec{F}\cdot d\vec{r} = (x^2 - y^2 + x)dx - (2xy + y)dy$

Given $y^2 = x \Rightarrow 2ydy = dx$

Along the curve C, y varies from 0 to 1.

$\int_c \vec{F}\cdot d\vec{r} = \int_0^1((y^2)^2 - y^2 + y^2)\,2ydy - (2(y^2)y + y)dy$

$= \int_0^1(2y^5 - 2y^3 + 2y^3 - 2y^3 - y)\,dy$

$= \int_0^1(2y^5 - 2y^3 - y)\,dy$

$$= [2\frac{y^6}{6} - 2\frac{y^4}{4} - \frac{y^2}{2}]_0^1$$

$$= \frac{2}{6} - \frac{2}{4} - \frac{1}{2} = -\frac{2}{3}$$

Example: Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} - z\vec{k}$ from $t = 0$ to $t = 1$ along the curve $x = 2t^2$, $y = t, z = 4t^3$.

Solution:

Given $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} - z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F} \cdot d\vec{r} = 3x^2dx + (2xz - y)dy - zdz$

Given $x = 2t^2$, $y = t,$ $z = 4t^3$

$dx = 4tdt$, $dy = dt,$ $dz = 12t^2dt$

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 48t^5dt + (16t^5 - t)dt - 48t^5\, dt$$

$$= \int_0^1 (16t^5 - t)dt$$

$$= [\frac{16\,t^6}{6} - \frac{t^2}{2}]_0^1 = \frac{16}{6} - \frac{1}{2} = \frac{13}{6}$$

Example: If $\vec{F} = (3x^2 + 6y)\vec{i} + 14yz\vec{j} + 20xz^2\vec{k}$, evaluate $\int_c \vec{F} \cdot d\vec{r}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve $x = t, y = t^2, z = t^3$.

Solution:

Given $\vec{F} = (3x^2 + 6y)\vec{i} + 14yz\vec{j} + 20xz^2\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F} \cdot d\vec{r} = (3x^2 + 6y)dx + 14yzdy + 20xz^2dz$

Given $x = t$, $y = t^2,$ $z = t^3$

$dx = dt$, $d\,y = 2tdt,$ $dz = 3t^2dt$

The point $(0, 0, 0)$ to $(1, 1,1)$ on the curve correspond to $t = 0$ and $t = 1$.

$$\int_c \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 + 6t^2)dt + 14t^5(2t\, dt) + 20t^7(3t^2)dt$$

$$= \int_0^1 (9t^2 + 28\, t^6 + 60t^9)\, dt$$

$$= [9\frac{t^3}{3} + 28\frac{t^7}{7} + 60\,\frac{t^9}{9}]_0^1$$

$= \frac{9}{3} + \frac{28}{7} + \frac{60}{10} = 3 + 4 + 6 = 13$units.

Example: Find $\int_c \vec{F} \cdot d\vec{r}$ where c is the circle $x^2 + y^2 = 4$ in the xy plane where $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$.

Solution:

Given $\vec{F} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$

In xy plane $z = 0 \Rightarrow dz = 0$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F} \cdot d\vec{r} = 2xydx + x^2dy$

Given C is $x^2 + y^2 = 4$

The parametric form of circle is

$x = 2\cos\theta$, $\quad y = 2\sin\theta$

$dx = -2\sin\theta d\theta$, $\quad dy = 2\cos\theta d\theta$

And θ varies from 0 to 2π

$\int_c \vec{F} \cdot d\vec{r} = \int_0^{2\pi} [2(2\cos\theta)(2\sin\theta)](-2\sin\theta d\theta) + (2\cos\theta)^2 2\cos\theta\, d\theta$

$= \int_0^{2\pi} -16\cos\theta sin^2\theta + 8\cos^3\theta\, d\theta$

$= \int_0^{2\pi} -16\cos\theta(1 - cos^2\theta) + 8\cos^3\theta\, d\theta$

$= \int_0^{2\pi} -16\cos\theta + 16\, cos^3\theta + 8\cos^3\theta\, d\theta$

$= -16\int_0^{2\pi} \cos\theta\, d\theta + 24\int_0^{2\pi} \cos^3\theta\, d\theta$

$= -16\int_0^{2\pi} \cos\theta\, d\theta + 24\int_0^{2\pi} \frac{3\cos\theta + \cos 3\theta}{4} d\theta$

$= 16\left[\sin\theta\right]_0^{2\pi} + \frac{24}{4}\left[3\sin\theta + \frac{sin3\theta}{3}\right]_0^{2\pi}$

$= 0 \qquad [\because \sin n\pi = 0, sin0 = 0]$

Example: State the physical interpretation of the line integral $\int_A^B \vec{F} \cdot d\vec{r}$.

Solution:

Physically $\int_A^B \vec{F} \cdot d\vec{r}$ denotes the total work done by the force $\vec{F}$, displacing a particle from A to B along the curve C.

Example: If $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^2z\vec{k}$, check whether the integral

$\int_c \vec{F} \cdot d\vec{r}$ **is independent of the path C.**

Solution:

Given $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^2z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F} \cdot d\vec{r} = (4xy - 3x^2z^2)dx + 2x^2dy - 2x^2zdz$

Then $\int_c \vec{F} \cdot d\vec{r}$ is independent of path C if $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(-6x^2z + 6x^2z) + \vec{k}(4x - 4x)$$

$$= \vec{0}$$

Hence the line integral is independent of path.

Example: Show that $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ is a conservative vector field.

Solution:

If $\vec{F}$ is conservative, then $\nabla \times \vec{F} = \vec{0}$.

$$\text{Now, } \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= \vec{0}$$

$\therefore$ $\vec{F}$ is a conservative vector field.

Surface Integral

The integral of the normal component of $\vec{F}$ is denoted by $\iint_S \vec{F} \cdot \vec{n}\, ds$ and is called the surface integral.

Evaluation of surface integral

Let R_1 be the projection of S on the $xy-$ plane, $\vec{k}$ is the unit vector normal to the $xy-$ plane then $ds = \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, ds = \iint_{R1} \vec{F} \cdot \vec{n} \frac{dx\,dy}{|\vec{n}\cdot\vec{k}|}$$

If R_2 be the projection of s on $yz-$ plane

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, ds = \iint_{R2} \vec{F} \cdot \vec{n} \frac{dx\,dy}{|\vec{n}\cdot\vec{i}|}$$

If R_3 be the projection of s on $xz-$ plane

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, ds = \iint_{R3} \vec{F} \cdot \vec{n} \frac{dx\,dy}{|\vec{n}\cdot\vec{j}|}$$

Example: Evaluate $\iint_S \vec{F} \cdot \vec{n}\, ds$ if $\vec{F} = (x+y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$ and s is the surface of the plane $2x+y+2z=6$ in the first octant.

Solution:

Given $\vec{F} = (x+y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$

Let $\varphi = 2x + y + 2z - 6$

Then $\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

$$= 2\vec{i} + 1\vec{j} + 2\vec{k}$$

$$|\nabla\varphi| = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2\vec{i} + 1\vec{j} + 2\vec{k}}{3}$$

$$\vec{F}\cdot\hat{n} = [(x+y^2)\vec{i} - 2x\vec{j} + 2yz\,\vec{k}] \cdot \left(\frac{2\vec{i} + 1\vec{j} + 2\vec{k}}{3}\right)$$

$$= \frac{1}{3}[2(x+y^2) - 2x + 4yz]$$

$$= \frac{2}{3}[y^2 + 2yz]$$

$$= \frac{2}{3} y[y + 2z]$$

$$= \frac{2}{3} y[y + 6 - 2x - y] \qquad [\because 2z = 6 - 2x - y]$$

$$= \frac{2}{3} y[6 - 2x]$$

$$= \frac{4}{3} y[3 - x]$$

Let R be the projection of S on the $xy -$ plane

$$\therefore ds = \frac{dx\, dy}{|\hat{n} \cdot \vec{k}|}$$

$$\hat{n} \cdot \vec{k} = \left(\frac{2\vec{i} + 1\vec{j} + 2\vec{k}}{3}\right) \cdot \vec{k} = \frac{2}{3}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_R \vec{F} \cdot \hat{n} \frac{dx\, dy}{|\hat{n} \cdot \vec{k}|}$$

$$= \iint_R \frac{4}{3} y(3 - x) \frac{dx\, dy}{\left(\frac{2}{3}\right)}$$

$$= 2 \int \int (3 - x) y\, dxdy$$

In $R_1 (2x + y = 6)$, x varies from 0 to $\frac{6-y}{2}$

y varies from 0 to 6

$$= 2 \int_0^6 \int_0^{\frac{6-y}{2}} y\, (3 - x)\, dxdy$$

$$= 2 \int_0^6 y \left[3x - \frac{x^2}{2}\right]_0^{\frac{6-y}{2}} dy$$

$$= 2 \int_0^6 y \left[3\left(\frac{6-y}{2}\right) - \frac{1}{2}\left(\frac{6-y}{2}\right)^2\right] dy$$

$$= 2 \int_0^6 \frac{1}{2}(18y - 3y^2) - \frac{1}{8}(6 - y)^2\, dy$$

$$= \frac{2}{2}\left[18 \frac{y^2}{2} - \frac{3y^3}{3} - \frac{1}{8}\frac{(6-y)^3}{3(-1)}\right]$$

$$= [9(6)^2 - (6)^3 + \frac{1}{12}(0)] - [0 - 0 + \frac{1}{12} (6)^3]$$

$$= 81 \text{ units}$$

Example: Show that $\iint_S (yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}) \cdot \hat{n}\, ds = \frac{3}{8}$ **where s is the surface of the sphere** $x^2 + y^2 + z^2 = 1$ **in the first octant.**

Solution:

Given $\vec{F} = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$

Let $\varphi = x^2 + y^2 + z^2 - 1$

$$\nabla\varphi = \vec{i}\,\frac{\partial\varphi}{\partial x} + \vec{j}\,\frac{\partial\varphi}{\partial y} + \vec{k}\,\frac{\partial\varphi}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2(1)$$

$\therefore$ The unit outward normal is $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2}$

$$\vec{F}\cdot\hat{n} = [yz\vec{i} + zx\vec{j} + xy\,\vec{k}]\cdot(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= 3xyz$$

Let R be the projection of S on xy −plane

$$\therefore ds = \frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

$$|\hat{n}\cdot\vec{k}| = (x\vec{i} + y\vec{j} + z\vec{k})\cdot\vec{k} = z$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{F}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

$$= \int\int 3xyz\,\frac{dxdy}{z}$$

$$= \int\int 3xy\,dxdy$$

In $R_1(x^2 + y^2 = 1)$, x varies from 0 to $\sqrt{1 - y^2}$

y varies from 0 to 1

$$= \int_0^1\int_0^{\sqrt{1-y^2}} 3xy\,dxdy$$

$$= 3\int_0^6 \left[y\,\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$$

$$= \frac{3}{2}\int_0^1 y(1 - y^2)dy$$

$$= \frac{3}{2}\int_0^1 y - y^3 dy$$

$$= \frac{3}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{8}$$

Volume integral

An integral which is evaluated over a volume bounded by a surface is called a volume integral.

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$ is a vector field in V, then the volume integral is defined by

$$\iiint_V \vec{F}\, dv$$

Example: If $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$, evaluate $\iiint_V \nabla \times \vec{F}\, dv$ where v is the volume of the region bounded by $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.

Solution:

Given $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \vec{i}(0 - 0) - \vec{j}(-4 + 3) + \vec{k}(-2y - 0)$$

$$= \vec{j} - 2y\vec{k}$$

For limits

Given $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$

$\therefore z : 0 \to 4 - 2x - 2y$

Put $z = 0 \Rightarrow 2x + 2y = 4\ (or)\ x + y = 4$

$\therefore y : 0 \to 2 - x$

Put $z = 0, y = 0 \Rightarrow 2x = 4\ (or)\ x = 2$

$\therefore x : 0 \to 2$

$$\therefore \iiint \nabla \times \vec{F}\, dv = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} (\vec{j} - 2y\vec{k})\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} (\vec{j} - 2y\vec{k})[z]_0^{4-2x-2y}\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} [(4 - 2x - 2y)\vec{j} - 2y(4 - 2x - 2y)\vec{k}]\, dy\, dx$$

$$= \int_0^2 \left\{\left[4y - 2xy - \frac{2y^2}{2}\right]\vec{j} - \left[4y^2 - 2xy^2 - \frac{4y^3}{3}\right]\vec{k}\right\}_0^{2-x} dx$$

$$= \int_0^2 \{[4(2 - x) - 2x(2 - x) - (2 - x)^2]\,\vec{j} - [4(2 - x)^2 - 2x(2 - x)^2 - \frac{4}{3}(2 - x)^3\vec{k}]\}\, dx$$

$= \int_0^2 [8 - 4x - 4x + 2x^2 - 4 + 4x - x^2]\, \vec{j} -$

$[16 - 16x + 4x^2 - 8x + 8x^2 - 2x^3 - \frac{4}{3}(8 - 12x + 6x^2 - x^3)\,\vec{k}]\ dx$

$= \int_0^2 [(4 - 4x + x^2)\,\vec{j} - \frac{\vec{k}}{3}(16 - 24x + 12x^2 - 2x^3)]\ dx$

$= [4x - 2x^2 + \frac{x^3}{3}]_0^2\,\vec{j} + \frac{\vec{k}}{3}[16x - 12x^2 + 4x^3 - \frac{x^4}{2}]_0^2$

$= (8 - 8 + \frac{8}{3})\,\vec{j} - \frac{\vec{k}}{3}(32 - 48 + 32 - 8)$

$= \frac{8}{3}(\vec{j} - \vec{k})$

Gauss Divergence and Stokes theorem

Green's Theorem

Green's theorem relates a line integral to the double integral taken over the region bounded by the closed curve.

Statement

If $M(x, y)$ and $N(x, y)$ are continuous functions with continuous, partial derivatives in a region R of the xy – plane bounded by a simple closed curve C, then

$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$,where C is the curve described in the positive direction.

Vector form of Green's theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\nabla \times \vec{F}) \cdot \vec{k}\, dR$$

STOKE'S THEOREM

Statement of Stoke's theorem

If S is an open surface bounded by a simple closed curve C if $\vec{F}$ is continuous having continuous partial derivatives in S and C, then

$$\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\, \vec{F} \cdot \hat{n}\, ds$$

(or)

$$\int_c \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds$$

$\hat{n}$ is the outward unit normal vector and C is traversed in the anti – clockwise direction.

GAUSS DIVERGENCE THEOREM

This theorem enables us to convert a surface integral of a vector function on a closed surface into volume integral.

Statement of Gauss Divergence theorem

If V is the volume bounded by a closed surface S and if a vector function $\vec{F}$ is continuous and has continuous partial derivatives in V and on S, then

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Where $\hat{n}$ is the unit outward normal to the surface S and $dV = dxdydz$

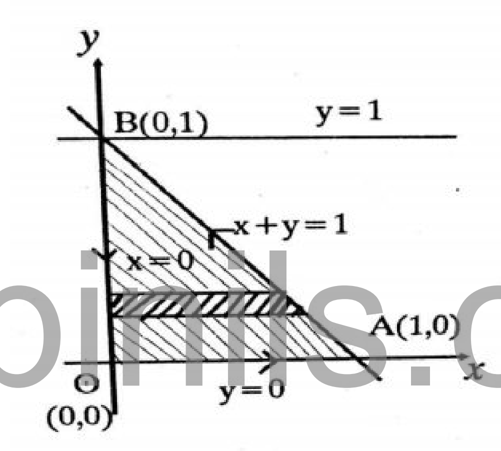
Verification and Application in evaluating line,surface and volume integrals

Example: Verify Green's theorem in the plane for $\int_c (3x^2 - 8y^2)dx + (4y - 6xy)dy$ where C is the boundary of the region defined by $x = 0, y = 0, x + y = 1$.

Solution:

We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = 3x^2 - 8y^2$ and $N = 4y - 6xy$



$\Rightarrow \frac{\partial M}{\partial y} = -16y \qquad \Rightarrow \frac{\partial N}{\partial x} = -6y$

$\therefore \int_c (3x^2 - 8y^2)dx + (4y - 6xy)dy = \int_c M\,dx + N\,dy$

By Green's theorem in the plane,

$$\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_0^{1-x} (10y)\,dy\,dx$$

$$= 10 \int_0^1 \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= 5 \int_0^1 (1-x)^2\,dx$$

$$= 5 \left[\frac{(1-x)^3}{-3}\right]_0^1 = \frac{5}{3} \ldots (1)$$

Consider $\int M\,dx + N\,dy = \int_{OA} + \int_{AB} + \int_{BO}$

Along $OA, y = 0 \Rightarrow dy = 0, x$ varies from 0 to 1

$$\therefore \int_{OA} M\,dx + N\,dy = \int_0^1 3x^2 dx = [x^3]_0^1 = 1$$

Along $AB, y = 1 - x \Rightarrow dy = -dx \text{ and } x$ varies from 1 to 0

$$\therefore \int_{AB} M\,dx + N\,dy = \int_1^0 [3x^2 - 8(1-x)^2 - 4(1-x) + 6x(1-x)]dx$$

$$= \left[\frac{3x^3}{3} - \frac{8(1-x)^3}{-3} - \frac{4(1-x)^2}{-2} + 3x^2 - 2x^3\right]_1^0$$

$$= \frac{8}{3} + 2 - 1 - 3 + 2 = \frac{8}{3}$$

Along $BO, x = 0 \Rightarrow dx = 0 \text{ and } y$ varies from 1 to 0

$$\therefore \int_{BO} M\,dx + N\,dy = \int_1^0 4y\,dy = [2y^2]_1^0 = -2$$

$$\therefore \int_c M\,dx + N\,dy = 1 + \frac{8}{3} - 2 = \frac{5}{3} \quad \ldots(2)$$

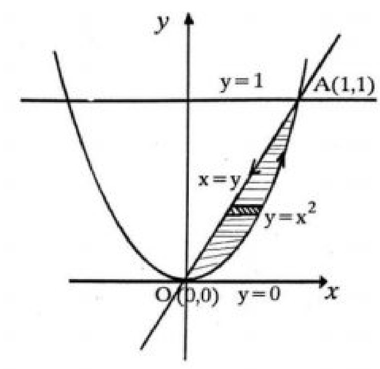
$\therefore$ From (1) and (2)

$$\therefore \int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Hence Green's theorem is verified.

Example: Verify Green's theorem in the XY −plane for $\int_c (xy + y^2)dx + x^2dy$ where C is the closed curve of the region bounded by $y = x, y = x^2$.

Solution:



We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = xy + y^2$ and $N = x^2$

$\Rightarrow \frac{\partial M}{\partial y} = x + 2y \qquad \Rightarrow \frac{\partial N}{\partial x} = 2x$

R.H.S = $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Limits:

x varies from y to $\sqrt{y}$

y varies from 0 to 1

$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_y^{\sqrt{y}} 2x - (x + 2y)\,dx\,dy$

$$= \int_0^1 \left[\frac{x^2}{2} - 2xy\right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 \left(\frac{y}{2} - 2y\sqrt{y}\right) - \left(\frac{y^2}{2} - 2y^2\right) dy$$

$$= \int_0^1 \left(\frac{y}{2} - 2y^{\frac{3}{2}} + 3\frac{y^2}{2}\right) dy$$

$$= \left[\frac{y^2}{2} - \frac{4y^{\frac{5}{2}}}{5} + \frac{y^3}{2}\right]_0^1$$

$$= \frac{1}{4} - \frac{4}{5} + \frac{1}{2} = -\frac{1}{20}$$

L.H.S = $\int_c M\,dx + N\,dy$

Consider $\int M\,dx + N\,dy = \int_{OA} + \int_{AO}$

Along $OA, y = x^2 \Rightarrow dy = 2x\,dx, x$ varies from 0 to 1

$$\therefore \int_{OA} M\,dx + N\,dy = \int_0^1 [(x(x^2) + (x^2)^2)dx + x^2 \cdot 2x\,dx]$$

$$= \int_0^1 (3x^3 + x^4)\,dx$$

$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Along $AO, y = x \Rightarrow dy = dx$ and x varies from 1 to 0

$$\therefore \int_{A0} M\,dx + N\,dy = \int_1^0 (x^2 + x^2)dx + x^2\,dx$$

$$= \int_1^0 3x^2 dx = [x^3]_1^0 = -1$$

$$\text{L.H.S} = \int_c M\,dx + N\,dy = \frac{19}{20} - 1 = -\frac{1}{20}$$

$\therefore$ L.H.S = R.H.S

Hence Green's theorem is verified.

Example: Verify Green's theorem in the plane for the integral $\int_c (x - 2y)dx + xdy$ taken around the circle $x^2 + y^2 = 1$.

Solution:

We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = x - 2y$ and $N = x$

$$\Rightarrow \frac{\partial M}{\partial y} = -2 \qquad \Rightarrow \frac{\partial N}{\partial x} = 1$$

$$\text{R.H.S} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1 + 2)dxdy$$

$$= 3 \iint_R dxdy$$

$= 3$ (Area of the circle)

$= 3\pi r^2$

$= 3\pi \quad (\because radius = 1)$

L.H.S $= \int_c M\,dx + N\,dy$

Given C is $x^2 + y^2 = 1$

The parametric equation of circle is

$x = \cos\theta,\ y = \sin\theta$

$dx = -\sin\theta d\theta,\ dy = \cos\theta\, d\theta$

Where θ varies from 0 to 2π

$$\therefore \int_c M\,dx + N\,dy = \int_0^{2\pi} (\cos\theta - 2\sin\theta)\,(-\sin\theta\, d\theta) + \cos\theta\,(\cos\theta\, d\theta)$$

$$= \int_0^{2\pi} (-\sin\theta\cos\theta + 2\sin^2\theta + \cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} (-\sin\theta\cos\theta + \sin^2\theta + 1)\, d\theta \quad (\because \sin^2\theta + \cos^2\theta = 1)$$

$$= \int_0^{2\pi} \left(-\frac{\sin 2\theta}{2} + \frac{1-\cos 2\theta}{2} + 1\right) d\theta$$

$$= \left[-\frac{1}{2}\left(-\frac{\cos 2\theta}{2}\right) + \frac{\theta}{2} - \frac{1}{2}\left(\frac{\sin 2\theta}{2}\right) + \theta\right]_0^{2\pi}$$

$$= \left[\frac{\cos(4\pi)}{4} + \frac{2\pi}{2} - \frac{\sin 4\pi}{4} + 2\pi\right] - \left[\frac{\cos 0}{4} + \frac{0}{2} - \frac{\sin 0}{4} + 0\right]$$

$$= \frac{1}{4} + \pi + 2\pi - \frac{1}{4} = 3\pi \quad [\because \sin n\pi = 0, \sin 0 = 0, \cos 0 = 1],$$

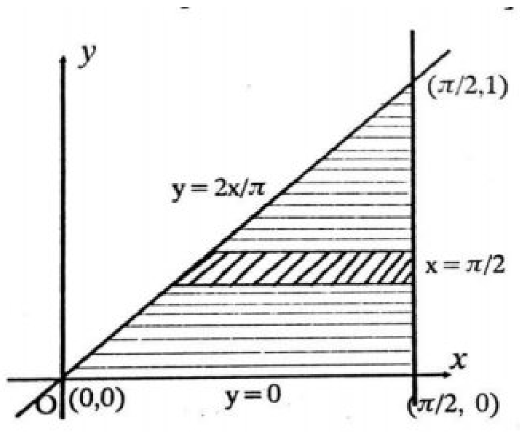
$[\cos n\pi = (-1)^n]$

$\therefore$ L.H.S = R.H.S

Hence Green's theorem is verified.

Example: Using Green's theorem evaluate $\int_c (y - \sin x)dx + \cos x\, dy$ where C is the triangle bounded by $y = 0, x = \frac{\pi}{2}, y = \frac{2x}{\pi}$

Solution:



We have to prove that $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here, $M = y - \sin x$ and $N = \cos x$

$\Rightarrow \frac{\partial M}{\partial y} = 1 - 0 \qquad \Rightarrow \frac{\partial N}{\partial x} = -\sin x$

Limits:

x varies from $\frac{y\pi}{2}$ to $\frac{\pi}{2}$

y varies from 0 to 1

Hence $\int_c (y - \sin x)dx + \cos x\,dy = \int_0^1 \int_{\frac{y\pi}{2}}^{\frac{\pi}{2}} (-\sin x - 1)\,dx\,dy$

$$= \int_0^1 (\cos x - x)_{\frac{y\pi}{2}}^{\frac{\pi}{2}}\,dy$$

$$= \int_0^1 \left[\left(\cos\frac{\pi}{2} - \frac{\pi}{2}\right) - \left(\cos\left(\frac{y\pi}{2}\right) - \frac{y\pi}{2}\right)\right] dy$$

$$= \int_0^1 \left[0 - \frac{\pi}{2} - \cos\frac{y\pi}{2} + \frac{y\pi}{2}\right] dy$$

$$= \left[-\frac{\pi}{2}y - \frac{\sin\frac{y\pi}{2}}{\frac{\pi}{2}} + \frac{\pi}{2}\frac{y^2}{2}\right]_0^1$$

$$= -\frac{\pi}{2} - \frac{2}{\pi}\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{4}$$

$$= -\frac{\pi}{2} - \frac{2}{\pi} + \frac{\pi}{4}$$

$$= -\frac{\pi}{4} - \frac{2}{\pi} = -\left[\frac{\pi}{4} + \frac{2}{\pi}\right]$$

Example: Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2}\int_c (xdy - ydx)$. Hence find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by using Green's theorem.

Solution:

By Green theorem, $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Let $M = -y$ and $N = x$

$\Rightarrow \frac{\partial M}{\partial y} = -1 \qquad \Rightarrow \frac{\partial N}{\partial x} = 1$

$\therefore \int_c (xdy - ydx) = \iint_R (1+1)\,dx\,dy$

$= 2\iint_R dx\,dy = 2$ (Area enclosed by C)

$\therefore$ Area enclosed by $C = \frac{1}{2}\int_c (xdy - ydx)$

Equation of ellipse in parametric form is $x = a\cos\theta$ and $y = b\sin\theta$ where $0 \le \theta \le 2\pi$.

$\therefore$ Area of the ellipse $= \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta)\,d\theta$

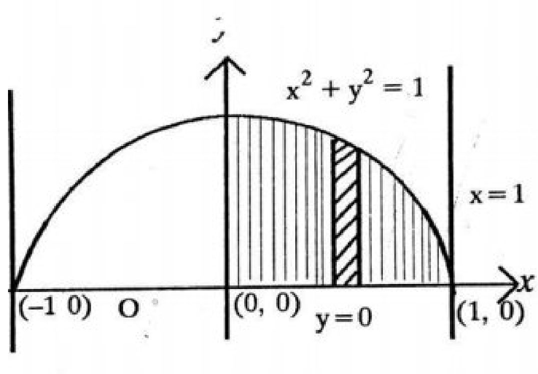
$= \frac{1}{2} ab \int_0^{2\pi} (cos^2\theta + sin^2\theta)\,d\theta$

$= \frac{1}{2} ab \int_0^{2\pi} d\theta = \frac{1}{2} ab\,[\theta]_0^{2\pi} = \pi ab$

Example: Evaluate the integral using Green's theorem

$\int_c (2x^2 - y^2)dx + (x^2 + y^2)dy$ **where C is the boundary in the** xy **– plane of the area enclosed by the** x **– axis and the semicircle** $x^2 + y^2 = a^2$ **in the upper half** xy **– plane.**

Solution:



In this figure $'a'$ is represented as 1

By Green theorem, $\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Let $M = 2x^2 - y^2$ and $N = x^2 + y^2$

$$\Rightarrow \frac{\partial M}{\partial y} = -2y \qquad \Rightarrow \frac{\partial N}{\partial x} = 2x$$

Limits:

y varies from 0 to $\sqrt{a^2 - x^2}$

x varies from $-a$ to a

$$\therefore \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = \int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} (2x + 2y)\,dy\,dx$$

$$= 2\int_{-a}^{a} \left[xy + \frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a} \left[x\sqrt{a^2 - x^2} + \frac{a^2 - x^2}{2}\right] dx$$

In the first integral, the function is odd function.

$\therefore$ The value is zero.

$\therefore$ we get $2\int_{-a}^{a} \frac{a^2 - x^2}{2} dx$

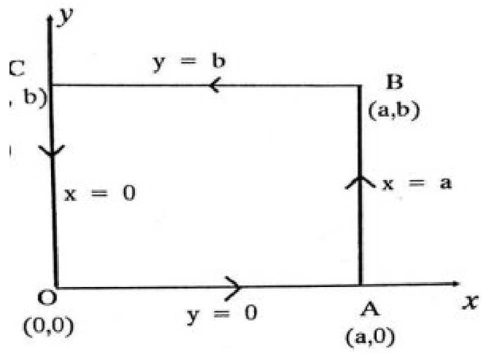
$$= \left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}$$

$$= \left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right)$$

$$= \frac{4a^3}{3}$$

Example: Verify stokes theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in a rectangular region in the xoy plane bounded by the lines $x = 0, x = a, y = 0, y = b$.

Solution:



By Stokes theorem, $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\, \vec{F} \cdot \hat{n} dS$

To evaluate: $\iint_S Curl\, \vec{F} \cdot \hat{n} dS$

Given $\vec{F} = (x^2 - y^2)\vec{i} + 2\,x\,y\vec{j}$

Curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$= \vec{i}\,(0) - \vec{j}\,(0 - 0) + \vec{k}[\,2y - (0 - 2y)]$

$= 4y\,\vec{k}$

Since the surface is a rectangle in the xy plane, $\hat{n} = \vec{k}$, $dS = dxdy$

Curl $\vec{F} \cdot \hat{n} = 4y\ \vec{k} \cdot \vec{k} = 4y$

Order of integration is $dxdy$

x varies from $x = 0$ to $x = a$

y varies from $y = 0$ to $y = b$

$$\Rightarrow \iint_S Curl\, \vec{F} \cdot \hat{n} dS = \int_0^b \int_0^a 4y\, dx\, dy$$

$$= \int_0^b 4y\, [x]_0^a dy$$

$$= \int_0^b 4aydy$$

$$= \left[\frac{4ay^2}{2}\right]_0^b$$

$$= 2ab^2$$

$$\Rightarrow \iint_S Curl\,\vec{F}\cdot\hat{n}\,dS = 2ab^2 \qquad \ldots (1)$$

Here the line integral over the simple closed curve C bounding the surface $OABCO$ consisting of the edges OA, AB ,BC and CO.

Curve	Equation	Limit
OA	$y = 0$	$x = 0$ to $x = a$
AB	$x = a$	$y = 0$ to $y = b$
BC	$y = b$	$x = a$ to $x = 0$
CO	$x = 0$	$y = b$ to $y = 0$

Therefore, $\int_c \vec{F}\cdot d\vec{r} = \int_{OABCO} \vec{F}\cdot d\vec{r}$

$$\int_c \vec{F}\cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$\vec{F}\cdot d\vec{r} = (x^2 - y^2) + 2xydy \qquad \ldots (2)$$

On OA: $y = 0, dy = 0, x$ varies from 0 to a

(2) $\Rightarrow \vec{F}\,.\,d\vec{r} = x^2dx$

$$\int_{OA} \vec{F}\cdot d\vec{r} = \int_0^a x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

On AB: $x = a, dx = 0, y$ varies from 0 to b

(2) $\Rightarrow \vec{F}\,.\,d\vec{r} = 2ay\,dy$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_0^b 2ay\,dy$$

$$= \left[\frac{2ay^2}{2}\right]_0^b = ab^2$$

On BC: $y = b, dy = 0, x$ varies from a to 0

(2) $\Rightarrow \vec{F}\,.\,d\vec{r} = (x^2 - b^2)dx$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_a^0 x^2 - b^2\, dx$$

$$= \left[\frac{x^3}{3} - b^2 x\right]_a^0$$

$$= -\frac{a^3}{3} + a\, b^2$$

On CO: $x = 0, dx = 0, y$ varies from b to 0

(2) $\Rightarrow \vec{F}.d\vec{r} = 0$

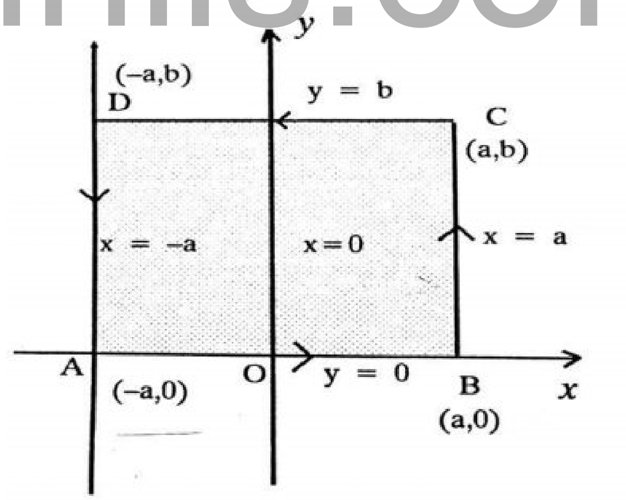
$$\int_{CO} \vec{F} \cdot d\vec{r} = 0$$

(2) $\Rightarrow \vec{F}.d\vec{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 = 2ab^2$ ... (3)

From (3) and (1) $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\, \vec{F} \cdot \hat{n} dS$

Hence Stokes theorem is verified.

Example: Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\,\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0, y = b$.



Solution:

By Stokes theorem, $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\, \vec{F} \cdot \hat{n} dS$

Given $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\,\vec{j}$

$$Curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}[0-0] + \vec{k}[-2y-2y]$$

$$= -4y\,\vec{k}$$

Since the region is in xoy plane we can take $\hat{n} = \vec{k}$ $and\ dS = dx\,dy$

Limits:

x varies from $-a$ to a.

y varies from 0 to b.

$$\therefore \iint_S Curl\ \vec{F}\cdot\hat{n}\,dS = -4\int_0^b \int_{-a}^{a} y\,dx\,dy$$

$$= -4\int_0^b [xy]_{-a}^{a}\,dy$$

$$= -8a\left[\frac{y^2}{2}\right]_0^b = -4ab^2 \quad \ldots (1)$$

$$\int_c \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB: $y = 0, dy = 0, x$ varies from $-a\ to\ a$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{2a^3}{3}$$

Along $BC, x = a, dx = 0,$ y varies from 0 $to\ b$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_0^b (-2ay)\,dy$$

$$= -a[y^2]_0^b = -ab^2$$

Along CD: $y = b, dy = 0, x$ varies from $a\ to - a$

$$\int_{CD} \vec{F}\,.\,d\vec{r} = \int_a^{-a} (x^2 + b^2)\,dx = \left[\frac{x^3}{3} + b^2x\right]_a^{-a}$$

$$= -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -\frac{2a^3}{3} - 2ab^2$$

Along DC: $x = -a, dx = 0, y$ varies from $b\ to\ 0$

$$\int_{DC} \vec{F} \cdot d\vec{r} = \int_b^0 2ay\, dy$$

$$= a[y^2]_b^0 = -b^2a$$

$$\therefore \int_c \vec{F} \cdot d\vec{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - b^2a$$

$$= -4ab^2 \quad \ldots.(2)$$

From (1) and (2) $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\, \vec{F} \cdot \vec{n}\, dS$

Hence Stoke's theorem is verified.

Example: Verify Stoke's theorem for $\vec{F} = y^2z\vec{i} + z^2x\vec{j} + x^2y\,\vec{k}$, where S is the open surface of the cube formed by the planes $x = \pm a$, $y = \pm a$, and $z = \pm a$ in which the plane $z = -a$ is a cut.

Solution:

Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\, \vec{F} \cdot \hat{n}\, ds$

Given $\vec{F} = y^2z\vec{i} + z^2x\vec{j} + x^2y\,\vec{k}$

$\vec{F} \cdot d\vec{r} = y^2zdx + z^2xdy + x^2y\,dz$

This square ABCD lies in the plane $z = -a \Rightarrow dz = 0$

$\therefore \vec{F} \cdot d\vec{r} = -ay^2dx + a^2x\,dy$

L.H.S $= \int_c \vec{F} \cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$

On AB: $y = -a \Rightarrow dy = 0$, x varies from $-a$ to a.

$$\Rightarrow \int_{AB} \vec{F} \cdot d\vec{r} = \int_{-a} -a^3\, dx$$

$$= -a^3\,[x]_{-a}^{a}$$

$$= -a^3(2a) = -2a^4$$

On BC: $x = a \Rightarrow dx = 0$, y varies from $-a$ to a.

$$\Rightarrow \int_{BC} \vec{F} \cdot d\vec{r} = \int_{-a} a^3\, dy$$

$$= a^3\,[y]_{-a}^{a}$$

$$= a^3(2a) = 2a^4$$

On CD: $y = a \Rightarrow dy = 0$, x varies from a to $-a$.

$$\Rightarrow \int_{CD} \vec{F} \cdot d\vec{r} = \int_a^{-a} -a^3\, dx$$

$$= -a^3\, [x]_a^{-a}$$

$$= -a^3(-2a) = 2a^4$$

On DA: $x = -a \Rightarrow dx = 0$, y varies from a to $-a$.

$$\Rightarrow \int_{DA} \vec{F} \cdot d\vec{r} = \int_a^{-a} -a^3\, dy$$

$$= -a^3\, [y]_a^{-a}$$

$$= -a^3(-2a) = 2a^4$$

$$\therefore \int_c \vec{F} \cdot d\vec{r} = -2a^4 + 2a^4 + 2a^4 + 2a^4 = 4a^4 \quad \ldots (1)$$

R.H.S $= \iint_S curl\, \vec{F} \cdot \hat{n}\, ds$

$$curl\, \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z & z^2 x & x^2 y \end{vmatrix}$$

$$= \vec{i}\,(x^2 - 2xz) - \vec{j}\,(y^2 - 2xy) + \vec{k}\,(z^2 - 2yz)$$

Given S is an open surface consisting of the 5 faces of the cube except, $z = -a$.

$$\int\int curl\, \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \cdots + \iint_{S5}$$

$$curl\, \vec{F} = 2y\vec{i} + z\,\vec{j} - x\,\vec{k}$$

Faces	Plane	ds	$\hat{n}$	Eqn	$curl\, \vec{F} \cdot \hat{n}$	$\nabla \times \vec{F} \cdot \hat{n}$
Top (S_1)	xy	$dxdy$	$\vec{k}$	$z = a$	$z^2 - 2yz$	$a^2 - 2ay$
Left (S_2)	xz	$dxdz$	$-\vec{j}$	$y = -a$	$y^2 - 2xy$	$a^2 + 2ax$
Right (S_3)	xz	$dxdz$	$\vec{j}$	$y = a$	$-(y^2 - 2xy)$	$-(a^2 - 2ax)$
Back (S_4)	yz	$dydz$	$-\vec{i}$	$x = -a$	$-(x^2 - 2xz)$	$-(a^2 + 2az)$
Front (S_5)	yz	$dydz$	$\vec{i}$	$x = a$	$x^2 - 2xz$	$a^2 - 2az$

On S_1: $\int_{-a}^{a}\int_{-a}^{a} (a^2 - 2ay)\, dxdy$

$$= \int_{-a}^{a} [(a^2x - 2ayx)]_{-a}^{a} \, dy$$

$$= \int_{-a}^{a} (a^3 - 2a^2y) - (-a^3 + 2a^2y) \, dy$$

$$= \int_{-a}^{a} 2a^3 - 4a^2y \, dy$$

$$= [2a^3y - 4a^2\frac{y^2}{2}]_{-a}^{a}$$

$$= (2a^4 - 2a^4) - (-2a^4 - 2a^4)$$

$$= 2a^4 - 2a^4 + 2a^4 + 2a^4$$

$$= 4a^4$$

On $S_2 + S_3$: $\int_{-a}^{a}\int_{-a}^{a} (a^2 + 2ax) \, dxdz + \int_{-a}^{a}\int_{-a}^{a} -(a^2 - 2ax) \, dxdz$

$$= \int_{-a}^{a}\int_{-a}^{a} (a^2 + 2ax - a^2 + 2ax) \, dx \, dz$$

$$= \int_{-a}^{a}\int_{-a}^{a} 4ax \, dx \, dz$$

$$= 4a \int_{-a}^{a} [\frac{x^2}{2}]_{-a}^{a} \, dz$$

$$= 2a^3 \int_{-a}^{a} dz$$

$$= 2a^3 \, [z]_{-a}^{a}$$

$$= 2a^3(0) = 0$$

On $S_4 + S_5$: $\int_{-a}^{a}\int_{-a}^{a} -(a^2 + 2az) \, dydz + \int_{-a}^{a}\int_{-a}^{a} (a^2 - 2az) \, dydz$

$$= \int_{-a}^{a}\int_{-a}^{a} (-a^2 - 2az + a^2 - 2az) \, dy \, dz$$

$$= \int_{-a}^{a}\int_{-a}^{a} -4az \, dy \, dz$$

$$= -4a \int_{-a}^{a} [zy]_{-a}^{a} \, dz$$

$$= -4a \int_{-a}^{a} z(2a) dz$$

$$= -6a^2 \, [\frac{z^2}{2}]_{-a}^{a}$$

$$= -3a^2(a^2 - a^2) = 0$$

$$\therefore \iint_S curl \, \bar{F} \cdot \hat{n} \, ds = 4a^4 + 0 + 0 = 4a^4 \quad \ldots (2)$$

From (1) and (2) $\int_c \bar{F} \cdot d\bar{r} = \iint_S curl \, \bar{F} \cdot \hat{n} \, ds$

Hence Stoke's theorem is verified.

Example: Evaluate $\int_c \vec{F} \cdot d\vec{r}$ by stoke's theorem, where $\vec{F} = y^2\vec{i} + x^2\vec{j} + (x+z)\vec{k}$, and C is the boundary of the triangle with vertices at $(0,0,0), (1,0,0)$ and $(1,1,0)$.

Solution:

Stoke's theorem is $\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\, \vec{F} \cdot \hat{n}\, ds \quad \ldots (1)$

Given $\vec{F} = y^2\vec{i} + x^2\vec{j} + (x+z)\vec{k}$

And C is triangle $(0,0,0), (1,0,0)$ and $(1,1,0)$.

Since z −coordinate of each vertex is zero the triangle lies in xy − plane with corners $(0,0), (1,0)$ and $(1,1)$.

To evaluate : $\iint_S curl\, \vec{F} \cdot \hat{n}\, ds$

$In\ xy - plane\ \hat{n} = \vec{k}, ds = dxdy$

$$curl\, \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(-1) + \vec{k}(2x - 2y)$$

$$= \vec{j} + 2(x-y)\vec{k}$$

$$curl\, \vec{F} \cdot \hat{n} = (\vec{j} + 2(x-y)\vec{k}) \cdot \vec{k}$$

$$= 2(x-y)$$

Limits:

x varies from y to 1.

y varies from 0 to 1.

$$\therefore \iint_S curl\, \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_y^1 2(x-y)dxdy$$

$$= 2\int_0^1 \left[\frac{x^2}{2} - xy\right]_y^1 dy$$

$$= 2\int_0^1 \left(\frac{1}{2} - y - \frac{y^2}{2} + y^2\right) dy$$

$$= 2\left[\frac{y}{2} - \frac{y^2}{2} - \frac{y^3}{6} + \frac{y^3}{3}\right]_0^1$$

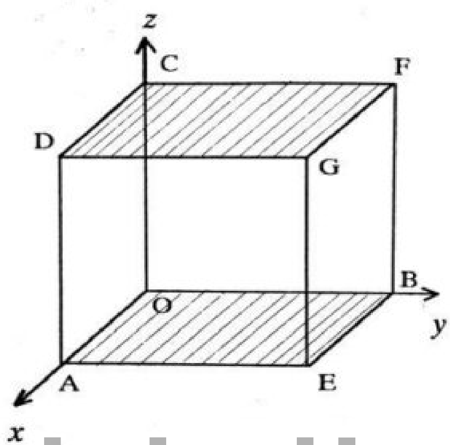
$$= 2\left[\frac{1}{2} - \frac{1}{2} - \frac{1}{6} + \frac{1}{3}\right]$$

$$= 2\left[\frac{1}{6}\right] = \frac{1}{3}$$

From (1), $\int_c \vec{F} \cdot d\vec{r} = \frac{1}{3}$

Example: Verify the G.D.T for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

Solution:



Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$

$$\nabla \cdot \vec{F} = 4z - 2y + y$$

$$= 4z - y$$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$$= \int_0^1 \int_0^1 \int_0^1 (4z - y)\,dxdydz$$

$$= \int_0^1 \int_0^1 [(4xz - yz)]_0^1\, dydz$$

$$= \int_0^1 \int_0^1 (4z - y)\,dydz$$

$$= \int_0^1 \left(4zy - \frac{y^2}{2}\right)_0^1 dz$$

$$= \int_0^1 \left(4z - \frac{1}{2}\right) dz$$

$$= \left[4\frac{z^2}{2} - \frac{1}{2}z\right]_0^1 = \left(2 - \frac{1}{2}\right) - 0 = \frac{3}{2}$$

Now, L.H.S $= \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	Plane	dS	$\hat{n}$	$\vec{F}\cdot\hat{n}$	Equation	$\vec{F}\cdot\hat{n}$ on S	$= \iint_S \vec{F}\cdot\hat{n}\,ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	$-yz$	$z=0$	0	$\int_0^1\int_0^1 0\,dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	yz	$z=1$	y	$\int_0^1\int_0^1 y\,dxdy$
$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	y^2	$y=0$	0	$\int_0^1\int_0^1 0\,dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	$-y^2$	$y=1$	-1	$\int_0^1\int_0^1 -1dxdz$
$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-4xz$	$x=0$	0	$\int_0^1\int_0^1 0\,dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	$4xz$	$x=1$	$4z$	$\int_0^1\int_0^1 4z\,dydz$

$(i)\iint_{S1}\vec{F}\cdot\hat{n}\,ds + \iint_{S2}\vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 0\,dxdy + \int_0^1\int_0^1 y\,dxdy$

$= 0 + \int_0^1\int_0^1 y\,dxdy$

$= \int_0^1 [yx]_0^1\,dy$

$= \int_0^1 y\,dy$

$= \left[\frac{y^2}{2}\right]_0^1 \quad = \frac{1}{2} - 0 \quad = \frac{1}{2}$

$(ii)\iint_{S3}\vec{F}\cdot\hat{n}\,ds + \iint_{S4}\vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 0\,dxdz + \int_0^1\int_0^1 -1\,dxdz$

$= 0 + \int_0^1\int_0^1 -1\,dxdz$

$= -\int_0^1 [x]_0^1\,dz$

$= -\int_0^1 dz$

$= -[z]_0^1 \quad = -[1]$

$$(iii) \iint_{S5} \vec{F} \cdot \hat{n}\, ds + \iint_{S6} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 0\, dydz + \int_0^1 \int_0^1 4z dydz$$

$$= 0 + \int_0^1 \int_0^1 4z\, dydz$$

$$= \int_0^1 [4zy]_0^1\, dz$$

$$= \int_0^1 4z\, dz$$

$$= 4\left[\frac{z^2}{2}\right]_0^1 = 4\left(\frac{1}{2} - 0\right) = 2$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

$$= \frac{1}{2} - 1 + 2 = \frac{3}{2}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Hence Gauss divergence theorem is verified.

Example: Verify the G.D.T for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$ over the rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$. (OR)

Verify the G.D.T for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$ over the rectangular parallelopiped bounded by $x = 0, x = a, y = 0, y = b, z = 0, z = c$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$

$$\nabla \cdot \vec{F} = 2x + 2y + 2z = 2(x + y + z)$$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$$= 2\int_0^c \int_0^b \int_0^a (x + y + z) dxdydz$$

$$= 2\int_0^c \int_0^b \left[\left(\frac{x^2}{2} + xy + xz\right)\right]_0^a dydz$$

$$= 2\int_0^c \int_0^b \left(\frac{a^2}{2} + ay + az\right) dydz$$

$$= 2\int_0^c \left(\frac{a^2y}{2} + \frac{ay^2}{2} + azy\right)_0^b dz$$

$$= 2\int_0^c \left(\frac{a^2b}{2} + \frac{ab^2}{2} + azb\right) dz$$

$$= 2\left[\frac{a^2bz}{2} + \frac{ab^2z}{2} + \frac{abz^2}{2}\right]_0^c$$

$$= 2\left(\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right)$$

$$= abc(a + b + c)$$

Now, L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	Plane	dS	$\hat{n}$	$\vec{F} \cdot \hat{n}$	Eqn	$\vec{F} \cdot \hat{n}$ on S	$= \iint_S \vec{F} \cdot \hat{n}\, ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	$-(z^2 - xy)$	$z = 0$	xy	$\int_0^b\int_0^a xy\, dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	$(z^2 - xy)$	$z = c$	$c^2 - xy$	$\int_0^b\int_0^a c^2 - xy\, dxdy$
$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	$-(y^2 - xz)$	$y = 0$	xz	$\int_0^c\int_0^a xz\, dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	$(y^2 - xz)$	$y = b$	$b^2 - xz$	$\int_0^c\int_0^a b^2 - xz\, dxdz$
$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-(x^2 - yz)$	$x = 0$	yz	$\int_0^c\int_0^b yz\, dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	$(x^2 - yz)$	$x = a$	$a^2 - yz$	$\int_0^c\int_0^b a^2 - yz\, dydz$

(i) $\iint_{S1} \vec{F} \cdot \hat{n}\, ds + \iint_{S2} \vec{F} \cdot \hat{n}\, ds = \int_0^b\int_0^a xy\, dxdy + \int_0^b\int_0^a c^2 - xy\, dxdy$

$$= \int_0^b\int_0^a c^2\, dxdy$$

$$= c^2 \int_0^a dx \int_0^b dy$$

$$= c^2[x]_0^a[y]_0^b = c^2ab$$

(ii) $\iint_{S3} \vec{F} \cdot \hat{n}\, ds + \iint_{S4} \vec{F} \cdot \hat{n}\, ds = \int_0^c\int_0^a xz\, dxdz + \int_0^c\int_0^a b^2 - xz\, dxdz$

$$= \int_0^c \int_0^a b^2 \, dxdz$$

$$= b^2 \int_0^a dx \int_0^c dz$$

$$= b^2 [x]_0^a [z]_0^c = b^2 ac$$

$$(iii) \iint_{S5} \vec{F} \cdot \hat{n} \, ds + \iint_{S6} \vec{F} \cdot \hat{n} \, ds = \int_0^c \int_0^b yz \, dydz + \int_0^c \int_0^b a^2 - yz \, dydz$$

$$= \int_0^c \int_0^b a^2 \, dydz$$

$$= a^2 \int_0^b dy \int_0^c dz$$

$$= a^2 [y]_0^b [z]_0^c = a^2 bc$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} \, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

$$= abc(a + b + c)$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv$$

Hence Gauss divergence theorem is verified.

Example: Verify divergence theorem for $\vec{F} = (2x - z)\vec{i} + x^2 y\vec{j} - xz^2\vec{k}$ over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n} \, ds = \iiint_V \nabla \cdot \vec{F} \, dv$

Given $\vec{F} = (2x - z)\vec{i} + x^2 y\vec{j} - xz^2\vec{k}$

$$\nabla \cdot \vec{F} = 2 + x^2 - 2xz$$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F} \, dv$

$$= \int_0^1 \int_0^1 \int_0^1 (2 + x^2 - 2xz) dxdydz$$

$$= \int_0^1 \int_0^1 \left[\left(2x + \frac{x^3}{3} - \frac{2zx^2}{2}\right)\right]_0^1 dydz$$

$$= \int_0^1 \int_0^1 \left(2 + \frac{1}{3} - z\right) dydz$$

$$= \int_0^1 \left(2y + \frac{1}{3}y - zy\right)_0^1 dz$$

$$= \int_0^1 \left(2 + \frac{1}{3} - z\right) dz$$

$$= \left[2z + \frac{1}{3}z - \frac{z^2}{2}\right]_0^1$$

$$= \left(2 + \frac{1}{3} - \frac{1}{2}\right) - 0 = \frac{11}{6}$$

Now, L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	Plane	dS	$\hat{n}$	$\vec{F}\cdot\hat{n}$	Equation	$\vec{F}\cdot\hat{n}$ on S	$= \iint_S \vec{F}\cdot\hat{n}\, ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	xz^2	$z = 0$	0	$\int_0^1\int_0^1 0\, dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	$-xz^2$	$z = 1$	$-x$	$\int_0^1\int_0^1 (-x)\, dxdy$
$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	$-x^2y$	$y = 0$	0	$\int_0^1\int_0^1 0\, dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	x^2y	$y = 1$	x^2	$\int^1 \int_0^1 x^2 dxdz$ 0
$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-(2x - z)$	$x = 0$	z	$\int_0^1\int_0^1 z\, dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	$(2x - z)$	$x = 1$	$2 - z$	$\int_0^1\int_0^1 2 - z\, dydz$

(i) $\iint_{S1} \vec{F}\cdot\hat{n}\, ds + \iint_{S2} \vec{F}\cdot\hat{n}\, ds = \int_0^1 \int_0^1 0\, dxdy + \int_0^1 \int_0^1 (-x)\, dxdy$

$$= \int_0^1 \int_0^1 (-x)\, dxdy$$

$$= -\int_0^1 \left[\frac{x^2}{2}\right]_0^1 dy$$

$$= -\int_0^1 \frac{1}{2}\, dy$$

$$= -\left[\frac{1}{2}y\right]_0^1 = -\left(\frac{1}{2} - 0\right) = \frac{-1}{2}$$

$(ii) \iint_{S3} \vec{F} \cdot \hat{n}\, ds + \iint_{S4} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 0\, dxdz + \int_0^1 \int_0^1 x^2\, dxdz$

$= \int_0^1 \int_0^1 x^2\, dxdz$

$= \int_0^1 \left[\frac{x^3}{3}\right]_0^1 dz$

$= \int_0^1 \frac{1}{3}\, dz$

$= \left[\frac{1}{3} z\right]_0^1 \quad = \left(\frac{1}{3} - 0\right) \quad = \frac{1}{3}$

$(iii) \iint_{S5} \vec{F} \cdot \hat{n}\, ds + \iint_{S6} \vec{F} \cdot \hat{n}\, ds = \int_0^1 \int_0^1 z\, dydz + \int_0^1 \int_0^1 (2 - z) dydz$

$= \int_0^1 \int_0^1 2\, dydz$

$= 2 \int_0^1 [y]_0^1\, dz$

$= 2 \int_0^1 dz$

$= 2\, [z]_0^1 \quad = 2$

$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

$= (i) + (ii) + (iii)$

$= -\frac{1}{2} + \frac{1}{3} + 2 \quad = \frac{11}{6}$

$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Hence Gauss divergence theorem is verified.

Example: Verify divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube bounded by $x = \pm 1, y = \pm 1, z = \pm 1$.

Solution:

Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$

$\nabla \cdot \vec{F} = 2x + y$

Now, R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(2x + y)dxdydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}\left[\left(2\frac{x^2}{2} + yx\right)\right]_{-1}^{1} dydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}[(1 + y) - (1 - y)]dydz$$

$$= \int_{-1}^{1}\int_{-1}^{1}[2y]dydz$$

$$= \int_{-1}^{1}\left(2\frac{y^2}{2}\right)_{-1}^{1} dz$$

$$= \int_{-1}^{1}[(1) - ((-1)^2)]\, dz$$

$$= \int_{-1}^{1}[0]\, dz$$

$$= 0$$

Now, L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$

Faces	Plane	dS	$\hat{n}$	$\vec{F}\cdot\hat{n}$	Equation	$\vec{F}\cdot\hat{n}$ on S	$= \iint_S \vec{F} \cdot \hat{n}\, ds$
S_1(Bottom)	xy	$dxdy$	$-\vec{k}$	$-yz$	$z = -1$	y	$\int_{-1}^{1}\int_{-1}^{1} y\, dxdy$
S_2(Top)	xy	$dxdy$	$\vec{k}$	yz	$z = 1$	y	$\int_{-1}^{1}\int_{-1}^{1} y\, dxdy$
$S_3(Left)$	xz	$dxdz$	$-\vec{j}$	$-z$	$y = -1$	$-z$	$\int_{-1}^{1}\int_{-1}^{1} -z\, dxdz$
$S_4(Right)$	xz	$dxdz$	$\vec{j}$	z	$y = 1$	z	$\int_{-1}^{1}\int_{-1}^{1} zdxdz$
$S_5(Back)$	yz	$dydz$	$-\vec{i}$	$-x^2$	$x = -1$	-1	$\int_{-1}^{1}\int_{-1}^{1} -1\, dydz$
$S_6(Front)$	yz	$dydz$	$\vec{i}$	x^2	$x = 1$	1	$\int_{-1}^{1}\int_{-1}^{1} dydz$

$(i) \iint_{S1} \vec{F} \cdot \hat{n}\, ds + \iint_{S2} \vec{F} \cdot \hat{n}\, ds = \int_{-1}^{1}\int_{-1}^{1} y\, dxdy + \int_{-1}^{1}\int_{-1}^{1} y\, dxdy$

$$= \int_{-1}^{1}\int_{-1}^{1} 2y\, dxdy$$

$$= 2\int_{-1}^{1} [xy]_{-1}^{1}\, dy$$

$$= 2\int_{-1}^{1}[(y) - (-y)]\, dy$$

$$= 2\int_{-1}^{1} 2y\,dy$$

$$= 4\left[\frac{y^2}{2}\right]_{-1}^{1} = 4\left[\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\right] = 0$$

$$(ii) \iint_{S3} \vec{F}\cdot\hat{n}\,ds + \iint_{S4} \vec{F}\cdot\hat{n}\,ds = \int_{-1}^{1}\int_{-1}^{1} -z\,dxdz + \int_{-1}^{1}\int_{-1}^{1} z\,dxdz$$

$$= \int_{-1}^{1}\int_{-1}^{1} 0\,dxdz$$

$$= 0$$

$$(iii) \iint_{S5} \vec{F}\cdot\hat{n}\,ds + \iint_{S6} \vec{F}\cdot\hat{n}\,ds = -\int_{-1}^{1}\int_{-1}^{1} dxdz + \int_{-1}^{1}\int_{-1}^{1} dxdz$$

$$= 0$$

$$\therefore \iint_{S} \vec{F}\cdot\hat{n}\,ds = \iint_{S1} + \iint_{S2} + \iint_{S3} + \iint_{S4} + \iint_{S5} + \iint_{S6}$$

$$= (i) + (ii) + (iii)$$

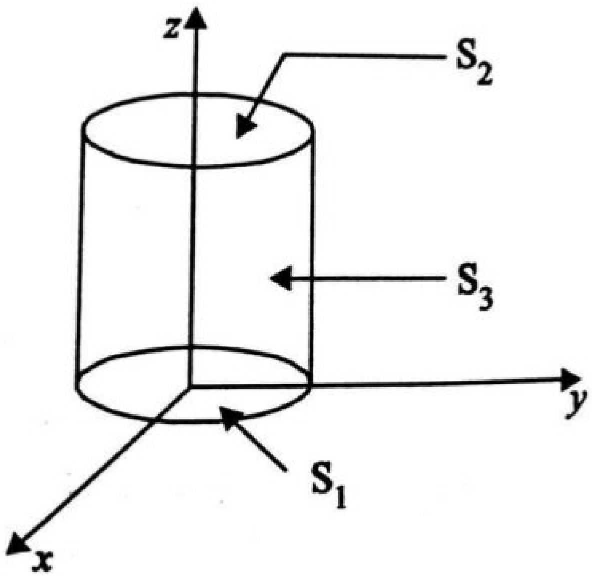
$$= 0$$

$$\therefore \iint_{S} \vec{F}\cdot\hat{n}\,ds = \iiint_{V} \nabla\cdot\vec{F}\,dv$$

Hence , Gauss divergence theorem is verified.

Example: Verify divergence theorem for the function $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the surface bounded by the cylinder $x^2 + y^2 = 4$ and $z = 0, z = 3$.

Solution:



Gauss divergence theorem is $\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$

Given $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$

$\nabla \cdot \vec{F} = 4 - 4y + 2z$

Limits:

$z = 0 \; to \; 3$

$x^2 + y^2 = 4 \Rightarrow y^2 = 4 - x^2$

$\Rightarrow y = \pm\sqrt{4 - x^2}$

$\therefore y = -\sqrt{4 - x^2} \; to \; \sqrt{4 - x^2}$

Put $y = 0 \Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

$\therefore y = -2 \; to \; 2$

$\therefore$ R.H.S $= \iiint_V \nabla \cdot \vec{F}\, dv$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{3}(4 - 4y + 2z)\, dzdydx$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[4z - 4yz + 2\frac{z^2}{2}\right]_0^3 dydx$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(12 - 12y + 9)dydx$

$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(21 - 12y)dydx$

$= 2\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} 21\, dydx$ $\quad \left[\because \int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x) \text{ if } f(x) \text{ is even}; = 0 \text{ if } f(x) \text{ is odd}\right]$

$= 42 \int_{-2}^{2} [y]_0^{\sqrt{4-x^2}}\, dx$

$= 42 \int_{-2}^{2} \sqrt{4-x^2}\, dx$

$= 42 \times 2 \int_0^2 \sqrt{4-x^2}\, dx$ $\quad [\because \text{even function}]$

$= 84 \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$

$= 84\,[0 + 2\sin^{-1}(1)]$

$= 84\,[2 \times \frac{\pi}{2}]$

$= 84\,\pi$

L.H.S $= \iint_S \vec{F} \cdot \hat{n}\, ds$

$= \iint_{S1} + \iint_{S2} + \iint_{S3}$

Along S_1 (bottom):

xy –plane $\Rightarrow z = 0, dz = 0$

And $ds = dxdy,\ \hat{n} = -\vec{k}$

$\therefore \vec{F} \cdot \hat{n} = (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}) \cdot (-\vec{k})$

$= -z^2 = 0$

$\therefore \iint_{S1} \vec{F} \cdot \hat{n}\, ds = \iint_{S1} 0 = 0$

Along S_2 (top):

xy –plane $\Rightarrow z = 3, dz = 0$

And $ds = dxdy,\ \hat{n} = \vec{k}$

$\therefore \vec{F} \cdot \hat{n} = (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}) \cdot (-\vec{k})$

$= z^2 = 9$

$\therefore \iint_{S2} \vec{F} \cdot \hat{n}\, ds = \iint_{S2} 9\, dxdy$

$= \iint_R 9\, dxdy$

$= 9\ (Area\ of\ the\ circle)$

$= 9\ (\pi r^2) \qquad [\because r = 2]$

$= 36\,\pi$

Along S_3 (curved surface):

Given $x^2 + y^2 = 4$

Let $\varphi = x^2 + y^2 - 4$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2} = 2\sqrt{4} = 4$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2(x\vec{i}+y\vec{j})}{4}$$

$$= \frac{x\vec{i}+y\vec{j}}{2}$$

The cylindrical coordinates are

$x = 2\cos\theta,\ y = 2\sin\theta\ ds = 2dzd\theta$

Where z varies from 0 to 3

θ varies from 0 to 2π

Now $\vec{F}\cdot\hat{n} = (4x\,\vec{i} - 2y^2\,\vec{j} + z^2\,\vec{k})\cdot(\frac{x\vec{i}+y\vec{j}}{2})$

$$= 2x^2 - y^3$$

$$= 2(2\cos\theta)^2 - (2\sin\theta)^3$$

$$= 8\cos^2\theta - 8\sin^3\theta$$

$$= 8\ [\frac{1+\cos 2\theta}{2} - (\frac{3\sin\theta - \sin 3\theta}{4})]$$

$$\therefore \iint_{S3} \vec{F}\cdot\hat{n}\,ds = 8\int_0^{2\pi}\int_0^3 (\frac{1}{2} + \frac{\cos 2\theta}{2} - \frac{3\sin\theta}{4} + \frac{\sin 3\theta}{4})\ 2dzd\theta$$

$$= 16\int_0^{2\pi}(\frac{1}{2} + \frac{\cos 2\theta}{2} - \frac{3\sin\theta}{4} + \frac{\sin 3\theta}{4})\ [z]_0^3\,d\theta$$

$$= 48\ [\frac{\theta}{2} + \frac{\sin 2\theta}{4} - \frac{3\cos\theta}{4} - \frac{\cos 3\theta}{12}]_0^{2\pi}$$

$$= 48\ [(\frac{2\pi}{2} + \frac{3}{4} - \frac{1}{12}) - (\frac{3}{4} - \frac{1}{12})]$$

$$= 48\,\pi$$

$$\text{L.H.S} = \iint_S \vec{F} \cdot \hat{n}\, ds = 0 + 36\,\pi + 48\pi$$

$$= 84\,\pi$$

$\therefore$ L.H.S = R.H.S

$$(i.e) \iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

Hence Gauss divergence theorem is verified.