Catalog

5.1 Electromagnetic wave generation and equations

One of the most fundamental equations to all of Electromagnetics is the wave equation, which shows that all waves travel at a single speed - the speed of light. On this page we'll derive it from Ampere's and Faraday's Law.

$$\nabla \times \nabla \times \mathbf{H} = \nabla(\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H}$$ **[Equation 1]**

The left side of Equation 1 is simply the curl of the curl of a vector field. On the right side, I can define the terms for you in the next couple of equations. The first term on the right side of Equation [1] is known as the "gradient of the divergence". However, since we know the divergence of the fields in question will be zero because we are in a source free region. Hence, this term is zero:

$$\nabla(\nabla \cdot \mathbf{H}) \quad \text{(this doesn't matter because it's zero)}$$ **[Equation 2]**

The second term on the right side of Equation [1] is known as the Laplacian. This is basically the sum of second-order partial derivatives, as seen in Equation [3]:

$$\nabla^2 \mathbf{H} = \nabla^2 \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial z^2} \\ \frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2} + \frac{\partial^2 H_y}{\partial z^2} \\ \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\partial^2 H_z}{\partial z^2} \end{bmatrix}$$ **[Equation 3]**

OK, so now we can rewrite Equation [1] as:

$$\nabla \times \nabla \times \mathbf{H} = -\nabla^2 \mathbf{H}$$

$$\nabla \times \nabla \times \mathbf{E} = -\nabla^2 \mathbf{E}$$

[Equation 4]

I've written Equation [4] out as two equations to show that this is true for both the Electric and Magnetic Fields, in source free regions.

If we start now with Farday's Law, and take the curl of both sides, we get:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H})$$

[Equation 5]

We can rewrite the left side of equation [5] (the curl of the curl of E) with the help of Equation [4]. And we can rewrite the right side of Equation [5] by substituting in Ampere's law:

$$\nabla \times \nabla \times \mathbf{E} = -\nabla^2 \mathbf{E} =$$

$$= -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) \quad \text{(substitute in Ampere's Law)}$$

$$= -\mu \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right) \quad \text{(}\mathbf{J}\text{ is zero because source free region)}$$

$$= -\mu\varepsilon \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{E}}{\partial t}\right)$$

[Equation 6]

$$\Rightarrow \boxed{\nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}} \quad \text{[The Vector Wave Equation]}$$

Equation [6] is known as the **Wave Equation** It is actually 3 equations, since we have an x-, y- and z- component for the **E** field.

To break down and understand Equation [6], let's imagine we have an E-field that exists in source-free region. Suppose we only have an E-field that is polarized in the x-direction, which means that Ey=Ez=0 (the y- and z- components of the E-field are zero). Further, let's assume that the field is travelling in the z-direction, and there is no variation in the x- and y-directions (this means the partial derivatives with respect to x- and y- are zero). Then Equation [6] simplifies to:

$$\nabla^2 E_x = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} \quad \text{[Scalar Wave Equation]}$$

[Equation 7]

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

The differential equation in Equation [7] actually has a very nice solution. It turns out any function that can be written as *f(z-ct)* or *f(z+ct)* will satisfy the differential equation [7]. This follows from the "chain rule" in calculus:

$$E_x = f(z-ct)$$

$$\begin{Bmatrix} \dfrac{\partial E_x}{\partial z} = \dfrac{\partial f(z-ct)}{\partial z} = f'(z-ct)\left(\dfrac{\partial (z-ct)}{\partial z}\right) = f'(z-ct) \\ \dfrac{\partial^2 E_x}{\partial z^2} = \dfrac{\partial}{\partial z}\left(\dfrac{\partial E_x}{\partial z}\right) = f''(z-ct)\left(\dfrac{\partial (z-ct)}{\partial z}\right) = f''(z-ct) \end{Bmatrix}$$

[Equation 8]

$$\begin{Bmatrix} \dfrac{\partial E_x}{\partial t} = \dfrac{\partial f(z-ct)}{\partial t} = f'(z-ct)\left(\dfrac{\partial (z-ct)}{\partial t}\right) = -c\, f'(z-ct) \\ \dfrac{\partial^2 E_x}{\partial t^2} = \dfrac{\partial}{\partial t}\left(\dfrac{\partial E_x}{\partial t}\right) = f''(z-ct)\left(\dfrac{\partial (z-ct)}{\partial t}\right) = c^2 f''(z-ct) \end{Bmatrix}$$

$$\Rightarrow \boxed{\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} \quad \text{if } E_x = f(z-ct) \text{ and } c = \frac{1}{\sqrt{\mu\varepsilon}}}$$

Equation [8] represents a profound derivation. First, it says that any function of the form *f(z-ct)* satisfies the wave equation. This means that Maxwell's Equations will allow waves of any shape to propagate through the universe! This allows the world to function: heat from the sun can travel to the earth in any form, and humans can send any type of signal via radio waves they want.

Second, a function of the form *f(z-ct)* represents a wave travelling in the +z direction at a speed of *c*. From Equation [8], we see this satisfies the wave equation for only one speed - and this is exactly the speed of light:

$$c = \frac{1}{\sqrt{\mu \varepsilon}} \quad \text{(speed of light)}$$

In Free Space (Vacuum):

$$\mu_0 = 4\pi \cdot 10^{-7} \quad \text{[H/m]}$$

[Equation 9]

$$\varepsilon_0 = 8.854 \cdot 10^{-12} \quad \text{[F/m]}$$

$$c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 299{,}795{,}638 \text{ [m/s]}$$

Maxwell's Equations has just told us something amazing. Electric and Magnetic Fields in "Free Space" - a region without charges or currents like air - can travel with any shape, and will propagate at a single speed - *c*. This is an amazing discovery, and one of the nicest properties that the universe could have given us. It is pretty cool.

5.2 Uniform Plane Wave in Lossless Dielectric

For a perfect or lossless dielectric the properties are given as, $\sigma = 0$, $\epsilon = \epsilon_o \epsilon_r$ and $\mu = \mu_o \mu_r$. In both free space medium and lossless dielectric medium $\sigma = 0$, so the analysis of the wave propagation is much similar in both cases. But as the permeability and permittivity values are different then expression in both cases gets varied.

The Velocity of propagation, $v = (1/\sqrt{(\mu \epsilon)})$

$$= (1/\sqrt{(\mu_o \mu_r \epsilon_o \epsilon_r)}) = 1/(\sqrt{(\mu_o \epsilon_o)} \sqrt{(\mu_r \epsilon_r)}) = 1/(\sqrt{(\mu_o \epsilon_o)}/ \sqrt{(\mu_r \epsilon_r)})$$

Therefore $v = c/ \sqrt{(\mu_r \epsilon_r)}$ m/s

The propagation constant,

$$\gamma = \sqrt{[jw\mu (\sigma + jw \epsilon)]} \ m^{-1}$$

By substituting $\sigma = 0$, $\epsilon = \epsilon_o \epsilon_r$ and $\mu = \mu_o \mu_r$ in the above equation for a perfect or lossless dielectric, we get

$$\gamma = +/- jw \sqrt{(\mu\epsilon)} \ m^{-1}$$

And also attenuation constant, $\alpha = 0$

The phase constant,

$$\beta = w \sqrt{(\mu \epsilon)} \text{ rad/m}$$

Intrinsic Impedance,

$$\eta = \sqrt{[(jw\mu) / (\sigma + jw \epsilon)]} \text{ ohms}$$

$$= \sqrt{(\mu_o/ \epsilon_o)} \sqrt{(\mu_r/ \epsilon_r)}$$

$$= \eta_o \sqrt{(\mu_r/ \epsilon_r)}$$

$$\eta = 377\sqrt{(\mu_r/ \epsilon_r)} \text{ ohms}$$

5.3 Uniform Plane Wave in Lossy Dielectric

A lossy dielectric is a poor insulator, in which free charges conducts to some extent. It is an imperfect conductor and imperfect dielectric (which is a partial conducting medium) with $\sigma \neq 0$.

The propagation constant is given as

$$\gamma = \sqrt{[jw\mu(\sigma + jw\epsilon)]}$$

Rearranging the terms, we get

$$\gamma = \sqrt{[jw\epsilon(1 + (\sigma / jw\epsilon))\, jw\mu]}$$

$$\text{Therefore, } \gamma = \alpha + j\beta = jw\sqrt{\mu\epsilon}\sqrt{(1 - j(\sigma / w\epsilon))}$$

The above equation gives the propagation constant for lossy dielectric medium which is different from lossless dielectric medium due to the presence of radical factor. The attenuation constant α and phase constant are calculated by substituting the values of w, μ, є, and σ in the above equation.

The attenuation constant α indicates the certain loss of the wave signal in the medium and hence this type of medium is called as lossy dielectric.

And also due to $\sigma \neq 0$, the intrinsic impedance becomes a complex quantity and is given as

$$\eta = \sqrt{[(jw\mu) / (\sigma + jw\epsilon)]}$$

$$\eta = |\eta| \angle \theta_n \text{ Ohms.}$$

Because of the complex quantity, η is represented in polar form as shown in the above equation where θ_n is the phase angle difference between electric and magnetic fields. Thus, in lossy dielectric medium there exist a phase difference between the electric and magnetic fields.

The intrinsic impedance can be expressed as

$$\eta = \sqrt{[(jw\mu) / (\sigma + jw\epsilon)]}$$

$$= \sqrt{[(jw\mu) / jw\epsilon (1 + (\sigma / jw\epsilon)]}$$

$$\eta = (\sqrt{(\mu/\epsilon)}) (1 / \sqrt{(1 - j (\sigma / w\epsilon)}) \text{ ohms}$$

And the angle Θ_n is given as

$$\Theta_n = \frac{1}{2} [(\pi/2) - \tan^{-1} (w\epsilon / \sigma)]$$

This angle depends on the frequency of the signal as well as properties of the lossy dielectric medium. Then, w becomes very small for a low frequency signal. Thus, the phase angle is given as

$$\Theta_n = (\pi/4)$$

For very high frequency signal, w becomes very large then,

$$\Theta_n = 0$$

So the range of Θn of a lossy dielectric for complete frequency range is 0 Θ_n (π/4).

5.4 EM Wave Polarization

It is important to know that the direction of the electric field vector changes with time for a uniform plane wave which decides the polarization of the wave. This is because some applications can only receive or transmit one type of polarized EM waves and the best example is different antennas in RF applications is designed for one type of polarized wave.

In a plane EM wave, the electric field oscillates in the x-z plane while the magnetic field oscillates in the y-z plane. Hence, it corresponds to a polarized wave. The plane in which the electric field oscillates is defined as the plane of polarization.

The polarization is nothing but a way in which an electric field varies with magnitude and direction. The polarization can be linear, or circular, or elliptical polarization. Let us consider that $\bar{E}x$ and $\bar{E}y$ are the electric fields directed along the x- axis and y-axis respectively, and also $\bar{E}$ be the resultant of $\bar{E}x$ and $\bar{E}y$.

Linear Polarization

If an electric field of an EM wave is parallel to the x- axis, then the wave is said to be linearly x- polarized. A straight wire antenna parallel to x-axis could generate this type of polarized wave. In a similar way, y-polarized waves are generated and defined along the y-axis.

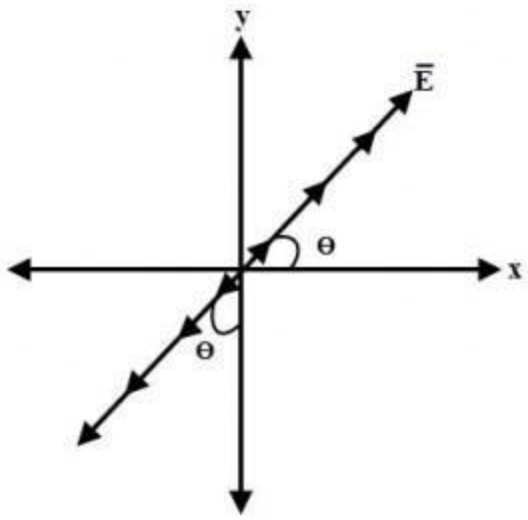
Suppose $\bar{E}$ has both E x and $\bar{E}y$ components which are in phase having different magnitudes. The magnitudes of $\bar{E}x$ and E y reach their maximum and minimum values simultaneously as $\bar{E}x$ and $\bar{E}y$ are in phase. So at any point on the positive z axis, the ratio of magnitudes both the components is constant.

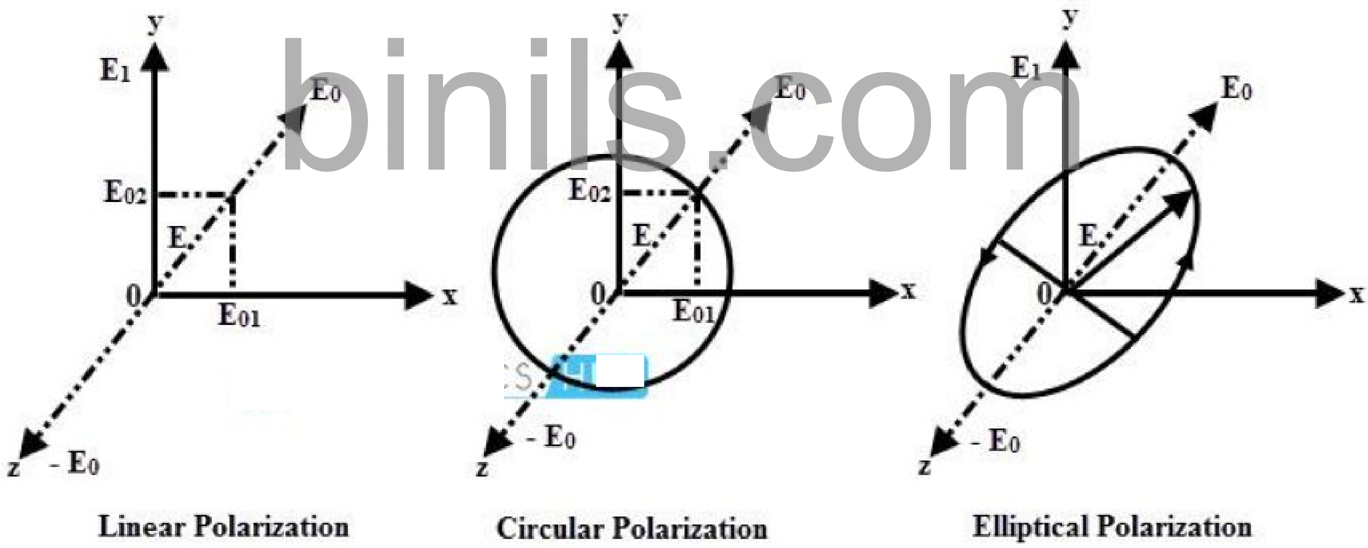
Therefore, the direction of the resultant electric field $\bar{E}$ depends on the relative magnitudes of E x and $\bar{E}y$. Thus the angle made by the $\bar{E}$ with x-axis is given by

$\Theta = \tan^{-1} Ey / Ex$

Where Ey and Ex are the magnitudes of the $\bar{E}y$ and $\bar{E}x$ respectively.

With respect to time, this angle is constant and hence the wave is said to be linearly polarized. Therefore, the polarization of the uniform plane propagating in a z- direction is linear when E̅x and E̅y are in phase either with unequal or equal magnitudes.





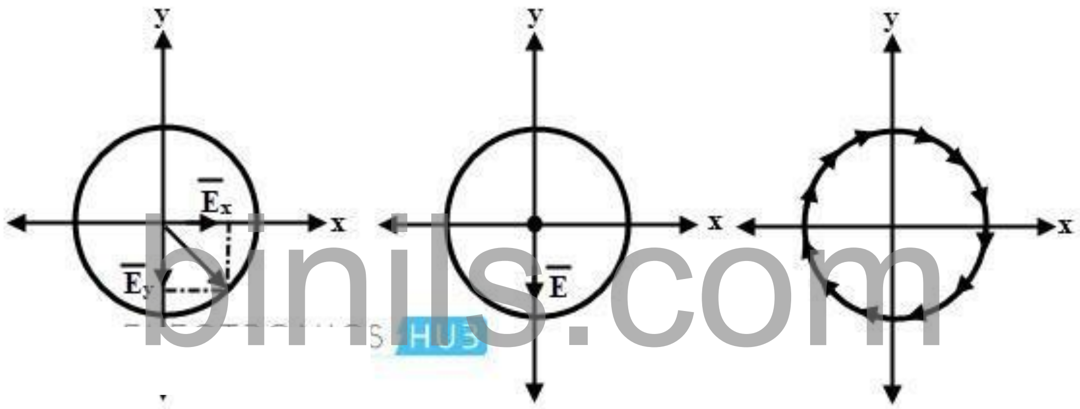
Circular Polarization

If the two planes E̅y and E̅x (which are orthogonally polarized) are of equal in amplitude but has 90 degrees phase difference between them, then the resulting wave is circularly polarized. In such case at any instant of time, if the amplitude of the any one component is maximum, then other component amplitude becomes zero due to the phase difference.

It is also described as if the amplitude of any one component gradually increases, then amplitude of other component gradually decreases and vice-versa. Thus the magnitude of the resultant vector E̅ is constant at any instant of time, but the direction is the function of angle between the relative amplitudes of E̅y and E̅x at any instant.

If the resultant electric field E̅ is projected on a plane perpendicular to the direction of propagation, then the locus of all such points is a circle with the center on the z- axis as shown in figure.

During the one wavelength span, the field vector E̅ rotate by 360 degrees or in other words, completes one cycle of rotation and hence such waves are said to be circularly polarized.



Circular polarization with 90 degrees phase shift between E̅x and E̅y which are having equal amplitudes

Circular polarization is generated as either right hand circular polarization (RHCP) or left hand circular polarization (LHCP). RHCP wave describes a wave with the electric field vector rotating clockwise when looking in the direction of propagation.

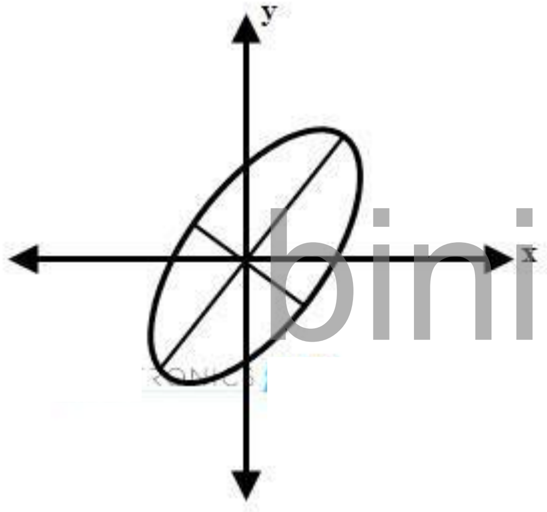
For a LHCP wave the vector field rotates in anticlockwise direction. Therefore, the polarization of a uniform plane wave is circular if the amplitudes of two components of electric field vector are equal and having a phase difference of 90 degrees between them.

Elliptical Polarization

In most of the cases, the components of the wave have different amplitudes and are at different phase angles other than 90 degrees. This results the elliptical polarization. Consider that electric field has both components $\bar{E}x$ and $\bar{E}y$ which are not equal in amplitude and are not in phase.

As the wave propagates, the maximum and minimum amplitude values of $\bar{E}x$ and $\bar{E}y$ not simultaneous and are occurring at different instants of the time. Thus the direction of resultant field vector varies with time.

If the locus of the end points of the field vector $\bar{E}$ traced then one can observe that the $\bar{E}$ moves elliptically on the plane. Hence such wave is called as elliptically polarized.



5.5 Poynting's Theorem

This theorem states that the cross product of electric field vector, E and magnetic field vector, H at any point is a measure of the rate of flow of electromagnetic energy per unit area at that point, that is

P = E x H

Here P → Poynting vector and it is named after its discoverer, J.H. Poynting. The direction of P is perpendicular to E and H and in the direction of vector E x H

With Maxwell's Equations, we now have the tools necessary to derive Poynting's Theorem, which will allow us to perform many useful calculations involving the direction of power flow in electromagnetic fields. We will begin with Faraday's Law, and we will take the dot product of H with both sides:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{H} \bullet (\nabla \times \mathbf{E}) = -\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t}\right)$$

Next, we will start with Ampere's Law and will take the dot product of **E** with both sides:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

$$\mathbf{E} \bullet (\nabla \times \mathbf{H}) = \mathbf{E} \bullet \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right)$$

$$\mathbf{H} \bullet (\nabla \times \mathbf{E}) - \mathbf{E} \bullet (\nabla \times \mathbf{H}) = -\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t}\right) - \mathbf{E} \bullet \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right)$$

We can now apply the following mathematical identity to the left side

$$\nabla \bullet (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \bullet (\nabla \times \mathbf{A}) - \mathbf{A} \bullet (\nabla \times \mathbf{B})$$

$$\nabla \bullet (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t}\right) - \mathbf{E} \bullet \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}\right)$$

Distributing the E across the right side gives

$$\nabla \bullet (\mathbf{E} \times \mathbf{H}) = -\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t}\right) - \mathbf{E} \bullet \frac{\partial \mathbf{D}}{\partial t} - \mathbf{E} \bullet \mathbf{J}$$

Now let's concentrate on the first time on the right side. Applying a constitutive equation,

$$\mathbf{H} \bullet \left(\frac{\partial \mathbf{B}}{\partial t} \right) = \mathbf{H} \bullet \mu \left(\frac{\partial \mathbf{H}}{\partial t} \right)$$

$$\frac{\partial \mathbf{H}^2}{\partial t} = \frac{\partial (\mathbf{H} \bullet \mathbf{H})}{\partial t} = 2\mathbf{H} \bullet \frac{\partial \mathbf{H}}{\partial t}$$

$$\mathbf{H} \bullet \frac{\partial \mathbf{H}}{\partial t} = \frac{1}{2} \frac{\partial (\mathbf{H} \bullet \mathbf{H})}{\partial t} = \frac{1}{2} \frac{\partial (H^2)}{\partial t}$$

Similarly,

$$\mathbf{E} \bullet \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2} \frac{\partial (\mathbf{E} \bullet \mathbf{E})}{\partial t} = \frac{1}{2} \frac{\partial (E^2)}{\partial t}$$

$$\nabla \bullet (\mathbf{E} \times \mathbf{H}) = -\frac{1}{2} \frac{\partial}{\partial t} \left(\mu H^2 + \epsilon E^2 \right) - \mathbf{E} \bullet \mathbf{J}$$

5.6 Skin Depth

The skin depth is that distance below the surface of a conductor where the current density has diminished to 1/*e* of its value at the surface. The thickness of the conductor is assumed to be several (perhaps at least three) times the skin depth. Imagine the conductor replaced by a cylindrical shell of the same surface shape but of thickness equal to the skin depth, with uniform current density equal to that which exists at the surface of the actual conductor. Then the total current in the shell and its resistance are equal to the corresponding values in the actual conductor.

The skin depth and the resistance per square (of any size), in meter-kilogram-second (rationalized) units, are

$\delta=(\lambda/\pi\sigma\mu c)1/2Rsq=1/\delta\sigma$

where,

δ = skin depth in meters,

R_{sq}= resistance per square in ohms,

c = velocity of light *in vacuo*

= 2.998×10^{8}meter/second,

$\mu = 4\pi\, x10^{-7} \times_r$henry/meter,

$1/\sigma = 1.724 \times 10^{-8} \rho/\rho_c$ohm-meter.