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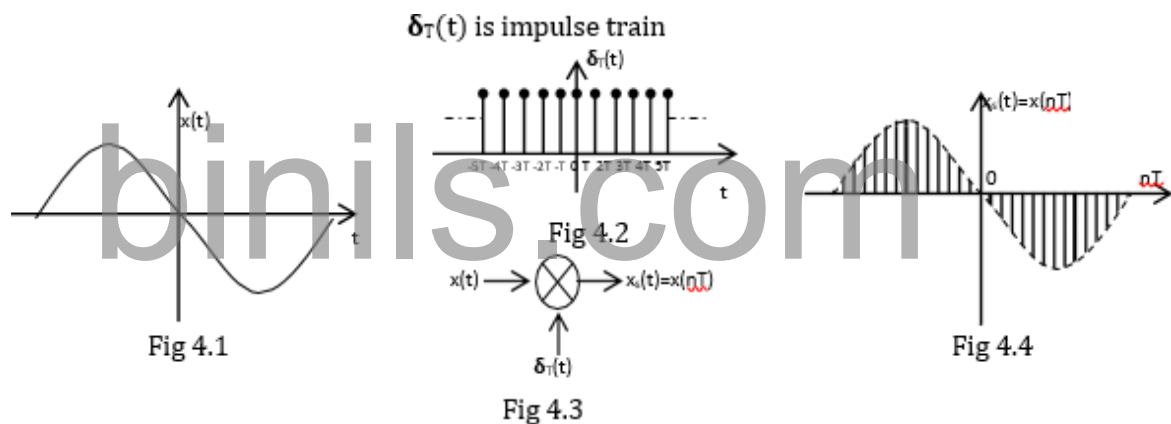
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4.1 SAMPLING THEOREM

It is one of useful theorem that applies to digital communication systems.

Sampling theorem states that “A band limited signal $x(t)$ with $X(\omega) = 0$ for $|\omega| \geq \omega_m$ can be represented into and uniquely determined from its samples $x(nT)$ if the sampling frequency $f_s \geq 2f_m$, where f_m is the frequency component present in it”.

(i.e) for signal recovery, the sampling frequency must be at least twice the highest frequency present in the signal.



Analog signal $x(t)$ is input signal as shown in Fig 4.1, $\delta_T(t)$ is the train of impulse shown in Fig 4.2. Sampled signal $x_s(t)$ is the product of signal $x(t)$ and impulse train $\delta_T(t)$ as shown in Fig 4.2

$$\therefore x_s(t) = x(t) \cdot \delta_T(t)$$

$$\text{we know } \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

$$\therefore x_s(t) = x(t) \cdot \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

Applying Fourier transform on both sides

$$X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F[x(t)e^{jn\omega_s t}]$$

$$X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

$$\text{where } \omega_s = 2\pi f_s = \frac{2\pi}{T}$$

$$\therefore X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X\left(\omega - \frac{2\pi n}{T}\right)$$

(or)

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \quad \text{where } f_s = \frac{1}{T}$$

Where $X(\omega)$ or $X(f)$ is Spectrum of input signal.

Where $X_s(\omega)$ or $X_s(f)$ is Spectrum of sampled signal.

Spectrum of continuous time signal $x(t)$ with maximum frequency ω_m is shown in

Fig 4.5

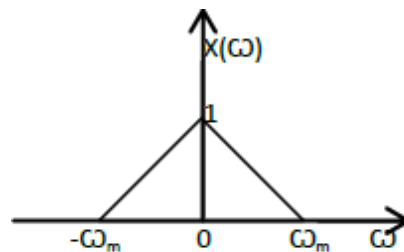


Fig 4.5 Spectrum of $x(t)$

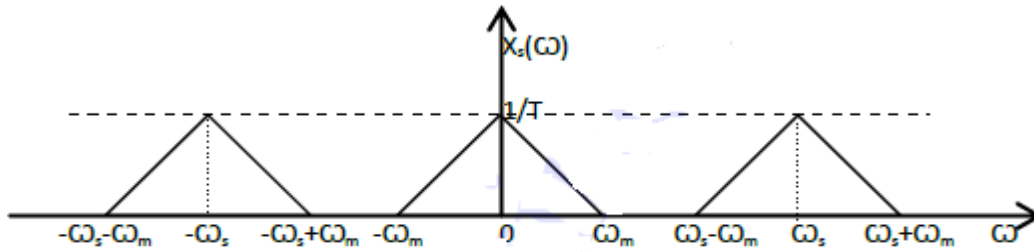


Fig 4.6 Spectrum of $x_s(t)$ when $\omega_s - \omega_m > \omega_m$

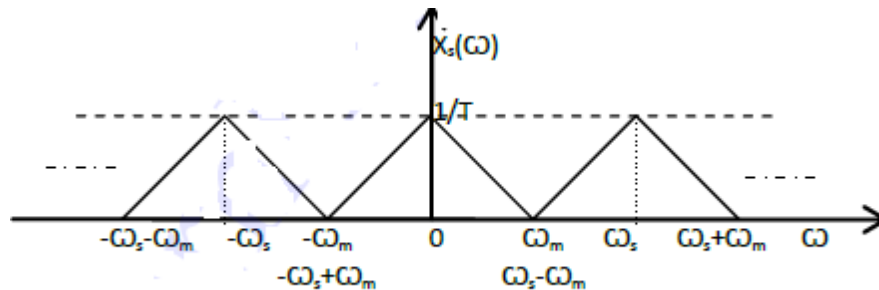


Fig 4.7 Spectrum of $x_s(t)$ when $\omega_s - \omega_m = \omega_m$

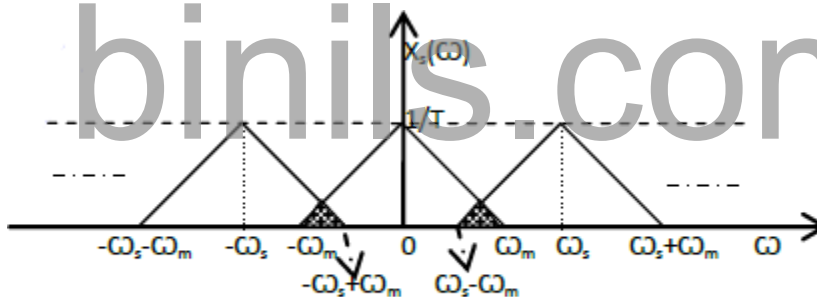


Fig 4.8 Spectrum of $x_s(t)$ when $\omega_s - \omega_m < \omega_m$

For $\omega_s > 2\omega_m$

The spectral replicates have a larger separation between them known as guard band which makes process of filtering much easier and effective. Even a non-ideal filter which does not have a sharp cut off can also be used.

For $\omega_s = 2\omega_m$

There is no separation between the spectral replicates so no guard band exists and $X(\omega)$ can be obtained from $X_s(\omega)$ by using only an ideal low pass filter (LPF) with sharp cutoff.

For $\omega_s < 2\omega_m$

The low frequency component in $X_s(\omega)$ overlap on high frequency components of $X(\omega)$ so that there is presence of distortion and $X(\omega)$ cannot be recovered from $X_s(\omega)$ by using any filter. This distortion is called aliasing.

So we can conclude that the frequency spectrum of $X_s(\omega)$ is not overlapped for $\omega_s - \omega_m \geq \omega_m$, therefore the Original signal can be recovered from the sampled signal.

For $\omega_s - \omega_m < \omega_m$, the frequency spectrum will overlap and hence the original signal cannot be recovered from the sampled signal.

∴ For signal recovery,

$$\omega_s - \omega_m \geq \omega_m \text{ (i. e) } \omega_s \geq 2\omega_m$$

(or)

$$f_s \geq 2f_m$$

i.e., Aliasing can be avoided if $f_s \geq 2f_m$

Aliasing effect (or) fold over effect

It is defined as the phenomenon in which a high frequency component in the frequency spectrum of signal takes identity of a lower frequency component in the spectrum of the sampled signal.

When $f_s < 2f_m$, (i.e) when signal is under sampled, the individual terms in equation

$$X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} x(\omega - n\omega_s)$$

get overlap. This process of spectral overlap is called frequency folding effect.

Occurrence of aliasing

Aliasing Occurs if

- i) The signal is not band-Limited to a finite range.
- ii) The sampling rate is too low.

To Avoid Aliasing

- i) $x(t)$ should be strictly band limited.

It can be ensured by using anti-aliasing filter before the sampler.

- ii) f_s should be greater than $2f_m$.

Nyquist Rate

It is the theoretical minimum sampling rate at which a signal can be sampled and still be reconstructed from its samples without any distortion

$$\text{Nyquist rate } f_N = 2f_m \text{ . Hz}$$

Data Reconstruction or Interpolation

The process of obtaining analog signal $x(t)$ from the sampled signal $x_s(t)$ is called data reconstruction or interpolation.

$$\begin{aligned} \text{we know } x_s(t) &= x(t) \cdot \delta_T(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ \delta(t - nT) &\text{ exist only at } t = nT \\ \therefore x_s(t) &= x(nt) \sum_{n=-\infty}^{\infty} \delta(t - nT) \end{aligned}$$

The reconstruction filter, which is assumed to be linear and time invariant, has unit impulse response $h(t)$.

The reconstruction filter, output $y(t)$ is given by convolution of $x_s(t)$ and $h(t)$.

$$\begin{aligned} \therefore y(t) &= x_s(t) * h(t) = \int_{-\infty}^{\infty} x(nT) \sum_{n=-\infty}^{\infty} \delta(\tau - nT) \cdot h(t - \tau) d\tau \\ &= \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \delta(\tau - nT) h(t - \tau) d\tau \end{aligned}$$

$\delta(\tau - nT)$ exist only at $\tau = nT$

$$\delta(\tau - nT) = 1 \text{ at } \tau = nT$$

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)h(t - nT)$$

Ideal Reconstruction filter

The sampled signal $x_s(t)$ is passed through an ideal LPF (Fig 4.9) with bandwidth greater than f_m and a pass band amplitude response of T , then the filter output is $x(t)$.

Transfer function of ideal reconstruction filter is

$$H(f) = T ; |f| < 0.5f_s$$

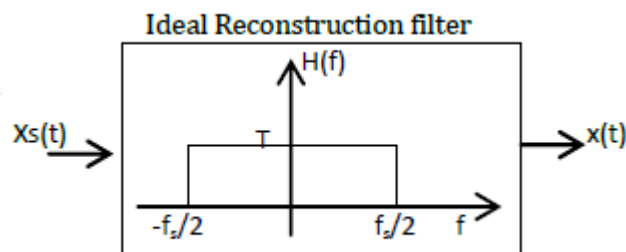


Fig 4.9

The impulse response of ideal reconstruction filter is

$$h(t) = \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} T e^{j\omega t} df$$

$$\begin{aligned}
 &= \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} T e^{j2\pi ft} df = T \left[\frac{e^{j2\pi ft}}{j2\pi t} \right]_{-\frac{f_s}{2}}^{\frac{f_s}{2}} = \frac{T}{j2\pi t} \left[e^{j2\pi \frac{f_s}{2} t} - e^{-j2\pi \frac{f_s}{2} t} \right] \\
 &= \frac{1}{f_s \pi t} \left[\frac{e^{j2\pi \frac{f_s}{2} t} - e^{-j2\pi \frac{f_s}{2} t}}{2j} \right] = \frac{1}{\pi f_s t} \sin \pi f_s t = \text{sinc } \pi f_s t \\
 &\therefore h(t - nT) = \text{sinc } \pi f_s (t - nT) \dots \dots \dots (1) \\
 &y(t) = \sum_{n=-\infty}^{\infty} x(nT) h(t - nT)
 \end{aligned}$$

Substitute equation 1 in above equation

$$\begin{aligned}
 \therefore y(t) &= \sum_{n=-\infty}^{\infty} x(nT) \text{sinc } \pi f_s (t - nT) = \sum_{n=-\infty}^{\infty} x(nT) \text{sinc } \pi \left(\frac{t}{T} - n \right) \\
 &\quad \left[\because f_s = \frac{1}{T} \right]
 \end{aligned}$$

4.2 DISCRETE TIME FOURIER TRANSFORM AND ITS PROPERTIES

The DTFT is a transformation that maps Discrete-time (DT) signal $x[n]$ into a complex valued function of the real variable namely:

$$F[x(n)] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

INVERSE DISCRETE FOURIER TRANSFORM

IDTFT is given by

$$x(n) = F^{-1}[X(e^{j\omega})] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \text{ for } n = -\infty \text{ to } \infty$$

Example 1: Find the DTFT of $x(n) = a^n u(n)$.

Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^{-n} = \frac{1}{1 - ae^{-j\omega}}.$$

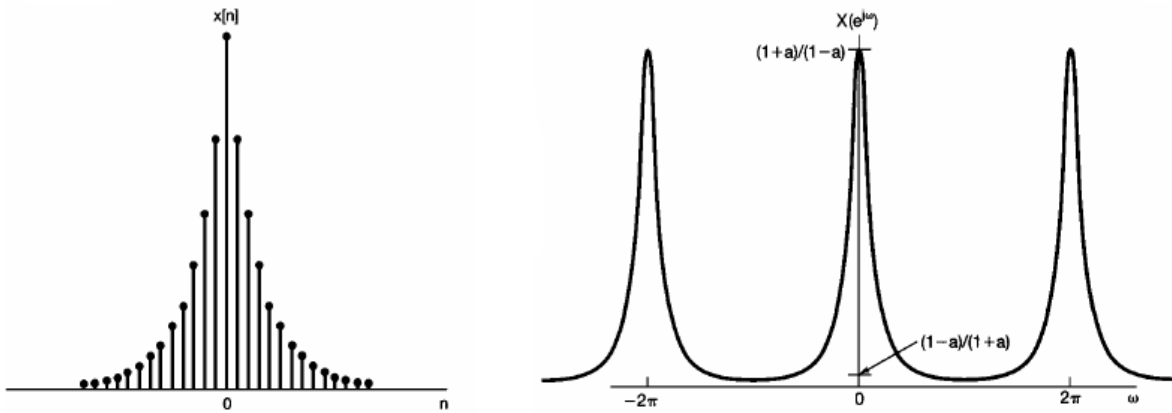
Example 2: Find the DTFT of $x(n) = a^{|n|}$, $|a| < 1$.

Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

Let $m = -n$ in the first summation, we obtain

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} a^{|n|} u[n] e^{-j\omega n} = \sum_{m=1}^{\infty} a^m e^{j\omega m} + \sum_{n=0}^{\infty} a^n e^{-j\omega n} \\ &= \frac{ae^{j\omega}}{1 - ae^{j\omega}} + \frac{1}{1 - ae^{-j\omega}} = \frac{1 - a^2}{1 - 2a \cos \omega + a^2} \end{aligned}$$



PROPERTIES OF DTFT:

Linearity:

$$\mathcal{F}_{DT}\{x_1(n) + x_2(n)\} = \mathcal{F}_{DT}\{x_1(n)\} + \mathcal{F}_{DT}\{x_2(n)\} = X_1(e^{j\omega}) + X_2(e^{j\omega})$$

Time Shifting:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{x(n - n_0)\} = e^{-j\omega n_0} X(e^{j\omega})$$

Proof:

$$\sum_{n=-\infty}^{\infty} x(n - n_0) e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x(m) e^{-j\omega(m+n_0)} = e^{-j\omega n_0} \sum_{m=-\infty}^{\infty} x(m) e^{-j\omega m}$$

Time Reversal:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{x(-n)\} = X(e^{-j\omega})$$

Convolution:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{x(n) * y(n)\} = X(e^{j\omega}) Y(e^{j\omega})$$

Proof:

$$\begin{aligned} G(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(m)y(n-m)e^{-jm\omega} = \sum_{n=-\infty}^{\infty} x(m) \sum_{m=-\infty}^{\infty} y(n-m)e^{-jm\omega} \\ &= \sum_{m=-\infty}^{\infty} x(m) \sum_{r=-\infty}^{\infty} y(r)e^{-j(m+r)\omega} = \sum_{m=-\infty}^{\infty} x(m)e^{-jm\omega} \sum_{r=-\infty}^{\infty} y(r)e^{-jr\omega} \end{aligned}$$

Frequency Shifting:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{e^{j\omega_0 n} x(n)\} = X(e^{j(\omega-\omega_0)}),$$

Time Multiplication:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{nx(n)\} = -z \frac{dX(z)}{dz} \Big|_{z=e^{j\omega}}$$

Parseval's Theorem:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

Proof:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_{n=-\infty}^{\infty} x(n) e^{-jn\omega} \right] Y^*(e^{j\omega}) d\omega = \sum_{n=-\infty}^{\infty} x(n) \frac{1}{2\pi} \int_{-\pi}^{\pi} Y^*(e^{j\omega}) e^{-j\omega n} d\omega$$

For the case $x(n) = y(n)$, then:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

Multiplication of Sequences:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$\begin{aligned} \mathcal{F}_{DT}\{x(n)y(n)\} &= \sum_{n=-\infty}^{\infty} x(n)y(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\lambda}) e^{j\lambda n} d\lambda \right] e^{-j\omega n} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\lambda}) d\lambda \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega-\lambda)n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\lambda}) X(e^{j(\omega-\lambda)}) d\lambda \\ &= \frac{1}{2\pi} Y(e^{j\omega}) * X(e^{j\omega}) \end{aligned}$$

Differentiation in the Frequency Domain:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\frac{dX(e^{j\omega})}{d\omega} = \frac{d}{d\omega} \left[\sum_{n=-\infty}^{\infty} x(n) e^{-jn\omega} \right] = -j \sum_{n=-\infty}^{\infty} nx(n) e^{-jn\omega} = -j \mathcal{F}_{DT}\{nx(n)\}$$