Catalog

4.1 SAMPLING THEOREM

It is one of useful theorem that applies to digital communication systems.

Sampling theorem states that "A band limited signal $\boldsymbol{x(t)}$ with $\boldsymbol{X}(\varpi) = \boldsymbol{0}$ for $|m| \geq \varpi\boldsymbol{m}$ can be represented into and uniquely determined from its samples $\boldsymbol{x(nT)}$ if the sampling frequency $\boldsymbol{fs} \geq \boldsymbol{2fm}$, where $\boldsymbol{fm}$ is the frequency component present in it".

(i.e) for signal recovery, the sampling frequency must be at least twice the highest frequency present in the signal.

δT(t) is impulse train

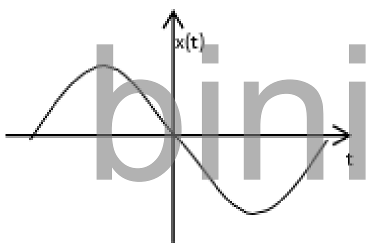


Fig 4.1

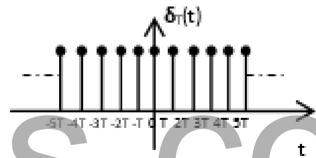


Fig 4.2

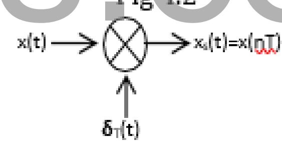


Fig 4.3

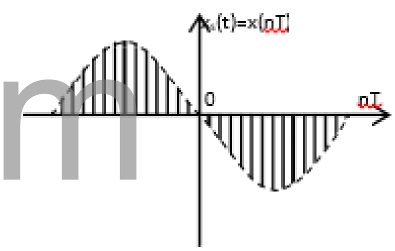


Fig 4.4

Analog signal $x(t)$ is input signal as shown in Fig 4.1, $\delta_T(t)$ is the train of impulse shown in Fig 4.2. Sampled signal $x_s(t)$ is the product of signal $x(t)$ and impulse train $\delta_T(t)$ as shown in Fig 4.2

$$\therefore x_s(t) = x(t).\delta_T(t)$$

$$we\ know\ \delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

$$\therefore x_s(t) = x(t).\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

Applying Fourier transform on both sides

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F[x(t)e^{jn\omega_s t}]$$

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

$$where\ \omega_s = 2\pi f_s = \frac{2\pi}{T}$$

$$\therefore X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - \frac{2\pi n}{T})$$

$$(or)$$

$$X_s(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \qquad where\ f_s = \frac{1}{T}$$

Where $X(\omega)$ or $X(f)$ is Spectrum of input signal.

Where $X_s(\omega)$ or $X_s(f)$ is Specturm of sampled signal.

Spectrum of continuous time signal x(t) with maximum frequency ωm is shown in Fig 4.5

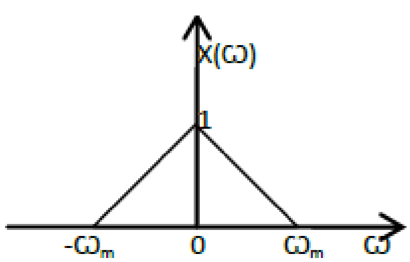


Fig 4.5 Spectrum of x(t)

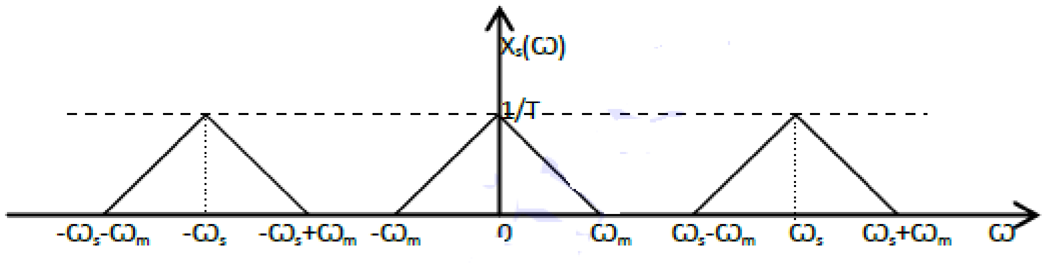


Fig 4.6 Spectrum of $xs\ t$ when $\boldsymbol{\omega s} - \boldsymbol{\omega m} > \boldsymbol{\omega m}$

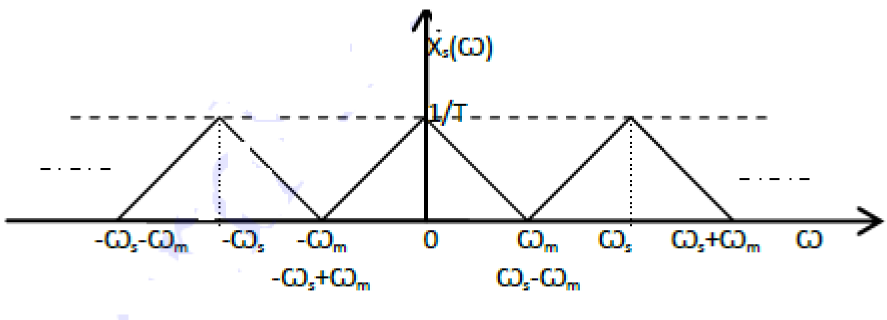


Fig 4.7 Spectrum of $xs\ t$ when $\boldsymbol{\omega s} - \boldsymbol{\omega m} = \boldsymbol{\omega m}$

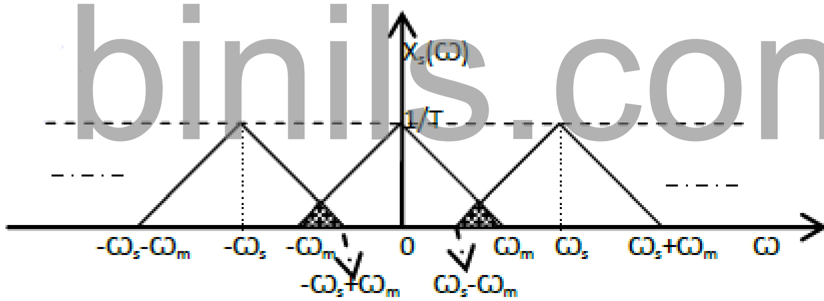


Fig 4.8 Spectrum of $x_s(t)$ when $\boldsymbol{\omega_s} - \boldsymbol{\omega_m} < \boldsymbol{\omega_m}$

For $\boldsymbol{\omega s} > 2\boldsymbol{\omega m}$

The spectral replicates have a larger separation between them known as guard band which makes process of filtering much easier and effective. Even a non-ideal filter which does not have a sharp cut off can also be used.

For $\boldsymbol{\omega s} = \boldsymbol{2\omega m}$

There is no separation between the spectral replicates so no guard band exists and $X(\omega)$ can be obtained from $Xs(\omega)$ by using only an ideal low pass filter (LPF) with sharp cutoff.

For $\omega s < 2\omega m$

The low frequency component in $Xs\ \omega$ overlap on high frequency components of $X\ \omega$ so that there is presence of distortion and $X\ \omega$ cannot be recovered from $Xs\ \omega$ by using any filter. This distortion is called aliasing.

So we can conclude that the frequency spectrum of $Xs(\omega)$ is not overlapped for $\omega s - \omega m \geq \omega m$, therefore the Original signal can be recovered from the sampled signal.

For $\omega s - \omega m < \omega m$, the frequency spectrum will overlap and hence the original signal cannot be recovered from the sampled signal.

∴ For signal recovery,

$$\omega_s - \omega_m \geq \omega_m (i.e)\ \ \omega_s \geq 2\omega_m$$
$$(or)$$
$$f_s \geq 2f_m$$

i.e., Aliasing can be avoided if $fs \geq 2fm$

Aliasing effect (or) fold over effect

It is defined as the phenomenon in which a high frequency component in the frequency spectrum of signal takes identity of a lower frequency component in the spectrum of the sampled signal.

When $fs < 2fm$, (i.e) when signal is under sampled, the individual terms in equation

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x(\omega - n\omega_s)$$

get overlap. This process of spectral overlap is called frequency folding effect.

Occurrence of aliasing

Aliasing Occurs if

i) The signal is not band-Limited to a finite range.

ii) The sampling rate is too low.

To Avoid Aliasing

i) $x(t)$ should be strictly band limited.

It can be ensured by using anti-aliasing filter before the sampler.

ii) f_S should be greater than $2f_m$.

Nyquist Rate

It is the theoretical minimum sampling rate at which a signal can be sampled and still be reconstructed from its samples without any distortion

$$Nyquist\ rate\ fN = 2fm\ .\ Hz$$

Data Reconstruction or Interpolation

The process of obtaining analog signal $x(t)$ from the sampled signal $xs(t)$ is called data reconstruction or interpolation.

$$we\ know\ x_s(t) = x(t).\delta_T(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$\delta(t-nT)\ exist\ \ only\ at\ t = nT$$

$$\therefore x_s(t) = x(nt)\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

The reconstruction filter, which is assumed to be linear and time invariant, has unit impulse response h(t).

The reconstruction filter, output $y(t)$ is given by convolution of $xs(t)$ and h(t).

$$\therefore y(t) = x_s(t) * h(t) = \int_{-\infty}^{\infty} x(nT) \sum_{n=-\infty}^{\infty} \delta(\tau - nT).\, h(t-\tau)d\tau$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \int_{-\infty}^{\infty} \delta(\tau - nT)\; h(t-\tau)d\tau$$

$$\delta(\tau - nT)\ exist\ only\ at\ \tau = nT$$

$$\delta(\tau - nT) = 1\ at\ \tau = nT$$

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)h(t - nT)$$

Ideal Reconstruction filter

The sampled signal $xs(t)$ is passed through an ideal LPF (Fig 4.9) with bandwidth greater than fm and a pass band amplitude response of T, then the filter output is $x(t)$.

Transfer function of ideal reconstruction filter is

$$H(f) = T\ ;\ \ |f| < 0.5f_s$$

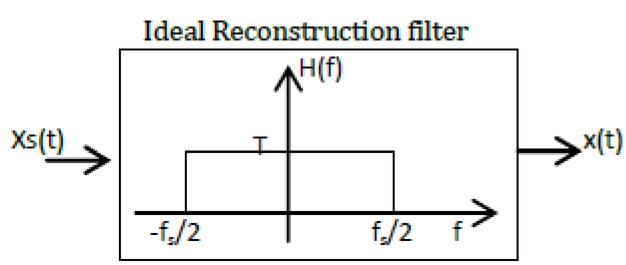


Fig 4.9

The impulse response of ideal reconstruction filter is

$$h(t) = \int_{\frac{-f_s}{2}}^{\frac{f_s}{2}} Te^{j\omega t}\, df$$

$$= \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} T\, e^{j2\pi ft}\, df = T\left[\frac{e^{j2\pi ft}}{j2\pi t}\right]_{-\frac{f_s}{2}}^{\frac{f_s}{2}} = \frac{T}{j2\pi t}\left[e^{j2\pi\frac{f_s}{2}t} - e^{-j2\pi\frac{f_s}{2}t}\right]$$

$$= \frac{1}{f_s\pi t}\left[\frac{e^{j2\pi\frac{f_s}{2}t} - e^{-j2\pi\frac{f_s}{2}t}}{2j}\right] = \frac{1}{\pi f_s t}\sin\pi f_s t = \sin c\,\pi f_s t$$

$$\therefore h(t-nT) = \sin c\,\pi f_s(t-nT) \quad \ldots\ldots\ldots\langle 1\rangle$$

$$y(t) = \sum_{n=-\infty}^{\infty} x(nT)h(t-nT)$$

Substitute equation 1 in above equation

$$\therefore y(t) = \sum_{n=-\infty}^{\infty} x(nT)\ \sin c\,\pi f_s(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\ \sin c\,\pi\left(\frac{t}{T} - n\right)$$

$$\left[\because f_s = \frac{1}{T}\right]$$

4.2 DISCRETE TIME FOURIER TRANSFORM AND ITS PROPERTIES

The DTFT is a transformation that maps Discrete-time (DT) signal $x[n]$ into a complex valued function of the real variable namely:

$$F[x(n)] = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

INVERSE DISCRETE FOURIER TRANSFORM

IDTFT is given by

$$x(n) = F^{-1}[X(e^{j\omega})] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega \text{ , for } n = -\infty \text{ to } \infty$$

Example 1: Find the DTFT of $x(n) = a^n u(n)$.

Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u[n]e^{-j\omega n} = \sum_{n=0}^{\infty}\left(ae^{-j\omega}\right)^{-n} = \frac{1}{1-ae^{-j\omega}}.$$

Example 2: Find the DTFT of $x(n) = a^{|n|}$, $|a| < 1$.

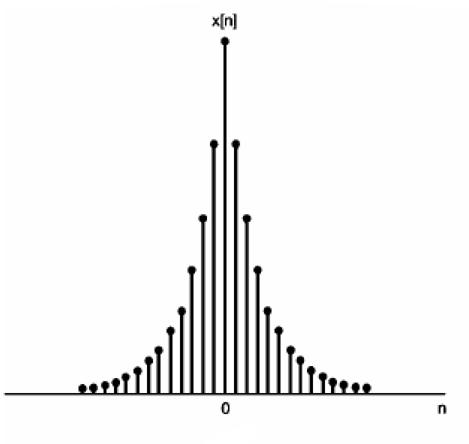
Solution:

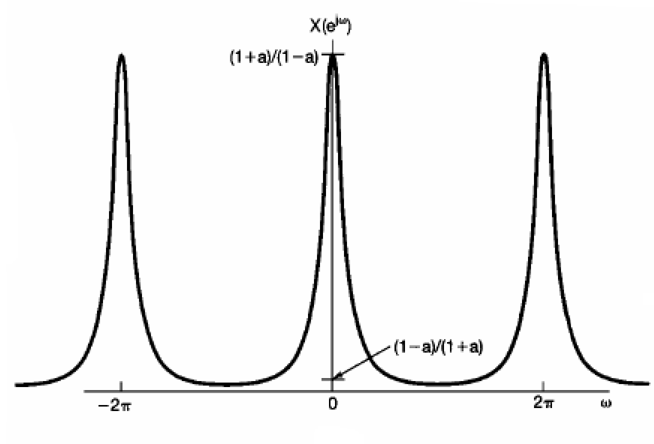
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|}u[n]e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n}e^{-j\omega n} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

Let $m = -n$ in the first summation, we obtain

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|}u[n]e^{-j\omega n} = \sum_{m=1}^{\infty} a^m e^{j\omega m} + \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \frac{ae^{j\omega}}{1-ae^{j\omega}} + \frac{1}{1-ae^{-j\omega}} = \frac{1-a^2}{1-2a\cos\omega + a^2}$$





PROPERTIES OF DTFT:

Linearity:

$$\mathcal{F}_{DT}\{x_1(n) + x_2(n)\} = \mathcal{F}_{DT}\{x_1(n)\} + \mathcal{F}_{DT}\{x_2(n) = X_1(e^{j\omega}) + X_2(e^{j\omega})$$

Time Shifting:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{x(n-n_0)\} = e^{-j\omega n_0} X(e^{j\omega})$$

Proof:

$$\sum_{n=-\infty}^{\infty} x(n-n_0)e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x(m)e^{-j\omega(m+n_0)} = e^{-j\omega n_0}\sum_{m=-\infty}^{\infty} x(m)e^{-j\omega m}$$

Time Reversal:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{x(-n)\} = X(e^{-j\omega})$$

Convolution:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$G(e^{j\omega}) = \mathcal{F}_{DT}\{x(n) * y(n)\} = X(e^{j\omega})\ Y(e^{j\omega})$$

Proof:

$$G\left(e^{j\omega}\right)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}x(m)\,y(n-m)\,e^{-jn\omega}=\sum_{n=-\infty}^{\infty}x(m)\sum_{m=-\infty}^{\infty}y(n-m)\,e^{-jn\omega}$$

$$=\sum_{m=-\infty}^{\infty}x(m)\sum_{r=-\infty}^{\infty}y(r)\,e^{-j(m+r)\omega}=\sum_{m=-\infty}^{\infty}x(m)\,e^{-jm\omega}\sum_{r=-\infty}^{\infty}y(r)\,e^{-jr\omega}$$

Frequency Shifting:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\left\{e^{j\omega_0 n}x(n)\right\}=X\left(e^{j(\omega-\omega_0)}\right),$$

Time Multiplication:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\left\{nx(n)\right\}=-z\left.\frac{dX(z)}{dz}\right|_{z=e^{j\omega}}$$

Parseval's Theorem:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$\sum_{n=-\infty}^{\infty}x(n)y^*(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X\left(e^{j\omega}\right)Y^*\left(e^{j\omega}\right)d\omega$$

Proof:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty}x(n)e^{-jn\omega}\right]Y^*(e^{j\omega})d\omega = \sum_{n=-\infty}^{\infty}x(n)\frac{1}{2\pi}\int_{-\pi}^{\pi}Y^*(e^{j\omega})e^{-j\omega}d\omega$$

For the case $x(n) = y(n)$, then:

$$\sum_{n=-\infty}^{\infty}|x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2 d\omega$$

Multiplication of Sequences:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}_{DT}\{y(n)\} = Y(e^{j\omega})$, then:

$$\mathcal{F}_{DT}\{x(n)y(n)\} = \sum_{n=-\infty}^{\infty}x(n)y(n)e^{-j\omega n} = \sum_{n=-\infty}^{\infty}x(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}Y(e^{j\lambda})e^{j\lambda n}d\lambda\right]e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}Y(e^{j\lambda})d\lambda\sum_{n=-\infty}^{\infty}x(n)e^{-j(\omega-\lambda)n} = \frac{1}{2\pi}\int_{-\pi}^{\pi}Y(e^{j\lambda})X(e^{j(\omega-\lambda)})d\lambda$$

$$= \frac{1}{2\pi}Y(e^{j\omega}) * X(e^{j\omega})$$

Differentiation in the Frequency Domain:

If $\mathcal{F}_{DT}\{x(n)\} = X(e^{j\omega})$, then:

$$\frac{dX(e^{j\omega})}{d\omega} = \frac{d}{d\omega}\left[\sum_{n=-\infty}^{\infty}x(n)e^{-j\omega n}\right] = -j\sum_{n=-\infty}^{\infty}nx(n)e^{-j\omega n} = -j\,\mathcal{F}_{DT}\{nx(n)\}$$