5.1. STIFFNESS METHOD.

5.1.1. INTRODUCTION:

Stiffness method is the more popular younger brother of Flexibility method.Although the two methods are the opposites to each other,they are akin to each other in several respects.

Like in Flexibility method,this also involves generating element matrices,assembling them to get the system matrix and inverting the same to solve for nodal displacements,member displacements and eventually member forces.In the solution of the structure the important result is the member forces.Displacements are generally of lesser importance.

However in stiffness method we get to the displacements first and thence to member forces.

5.1.2. PROPERTIES OF THE STIFFNESS MATRIX:

It is a symmetric matrix and the sum of elements in any columns must be equal to zero.It is an unstable element therefore the the determinate is equal to zero.

The given indeterminate structure is first made kinematically determinate by introducing constraints at the nodes. The required number of constraints is equal to degrees of freedom at the nodes that is kinematic indeterminacy k. The kinematic ally determinate structure comprises of fixed ended members, hence, all nodal displacements are zero. These results in stress resultant discontinuities at these nodes under the action of applied loads or in other words the clamped joints are not in equilibrium.

Inorder to restore the equilibrium of stress resultants at the nodes the nodes are imparted suitable unknown displacements. The number of simultaneous equations representing joint equilibrium of forces is equal to kinematic indeterminacy k. Solution of these equations gives unknown nodal displacements. Using stiffness properties of members the member end forces are computed and hence the internal forces through out the structure.

Since nodal displacements are unknowns, the method is also called displacement method. Since equilibrium conditions are applied at the joints the method is also called equilibrium method. Since stiffness properties of members are used the method is also called stiffness method.

5.1.3. ELEMENT AND GLOBAL STIFFNESS MATRICES

Local co ordinates

In the analysis for convenience we fix the element coordinates coincident with the member axis called element (or) local coordinates (coordinates defined along the individual member axis)

Global co ordinates

It is normally necessary to define a coordinate systemdealing with the entire structure is called system on global coordinates (Common coordinate system dealing with the entire structure)

Transformation matrix

The connectivity matrix which relates the internal forces Q and the external forces R is known as the force transformation matrix. Writing it in a matrix form,

$$\{Q\} = [b]\{R\}$$

Where; Q = member force matrix/vector,

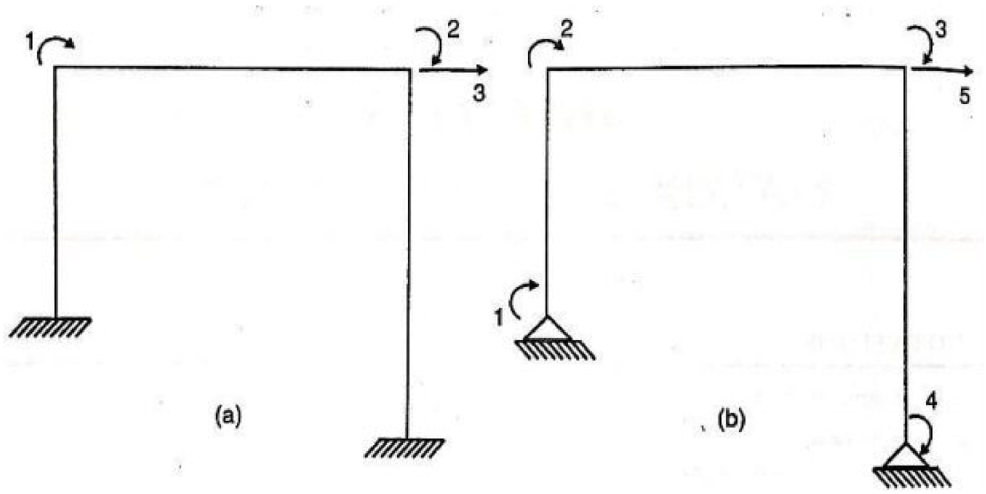
b = force transformation matrix

R = external force/load matrix/ vector

5.1.4. RESTRAINED STRUCTURE

In the Flexibility methods the difficulty of solving a structure increases with the static indeterminacy of a structure.

In stiffness methods the difficulty increases with its kinematic indeterminacy.Thus, structures with more constraints(supports,fixities etc.) are more easily solved then structures with more freedom.Strangely,the structure in fig,(a) is easier to tackle than the structure in fig,(b).Thus we have to get familiar with kinematic indeterminacies or freedoms.



5.1.5. PIN JOINTED FRAMES

In the case of pin jointed plane frames,we have to assign two degrees of freedom to each node.

The elements in trusses are very distinct.Each element shall have one degree of freedom for each end except the ends that are restrained.Normally the questions of forces not at co-ordinates will not arise in trusses.

An introduction to the stiffness method was given in the previous Page. The basic principles involved in the analysis of beams, trusses were discussed. The problems were solved with hand computation by the direct application of the basic principles.

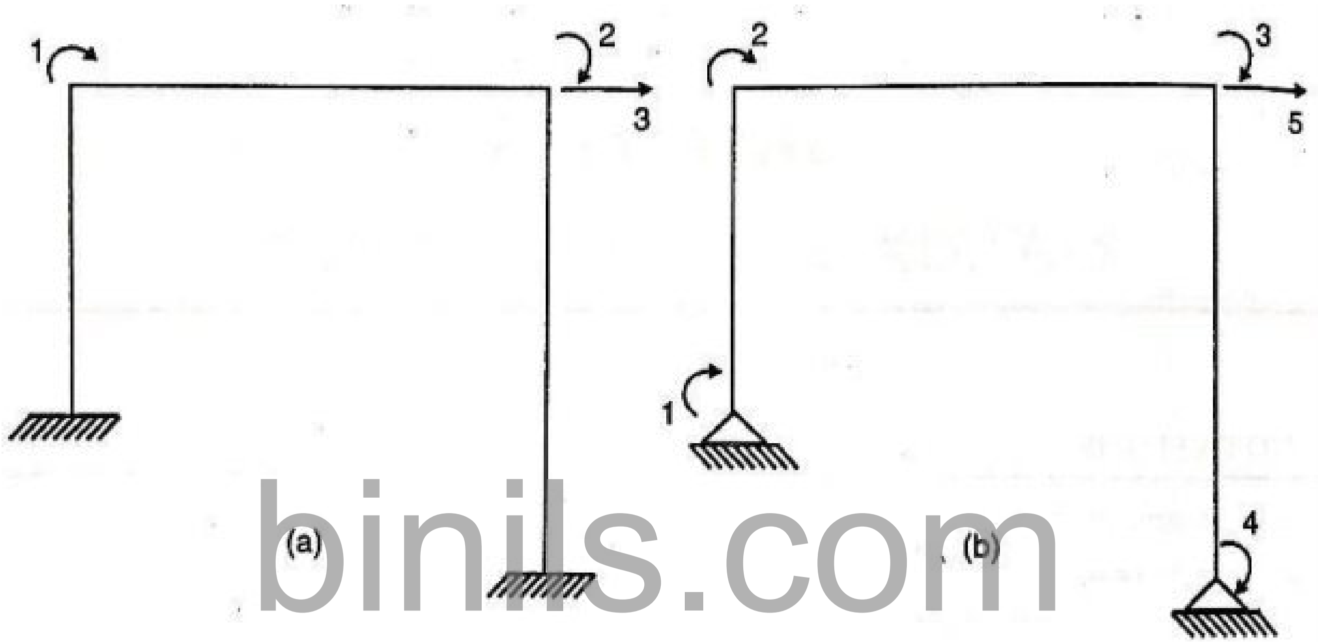
In this session a formal approach has been discussed which may be readily programmed on a computer. In this less on the direct stiffness method as applied to planar truss structure is discussed.

Planetrusses are made up of short thin members inter connected a thin gesto form triangulated patterns. Ahinge connection can only transmit forces from one member to another member but not the moment. For analysis purpose, the truss is loaded at the joints. Hence, a truss member is subjected to only axial forces and the forces remain constant along the length of the member. The forces in the member at its two ends must be of the same magnitude but actin the opposite directions for equilibrium.

5.2. FORMATION OF STIFFNESS MATRICES

The *n x n* stiffness matrix of a structure with a specified set of *n* co-ordinates is determined by applying one unit displacement at a time and determining the forces at each co-ordinate to sustain that displacement.

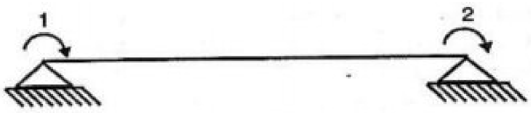
For example if we want to determine the 3 x 3 stiffness matrix for the structure in this fig.5.1,.



- Find the forces at 1,2 and 3 when displacements at 1 is unity and displacements at 2 and 3 are zero i.e., find P1,P2 and P3 when $\delta_1 = 1$ and $\delta_2 = \delta_3 = 0$.These 3 Forces constitute the first column of the stiffness matrix [k_1].
- Find the 3 forces at 1,2 and 3 when $\delta_2 = 1$ and $\delta_1 = \delta_3 = 0$.These 3 Forces constitute the second column of the stiffness matrix [k_1].
- Find forces at 1,2 and 3 when $\delta_3 = 1$and $\delta_1 = \delta_2 = 0$.These 3 forces make the third column of [k_1].

Example 5.2.1

Determine the 2 x 2 stiffness matrix of the beam system shown in fig.5.7

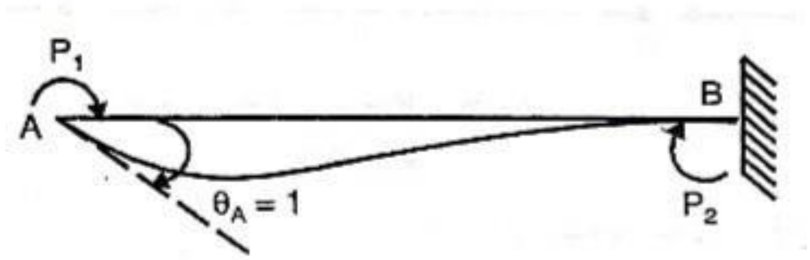


$l = 3$, $A = 1$, $E = 2$, $I = 3$

Fig. 5.7

Solution:

Step 1.To find the first column of [k] apply a unit displacement at 1 only and restrained 2 from rotating



If $\theta = 1$,

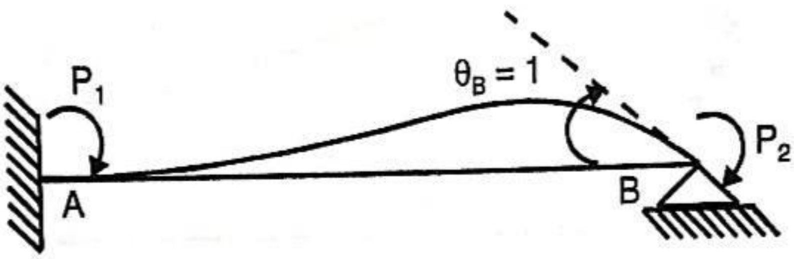
$P_1 = 4EI\theta_A / L = 4 \times 2 \times 3 / 3 = 8$

$P_2 = 2EI\theta_A / L = 2 \times 2 \times 3 / 3 = 4$

Hence;

$$\begin{Bmatrix} k_{11} \\ k_{21} \end{Bmatrix} = \begin{Bmatrix} 8 \\ 4 \end{Bmatrix}$$

Step 2.To get the second column of [k] apply a unit rotation at B and restrain A



$P_2 = 4Ei\theta_B / L = 4 \times 2 \times 3 / 3 = 8$

$P_1 = 2Ei\theta_B / L = 2 \times 2 \times 3 / 3 = 4$

Hence;

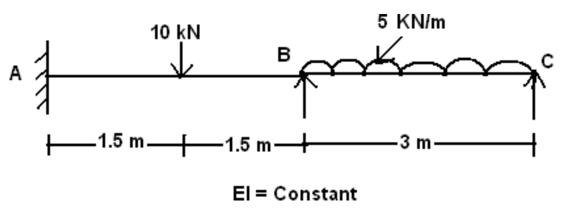
$$\begin{Bmatrix} k_{12} \\ k_{22} \end{Bmatrix} = \begin{Bmatrix} 4 \\ 8 \end{Bmatrix} \text{ and } [k] = \begin{bmatrix} 8 & 4 \\ 4 & 8 \end{bmatrix}$$

5.3. ANALYSIS OF CONTINUOUS BEAMS BY STIFFNESS METHOD

5.3.1.NUMERICAL PROBLEMS ON CONTINUOUS BEAMS;

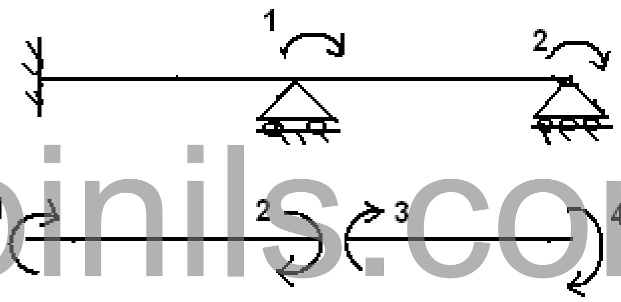
PROBLEM NO:01

Analysis the continuous beam by Stiffness Method.And find the final moments.



Solution:

- **Assigned co-ordinates:**



- **Fixed End Moments:**

  MFAB = -Wl/8 = -10X3/8 = -3.75 kNm

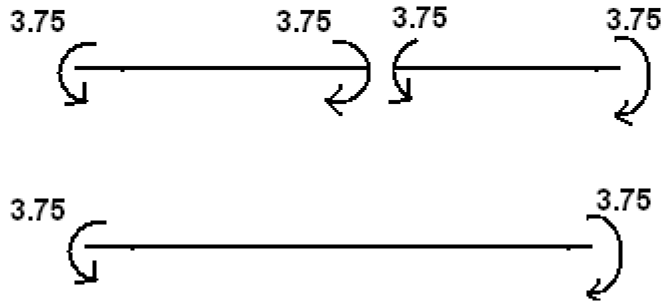
  MFBA = Wl/8 = 10X3/8 = 3.75 kNm

  MFBC = $-Wl^2/12 = -5x3^2/12$ = -3.75 kNm

  MFCB = $Wl^2/12 = 5x3^2/12$ = 3.75 kNm

- **Fixed End Moments Diagrams:**

$$W^O = \begin{bmatrix} 0 \\ 3.75 \end{bmatrix}$$



- **Formation of (A) Matrix:**

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Stiffness Matrix (K):**

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix}$$

- **System Stiffness Matrix (J):**

$J = A^T . K . A$

$$= EI \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= EI \begin{bmatrix} 0.67 & 1.33 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J = EI \begin{bmatrix} 2.6 & 0.67 \\ 0.67 & 1.33 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI} \begin{bmatrix} 0.431 & -0.217 \\ -0.217 & 0.861 \end{bmatrix}$$

Displacement Matrix

(Δ):Δ = J^{-1} . W

= J^{-1} [W^* - W^0]

$$= \frac{1}{EI} \begin{bmatrix} 0.431 & -0.217 \\ -0.217 & 0.861 \end{bmatrix} \left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 0 \\ 3.75 \end{Bmatrix} \right]$$

$$\Delta = \frac{1}{EI} \begin{bmatrix} 0.814 \\ -3.228 \end{bmatrix}$$

- **Element Force (P):**

P = K . A . Δ

$$= \frac{EI}{EI} \begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \frac{EI}{EI}\begin{bmatrix} 1.33 & 0.67 & 0 & 0 \\ 0.67 & 1.33 & 0 & 0 \\ 0 & 0 & 1.33 & 0.67 \\ 0 & 0 & 0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0.814 \\ -3.228 \end{bmatrix}$$

$$= \begin{bmatrix} 0.67 & 0 \\ 1.33 & 0 \\ 1.33 & 0.67 \\ 0.67 & 1.33 \end{bmatrix}\begin{bmatrix} 0.814 \\ -3.228 \end{bmatrix}$$

$$P = \begin{bmatrix} 0.545 \\ 1.082 \\ -1.081 \\ -3.75 \end{bmatrix}$$

Final Moments (M):

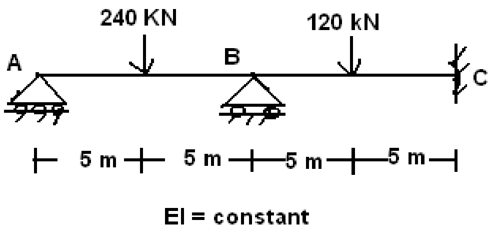
M = μ + P

$$= \begin{bmatrix} -3.75 \\ 3.75 \\ -3.75 \\ 3.75 \end{bmatrix} + \begin{bmatrix} 0.545 \\ 1.082 \\ -1.081 \\ -3.75 \end{bmatrix}$$

$$M = \begin{bmatrix} -3.205 \\ 4.832 \\ -4.832 \\ 0 \end{bmatrix}$$

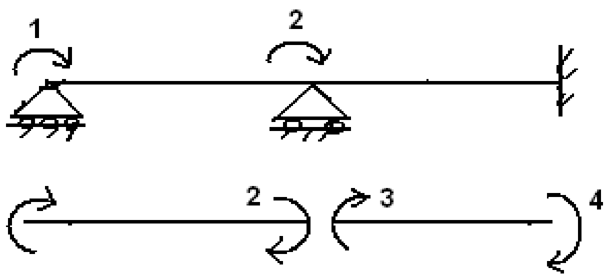
PROBLEM NO:02

Analysis the continuous beam by Stiffness Method.And find the final moments.



Solution:

- **Assigned co-ordinates:**



- **Fixed End Moments:**

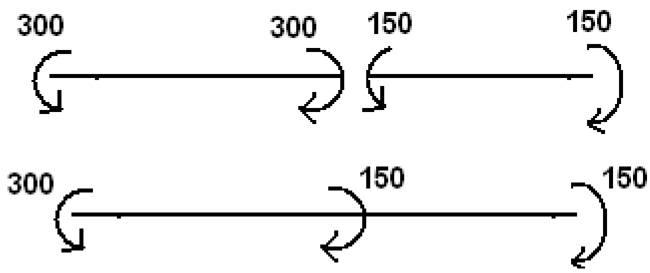
MFAB = -Wl/8 = -240X10/8 = -300 kNm

MFBA = Wl/8 = 240X10/8 = 300 kNm

MFBC = -Wl/8 = -120X10/8 = -150 kNm

MFCB = Wl/8 = 120X10/8 = 150 kNm

- **Fixed End Moments Diagrams:**



$$W^{O} = \begin{bmatrix} -300 \\ 150 \end{bmatrix}$$

- **Formation of (A) Matrix:**

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Stiffness Matrix (K):

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 & 0.4 \end{bmatrix}$$

- **System Stiffness Matrix (J):**

J = A^T . K . A

$$= EI\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= EI\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0.4 & 0.2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI\begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 2.86 & -0.71 \\ -0.71 & 1.43 \end{bmatrix}$$

- **Displacement Matrix (Δ):**

$\Delta = J^{-1} . W$

$= J^{-1} [W^* - W^0]$

$$= \frac{1}{EI}\begin{bmatrix} 2.86 & -0.71 \\ -0.71 & 1.43 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -300 \\ 150 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 964.5 \\ -427.5 \end{bmatrix}$$

- **Element Force (P):**

$P = K . A . \Delta$

$$= \frac{EI}{EI}\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.2 \\ 0 & 0 & 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 964.5 \\ -427.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 & 0.2 \\ 0.2 & 0.4 \\ 0 & 0.4 \\ 0 & 0.2 \end{bmatrix}\begin{bmatrix} 964.5 \\ -427.5 \end{bmatrix}$$

$$P = \begin{bmatrix} 300 \\ 21.9 \\ -171 \\ -85.5 \end{bmatrix}$$

- **Final Moments (M):**

  M = μ + P

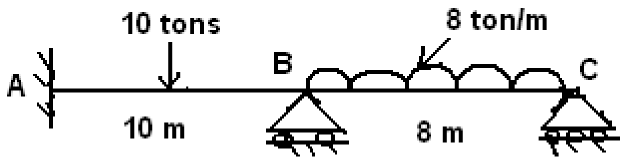
$$= \begin{bmatrix} -300 \\ 300 \\ -150 \\ 150 \end{bmatrix} + \begin{bmatrix} 300 \\ 21.9 \\ -171 \\ -85.5 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 \\ 321.9 \\ -321 \\ 64.5 \end{bmatrix}$$

PROBLEM NO:03

A two span continuous beam ABC is fixed at A and simply supported over the supports B and C. AB = 10 m and BC = 8 m. moment of inertia is constant throughout. A single central concentrated load of 10 tons acts on AB and a uniformly distributed load of 8 ton/m acts over BC. Analyse the beam by stiffness matrix method.

Solution:



- **Fixed End Moments:**

  MFAB = -Wl/8 = -10X10/8 = -12.5 kNm

  MFBA = Wl/8 = 10X10/8 = 12.5 kNm

  MFBC = $-Wl^2/12 = -8x8^2/12$ = -42.67 kNm

  MFCB = $Wl^2/12 = 8x8^2/12$ = 42.67 kNm

-
-
- **Formation of (A) Matrix:**

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- **Stiffness Matrix (K):**

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$K = EI\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix}$$

- **System Stiffness Matrix (J):**

J = A^T . K . A

$$= EI\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 1.29 & -0.65 \\ -0.65 & 2.32 \end{bmatrix}$$

- **Displacement Matrix (Δ):**

$\Delta = J^{-1} . W$

$= J^{-1} [W^{*} - W^{0}]$

$$= \frac{1}{EI}\begin{bmatrix} 1.29 & -0.65 \\ -0.65 & 2.32 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -30.17 \\ 42.67 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 66.65 \\ -118.60 \end{bmatrix}$$

- **Element Force (P):**

$P = K . A . \Delta$

$$= \frac{EI}{EI}\begin{bmatrix} 0.4 & 0.2 & 0 & 0 \\ 0.2 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 66.65 \\ -118.60 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 0 \\ 0.4 & 0 \\ 0.5 & 0.25 \\ 0.25 & 0.5 \end{bmatrix}\begin{bmatrix} 66.65 \\ -118.60 \end{bmatrix}$$

$$P = \begin{bmatrix} 13.33 \\ 26.66 \\ 3.68 \\ -42.64 \end{bmatrix}$$

- **Final Moments (M):**

  M = μ + P

$$= \begin{bmatrix} -12.5 \\ 12.5 \\ -42.67 \\ 42.67 \end{bmatrix} + \begin{bmatrix} 13.33 \\ 26.66 \\ 3.68 \\ -42.64 \end{bmatrix}$$

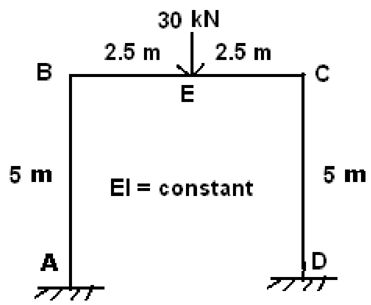
$$M = \begin{bmatrix} 0.83 \\ 39.16 \\ -39 \\ 0 \end{bmatrix}$$

5.4. ANALYSIS OF RIGID FRAMES BY STIFFNESS MATRICES METHOD

5.4.1.NUMERICAL PROBLEMS ON RIGID FRAMES;

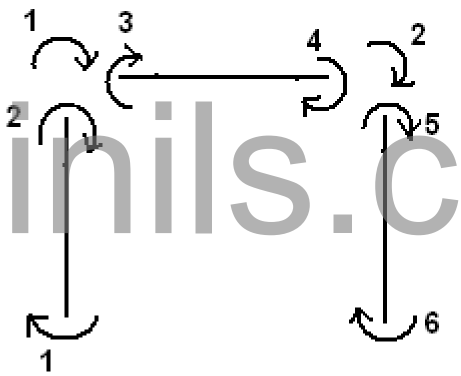
PROBLEM NO:01

Analysis the rigid portal frame ABCD shown in fig,by using Stiffness method.



Solution:

- **Assigned Co-Ordinates:**



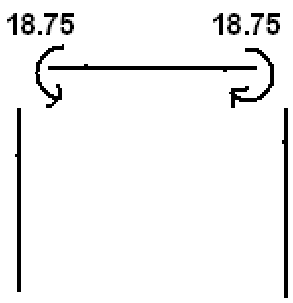
- **Fixed End Moments:**

MFBC = -Wl/8 = -30x5/8 = -13.75 kNm

MFCB = Wl/8 = 30x5/8 = 13.75 kNm

MFAB = MFBA = MFCD = MFDC = 0

- **Fixed End Moments Diagrams:**



$$W^{o} = \begin{bmatrix} -18.75 \\ 18.75 \end{bmatrix}$$

- **Formation of (A) Matrix:**

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^{T} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

- **Stiffness Matrix(K):**

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \end{bmatrix}$$

- **System Stiffness Matrix(J):**

J = A^{T} . K . A

$$= EI\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI\begin{bmatrix} 1.6 & 0.4 \\ 0.4 & 1.6 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 0.67 & -0.17 \\ -0.17 & 0.67 \end{bmatrix}$$

- **Displacement Matrix(Δ):**

$\Delta = J^{-1} . W$

$= J^{-1} [W^* - W^0]$

$$= \frac{1}{EI}\begin{bmatrix} 0.67 & -0.17 \\ -0.17 & 0.67 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -18.75 \\ 18.75 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 15.75 \\ -15.75 \end{bmatrix}$$

- **Element Force (P):**

$P = K . A . \Delta$

$$= \frac{EI}{EI}\begin{bmatrix} 0.8 & 0.4 & 0 & 0 & 0 & 0 \\ 0.4 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.4 & 0 & 0 \\ 0 & 0 & 0.4 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \\ 0 & 0 & 0 & 0 & 0.8 & 0.4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 15.75 \\ -15.75 \end{bmatrix}$$

$$P = \begin{bmatrix} 6.3 \\ 12.6 \\ 6.3 \\ -6.3 \\ -12.6 \\ -6.3 \end{bmatrix}$$

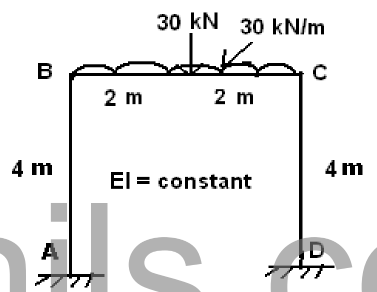
- **Final Moments (M):**

M = μ + P

$$= \begin{bmatrix} 0 \\ 0 \\ -18.75 \\ 18.75 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 6.3 \\ 12.6 \\ 6.3 \\ -6.3 \\ -12.6 \\ -6.3 \end{bmatrix} = \begin{bmatrix} 6.3 \\ 12.6 \\ -12.5 \\ 12.5 \\ -12.6 \\ -6.3 \end{bmatrix}$$

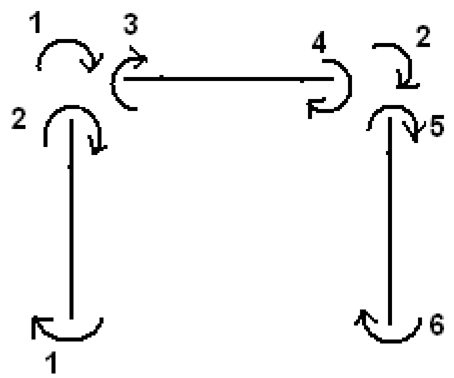
PROBLEM NO:02

Analysis the portal rigid frame ABCD using stiffness method and find the support moments.



Solution:

- **Assigned Co-Ordinates:**



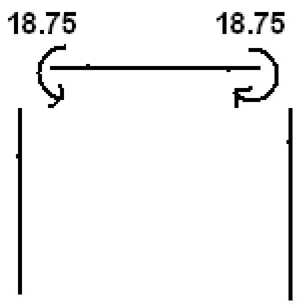
- **Fixed End Moments:**

  MFBC = -[Wl/8 + Wl^2/12] = -[30 x 4/8 + 30 x 4^2/12]= -55 kNm

  MFCB = [Wl/8 + Wl^2/12] = [30 x 4/8 + 30 x 4^2/12] = 55 kNm

  MFAB = MFBA = MFCD = MFDC = 0

- **Fixed End Moments Diagrams:**



$$W^{o} = \begin{bmatrix} -55 \\ 55 \end{bmatrix}$$

- **Formation of (A) Matrix:**

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^{T} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

- **Stiffness Matrix(K):**

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}$$

- **System Stiffness Matrix(J):**

$J = A^{T} . K . A$

$$= EI \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI \begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI} \begin{bmatrix} 0.53 & -0.13 \\ -0.13 & 0.53 \end{bmatrix}$$

- **Displacement Matrix(Δ):**

$$\Delta = J^{-1} . W$$

$$= J^{-1} [W^{*} - W^{0}]$$

$$= \frac{1}{EI} \begin{bmatrix} 0.53 & -0.13 \\ -0.13 & 0.53 \end{bmatrix} \left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -55 \\ 55 \end{Bmatrix} \right]$$

$$\Delta = \frac{1}{EI} \begin{bmatrix} 36.3 \\ -36.3 \end{bmatrix}$$

- **Element Force (P):**

$$P = K . A . \Delta$$

$$= \frac{EI}{EI} \begin{bmatrix} 1 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 36.3 \\ -36.3 \end{bmatrix}$$

$$P = \begin{bmatrix} 18.15 \\ 36.3 \\ 18.15 \\ -18.15 \\ -36.3 \\ -18.15 \end{bmatrix}$$

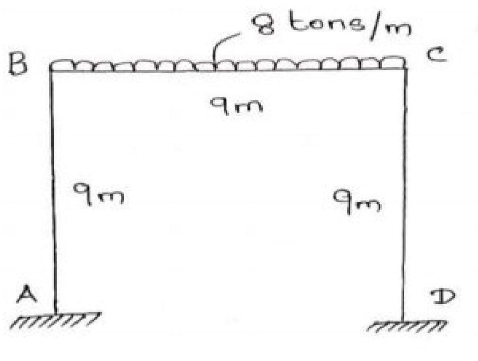
- **Final Moments (M):**

M = μ + P

$$= \begin{bmatrix} 0 \\ 0 \\ -55 \\ 55 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 18.15 \\ 36.3 \\ 18.15 \\ -18.15 \\ -36.3 \\ -18.15 \end{bmatrix} = \begin{bmatrix} 18.15 \\ 36.3 \\ -36.3 \\ 36.45 \\ -36.3 \\ -18.15 \end{bmatrix}$$

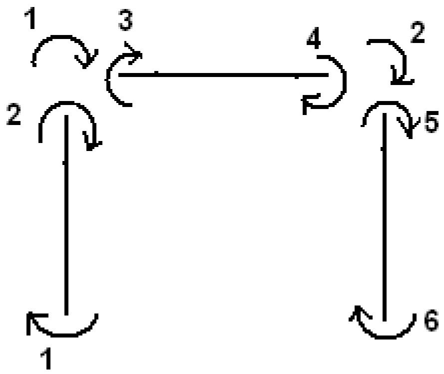
PROBLEM NO:03

A portal frame ABCD with supports A and D are fixed at same level carries a uniformly distributed load of 8 tons/m on the span AB. Span AB = BC = CD = 9 m. EI is constant throughout. Analyse the frame by stiffness matrix method.



Solution:

- **Assigned Co-Ordinates:**



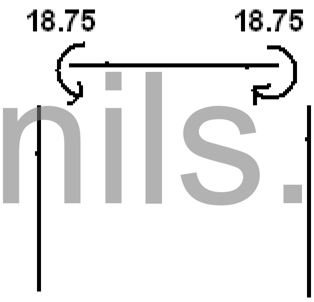
- **Fixed End Moments:**

MFBC = -Wl^2/12 = -8 x 9^2/12 = -54 ton.m

MFCB = Wl^2/12 = -8 x 9^2/12 = 54 ton.m

MFAB = MFBA = MFCD = MFDC = 0

- **Fixed End Moments Diagrams:**



$$W^o = \begin{bmatrix} -54 \\ 54 \end{bmatrix}$$

- **Formation of (A) Matrix:**

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 1\,1\,0 & 0 & 0 \\ 0 & 0 & 0\,1 & 1 & 0 \end{bmatrix}$$

- **Stiffness Matrix(K):**

$$K = \frac{EI}{L}\begin{bmatrix} 4 & 2 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix} = EI\begin{bmatrix} 0.44 & 0.22 & 0 & 0 & 0 & 0 \\ 0.22 & 0.44 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.44 & 0.22 & 0 & 0 \\ 0 & 0 & 0.22 & 0.44 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.44 & 0.22 \\ 0 & 0 & 0 & 0 & 0.22 & 0.44 \end{bmatrix}$$

-

- **System Stiffness Matrix(J):**

J = A^T . K . A

$$= EI\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0.44 & 0.22 & 0 & 0 & 0 & 0 \\ 0.22 & 0.44 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.44 & 0.22 & 0 & 0 \\ 0 & 0 & 0.22 & 0.44 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.44 & 0.22 \\ 0 & 0 & 0 & 0 & 0.22 & 0.44 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J = EI\begin{bmatrix} 0.88 & 0.22 \\ 0.22 & 0.88 \end{bmatrix}$$

$$J^{-1} = \frac{1}{EI}\begin{bmatrix} 1.212 & -0.303 \\ -0.303 & 1.212 \end{bmatrix}$$

- **Displacement Matrix(Δ):**

Δ = J^{-1} . W

= J^{-1} [W* - W^0]

$$= \frac{1}{EI}\begin{bmatrix} 1.212 & -0.303 \\ -0.303 & 1.212 \end{bmatrix}\left[\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} -54 \\ 54 \end{Bmatrix}\right]$$

$$\Delta = \frac{1}{EI}\begin{bmatrix} 81.81 \\ -81.81 \end{bmatrix}$$

- **Element Force (P):**

P = K . A . Δ

$$= \frac{EI}{EI}\begin{bmatrix} 0.44 & 0.22 & 0 & 0 & 0 & 0 \\ 0.22 & 0.44 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.44 & 0.22 & 0 & 0 \\ 0 & 0 & 0.22 & 0.44 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.44 & 0.22 \\ 0 & 0 & 0 & 0 & 0.22 & 0.44 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 81.81 \\ -81.81 \end{bmatrix}$$

$$= \begin{bmatrix} 0.22 & 0 \\ 0.44 & 0 \\ 0.44 & 0.22 \\ 0.22 & 0.44 \\ 0 & 0.44 \\ 0 & 0.22 \end{bmatrix}\begin{bmatrix} 81.81 \\ -81.81 \end{bmatrix}$$

$$P = \begin{bmatrix} 18 \\ 36 \\ 18 \\ -18 \\ -36 \\ -18 \end{bmatrix}$$

- **Final Moments (M):**

M = μ + P

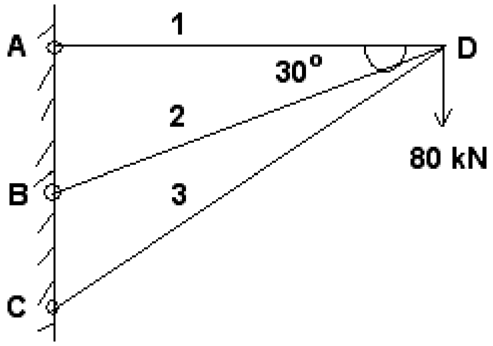
$$= \begin{bmatrix} 0 \\ 0 \\ -54 \\ 54 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 18 \\ 36 \\ 18 \\ -18 \\ -36 \\ -18 \end{bmatrix} = \begin{bmatrix} 18 \\ 36 \\ -36 \\ 36 \\ -36 \\ -18 \end{bmatrix}$$

5.5. ANALYSIS OF RIGID FRAMES BY STIFFNESS MATRICES METHOD

5.5.1. NUMERICAL PROBLEMS ON PIN JOINTED FRAMES;

PROBLEM NO:01

Using matrix stiffness method, analyze the truss for the member forces in the truss loaded as shown in figure. AE and L are tabulated below for all the three members.

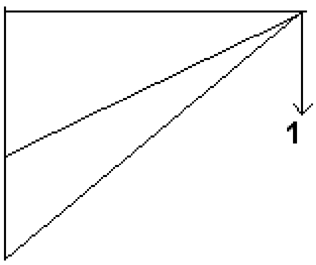


Member	AE	L
AD	400	400
BD	461.9	461.9
CD	800	800

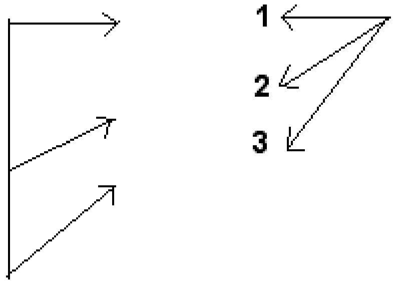
Solution:

- **Assigned Co-Ordinates:**

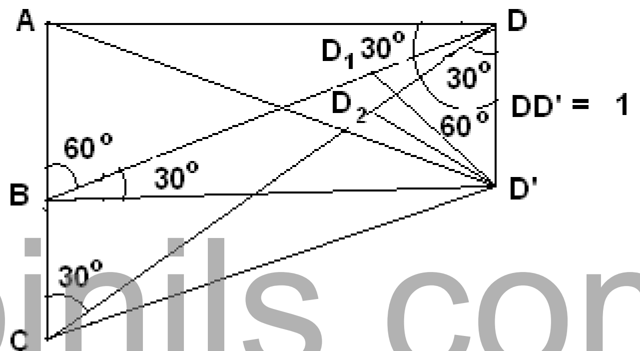
Global Co-Ordinates



Local Co-Ordinates



- **Displacement Diagrams:**



- **Formation of [A] Matrix:**

  Apply unit displacement in DD'. Displacement along 1, AD = 0

  Displacement along 2 and 3,

  $DD_1 = \cos 60^o = 0.5$ and $DD_2 = \cos 30^o = 0.866$

$$A = \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix}$$

- **Stiffness Matrix [K]:**

$$K = \frac{AE}{L}\begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_2 & 0 \\ 0 & 0 & K_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **System Stiffness Matrix(J):**

$$\mathbf{J = A^T . K . A}$$

$$= \begin{matrix} 0 & -0.5 & -0.866 \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix}$$

$$= \begin{matrix} 0 & -0.5 & -0.866 \end{matrix} \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix}$$

$$J = 1$$

$$J^{-1} = 1$$

- **Displacement Matrix(Δ):**

$$\mathbf{\Delta = J^{-1} . W}$$

$$\mathbf{= 1 \times 80 = 80\ mm}$$

- **Element Force (P):**

$$\mathbf{P = K . A . \Delta}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix} \quad 80$$

$$= \begin{bmatrix} 0 \\ -0.5 \\ -0.866 \end{bmatrix} \quad 80$$

- **Final Force (P):**

$$= \begin{bmatrix} 0 \\ -40 \\ -69.28 \end{bmatrix}$$