ENGINEERING MATHEMATICS - I

DIPLOMA COURSE IN ENGINEERING & TECHNOLOGY

FIRST SEMESTER

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Untouchability is a sin
Untouchability is a crime
Untouchability is an inhuman

DIRECTORATE OF TECHNICAL EDUCATION
GOVERNMENT OF TAMILNADU

First Edition – 2020

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THE NATIONAL ANTHEM

FULL VERSION

Jana-gana-mana-adhinayaka jaya he
Bharata-bhagya-vidhata
Punjaba-Sindhu-Gujarata-Maratha-
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchhala-jaladhi-taranga
Tava Subha name jage,TavaSubhaasisa mage,
Gahe tava jaya-gatha.
Jana-gana-mangala-dayaka jaya he
Bharata-bhagya-vidhata.
Jaya he, jaya he, jaya he,
Jaya jaya jaya jaya he.

\- **Rabindranath Tagore**

SHORT VERSION

Jana-gana-mana-adhinayaka jaya he
Bharata-bhagya-vidhata.
Jaya he, jaya he, jaya he,
Jaya jaya jaya jaya he.

AUTHENTIC ENGLISH TRANSLATION OF THE NATIONAL ANTHEM

Thou art the ruler of the minds of all people,
Thou dispenser of India's destiny.
Thy name rouses the hearts of the Punjab, Sind,
Gujarat and Maratha, of Dravida, Orissa and Bengal
It echoes in the hills of the Vindhyas and Himalayas,
mingles in the music of the Yamuna and Ganges
and is chanted by the waves of the Indian Sea.
They pray for Thy blessings and sing Thy praise
The saving of all people waits in Thy hand,
Thou dispenser of India's destiny.
Victory, Victory, Victory to Thee

THE NATIONAL INTEGRATION PLEDGE

"I solemnly pledge to work with dedication to preserve and strengthen the freedom and integrity of the nation."

"I further affirm that I shall never resort to violence and that all differences and disputes relating to religion, language, region or other political or economic grievances should be settled by peaceful and constitutional means."

INVOCATION TO GODDESS TAMIL

Bharat is like the face beauteous of Earth clad in wavy seas;
Deccan is her brow crescent-like on which the fragrant 'Tilak' is the
blessed Dravidian land.
Like the fragrance of that 'Tilak' plunging the world in joy supreme
reigns Goddess Tamil with renown spread far and wide.
Praise unto 'You, Goddess Tamil, whose majestic youthfulness, inspires
awe and ecstasy'.

PREFACE

We take great pleasure in presenting this book of mathematics to the students of polytechnic colleges. This book is prepared in accordance with the new syllabus under 'N' scheme framed by the Directorate of Technical Education, Chennai.

This book has been prepared keeping in mind the aptitude and attitude of the students and modern methods of education. The lucid manner in which the concepts are explained, make the teaching and learning process more easy and effective. Each chapter in this book is prepared with strenuous effort to present the principles of the subject in the most easy to understand and the most easy to workout manner.

Each chapter and section is presented with QR code, an introduction, learning objective, definitions, theorems , explanation, worked examples, summary and exercises with answer given are for better understanding of concepts and in the exercises, problems have been given in view of enough practice for mastering the concept.

We hope that the book serve the purpose keeping in mind the changing needs of the society to make it lively and vibrating. The language used is very clear and simple which is up to the level of comprehension of students.

We extend our deep sense of gratitude to Thiru. K.Vivekanandan I.A.S., the Chairperson for giving valuable inputs and suggestions in bringing out this text book for the benefit of the student community. We also thank the Co-ordinator Dr. M.S. Padmanabhan, Principal(i/c), Central Polytechnic College, Chennai and Conveners who took sincere efforts in preparing and reviewing this book.

Valuable suggestions and corrections for the improvement of this book is most welcome and will be acknowledge most gratefully. Mail your suggestions to dote.nscheme@gmail.com.

- AUTHORS

ANNEXURE- I

STATE BOARD OF TECHNICAL EDUCATION &TRAINING, TAMILNADU

DIPLOMA IN ENGINEERING / TECHNOLOGY SYLLABUS

N-SCHEME

(Implemented from the Academic year 2020 - 2021 onwards)

Course Name : All branches of Diploma in Engineering and Technology and Special Programs except DMOP, HMCT and Film &TV.

Subject Code : 40012

Semester : I

Subject Title : ENGINEERING MATHEMATICS - I

TEACHING AND SCHEME OF EXAMINATION

No. of weeks per semester: 16 weeks

Subject	Instructions		Examination			
	Hours / Week	Hours / Semester	Marks			Duration
			Internal Assessment	Board Examinations	Total	
ENGINEERING MATHEMATICS I	5	80	25	100*	100	3 Hrs.

*** Examinations will be conducted for 100 marks will be reduced to 75 marks.**

TOPICS AND ALLOCATION OF HOURS:

Sl. No.	Topics	Time (Hrs)
1	Algebra	15
2	Complex Number	15
3	Trigonometry	14
4	Differential Calculus – I	15
5	Differential Calculus – II	14
	Test & Model Exam	7
	TOTAL	**80**

40012 ENGINEERING MATHEMATICS – I

DETAILED SYLLABUS

Contents: Theory

UNIT	NAME OF THE TOPICS	HOURS
I	**ALGEBRA**	
	1.1 MATRICES AND DETERMINANTS:	3
	MATRICES: Definition, Concept and Types of Matrices.	
	OPERATIONS ON MATRICES: Multiplication of a Matrix by a scalar, Addition/Subtraction of two Matrices. Multiplication of two Matrices - properties. Reducing a Matrix into triangular and echelon form. Transpose of a Matrix and its properties.	
	DETERMINANTS: Definition and Evaluation of 2^{nd} and 3^{rd} order Determinants. Properties of determinants, product of Determinants. Determinant of a square Matrix – singular and non – singular Matrices - simple problems.	
	1.2 APPLICATIONS OF MATRICES AND DETERMINANTS:	7
	Co-factor, Adjoint of Matrix, Inverse of Matrix and Rank of a matrix – Simple problems.	
	Solution of simultaneous equations using Cramer's rule - Matrix Inversion method - Gaussian Elimination method – simple problems.	
	Characteristic Equation – Eigen Values and Eigen Vectors of a real matrix – consistency and inconsistency of system of linear equations.	
	1.3 BINOMIAL THEOREM:	5
	Introduction – Factorial, Permutation and Combinations – Values of nP_r and nC_r.	
	Statement of Binomial theorem for positive integral index. Expansion of Binomial - Finding general term – Middle term – Coefficient of x^n and Term independent of x – Binomial Theorem for rational index up to -3.	
	Applications of binomial theorem – Finding the remainder, digits of a number and greatest term – simple problems.	
II	**COMPLEX NUMBERS**	
	2.1 ALGEBRA OF COMPLEX NUMBERS	6
	Introduction – Complex Numbers – Conjugates – Algebra of complex numbers (without geometrical proof), Properties of complex conjugates - Modulus and Amplitude - Polar and Euler form of a complex number – Simple problems.	
	Argand Diagram – Collinear points, four points forming square, rectangle, rhombus and parallelogram only - Simple problems.	

UNIT	NAME OF THE TOPICS	HOURS
	2.2 DE MOIVRE'S THEOREM De Moivre's Theorem (Statement & Applications) – related simple problems.	**5**
	2.3 ROOTS OF COMPLEX NUMBERS Finding the n^{th} roots of unity – solving the equations of the form $x^n \pm 1 = 0$ where $n \leq 7$ - Simple problems. APPLICATIONS OF COMPLEX NUMBERS An application of Complex numbers: AC Circuits - Definitions – Impedance and Admittance - Simple Problems	**4**
III	**TRIGONOMETRY**	
	3.1 TRIGONOMETRIC FUNCTIONS & ALLIED ANGLES Trigonometric functions – Properties of Trigonometric functions - Relation between Degree & Radian Measure – Simple problems. Applications of Radian Measure – Length of an arc of a sector – Linear and angular velocity - Trigonometric Ratios of Allied angles – Simple problems.	**5**
	3.2 TRIGONOMETRIC IDENTITIES Trigonometric Ratios of sum & difference of two angles – Multiple and Sub multiple angles – Functions of 3A angles – Sum and Product Identities - Simple problems.	**5**
	3.3 PROPERTIES OF TRIANGLE & INVERSE TRIGONOMETRIC FUNCTIONS Properties of Triangle – Law of Sines and Law of Cosines - Inverse Trigonometric Functions – Principal value – Properties of Inverse Trigonometric functions – simple problems.	**4**
IV	**DIFFERENTIAL CALCULUS - I**	
	4.1 LIMITS Introduction to Calculus - The calculation of limits – Theorems on limits – Limits at infinity – Limits of rational functions – Trigonometrical limits – other limits – Applications of limits – Simple problems.	**5**
	4.2 DIFFERENTIATION The derivative of a Function – Differentiation of constant, x^n, $sinx$, $cosx$, $tanx$, $cotx$, $secx$, $cosecx$, $logx$, e^x, a^x, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$, $cosec^{-1} x$ (Formulae only) - Differentiation Rules: $u \pm v, uv, uvw, \frac{u}{v}$ & Chain rule – Simple problems.	**5**
	4.3 DIFFERENTIATION METHODS Differentiation by Substitution method – Differentiation of Implicit functions – Logarithmic differentiation – Derivatives of parametric functions – Differentiation of one function with respect to another function – Simple problems.	**5**

UNIT	NAME OF THE TOPICS	HOURS
V	**DIFFERENTIAL CALCULUS – II**	
	5.1 SUCCESSIVE DIFFERENTIATION	5
	Successive differentiation upto second order (parametric form not included). Definition of differential equation, order and degree, formation of differential equation. Simple problems	
	5.2 GEOMETRICAL APPLICATIONS	5
	Curvature and Radius of curvature (cartesian form only) - Envelope of family of curves – Simple problems.	
	5.3 PARTIAL DIFFERENTIATION	4
	Definition - Partial Differentiation of two variables upto second order only - simple problems. Jacobian and its properties. Euler's theorem for homogeneous function – Simple problems.	

Reference Book

1. Higher Secondary +1 Mathematics volume I&II. Tamil Nadu Text book corporation.
2. Higher Secondary +2 Mathematics Volume I&II. Tamil Nadu Text book corporation.
3. Engineering Mathematics V. Sundaram, R. Balasubramanian
4. Engineering Mathematics – I C.B.Gupta ,A.K.Malik, New age international Publishers, 1st edition – 2008.
5. Differential Calculus S. Balachandra Rao, CK Shantha New age Publishers
6. Probability Theory and Stochastic Process B.Prabhakara Rao, TSR Murthy, BS Publishers.
7. Vectors and Geometry GS. Pandey, RR Sharma, New age international publishers.
8. Engineering Mathematics – I Guruprasad Samanta, New age international publishers, 2nd edition 2015.
9. Engineering Mathematics Reena Garg, Khanna publishing house, New Delhi, Revised edn. – 2018.
10. Engineering Mathematics Volume I P. Kandasamy and K. Thilagavathy, S. Chand & Company Ltd.

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UNIT - I

ALGEBRA

Chapter 1.1 MATRICES AND DETERMINANTS

Introduction:

Matrix and its applications are very important part of Mathematics. Also it is one of the most powerful tools in Mathematics.

Matrix notation and operations are used in Electronic spread sheet programs on personal computer which are used in like business, budgeting, sales projection, cost estimation, analysing the results of an experiment etc. Also many physical operations such as magnification, rotation and reflection through a plane can be represented mathematically by matrices. Also matrices are used in Cryptography.

Matrix:

A Matrix is represented by a rectangular array of numbers (or) functions arranged in rows and columns, put within a bracket.

The numbers (or) functions which are entries in the matrix are called as the element of the matrix.

Examples of Matrices:

Raju has 10 notebooks and 15 pens. Mani has 5 notebooks and 2 pens. Malar has 9 note books and 5 pens.

The above information may be represented in the form of matrix as follows.

$$\begin{array}{l c} & \begin{array}{cc} \text{Note Books} & \text{Pens} \end{array} \\ \begin{array}{l} \text{Raju} \\ \text{Mani} \\ \text{Malar} \end{array} & \begin{bmatrix} 10 & 15 \\ 5 & 2 \\ 9 & 5 \end{bmatrix} \end{array}$$

Consider the linear equations with 3 unknowns a, b, c.

$$a + b + c = 3$$

$$2a - b + c = 2 \text{ and}$$

$$3a + 2b - 2c = 3$$

The above equations can be represented in the form of matrix A by writing the co-efficients of a, b, c in the order which they occur and enclose them within a bracket.

Then we get

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 3 & 2 & -2 \end{bmatrix}$$

Here A is Matrix

The horizontal lines of elements are called as Row of the matrix.

i.e. $[1 \ 1 \ 1] \longrightarrow$ I Row of the matrix A

The vertical lines of elements are called column of the matrix.

i.e. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \downarrow$ I Column

Order of a matrix:

If a matrix has m rows and n columns then the order of the matrix is m x n (read as m by n)

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Here A has 3 rows and 2 columns. So the order of matrix A is 3 x 2.

Problems:

Find the order of the following matrix.

(i) $A = \begin{bmatrix} 1 & 3 & -1 \\ 5 & 0 & 2 \\ 7 & 5 & 8 \end{bmatrix}$

(ii) $B = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

(iii) $C = \begin{bmatrix} 7 & 0 & 2 \end{bmatrix}$

(iv) $D = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$

Types of Matrices:

1. **Row Matrix:** A matrix is said to be a row matrix, if it has only one row and any number of columns.

   e.g, $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ is a row matrix of order 1 x 3.

2. **Column Matrix:**

   A matrix is said to be a column matrix, if it has only one column and any number of rows.

   e.g., $B = \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix}$ is a column matrix of order 3 x 1.

3. Null (or) Zero Matrix:

If all the elements of a matrix are zero, then it is called a Null (or) Zero matrix. It is denoted by O.

Eg. $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is zero matrix of order 2 x 2.

4. Square Matrix:

In a matrix, if the number of rows and the number of columns of a matrix are equal then the matrix is called a square matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Here, number of rows = number of columns = 3

A is a square matrix of order 3 x 3.

5. Triangular Matrix:

(a) Upper Triangular Matrix:

In a square matrix if all the elements below the leading diagonal are zero then it is called an upper triangular matrix.

Eg. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$ → Leading diagonal

(b) Lower Triangular Matrix:

In a square matrix if all the elements above the leading diagonal are zero, then it is called a Lower Triangular Matrix.

Eg. $B = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}$ → Leading diagonal

6. Transpose of Matrix

Let A be any matrix. The transpose of matrix A is obtained by interchanging either rows into columns or columns into rows of A. It is denoted by A^T or A'.

Eg. If $A = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 5 & 6 \\ 2 & -1 & 9 \end{bmatrix}$

Then $A^T = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 5 & -1 \\ 3 & 6 & 9 \end{bmatrix}$

Note:

(i) If a matrix A is of order m x n then the order of A^T is n x m.

(ii) $(A^T)^T = A$.

7. Symmetric Matrix:

The square matrix A is called a symmetric matrix if $A = A^T$.

For Example:

$$\text{If } A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \quad \text{then } A^T = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

$\therefore \; A = A^T$

$\therefore$ A is symmetric.

8. Skew Symmetric Matrix:

The square matrix A is called a skew symmetric matrix if $A = -A^T$.

For Example:

$$\text{If } A = \begin{bmatrix} a & h & g \\ -h & b & f \\ -g & -f & c \end{bmatrix} \text{ then } -A^T = \begin{bmatrix} a & h & g \\ -h & b & f \\ -g & -f & c \end{bmatrix}$$

Here $A = -A^T$

$\therefore$ A is skew symmetric.

9. Diagonal Matrix:

In a square matrix, if all the elements other than the elements of the leading Diagonal (or) main diagonal are zero then the matrix is called Diagonal matrix.

$$\text{Eg.} \quad A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \longrightarrow \text{Leading diagonal}$$

10. Scalar Matrix:

A diagonal matrix in which all the elements are equal to a scalar is called a scalar matrix.

$$\text{Eg.} \quad A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$$

11. Unit Matrix:

A square matrix in which all the elements of the leading diagonal are 1 and other elements are zero, is called a Unit Matrix.

It is denoted by I.

Eg. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3.

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2.

Operations on Matrices:

i) Addition and subtraction of matrices

ii) Multiplication of matrix by a scalar

iii) Multiplication of two matrices

i) Addition and Subtraction of Matrices:

Two Matrices can be added (or) subtracted provided both the matrices are of same order. We can add (or) subtract the corresponding elements of two matrices of same order.

Example: 1

If $A = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 4 & 5 \\ 3 & 1 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 1 & 7 & 5 \end{bmatrix}$ then

Find A + B

Solution:

$$A + B = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 4 & 5 \\ 3 & 1 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \\ 1 & 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 2+3 & 7+1 \\ 0+2 & 4+4 & 5+0 \\ 3+1 & 1+7 & 6+5 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 8 \\ 2 & 8 & 5 \\ 4 & 8 & 11 \end{bmatrix}$$

Example: 2

If $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & 7 \\ 1 & 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 & 4 \\ 1 & -1 & 5 \\ 0 & 2 & 4 \end{bmatrix}$ then

Find A - B

Solution:

$$A - B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 0 & 7 \\ 1 & 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 3 & 4 \\ 1 & -1 & 5 \\ 0 & 2 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1-7 & 3-3 & 5-4 \\ 2-1 & 0-(-1) & 7-5 \\ 1-0 & 5-2 & 2-4 \end{bmatrix}$$

$$A - B \Rightarrow \begin{bmatrix} -6 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & -2 \end{bmatrix}$$

ii) Multiplication of a matrix by a scalar:

We can multiply the matrix by any non-zero scalar K. To multiply the matrix by a scalar K, multiply all the elements by the same scalar K.

i.e., If $A = [a_{ij}]_{m\times n}$ then $KA = [Ka_{ij}]_{m\times n}$

for all $i = 1, 2, \ldots\ldots m$

$j = 1, 2, \ldots\ldots n$

For Example:

If $A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 1 & 0 \\ 7 & 2 & 8 \end{bmatrix}$ & $B = \begin{bmatrix} -3 & 1 & 0 \\ 2 & 7 & 1 \\ 4 & 3 & 5 \end{bmatrix}$

Then find 2A and 7B

Solution:

Given $A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 1 & 0 \\ 7 & 2 & 8 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 1 & 0 \\ 2 & 7 & 1 \\ 4 & 3 & 5 \end{bmatrix}$

$$2A = 2\begin{bmatrix} 4 & 3 & 2 \\ 5 & 1 & 0 \\ 7 & 2 & 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 \text{ x } 4 & 2 \text{ x } 3 & 2 \text{ x } 2 \\ 2 \text{ x } 5 & 2 \text{ x } 1 & 2 \text{ x } 0 \\ 2 \text{ x } 7 & 2 \text{ x } 2 & 2 \text{ x } 8 \end{bmatrix}$$

$$2A \Rightarrow \begin{bmatrix} 8 & 6 & 4 \\ 10 & 2 & 0 \\ 14 & 4 & 16 \end{bmatrix}$$

$$7B = 7\begin{bmatrix} -3 & 1 & 0 \\ 2 & 7 & 1 \\ 4 & 3 & 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 7 \text{ x } -3 & 7 \text{ x } 1 & 7 \text{ x } 0 \\ 7 \text{ x } 2 & 7 \text{ x } 7 & 7 \text{ x } 1 \\ 7 \text{ x } 4 & 7 \text{ x } 3 & 7 \text{ x } 5 \end{bmatrix}$$

$$7B = \begin{bmatrix} -21 & 7 & 0 \\ 14 & 49 & 7 \\ 28 & 21 & 35 \end{bmatrix}$$

Example: 2

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 7 \\ 0 & 4 \end{bmatrix}$ then find 4A - 2B

Solution:

Given $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 7 \\ 0 & 4 \end{bmatrix}$

$$4A = 4 \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

$$4A \Rightarrow \begin{bmatrix} 4 & 8 \\ 12 & 20 \end{bmatrix} \qquad \text{-------} \ (1)$$

$$2B = 2 \begin{bmatrix} -5 & 7 \\ 0 & 4 \end{bmatrix}$$

$$2B = \begin{bmatrix} -10 & 14 \\ 0 & 8 \end{bmatrix} \qquad \text{-------} \ (2)$$

$$(1) - (2) \Rightarrow \quad 4A - 2B = \begin{bmatrix} 4 & 8 \\ 12 & 20 \end{bmatrix} - \begin{bmatrix} -10 & 14 \\ 0 & 8 \end{bmatrix}$$

$$4A - 2B \Rightarrow \begin{bmatrix} 14 & -6 \\ 12 & 12 \end{bmatrix}$$

iii) Multiplication of Matrices:

Let A and B be any two Matrices. Multiplication of two matrices possible only when the number of columns of A must be equal to the number of rows of B.

Let A, B, C be any three matrices of same order.

i.e $\quad A = \begin{bmatrix} a_1 & b_1 \\ a_2 & a_2 \end{bmatrix}, \qquad B = \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix}, \qquad C = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$

then, i) $AB = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix}$

$$AB \Rightarrow \begin{bmatrix} a_1p_1 + b_1p_2 & a_1q_1 + b_1q_2 \\ a_2p_1 + b_2p_2 & a_2q_1 + b_2q_2 \end{bmatrix}$$

ii) $BC = \begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$

$$BC \Rightarrow \begin{bmatrix} p_1x_1 + q_1x_2 & p_1y_1 + q_1y_2 \\ p_2x_1 + q_2x_2 & p_2y_1 + q_2y_2 \end{bmatrix}$$

Example: 1

If $A = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 4 \\ 5 & 0 \end{bmatrix}$ then find AB

Solution:

$$AB = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} (1 \text{ x } 3) + (-1 \text{ x } 5) & (1 \text{ x } 4) + (-1 \text{ x } 0) \\ (1 \text{ x } 3) + (2 \text{ x } 5) & (1 \text{ x } 4) + (2 \text{ x } 0) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 - 5 & 4 + 0 \\ 3 + 10 & 4 + 0 \end{bmatrix}$$

$$AB \Rightarrow \begin{bmatrix} -2 & 4 \\ 13 & 4 \end{bmatrix}$$

Example: 2

If $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ verify AB = BA

Solution:

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1 & 0+3 \\ 2+3 & 0+9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 3 \\ 5 & 9 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4+0 & 2+0 \\ 2+3 & 1+9 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 2 \\ 5 & 10 \end{bmatrix}$$

$$\therefore\ AB \neq BA$$

Example: 3

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 7 & 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 3 & -1 \\ 2 & -4 & 4 \\ 5 & 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -4 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 1 & 1 \end{bmatrix}$

then find A (B + C)

Solution:

$$B + C = \begin{bmatrix} 8 & 3 & -1 \\ 2 & -4 & 4 \\ 5 & 3 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 5 & -1 \\ 2 & -1 & 8 \\ 10 & 4 & 2 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 7 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 5 & -1 \\ 2 & -1 & 8 \\ 10 & 4 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 4+4+30 & 5-2+12 & -1+16+6 \\ 0+10+40 & 0-5+16 & 0+40+8 \\ 28+4+40 & 35-2+16 & -7+16+8 \end{bmatrix}$$

$$A(B+C) = \begin{bmatrix} 38 & 15 & 21 \\ 50 & 11 & 48 \\ 72 & 49 & 17 \end{bmatrix}$$

Exercise

(1) If $A = [1 \quad 0 \quad 5]$, $B = \begin{bmatrix} 7 & -2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$ & $C = \begin{bmatrix} 4 & 3 \\ -1 & 1 \end{bmatrix}$

then verify (AB) C = A(BC)

(2) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 6 \\ 0 & 3 \end{bmatrix}$, $C = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}$

verify that A (B + C) = AB + BC

Properties of Matrices:

i) Commutative Property in Matrix Addition:

For any matrices A & B of same order,

a) Matrix Addition is commutative:

A + B = B + A ,

b) Matrix Multiplication is not commutative in general.

$\therefore$ AB $\neq$ BA

ii) a) Associative Property in Matrix Addition:

(A + B) + C = A + (B + C), for any matrices A, B & C of same order.

b) Associative Property in Matrix Multiplication:

A(BC) = (AB)C , for any matrices A, B & C such that A(BC) and (AB)C are of same order.

iii) a) Identity Property in Matrix Addition:

For any matrix A, there exist a matrix O of same order such that A + O = O + A = A

b) Identity Property in Matrix Multiplication:

For any matrix A, there exist an Identity matrix I such that AI = IA = A

Here, I is a Unit Matrix.

iv) Inverse Property:

For any matrix A, – A is the additive inverse of A such that A + (-A) = O = (-A) + A

v) Distributive Property:

a) Matrix multiplication is left distributive over addition

A (B + C) = AB + AC

b) Matrix multiplication is right distributive over addition

(A + B) C = AC + BC

c) Scalar multiplication is distributive over addition

$(a + b) A = aA + bA$

$a (A + B) = aA + aB$

Here, a, b are any scalar

A, B are any matrices of same order.

Worked Examples

1) If $A = \begin{bmatrix} 3 & 4 \\ 8 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix}$

Then verify that i) $A + B = B + A$

ii) $A + (- A = (-A) + A = O$

Solution:

i) $A = \begin{bmatrix} 3 & 4 \\ 8 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix}$

$A + B = \begin{bmatrix} 3 & 4 \\ 8 & -3 \end{bmatrix} + \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 6 & 7 \\ 9 & -3 \end{bmatrix}$ ----- (1)

$B + A = \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 8 & -3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 6 & 7 \\ 9 & -3 \end{bmatrix}$ ----- (2)

$\therefore A + B = B + A$

ii) $A = \begin{bmatrix} 3 & 4 \\ 8 & -3 \end{bmatrix}$, $-A = \begin{bmatrix} -3 & -4 \\ -8 & 3 \end{bmatrix}$

$A + (-A) = \begin{bmatrix} 3 & 4 \\ 8 & -3 \end{bmatrix} + \begin{bmatrix} -3 & -4 \\ -8 & 3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ ----- (1)

$-A + A = \begin{bmatrix} -3 & -4 \\ -8 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 8 & -3 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$ ----- (2)

$\therefore (1) = (2)$

$A + (-A) = (-A) + A = 0$

2) If $A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & 0 \\ 1 & 3 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 8 & 3 & 1 \\ 0 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix}$

Then verify that $A + (B + C) = (A + B) + C$

Solution:

Given,

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & 0 \\ 1 & 3 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 8 & 3 & 1 \\ 0 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

L.H.S.

$$B + C = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 8 & 3 & 1 \\ 0 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

$$B + C = \begin{bmatrix} 10 & 6 & 2 \\ -1 & 4 & 6 \\ 5 & 3 & 8 \end{bmatrix}$$

$$A + (B + C) = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & 0 \\ 1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 10 & 6 & 2 \\ -1 & 4 & 6 \\ 5 & 3 & 8 \end{bmatrix}$$

$$A + (B + C) \Rightarrow \begin{bmatrix} 14 & 9 & 3 \\ 1 & 6 & 6 \\ 6 & 6 & 13 \end{bmatrix} \qquad \text{----- (1)}$$

R.H.S.

$$A + B = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 2 & 0 \\ 1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 3 & 1 \\ -1 & -1 & 2 \\ 4 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 6 & 2 \\ 1 & 1 & 2 \\ 5 & 4 & 10 \end{bmatrix}$$

$$(A + B) + C = \begin{bmatrix} 6 & 6 & 2 \\ 1 & 1 & 2 \\ 5 & 4 & 10 \end{bmatrix} + \begin{bmatrix} 8 & 3 & 1 \\ 0 & 5 & 4 \\ 1 & 2 & 3 \end{bmatrix}$$

$$(A + B) + C) = \begin{bmatrix} 14 & 9 & 3 \\ 1 & 6 & 6 \\ 6 & 6 & 13 \end{bmatrix} \qquad \text{----- (2)}$$

(1) = (2)

$\therefore$ A + (B + C) = (A + B) + C

Hence proved.

Exercise Problems

1) Verify the property A (B + C) = AB + AC for the following matrices A, B and C.

$$A = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 4 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 4 & 7 \\ 2 & 1 \\ 1 & -1 \end{bmatrix}$$

2) Check the Associative property of matrix multiplication to the following matrices A, B, C.

$$A = \begin{bmatrix} 5 & 0 \\ 4 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ 5 & 3 \end{bmatrix}$$

3) If $A = [2 \ \ 3 \ \ -1]$ $B = \begin{bmatrix} 3 & 4 \\ 1 & 0 \\ 5 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 7 \\ 0 & -1 \end{bmatrix}$

Show that (AB) C = A (BC)

4) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. $B = \begin{bmatrix} 4 & 0 \\ 1 & 5 \end{bmatrix}$. $C = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

Show that (A – B) C = AC – BC

5) Show that the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix}$ Satisfy commutative property

AB = BA.

Reducing a Matrix into Triangular and Row Echelon Form:

Using the row elementary operations, We can transform a given non zero matrix to a simplified form called a Row-echelon Form.

In a Row – echelon form, we may have rows all of whose entries are zero, such rows are called zero rows.

Simply said,

If a non zero matrix is in row-echelon form, then all the entries below the leading diagonal [ie., a_{11}, a_{22}, a_{33} …………] are zeros.

As similar way to propose for the triangular form.

There is two type of triangular form.

i.e. Upper Triangular Form:

If a matrix is said to be a Upper Triangular Form which all the elements below the leading diagonal are zero.

Eg. $$A = \begin{bmatrix} a & h & g \\ 0 & b & f \\ 0 & 0 & c \end{bmatrix}$$

i.e. Lower Triangular Form:

If a matrix is said to be a Lower Triangular Form which all the elements above the leading diagonal are zero.

Eg. $B = \begin{bmatrix} a & 0 & 0 \\ e & b & 0 \\ f & h & c \end{bmatrix}$

Worked Problem

1) Reduce the Matrix $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ to a row-echelon form.

Solution:

Given, $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

$R_2 \to R_2 + 2R$, $\quad R_3 \to R_3 + R_1$

$$\Rightarrow \begin{bmatrix} 3 & -1 & 2 \\ 0 & 0 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

This is the required Row – echelon form.

2) Reduce the matrix $\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$ into Row-echelon form.

Solution:

Given, $\begin{bmatrix} 0 & 3 & 1 & 6 \\ -1 & 0 & 2 & 5 \\ 4 & 2 & 0 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 4 & 2 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 4R_1$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 2 & 8 & 20 \end{bmatrix}$$

$R_3 \to 3R_3 - 2R_2$

$$\Rightarrow \begin{bmatrix} -1 & 0 & 2 & 5 \\ 0 & 3 & 1 & 6 \\ 0 & 0 & 22 & 48 \end{bmatrix}$$

This is the required row-echelon form.

3) Reduce the matrix $\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$ into a Triangular Form.

Solution:

Given, $\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & -2 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1,\quad R_3 \rightarrow R_3 - 3R_1$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 7 & 1 \\ 0 & 5 & -5 \end{bmatrix}$$

$R_3 \rightarrow 7R_3 - 5R_2$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 7 & 1 \\ 0 & 0 & -40 \end{bmatrix}$$

This is the required Triangular form.

4) Reduce the matrix $\begin{bmatrix} 4 & 3 & 6 & 25 \\ 1 & 5 & 7 & 13 \\ 2 & 9 & 1 & 1 \end{bmatrix}$ into a Triangular Form.

Solution:

Given, $\begin{bmatrix} 4 & 3 & 6 & 25 \\ 1 & 5 & 7 & 13 \\ 2 & 9 & 1 & 1 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 7 & 13 \\ 4 & 3 & 6 & 25 \\ 2 & 9 & 1 & 1 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 4R_1,\quad R_3 \rightarrow R_3 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & -17 & -22 & -27 \\ 0 & -1 & -13 & -25 \end{bmatrix}$$

$R_2 \rightarrow R_2 \div (-1),\quad R_3 \rightarrow R_3 \div (-1)$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & +17 & +22 & +27 \\ 0 & 1 & 13 & 25 \end{bmatrix}$$

$R_3 \rightarrow 17R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 5 & 7 & 13 \\ 0 & 17 & 22 & 27 \\ 0 & 0 & 199 & 398 \end{bmatrix}$$

This is the required Triangular Form.

Exercise

(1) Reduce the following Matrix into Row-echelon Form.

a) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(2) Reduce the following Matrix into Triangular Form.

c) $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

d) $\begin{bmatrix} 3 & 3 & -1 & 11 \\ 2 & -1 & 2 & 9 \\ 4 & 3 & 2 & 25 \end{bmatrix}$

Transpose of a matrix and its properties:

The transpose of a matrix is obtained by interchanging their rows and columns of the matrix A and it is denoted by A^T.

More precisely, if $A = [a_{ij}]_{m\times n}$

Then $A^T = [b_{ij}]_{n\times m}$, where bij = aji

For instance,

$A = \begin{bmatrix} 1 & \sqrt{2} & 4 \\ -8 & 0 & 0.2 \end{bmatrix}$ implies $A^T = \begin{bmatrix} 1 & -8 \\ \sqrt{2} & 0 \\ 4 & 0.2 \end{bmatrix}$, Here $(i, j)^{th}$ entry of A^T is aji.

Results on Transpose of a Matrix:

For any two matrices A and B of suitable orders

(i) $(A^T)^T = A$

(ii) $(KA)^T = KA^T$ (Where K is any scalar)

(iii) $(A + B)^T = A^T + B^T$

(iv) $(AB)^T = B^T A^T$ [Reversal law on Transpose]

Examples

1. If $A = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

Verify i) $(AB)^T = B^T A^T$ ii) $(A+B)^T = A^T + B^T$

iii) $(A-B)^T = A^T - B^T$ iv) $(3A)^T = 3A^T$

Solution:

i) $$AB = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 2 & 22 \\ -2 & 9 & 9 \\ 7 & 1 & 14 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \quad \text{----- (1)}$$

$$B^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix}, \quad A^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -2 & 7 \\ 2 & 9 & 1 \\ 22 & 9 & 14 \end{bmatrix} \quad \text{----- (2)}$$

From (1) and (2),

$(AB)^T = B^T A^T$

ii) $$A + B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 1 \\ 3 & 0 & 9 \\ -1 & 5 & 3 \end{bmatrix}$$

$$(A + B)^T = \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \quad \text{----- (3)}$$

$$A^T + B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 & -1 \\ 7 & 0 & 5 \\ 1 & 9 & 3 \end{bmatrix} \quad \text{----- (4)}$$

From (3) and (4),

$(A+B)^T = A^T + B^T$

iii) $A - B = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1 & 5 \\ 0 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & -1 \\ 3 & -1 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & 5 & 3 \\ -3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$(A\text{-}B)^T = \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ ----- (5)

$A^T - B^T = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 & -1 \\ 1 & -1 & 2 \\ -1 & 4 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4 & -3 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ ----- (6)

From (5) and (6), $(A - B)^T = A^T - B^T$

iv) $3A = \begin{bmatrix} 12 & 18 & 6 \\ 0 & 3 & 15 \\ 0 & 9 & 6 \end{bmatrix}$

$(3A)^T = \begin{bmatrix} 12 & 0 & 0 \\ 18 & 3 & 9 \\ 6 & 15 & 6 \end{bmatrix} = 3\begin{bmatrix} 4 & 0 & 0 \\ 6 & 1 & 3 \\ 2 & 5 & 2 \end{bmatrix}$

$= 3(A^T)$

2. If $A = \begin{bmatrix} 2 & -3 & 8 \\ 21 & 6 & -6 \\ 4 & -33 & 19 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -29 & -8 \\ 2 & 0 & 3 \\ 17 & 15 & 4 \end{bmatrix}$

Verify (i) $(A+B)^T = A^T + B^T$ (ii) $(AB)^T = B^T A^T$

i) $A + B = \begin{bmatrix} 2 & -3 & 8 \\ 21 & 6 & -6 \\ 4 & -33 & 19 \end{bmatrix} + \begin{bmatrix} 1 & -29 & -8 \\ 2 & 0 & 3 \\ 17 & 15 & 4 \end{bmatrix}$

$= \begin{bmatrix} 3 & -32 & 0 \\ 23 & 6 & -3 \\ 21 & -18 & 23 \end{bmatrix}$

$(A + B)^T = \begin{bmatrix} 3 & 23 & 21 \\ -32 & 6 & -18 \\ 0 & -3 & 23 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & 21 & 4 \\ -3 & 6 & -33 \\ 8 & -6 & 19 \end{bmatrix}$, $B^T = \begin{bmatrix} 1 & 2 & 17 \\ -29 & 0 & 15 \\ -8 & 3 & 4 \end{bmatrix}$

$A^T + B^T = \begin{bmatrix} 2 & 21 & 4 \\ -3 & 6 & -33 \\ 8 & -6 & 19 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 17 \\ -29 & 0 & 15 \\ -8 & 3 & 4 \end{bmatrix}$

$= \begin{bmatrix} 3 & 23 & 21 \\ -32 & 6 & -18 \\ 0 & -3 & 23 \end{bmatrix}$

So we can observe that $(A + B)^T = A^T + B^T$.

ii)

$$AB = \begin{bmatrix} 2 & -3 & 8 \\ 21 & 6 & -6 \\ 4 & -33 & 19 \end{bmatrix} + \begin{bmatrix} 1 & -29 & -8 \\ 2 & 0 & 3 \\ 17 & 15 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 132 & 62 & 7 \\ -69 & -699 & -174 \\ 261 & 169 & -55 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 132 & -69 & 261 \\ 62 & -699 & 169 \\ 7 & -174 & -55 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 1 & 2 & 17 \\ -29 & 0 & 15 \\ -8 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 21 & 4 \\ -3 & 6 & -33 \\ 8 & -6 & 19 \end{bmatrix}$$

$$= \begin{bmatrix} 132 & -69 & 261 \\ 62 & -699 & 169 \\ 7 & -174 & -55 \end{bmatrix}$$

$\therefore (AB)^T = B^T A^T$.

3. If $A = \begin{bmatrix} 9 & 8 \\ 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}$

Show that i) $(AB)^T = B^T A^T$ ii) $(AB)^T \neq A^T B^T$

Solution:

$$AB = \begin{bmatrix} 9 & 8 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 44 & 18 \\ 5 & 4 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 44 & 5 \\ 18 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 9 & 2 \\ 8 & -3 \end{bmatrix}, \qquad B^T = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix}$$

i) $B^T A^T = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 9 & 2 \\ 8 & -3 \end{bmatrix} = \begin{bmatrix} 44 & 5 \\ 18 & 4 \end{bmatrix}$

$B^T A^T = (AB)^T$

ii) $A^T B^T = \begin{bmatrix} 9 & 2 \\ 8 & -3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 40 & 9 \\ 26 & 8 \end{bmatrix}$

$A^T B^T \neq (AB)^T$

We can clearly observe from here that $(AB)^T \neq A^T B^T$

Exercise

1. If $A = \begin{bmatrix} 4 & 5 \\ -1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 7 \\ 9 & -1 \\ 1 & -2 \end{bmatrix}$

   Verify the following

   i) $(A + B)^T = A^T + B^T$ ii) $(A - B)^T = A^T - B^T$

   iii) $(B^T)^T = B$

2. If $P = \begin{bmatrix} 2 & 8 & 9 \\ 11 & -15 & -13 \end{bmatrix}$ and K is a constant, then verify $(KP)^T = KP^T$

3. If A is a 3 x 4 matrix and B is a matrix such that both A^TB and BA^T are defined, what is the order of the matrix B.

Determinants:

Introduction of Determinants

The method of solving simultaneous linear equations was instrumental to the origin of the topic determinants. The theory of determinants began with Leibnitz who solved, the simultaneous linear equation.

Definition of a Determinant:

The determinant is a scalar value that can be computed from the elements of a square matrix A.

It is denoted by det (A)

Also $\Delta A = \det(A) = |A|$

First Order Determinant:

Let A = [a] be the matrix of order 1 Then the determinant of A is defined as "a".

Second Order Determinant:

If a Determinant consists of two rows and two columns then it is called a second order determinant.

Ex: $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Third Order Determinant:

If a Determinant consist of three rows and three columns then it is called a third order determinant.

Ex: $|A| = \begin{vmatrix} 1 & -1 & 2 \\ 2 & -1 & 0 \\ 1 & -2 & 3 \end{vmatrix} = 1\,(-3 + 0) + 1\,(6 - 0) + 2\,(-4 + 1)$

$|A| = -3$

Properties of Determinants:

Property: 1

The value of the determinant is unaltered by changing rows into columns and vice versa. i.e. $|A| = |A^T|$

Proof:

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow a_1(b_2 c_3 - c_2 b_3) - b_1(a_2 c_3 - c_2 a_3) + c_1(a_2 b_3 - a_3 b_2)$$

$$|A^T| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1(b_2 c_3 - c_2 b_3) - a_2(b_1 c_3 - c_1 b_3) + a_3(b_1 c_2 - c_1 b_2)$$

$$= a_1(b_2 c_3 - c_2 b_3) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$$

$$|A| = |A^T|$$

Proved.

Property: 2

If any two rows / columns of a determinant are interchanged, then the value of the determinant changes in sign but its absolute value remains unaltered.

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$|A| = a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$$

$R_2 \leftrightarrow R_3$

$$|A_1| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_3 c_2 - b_2 c_3) - b_1(a_3 c_2 - a_2 c_3) + c_1(a_3 b_2 - a_2 b_3)$$

$$= -a_1(b_2 c_3 - b_3 c_2) + b_1(a_2 c_3 - a_3 c_2) - c_1(a_2 b_3 - a_3 b_2)$$

$$= -[a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)]$$

$$|A_1| = -|A|$$

Hence proved.

Note:

If there are n interchanges of rows (columns) of a matrix A then the resulting determinant is $(-1)^n |A|$.

Property: 3

If any two rows (or) two columns of a matrix are identical, then the value of the determinant is zero.

Proof:

Let $|A| = \begin{vmatrix} 2 & 5 & 1 \\ 2 & 5 & 1 \\ 0 & 2 & 4 \end{vmatrix}$

Here R_1, R_2 are identical.

$|A| = 2(20-2) - 5(8-0) + 1(4-0)$

$= 2(18) - 5(8) + 4$

$= 36 - 40 + 4$

$|A| = 0$

Property: 4

If each element of a row (or column) is multiplied by any scalar K, then the value of the determinant is also multiplied by the same scalar K.

i.e. If $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

then $|A_1| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Here R_1 is multiplied K

$\therefore |A_1| = k|A|$

Consider

$|A| = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 2 & 4 \\ 3 & 9 & 4 \end{vmatrix}$

$\Rightarrow 1(8-36) - 0(-4-12) + 1(-9-6)$

$= -28 - 15 = -43$

Multiply the Row 1 by 2

$|A_1| = \begin{vmatrix} 2 & 0 & 2 \\ -1 & 2 & 4 \\ 3 & 9 & 4 \end{vmatrix}$

$\Rightarrow 2(8-36) - 0(-4-12) + 2(-9-6)$

$\Rightarrow 2(-28) - 0 + 2(-15)$

$= -56 - 30$

$= -86$

$= 2\,(-43)$

$|A_1| = 2\,|A|$

Property: 5

If each element of a row (or column) of a determinant is expressed as sum of two or more terms then the whole determinant can be expressed as the sum of two (or) more determinants of the same order.

i.e. $$\begin{vmatrix} a_1 & b_1+m_1 & c_1 \\ a_3 & b_2+m_2 & c_3 \\ a_2 & b_3+m_3 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & m_1 & c_1 \\ a_2 & m_2 & c_2 \\ a_3 & m_3 & c_3 \end{vmatrix}$$

Property: 6

A determinant is unaltered when to each element of any row (or column) are added those of several other rows or columns multiplied respectively by constant factors.

i.e. if $$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$|A_1| = \begin{vmatrix} a_1+pa_2+qa_3 & b_1+pb_2+qb_3 & c_1+pc_2+qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then $|A| = |A_1|$

Proof:

$$|A_1| = \begin{vmatrix} a_1+pa_2+qa_3 & b_1+pb_2+qb_3 & c_1+pc_2+qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

By property (5),

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pa_2 & pb_2 & pc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} qa_3 & qb_3 & qc_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

By property (4)

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + p \begin{vmatrix} a_2 & b_2 & c_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + q \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

By property (3)

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + p\,(0) + q\,(0)$$

$$\Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \therefore \ |A_1| \Rightarrow |A|$$

Problems using Properties of Determinants:

1. Evaluate: $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

Solution:

Let $\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

Effect $C_2 = C_2 + C_3$

$$= \begin{vmatrix} 1 & a+b+c & b+c \\ 1 & b+c+a & c+a \\ 1 & c+a+b & a+b \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & b+c \\ 1 & 1 & c+a \\ 1 & 1 & a+b \end{vmatrix}$$

$$= (a+b+c)\,0$$

$$= 0 \quad \therefore [C_1 \equiv C_2)$$

2. Prove that $\begin{vmatrix} 2a+b & a & b \\ 2b+c & b & c \\ 2c+a & c & a \end{vmatrix} = 0$

Solution:

LHS $= \begin{vmatrix} 2a+b & a & b \\ 2b+c & b & c \\ 2c+a & c & a \end{vmatrix}$

$$= \begin{vmatrix} 2a & a & b \\ 2b & b & c \\ 2c & c & a \end{vmatrix} + \begin{vmatrix} b & a & b \\ c & b & c \\ a & c & a \end{vmatrix}$$

$$= 2\begin{vmatrix} a & a & b \\ b & b & c \\ c & c & a \end{vmatrix} + 0 \qquad [\because C_1 \equiv C_3]$$

$$= 2\,(0) \quad [\because C_1 \equiv C_2]$$

$$= 0$$

$= $ RHS

3. Prove that $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix} = (x-y)(y-z)(z-x)$

Solution:

$$\text{LHS} = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

$R_1 \to R_1 - R_2$

$R_2 \to R_2 - R_3$

$$= \begin{vmatrix} 0 & x-y & yz-zx \\ 0 & y-z & zx-xy \\ 1 & z & xy \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x-y & -z(x-y) \\ 0 & y-z & -x(y-z) \\ 1 & z & xy \end{vmatrix}$$

$$= 0 + 0 + 1 \begin{vmatrix} x-y & -z(x-y) \\ y-z & -x(y-z) \end{vmatrix}$$

(Expand along the first column)

$$= (x-y)(y-z) \begin{vmatrix} 1 & -z \\ 1 & -x \end{vmatrix}$$

$= (x-y)(y-z)(z-x)$

$=$ RHS

Product of Determinants:

While multiplying two matrices "row-by column" rule alone can be followed. The process of interchanging the rows and columns will not affect the value of the determinant i.e. we can also apply the following procedures for multiplication of two determinants.

(i) Row by row multiplication rule

(ii) Row by column multiplication rule

(iii) Column by column multiplication rule

(iv) Column by row multiplication rule

Note:

If A and B are square matrices of same order n, then $|AB| = |A||B|$.

In Matrices, $AB \neq BA$ in general, we can $|AB| = |BA|$

Worked Examples

(1) If $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$, $|B| = \begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix}$, then find the product of Determinant?

Solution:

By Row – Column Multiple,

$$|A||B| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}\begin{vmatrix} 1 & 0 \\ 3 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+6 & 0-4 \\ 3+12 & 0-8 \end{vmatrix}$$

$$|A||B| = \begin{vmatrix} 7 & -4 \\ 15 & -8 \end{vmatrix}$$

We know $|A||B| = |AB|$

$$\text{So } |AB| = \begin{vmatrix} 7 & -4 \\ 15 & 8 \end{vmatrix}$$

(2) If A_i, B_i, C_i are the co-factors of a_i, b_i, c_i respectively, i = 1 to 3 in

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ show that } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2$$

Solution:

Consider the product

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$

By the Row – Row Multiplication

$$\Rightarrow \begin{vmatrix} a_1A_1 + b_1B_1 + c_1C_1 + & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{vmatrix}$$

$\Rightarrow$ $|A|[|A|^2 – 0]$

$\Rightarrow$ $|A|^3$

$$\text{i.e.} \quad |A| \text{ x } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^3$$

$$\Rightarrow \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = |A|^2$$

Hence proved.

(3) Verify that, $|AB| = |A||B|$ if $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Solution:

$$AB = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta \sin\theta - \sin\theta\cos\theta \\ \cos\theta\sin\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 1 - 0$$

$$|AB| = 1$$

$$|A| = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$$|A| = \cos^2\theta + \sin^2\theta$$

$$|A| = 1$$

$$|B| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$$

$$= \cos^2\theta + \sin^2\theta$$

$$|B| = 1$$

$$\therefore |A||B| = 1 . 1 = 1$$

$$\because |A||B| = 1 . 1 = 1$$

Hence proved.

Exercise

1) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then find the product of determinants by Row – Row Multiplication.

2) If $A = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$. $B = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ then find $|AB|$

3) Show that $\begin{vmatrix} o & c & b \\ c & o & a \\ b & a & o \end{vmatrix}^2 = \begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ac & bc & a^2+b^2 \end{vmatrix}$

Chapter 1.2 APPLICATIONS OF MATRICES AND DETERMINANTS

Minor of an element:

Minor of an element is the determinant obtained by deleting the row and column in which that element occurs.

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Minor of $a_1 \Rightarrow \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

Minor of $a_1 \Rightarrow b_2 c_3 - c_2 b_3$

Example: 1

Find the minor of 2 to the matrix $\begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \\ 7 & 8 & -2 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 3 & 4 \\ 7 & 8 & -2 \end{vmatrix}$

$\because$ Minor of 2 $= \begin{vmatrix} 0 & -1 \\ 8 & -2 \end{vmatrix}$

$= 0 + 8$

Minor of 2 = 8

Co-factor of an Element:

Co-factor of an element is defined as the signed minor.

$\because$ Co-factor of aij $= (-1)^{i+j}$ minor of a_{ij}

Here, a_{ij} is an element which is i^{th} row and j^{th} column.

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Co-factor of $a_1 = (-1)^{1+1}$ Minor of a_1

$= (-1)^2 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

$= (+1)\,[b_2 c_3 - b_3 c_2]$

Co-factor of $a_1 = b_2 c_3 - b_3 c_2$

Adjoint of Matrix:

The Adjoint of a square matrix A is the transpose of the matrix which is formed by replacing each element with the corresponding cofactor.

Method to find adjoint of Matrix of order 3 (order 2)

If A is square Matrix of order 3 (order 2)

i) Find the co-factor of all the elements of A.

ii) Form the matrix by replacing all the elements of A by the corresponding co-factor of A.

iii) Then take the Transpose of that matrix, then we get adj. A.

Example: 1 Find the Adjoint of the matrix $\begin{bmatrix} 1 & 3 \\ 6 & 5 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 3 \\ 6 & 5 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 3 \\ 6 & 5 \end{vmatrix} = 5 - 18 = -13$

Co-factor of Matrix A:

Co-factor of 1 = $(-1)^{1+1}5 = 5$

Co-factor of 3 = $(-1)^{1+2}6 = -6$

Co-factor of 6 = $(-1)^{2+1}3 = -3$

Co-factor of 5 = $(-1)^{2+2}1 = 1$

$\therefore$ Co-factor matrix $= \begin{bmatrix} 5 & -6 \\ -3 & 1 \end{bmatrix}$

Adj A $= \begin{bmatrix} 5 & -3 \\ -6 & 1 \end{bmatrix}$

Example: 2

Find the adjoint of the matrix $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}$

Solution:

$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 2 \end{vmatrix}$

$\Rightarrow 2(4-3) - 3(2+3) + 4(1+2)$

$\Rightarrow 2 - 3(5) + 4(3)$

$\Rightarrow 2 - 15 + 12$

$|A| = -1 \neq 0$

Co-factor 2 = $(-1)^{1+1}(4-3) = 1$

Co-factor 3 = $(-1)^{1+2}(2+3) = -5$

Co-factor 4 = $(-1)^{1+3}(1+2) = 3$

Co-factor 1 = $(-1)^{2+1}(6-4) = -2$

Co-factor 2 = $(-1)^{2+2}(4+4) = 8$

Co-factor $3 \Rightarrow (-1)^{2+3}(2+3) = -5$

Co-factor $-1 = (-1)^{3+1}(9-8) = 1$

Co-factor of $1 = (-1)^{3+2}(6-4) = -2$

Co-factor of $2 = (-1)^{3+3}(4-3) = 1$

$$\text{Co-factor matrix} = \begin{bmatrix} 1 & -5 & 3 \\ -2 & 8 & -5 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\text{Adj } A = [\text{co-factor of } A]^T$$

$$\text{Adj. } A = \begin{bmatrix} 1 & -2 & 1 \\ -5 & 8 & -2 \\ 3 & -5 & 1 \end{bmatrix}$$

Singular matrix and Non-singular matrix

A square matrix A is said to be singular matrix if $|A| = 0$.

A square matrix A is said to be non-singular matrix if $|A| \neq 0$.

Example: 1

Show that the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ is a non-singular matrix.

Solution:

$$\text{Let } A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$$

$$= 10 - 12$$

$$|A| = -2 \neq 0 \qquad \therefore A \text{ is non singular matrix .}$$

Example: 2

Prove that the matrix $\begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$ is a singular matrix.

Solution:

$$\text{Let } B = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$$

$$|B| = \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix}$$

$$= 4 - 4$$

$$|B| = 0 \qquad \therefore B \text{ is a singular matrix.}$$

Inverse of a Matrix:

Let A be a non-singular matrix. If there exist a square matrix B, such that,

AB = BA = I then B is called the inverse of Matrix A.

Where I is the unit Matrix of same order

Also it is denoted by A^{-1}.

Inverse of square matrix A is defined as $A^{-1} = \frac{1}{|A|}$ adjA, $|A| \neq 0$

Note:

i) Inverse of a Matrix is unique

ii) $AA^{-1} = A^{-1}A = I$

iii) $(AB)^{-1} = B^{-1}A^{-1}$

iv) $(A^T)^{-1} = (A^{-1})^T$

Example: 1

If $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$, then find the inverse of A

Solution:

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix}$$

$$= 4 + 3$$

$$|A| = 7$$

$$\text{adj } A = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{7}\begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$$

Example: 2

If $B = \begin{bmatrix} -1 & 2 \\ 1 & -5 \end{bmatrix}$ then find B^{-1}

Solution:

$$B = \begin{bmatrix} -1 & 2 \\ 1 & -5 \end{bmatrix}$$

$$|B| = \begin{vmatrix} -1 & 2 \\ 1 & -5 \end{vmatrix}$$

$$|B| = 5 - 2 = 3$$

$$\text{Adj } B = \begin{bmatrix} -5 & -2 \\ -1 & -1 \end{bmatrix}$$

$$B^{-1} = \frac{1}{|B|} \text{ adj } B$$

$$B^{-1} = \frac{1}{3}\begin{bmatrix} -5 & -2 \\ -1 & -1 \end{bmatrix}$$

3. Find the inverse of $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

Solution:

$$\text{Let } A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix}$$

$$= 2\,(12 - 2) - 3\,(16 - 1) + 4\,(8 - 3)$$

$$= 20 - 45 + 20$$

$$|A| = -5 \neq 0$$

$\therefore$ Inverse of A exist.

Cofactors of Matrix A:

$$A_{11} = +\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 12 - 2 = 10$$

$$A_{12} = -\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = -(16 - 1) = -15$$

$$A_{13} = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 8 - 3 = 5$$

$$A_{21} = -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -(12 - 8) = -4$$

$$A_{22} = \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} = 8 - 4 = 4$$

$$A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = -(4 - 3) = -1$$

$$A_{31} = \begin{vmatrix} 3 & 4 \\ 3 & 1 \end{vmatrix} = 3 - 12 = -9$$

$$A_{32} = -\begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = -(2 - 16) = 14$$

$$A_{33} = \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6 - 12 = -6$$

$\therefore$ Adj (A) $= [Aij]^T$

$$= \begin{bmatrix} 10 & -15 & 5 \\ -4 & 4 & -1 \\ -9 & 14 & -6 \end{bmatrix}^T$$

$$= \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

$\therefore A^{-1} = \frac{1}{|A|}$ adjA

$$= \frac{1}{-5}\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

4. Find the inverse of $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 4 & 2 & 2 \end{bmatrix}$

Solution:

Let A $= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 4 & 2 & 2 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 4 & 2 & 2 \end{vmatrix}$

$= 2\,(0 - 4) - 1\,(2 - 8) + 1\,(2 - 0)$

$= 2(-4) - 1\,(-6) + 2$

$= -8+6+2$

$|A| = 0$

$\therefore$ A is a singular matrix $\Rightarrow$ Inverse of A does not exist.

Exercise

1) Find the Adjoint of the matrix $\begin{bmatrix} 5 & 2 \\ 10 & 4 \end{bmatrix}$

2) Find the Adjoint of the matrix $\begin{bmatrix} 2 & 5 & 7 \\ 7 & 1 & 6 \\ 5 & -4 & -1 \end{bmatrix}$

3) For any two Matrix A = $\begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix}$ B = $\begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}$ prove that $(AB)^T = B^TA^T$

4) Find the inverse of $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & -3 \\ 6 & -2 & -1 \end{bmatrix}$

5) Find the inverse of $\begin{bmatrix} 5 & 0 \\ 2 & 3 \end{bmatrix}$

Rank of Matrix:

A positive integer 'r' is said to be the Rank of a non zero matrix if

(i) At least one minor of order r is non-zero.

(ii) All minors of higher order than r are zero. It is denote by $\rho(A)$

Note:

From the definition of rank of the matrix, it follows,

1. The rank of non-singular matrix of order n is n. If the matrix is singular, its rank is less than n.
2. The rank of a m x n matrix 'A' can at most be equal to the smaller of numbers m and n but it may be less.

   $\rho(A) \leq$ minimum of m and n.
3. If there is a non-zero minor of order 'r' then rank is $\geq$ r.
4. The rank of the null matrix is zero and rank of non zero matrix is ≥ 1.
5. The rank of In, the unit matrix of order n is equal to n.

   i.e. $\rho(In) = n$

   $\rho(I_2) = 2$

   $\rho(I_3) = 3$ etc.
6. $\rho(A) = \rho(A^T)$

Example

1) Find the rank of $\begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix}$

Solution: Let A = $\begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix}$

Order of A = 2 x 2

$\therefore \rho(A) \leq 2.$

The highest order of minor of A = 2.

The minor is $\begin{vmatrix} 5 & 2 \\ 6 & 3 \end{vmatrix} = 15 - 12 = 3 \neq 0$

$\therefore$ Rank of A = $\rho(A) = 2$.

2) Find the rank of $\begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$

Solution: A = $\begin{bmatrix} 3 & -6 \\ -1 & 2 \end{bmatrix}$

Order of A = 2 x 2

$\therefore \rho(A) \leq 2.$

The highest order of minor of A = 2.

i.e. $\begin{vmatrix} 3 & -6 \\ -1 & 2 \end{vmatrix} = 6 - 6 = 0$

$\therefore \rho(A) \neq 2.$

To find at least one non zero first order minor.

Atleast non zero element exist in A.

$\therefore \rho(A) = 1$

3) Find the rank of $\begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 1 \\ 4 & 5 & 2 \end{bmatrix}$

Solution:

Let A $= \begin{bmatrix} -1 & 2 & 3 \\ 0 & 3 & 1 \\ 4 & 5 & 2 \end{bmatrix}$

Order of A = 3 x 3

$\therefore \rho(A) \leq 3.$

The highest order of minor of A = 3.

The minor is $\begin{vmatrix} -1 & 2 & 3 \\ 0 & 3 & 1 \\ 4 & 5 & 2 \end{vmatrix}$

$= -1 \begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} -2 \begin{vmatrix} 0 & 1 \\ 4 & 2 \end{vmatrix} + 2 \begin{vmatrix} 0 & 3 \\ 4 & 5 \end{vmatrix}$

$= -1 (6 – 5) – 2 (0 – 4) + 3 (0 – 12)$

$= -1 + 8 – 36 = - 29 \neq 0$

$\therefore$ Rank of A = $\rho(A) = 3.$

4) Find the rank of $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{bmatrix}$

Solution:

Let A = $\begin{bmatrix} 1 & -3 & 4 & 7 \\ 9 & 1 & 2 & 0 \end{bmatrix}$

Order of A = 2 x 4

$\therefore$ Rank of A = $\rho(A) \leq$ Min {2, 4} = 2.

The highest order of minors of A = 2.

To find a non zero minor of order 2.

$\begin{vmatrix} -2 & 4 \\ 1 & 2 \end{vmatrix} = -6 – 4 = - 10 \neq 0.$

$\therefore$ A has a non zero minor of order 2

$\therefore$ $\rho(A) = 2.$

5) Find the rank of $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Order of A = 3 x 4.

$\therefore$ Rank of A = $\rho(A) \leq$ Min {3, 4} = 3.

The highest order of minors of A = 3.

A has the following minors of order 3.

$$A_1 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 1 & 3 & 4 \end{vmatrix}$$

$$= 1\,(12 - 15) - 2\,(8 - 5) + 3\,(6 - 3)$$
$$= -3 - 6 + 9 = 0$$

$$A_2 = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 3 & 1 \\ 1 & 3 & 5 \end{vmatrix}$$

$$= 1\,(15 - 3) - 2\,(10 - 1) + 2\,(6 - 3)$$
$$= 12 - 18 + 6 = 0.$$

$$A_3 = \begin{vmatrix} 2 & 3 & 2 \\ 3 & 5 & 1 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= 2\,(25 - 4) - 3\,(15 - 3) + 2\,(12 - 15)$$

$$= 42 - 36 - 6 = 0.$$

$$A_4 = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 1 & 4 & 5 \end{vmatrix}$$

$$= 1\,(25 - 4) - 3\,(10 - 1) + 2\,(8 - 5)$$

$$= 21 - 27 + 6 = 0.$$

All third order minors vanish

$\therefore \rho(A) < 3.$

To find at least a non zero minor of order 2.

$$\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = 5 - 6 = -1 \neq 0.$$

$\therefore$ A has at least one zero minor of order 2

$\therefore$ $\rho(A) = 2.$

Exercise

1) The Rank of null Matrix is _________.

2) The Rank of a unit matrix of order 3 is __________.

3) Find the Rank of the following Matrices:

(i) $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & -1 \\ 5 & 4 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

(iv) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & -5 \\ 3 & 1 & 4 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 2 & 3 \\ -4 & 0 & 5 \end{bmatrix}$ (vi) $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 5 & 6 \end{bmatrix}$

(vii) $\begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$

4) Find the Rank of the following Matrices:

(i) $\begin{bmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{bmatrix}$

(ii) $\begin{bmatrix} 3 & 11 & 1 & 5 \\ 5 & 13 & -1 & 11 \\ -2 & 2 & 4 & -8 \end{bmatrix}$

(iii) $\begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & -1 \\ 2 & 5 & 1 & 5 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 1 & 5 & 4 \\ 3 & -2 & 2 & 2 \\ 5 & -8 & -4 & 1 \end{bmatrix}$

(vi) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$

Solution of simultaneous equation using Cramer's Rule:

Consider the following system of equations with unknown variables x, y and z.

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

We have to write in the form of matrix Ax = B

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$A\ X = B$

Then Cramer's Rule:

$$x = \frac{\Delta_x}{\Delta}, \qquad y = \frac{\Delta_y}{\Delta}, \qquad z = \frac{\Delta_z}{\Delta}$$

Where, $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$

$$\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Worked Example:

1. Solve the following equations using Cramer's rule $x + y - z = 4$, $3x - y + z = 4$, $2x - 7y + 3z = -6$

Solution:

$x + y - z = 4$ ----- (1)

$3x - y + z = 4$ ----- (2)

$2x - 7y + 3z = -6$ ----- (3)

$$\begin{bmatrix} 1 & 1 & -1 \\ 3 & -1 & 1 \\ 2 & -7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ -6 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 1 & 1 & -1 \\ 3 & -1 & 1 \\ 2 & -7 & 3 \end{vmatrix}$$

$\Rightarrow$ $1(-3+7) - 1(9-2) - 1(-21+2)$

$\Rightarrow$ $1(4) - 1(7) - 1(-19)$

$\Rightarrow$ $4 - 7 + 19$

$\boxed{\Delta = 16}$

$$\Delta_x = \begin{vmatrix} 4 & 1 & -1 \\ 4 & -1 & 1 \\ -6 & -7 & 3 \end{vmatrix} = 4(-3+7) - 1(12+6) - 1(-28-6)$$

$\Rightarrow$ $4(4) - 1(18) - 1(-34)$

$\Rightarrow$ $16 - 18 + 34$

$\boxed{\Delta_x = 32}$

$$\Delta_y = \begin{vmatrix} 1 & 4 & -1 \\ 3 & 4 & 1 \\ 2 & -6 & 3 \end{vmatrix} \Rightarrow 1\,(12 + 6) - 4\,(9-2) - 1\,(-18 - 8)$$

$\Rightarrow 18 - 4\,(7) - 1\,(-26)$

$\Rightarrow 18 - 28 + 26 \quad \Rightarrow \boxed{\Delta_y = 16}$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 4 \\ 3 & -1 & 4 \\ 2 & -7 & -6 \end{vmatrix} \Rightarrow 1\,(6 + 28) - 1\,(-18 - 8) + 4\,(-21 + 2)$$

$\Rightarrow 34 + 26 - 76$

By Cramer's Rule $\Rightarrow \boxed{\Delta_z = -16}$

$x = \frac{\Delta_x}{\Delta} = \frac{32}{16} = 2$

$y = \frac{\Delta_y}{\Delta} = \frac{16}{16} = 1$

$z = \frac{\Delta_z}{\Delta} = \frac{-16}{16} = -1$

$\therefore$ Solution: {2, 1, -1}

2. Solve the following equation using Cramer's Rule

$4x + y + z = 6, \qquad 2x - y - 2z = -6, \qquad x + y + z = 3$

Solution:

$4x + y + z = 6$

$2x - y - 2z = -6$

$x + y + z = 3$

Equation can be written as AX = B

$$\begin{bmatrix} 4 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 3 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 4 & 1 & 1 \\ 2 & -1 & -2 \\ 1 & 1 & 1 \end{vmatrix} \Rightarrow 4\,(-1 + 2) - 1\,(2 + 2) + 1\,(2 + 1)$$

$\Rightarrow 4(1) - 1\,(4) + 1\,(3)$

$\boxed{\Delta = 3}$

$$\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ -6 & -1 & -2 \\ 3 & 1 & 1 \end{vmatrix} \Rightarrow 6\,(-1 + 2) - 1\,(-6+6) + 1\,(-6+ 3)$$

$\Rightarrow 6(1) - 1\,(0) + 1\,(-3)$

$\Rightarrow 6 - 3$

$\boxed{\Delta_x \Rightarrow 3}$

$$\Delta_y = \begin{vmatrix} 4 & 6 & 1 \\ 2 & -6 & -2 \\ 1 & 3 & 1 \end{vmatrix} \Rightarrow 4(-6+6) - 6(2+2) + 1(6+6)$$

$\Rightarrow 4(0) - 6(4) + 1(12)$

$\Rightarrow -24 + 12$

$$\boxed{\Delta_y \Rightarrow -12}$$

$$\Delta_z = \begin{vmatrix} 4 & 1 & 6 \\ 2 & -1 & -6 \\ 1 & 1 & 3 \end{vmatrix} = 4(-3+6) - 1(6+6) + 6(2+1)$$

$\Rightarrow 4(3) - 1(12) + 6(3)$

$\Rightarrow 12 - 12 + 18$

$\Delta_z = 18$

By Cramer's Rule

$x = \frac{\Delta_x}{\Delta} = \frac{3}{3} = 1$

$y = \frac{\Delta_y}{\Delta} = \frac{-12}{3} = -4$

$z = \frac{\Delta_z}{\Delta} = \frac{18}{3} = 6$

Exercise

1) Solve the following equations by Cramer's rule:

   i) $3x + 3y - z = 11$, $2x - y + 2z = 9$, $4x + 3y + 2z = 25$

   ii) $\frac{3}{x} - \frac{4}{y} - \frac{2}{z} - 1 = 0$, $\frac{1}{x} - \frac{2}{y} + \frac{1}{z} = 0$, $\frac{2}{x} - \frac{5}{y} - \frac{4}{z} + 1 = 0$,

   iii) $x + 2y - z = -3$, $3x + y + z = 4$, $x - y + 2z = 6$

   iv) $x + 2y + 5z = 4$, $3x + y + 4z = 6$, $-x + y + z = -1$

   v) $4x + y + z = 6$, $2x - y - 2z = -6$, $x + y + z = 3$

2) A fish tank can be filled in 10 minutes using both pumps A and B simultaneously. However pump B can pump water in or out at the same rate. If pump B is inadvertently run in reverse, then the tank will be filled in 30 minutes. How long would it take each pump to fill the tank by itself?

   [use Cramer's Rule)

3) A chemist has one solution which is 50% acid and another solution which is 25% acid. How much each should be mixed to make 10 litres of a 40% acid solution?

   [use Cramer's Rule)

Matrix Inversion Method:

This method can be applied only when the co-efficient matrix is a square matrix and non-singular.

Consider the matrix equation in the form, $\boxed{A\,x = B}$ ----- (1)

Here A is a square matrix and it is non-singular.

Since A is non-singular, A^{-1} exists and $A^{-1}A = AA^{-1} = I$. ----- (2)

pre-multiply A^{-1} on both sides of (1),

$$\Rightarrow \quad A^{-1}(AX) = A^{-1}(B)$$

$$(A^{-1}A)\,x = A^{-1}(B)$$

$$\boxed{X = A^{-1}B}$$

This is the formula for Matrix inversion method. This method also can be used for finding the solution of linear equations.

Worked Examples

1) Solve the following system of linear equations by matrix – inversion method.

$2x + 5y = -2, \quad x + 2y = -3$

Solution:

Given $2x + 5y = -2$ ----- (1)

$x + 2y = -3$ ----- (2)

It can be written in the form A X = B.

$$\begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

By Matrix – inversion formula, $\boxed{X = A^{-1}B}$

$$A^{-1} = \frac{1}{|A|}\,\text{adj } A$$

$$|A| = \begin{vmatrix} 2 & 5 \\ 1 & 2 \end{vmatrix} = 4 - 5 = -1$$

$$\text{adj } A = \begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$$

$$\because A^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & -5 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & +5 \\ +1 & -2 \end{bmatrix}$$

$$A^{-1}B = \begin{bmatrix} -2 & 5 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 - 15 \\ -2 + 6 \end{bmatrix}$$

$$A^{-1}B = \begin{bmatrix} -11 \\ 4 \end{bmatrix}$$

$$\therefore x = A^{-1} B = \begin{bmatrix} -11 \\ 4 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \end{bmatrix}$$

$\therefore$ Solution (x, y) = (-11, 4)

2) Solve the following system of equations using Matrix inversion – Method.

$2x + 3y + 3z = 5,\qquad x - 2y + z = -4,\qquad 3x - y - 2z = 3.$

Solution:

Given, $2x + 3y + 3z = 5$

$x - 2y + z = -4$

$3x - y - 2z = 3$

It can be written in matrix form AX = B

$$\begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

By the Matrix – inversion formula, $\boxed{X = A^{-1}B}$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$|A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix}$$

$= 2(4 + 1) - 3(-2 - 3) + 3(-1 + 6)$

$\Rightarrow 2(5) - 3(-5) + 3(5)$

$\Rightarrow 10 + 15 + 15$

$|A| \Rightarrow 40$

Adj A = $[Co - factor\ matrix\ \ of\ A]^{T}$

Co-factor of 2 = $\begin{vmatrix} -2 & 1 \\ -1 & -2 \end{vmatrix}$ = 4 + 1 = 5

Co-factor of 3 = - $\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}$ = - (-2 – 3) = + 5

Co-factor of 3 = $\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}$ = -1 + 6 = 5

Co-factor of 1 = - $\begin{vmatrix} 3 & 3 \\ -1 & -2 \end{vmatrix}$ = - (-6 + 3) = 3

Co-factor of -2 = $\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix}$ = -4 -9 = -13

Co-factor of 1 = - $\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix}$ = - (- 2 – 9) = 11

Co-factor of 3 = $\begin{vmatrix} 3 & 3 \\ -2 & 1 \end{vmatrix}$ = 3 + 6 = 9

Co-factor of -1 = $-\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix}$ = - (2 – 3) = 1

Co-factor of -2 = $\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix}$ = - 4 – 3 = - 7

adj A = [co-factor matrix of A]^T = $\begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}^T$

$$\text{adj } A = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\because\ A^{-1} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

Then X = A^{-1} B

$$X = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

So the solution x = 1, y = 2, z = -1

3) Solve the following linear equations by Matrix inversion method.

x + y + z – 2 = 0, 6x – 4y + 5z – 31 = 0, 5x + 2y + 2z = 13

Solution: Given, x + y + z = 2

6x – 4y + 5z = 31

5x + 2y + 2z = 13

We can write in the form AX = B

$$\begin{bmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$$

By the Matrix – inversion formula,

$$X = A^{-1}B$$

$$A^{-1} = \frac{1}{|A|}\text{ adj } A$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 6 & -4 & 5 \\ 5 & 2 & 2 \end{vmatrix}$$

= 1 (- 8 - 10) – 1(12 – 25) + 1 (12 + 20)

$\Rightarrow 1\,(-18) - 1(-13) + 1\,(32)$

$|A| \Rightarrow -18 + 13 + 32$

$\boxed{|A| \Rightarrow 27}$

Co-factor of 1= $\begin{vmatrix} -4 & 5 \\ 2 & 2 \end{vmatrix}$ = - 8 – 10 = - 18

Co-factor of 1 = $-\begin{vmatrix} 6 & 5 \\ 5 & 2 \end{vmatrix}$ = - (12 – 25) = 13

Co-factor of 1 = $\begin{vmatrix} 6 & -4 \\ 5 & 2 \end{vmatrix}$ = 12 + 20 = 32

Co-factor of 6 = $-\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$ = - (2 – 2) = 0

Co-factor of - 4 = $\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix}$ = 2 – 5 = - 3

Co-factor of 5 = $-\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix}$ = - (2 – 5) = 3

Co-factor of 5 = $\begin{vmatrix} 1 & 1 \\ -4 & 5 \end{vmatrix}$ = 5 + 4 = 9

Co-factor of 2 = $-\begin{vmatrix} 1 & 1 \\ 6 & 5 \end{vmatrix}$ = - (5 – 6) = 1

Co-factor of 2 = $\begin{vmatrix} 1 & 1 \\ 6 & -4 \end{vmatrix}$ = - 4 – 6 = - 10

Co-factor of A $= \begin{bmatrix} -18 & 13 & 32 \\ 0 & -3 & 3 \\ 9 & 1 & -10 \end{bmatrix}$

adj A = [Co-factor of A]^T $= \begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}$

$X = A^{-1} B$

$= \frac{1}{27}\begin{bmatrix} -18 & 0 & 9 \\ 13 & -3 & 1 \\ 32 & 3 & -10 \end{bmatrix}\begin{bmatrix} 2 \\ 31 \\ 13 \end{bmatrix}$

$= \frac{1}{27}\begin{bmatrix} -36 + 0 + 117 \\ 26 - 93 + 13 \\ 64 + 93 - 130 \end{bmatrix}$

$= \frac{1}{27}\begin{bmatrix} 81 \\ -54 \\ 27 \end{bmatrix}$

$X = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ So the solution x = 3, y = -2, z = 1

Exercise

1) Solve the following system of linear equations by Matrix Inversion Method.
   a) 2x – y = 8, 3x + 2y = -2
   b) 2x + 3y – z = 9, x + y + z = 9, 3x – y – z = -1
   c) x + y + 2z = 1, 3x + 2y + z = 7, 2x + y + 3z = 2
2) A man is appointed in a job with the monthly salary of certain amount and a fixed amount of annual increment. If his salary was ` 19,800 per month at the end of the first month after 3 years of service and ` 23,400 per month at the end of first month after 9 years of service, find his starting salary and his annual increment. (Use Matrix inversion method)
3) Four men and 4 women can finish a piece of work jointly in 3 days while 2 men 5 woman can finish the same work jointly in 4 days. Find the time taken by one man alone and that of one woman alone to finish the same work by using matrix inversion method.

Gaussian Elimination Method:

This method can be applied even if the co-efficient matrix is singular matrix and rectangular matrix. It is essentially the method of substitution which we have already seen.

In this method we transform the augmented matrix of the system of linear equations into row-echelon form and then by back-substitution. Then we get a solution.

Note:

AX = B is a matrix equation involving matrices & it is called as matrix form of the system of linear equations. Then the matrix [A | B] is called augmented matrix.

Worked Examples

1) Solve the following system of linear equation by Gauss – elimination method.

4x + 3y + 6z = 25, x + 5y + 7z = 13, 2x + 9y + z = 1

Given:

4x + 3y + 6z = 25

x + 5y + 7z = 13

2x + 9y + z = 1

We have to transforming the augmented matrix to echelon form,

We get,

$$\Rightarrow \left[\begin{array}{ccc|c} 4 & 3 & 6 & 25 \\ 1 & 5 & 7 & 13 \\ 2 & 9 & 1 & 1 \end{array}\right]$$

$\boxed{R_1 \leftrightarrow R_2}$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 4 & 3 & 6 & 25 \\ 2 & 9 & 1 & 1 \end{array}\right]$$

$R_2 \rightarrow R_2 - 4R_1$, $R_3 \rightarrow R_3 - 2R_2$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & -17 & -22 & -27 \\ 0 & -1 & -13 & -25 \end{array}\right]$$

$R_2 \rightarrow R_2 \div (-1)$, $R_3 \rightarrow R_3 \div (-1)$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & +17 & +22 & +27 \\ 0 & +1 & +13 & +25 \end{array}\right]$$

$R_3 \rightarrow 17R_3 - R_2$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 7 & 13 \\ 0 & 17 & 22 & 27 \\ 0 & 0 & 199 & 398 \end{array}\right]$$

The equivalent system is written by using echelon form

$x + 5y + 7z = 13$ ----- (1)

$17y + 22z = 27$ ----- (2)

$199z = 398$ ----- (3)

From (3) we get

$199z = 398$

$z = 398/199$

$\boxed{Z = 2}$

Sub z value in (2)

(1) $\Rightarrow$ $17y + 22(2) = 27$

$17y + 44 = 27$

$17y = 27 - 44$

$17y = -17$

$y = -17/17$

$\boxed{y = -1}$

Sub: $y = -1$, $z = 2$ in (1)

$x + 5y + 7z = 13$

$x + 5(-1) + 7(2) = 13$

$x + (-5) + 14 = 13$

$x + 9 = 13$

$x = 13 - 9$

$$\boxed{x = 4}$$

So the solution is x = 4, y = -1, z = 2.

2) Solve the following system of linear equation by Gaussian elimination method.

$2x - 2y + 3z = 2, \quad x + 2y - z = 3, \quad 3x - y + 2z = 1$

Solution:

Given:

$$2x - 2y + 3z = 2$$
$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$

Transforming the augmented matrix to echelon form,

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{array}\right]$$

$R_2 \rightarrow R_2 - 2R_1, \quad R_3 \rightarrow R_3 - 3R_1$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{array}\right]$$

$R_3 \rightarrow 7R_2 - 6R_3$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & 0 & 5 & 20 \end{array}\right]$$

The equivalent system is written by using echelon form

$x + 2y - z = 3$ ----- (1)

$-6y + 5z = -4$ ----- (2)

$5z = 20$ ----- (3)

From (3) we get, $5z = 20$

$z = 20/5$

$$\boxed{Z = 4}$$

Sub z value in (2)

$-6y + 5(4) = -4$

$-6y = -4 - 20$

$-6y = -24$

$y = -24/-6$

$\boxed{y = 4}$

Sub: $y = 4$, $z = 4$ in (1)

$(1) \Rightarrow \quad x + 2y - z = 3$

$x + 2(4) - 4 = 3$

$x + 4 = 3$

$x = 3 - 4$

$\boxed{x = -1}$

So the solution $\quad x = -1$

$y = 4$

$z = 4$

Exercise

1) Solve the following system of linear equation by Gaussian elimination method.

i) $2x + 4y + 6z = 22$, $\quad 3x + 8y + 5z = 27$, $\quad -x + y + 2z = 2$

ii) $3x - 4y - 2z = 1$, $\quad x + 2y + z = 2$, $\quad 2x - 5y - 4z = -1$

iii) $3x + 3y - z = 11$, $\quad 2x - y + 2z = 9$, $\quad 4x + 3y + 2z = 25$

2) A boy is walking along the path $y = ax^2 + bx + c$ through the points (-6, 8) (-2, -12) and (3, 8). He wants to meet his friend at p (7, 60) will he meet his friend?

[Use Gaussian elimination method]

Characteristic Equation of a Matrix:

Consider the linear transformation $Y = AX$

In general, this transformation transforms a column vector.

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ into another column vector $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ by means of the square matrix A where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

If a vector x is transformed into a scalar multiple of the same vector.

i.e. x is transformed into λx,

then $y = \lambda x = AX$

i.e. $AX = \lambda X$

$AX = \lambda IX$,

Where I is the unit matrix of order 'n'.

$AX - \lambda I\, x = 0$

$(A - \lambda I)\, x = 0$

Then the system of equations will have a non-trivial solution, if

$| A - \lambda I | = 0$

This is called the characteristic equation of A.

Worked Examples

1. Find the characteristic equation of the matrix $\begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

The characteristic equation of A is $| A - \lambda I | = 0$

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix}$$

$$| A - \lambda I | = \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$(1 - \lambda)(4 - \lambda) + 2 = 0$

$4 - \lambda - 4\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$\therefore$ The required characteristic equation is $\boxed{\lambda^2 - 5\lambda + 6 = 0}$

2. Find the characteristic equation of $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Solution:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

The characteristic equation of A is $| A - \lambda I | = 0$

$$\left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)\,[\,(1 - \lambda)(1 - \lambda)\,] - 0 = 0$

$(1 - \lambda)^3 = 0$

$-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = 0$

$\therefore$ The required characteristic equation is $\boxed{\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0}$

Exercise

1. Find the characteristic equation of the following matrices:

i) $\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

iii) $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ iv) $\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

v) $\begin{bmatrix} 2 & -2 \\ -2 & 1 \end{bmatrix}$ vi) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

vii) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ viii) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

ix) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ x) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Eigenvalues and Eigenvectors of a real Matrix:

Eigen values (or) characteristic roots:

Let A = $[a_{ij}]$ be a square matrix of order of n.

The characteristic equation of A is $|A - \lambda I| = 0$. The roots of this equation are called characteristic roots of the Matrix A. Characteristic roots are also called Eigen values.

Eigen Vector (or) Characteristic Vectors:

Let A = $[a_{ij}]$ be a square Matrix of order 'n', Let X be any non-zero column vector.

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Then this equation $AX = \lambda X$ has a non-zero solution of X corresponding to each value of λ is called Eigen Vector or Characteristic Vector (or) latent vector of A.

Worked Examples

1. Find the Eigen values of $\begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

Solution:

Let A = $\begin{bmatrix} a & h & g \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} a-\lambda & h & g \\ 0 & b-\lambda & 0 \\ 0 & 0 & c-\lambda \end{vmatrix} = 0$$

$(a - \lambda)\,[\,(b - \lambda)(c - \lambda) - 0\,] - h(0) + g(0) = 0$

$(a - \lambda)(b - \lambda)(c - \lambda) = 0$

$\therefore \lambda = a, b, c$

2. Find the Eigen values of Eigen vectors of the matrix $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Solution:

$$A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

The characteristic equation is A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$(2 - \lambda)[(1 - \lambda)(-1-\lambda) - 3] + 2[-1 - \lambda - 1] + 2[3 - (1 - \lambda)] = 0$

$(2 - \lambda)(\lambda - 2)(\lambda + 2) = 0$

$\therefore$ The Eigen values of A are $\lambda = -2, 2, 2.$

Case: 1

$\lambda = -2$

The Eigenvector is given by

$$\begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 1 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{-8} = \frac{x_2}{-2} = \frac{x_3}{14}$$

$$\frac{x_1}{-4} = \frac{x_2}{-1} = \frac{x_3}{7}$$

$$X_1 = \begin{bmatrix} -4 \\ -1 \\ 7 \end{bmatrix}$$

Case: 2

$\lambda = 2$

The Eigen vector is given by

$$\begin{bmatrix} 0 & -2 & 2 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\frac{x_1}{0} = \frac{x_2}{4} = \frac{x_3}{4}$$

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore X_2 = X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Exercise

1. Define Eigen values of a Matrix.
2. Define Eigen vectors of a Matrix.
3. If two Eigen values are equal then the Eigen vectors are linearly __________.
4. The characteristic roots of a real Symmetric Matrix are __________.

5. Find the Eigen vectors of A = $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

6. Find the Eigen roots of the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and prove that the product of Eigen value is unity.

7. Find the Eigen values and Eigen vectors of the following matrices.

a) $\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 2 & -2 \\ 0 & 1 & 1 \\ 2 & -1 & 4 \end{bmatrix}$

a) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Consistency and inconsistency of system of linear Algebraic equations:

Let us consider a system of linear Algebraic equations.

$$a_{11}\,x_1 + a_{12}\,x_2 + \ldots\ldots\ldots\ldots.\; a_{1n}\,x_n = b_1$$
$$a_{21}\,x_1 + a_{22}\,x_2 + \ldots\ldots\ldots\ldots.\; a_{2n}\,x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}\,x_1 + a_{m2}\,x_2 + \ldots\ldots\ldots\ldots.\; a_{mn}\,x_n = b_m$$

The equations can be written in the form of matrix equation AX = B.

Where A, X, B are matrices of order mxn, nx1 and mx1 respectively.

A set of values of x_1, x_2 ……… x_n, which satisfy all the given m equations is known as solution of the equation.

When the system of equations has a solution, it is said to be consistent. Otherwise the system is said to be inconsistent.

A consistent system may have either only one (or) infinitely many solutions. When the system has only one solution, it is called the Unique Solution.

Condition for Consistency

Rouche's Theorem:

The system of equations AX = B is consistent, if and only if the coefficient Matrix A and the augmented matrix [A, B] are of the same rank.

Note:

The necessary and sufficient condition for the consistency of a system of linear non-homogeneous equations is provided by a theorem, called Rouches's theorem.

Working Rule for finding the solution.

Consider the equation AX = B , m equations and n unknowns.

i) Find Rank A and Rank [A, B]

ii) If Rank A ≠ Rank [A, B].

The equations are inconsistent i.e. they have no solution.

iii) If Rank A = Rank [A, B] = r (say).

The equations are consistent i.e. they possess a solution

If r = n then the solution is unique.

If r < n then there are infinite number of solutions.

1) Verify whether the given system of equations is consistent. If it is consistent, solve them:

$x - y + z = 5, \quad -x + y - z = -5, \quad 2x - 2y + 2z = 10$

Solution:

The matrix equation corresponding to the given system is

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$$

$$AX = B$$

The augmented matrix is

$$[A, B] = \begin{bmatrix} 1 & -1 & 1 & 5 \\ -1 & 1 & -1 & -5 \\ 2 & -2 & 2 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 - 2R_1$

in the last equivalent matrix, there is only one non-zero, $\therefore \rho\,[A, B] = 1$ and $\rho\,(A) = 1$.

Thus $\rho\,(A) = \rho\,(A, B) = 1$.

$\therefore$ The given system is consistent, since the rank is less than the number of unknowns. There are infinitely many solutions. The given system is equivalent to the matrix equation.

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$$

$x - y + z = 5$; taking $y = k_1$, $z = k_2$, we have $x = 5 + k_1 - k_2$, for various values of K_1 and K_2. We have infinitely many solutions, $k_1, k_2 \in R$.

2) Solve the following homogeneous linear equations.

$x + 2y - 5z = 0, \quad 3x + 4y + 6z = 0, \quad x + y + z = 0$

Solution:

The given system of equations can be written in the form of matrix equation.

$$\begin{bmatrix} 1 & 2 & -5 \\ 3 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = B$$

The augmented matrix is

$$[A, B] = \begin{bmatrix} 1 & 2 & -5 & 0 \\ 3 & 4 & 6 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & -2 & 21 & 0 \\ 0 & -1 & 6 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & -2 & 21 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$= \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & -1 & 6 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

This is in the echelon form clearly ρ (A, B) = 3 and ρ (A) = 3.

$\therefore$ ρ (A) = ρ [A, B] = 3 = number of unknowns.

$\therefore$ The given system of equations is consistent and has a unique solution.

i.e. trivial solution,

$\therefore$ x = 0, y = 0 and z = 0.

3) Test for the consistency of the following system of equation $x_1 - 2x_2 - 3x_3 = 2$, $3x_1 - 2x_2 = -1$, $-2x_2 - 3x_3 = 2$ and $x_2 + 2x_3 = 1$.

Solution:

The system can be put as

$$\begin{bmatrix} 1 & -2 & -3 \\ 3 & -2 & 0 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

i.e AX = B (say)

$$(A, B) = \begin{bmatrix} 1 & -2 & -3 & 2 \\ 3 & -2 & 0 & -1 \\ 0 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 4 & 9 & -7 \\ 0 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

$(R_2 \to R_2 - 3R_1)$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 9 & -7 \\ 0 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \quad (C_2 \to C_2 + 2C_1, \quad C_3 \to C_3 + 3C_1, \quad C_4 \to C_4 - 2C_1)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -3 & 2 \\ 0 & 4 & 9 & -7 \end{bmatrix} \quad (R_2 \leftrightarrow R_4)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 4 & 9 & -7 \end{bmatrix} \quad (R_3 \to R_3 + 2R_2)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & -11 \end{bmatrix} \quad (C_3 \to C_3 - 2C_2, \ C_4 \to C_4 - C_2)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -15 \end{bmatrix} \quad (R_4 \to R_4 - R_3)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -15 \end{bmatrix} \quad (C_4 \to C_4 - 4C_3)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (R_4 \to -\frac{1}{15} R_4)$$

$\therefore$ R [A, B] = 4

But R (A) $\neq$ 4, as A is a (4 x 3) matrix,

The value of the minor

$$\begin{vmatrix} 3 & -2 & 0 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \end{vmatrix} \neq 0$$

Thus R(A) $\neq$ R [A, B]

$\therefore$ the given system is inconsistent.

4) Test for consistency, if possible, balance the chemical reaction.

Equation

$C_5 H_8 + O_2 \to CO_2 + H_2O$

The above is the reaction that is taking place in the burring of organic compound called isoprene.

Solution:

Let x_1, x_2, x_3 & x_4 are positive integers such that

$x_1 C_5 H_8 + x_2 O_2 = x_3 Co_2 + x_4 H_2O$ ----- (1)

The number of carbon atoms on the LHS of (1) should be equal to the number of carbon atoms on RHS of (1).

We get,

$5x_1 = x_3 \Rightarrow 5x_1 - x_3 = 0$ ----- (2)

Similarly, considering hydrogen and oxygen atoms, we get

$8x_1 = 2x_4 \Rightarrow 4x_1 - x_4 = 0$ ----- (3)

$2x_2 = 2x_3 + x_4 \Rightarrow 2x_2 - 2x_3 - x_4 = 0$ ----- (4)

$$[A \mid B] = \left[\begin{array}{cccc|c} 5 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$$

$$[A \mid B] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 5 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right]$$

$$[A \mid B] = \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 5 & 0 & -1 & 0 & 0 \end{array}\right] \quad R_2 \leftrightarrow R_3$$

$$= \left[\begin{array}{cccc|c} 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & -4 & 5 & 0 \end{array}\right] \quad R_3 \leftrightarrow 4R_3 - 5R_1$$

$\therefore \rho(A) = \rho(A, B) = 3 < 4$, No. of unknowns.

The system is consistent and has infinite number of solutions.

$4x_1 - x_4 = 0,\ 2x_2 - 2x_3 - x_4 = 0,\ -4x_3 + 5x_4 = 0$

So, one of the unknowns should be chosen arbitrarily as a non – zero real number.

Let us choose $x_4 = t,\ t \neq 0$.

$\therefore$ We get $x_1 = \frac{t}{4},\ x_2 = \frac{7t}{4},\ x_3 = \frac{5t}{4}$

Exercise

Test for consistency of the following system of equations and solve, if consistent:

a) $2x - y + 3z = 9;\ -x + 2y - z = 4;\ 3x + y - 4z = 0.$

b) $x + 2y - z = 6;\ -3x - 2y + 5z = -12;\ x - 2z = 3.$

c) $x + 2y - z = 5;\ x - y + z = -2;\ -5x - 4y + z = -11.$

d) $x - 2y + 3z = 2;\ 2x - 3z = 3;\ x + y + z = 0$

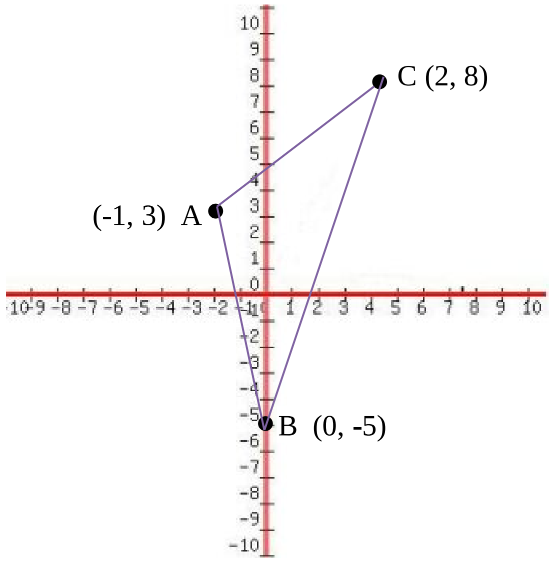
e) $3x + y - 3z = 1;\ -2x - y + 2z = 1;\ -x - y + z = 2$

f) $x + y - z = 1,\ 2x + 2y - 2z = 2,\ -3x - 3y + 3z = -3.$

Practical Problems

1. Find the Area of the Triangle formed by the points (-1, 3), (0, -5) and (2, 8)

Solution:



Given, A (-1, 3) = (a_1, b_1)

B (0, -5) = (a_2, b_2)

C (2, 8) = (a_3, b_3)

Area of the Triangle $= \pm\frac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_1 & 1 \end{vmatrix}$

$= \pm\frac{1}{2}\begin{vmatrix} -1 & 3 & 1 \\ 0 & -5 & 1 \\ 2 & 8 & 1 \end{vmatrix}$

$= \pm\frac{1}{2}\ [-1(-5-8) - 3(0-2) + 1(0+10)]$

$= \pm\frac{1}{2}\ [-1(-13) - 3(-2) + 1(10)]$

$= \pm\frac{1}{2}\ [13 + 6 + 10]$

$= \pm\frac{1}{2}\ [29]$

Area of the triangle = ± 14.5 square units

2. Find the final grades for Alexandra, Megan and Brigida.

Student	Tests	Projects	Home work	Quizzes
Alexandra	92	100	89	80
Megan	72	85	80	75
Brigida	88	78	85	92

Weightages for test 40%, project 15%, homework 25% & Quizzes 20%.

Solution:

Form a Matrix to the Given Data

$$\text{Let } A = \begin{bmatrix} 92 & 100 & 89 & 80 \\ 72 & 85 & 80 & 75 \\ 88 & 78 & 85 & 92 \end{bmatrix}$$

Along the weightages for test, project, homework, Quizzes

$$\text{Let } B = \begin{bmatrix} 40\% \\ 15\% \\ 25\% \\ 20\% \end{bmatrix} \Rightarrow \begin{bmatrix} 0.4 \\ 0.15 \\ 0.25 \\ 0.20 \end{bmatrix}$$

By Matrix Multiplication,

$$AB = \begin{bmatrix} 92 & 100 & 89 & 80 \\ 72 & 85 & 80 & 75 \\ 88 & 78 & 85 & 92 \end{bmatrix} \begin{bmatrix} 0.4 \\ 0.15 \\ 0.25 \\ 0.20 \end{bmatrix}$$

$$AB = \begin{bmatrix} 36.8 + 15 + 22.25 + 16 \\ 28.8 + 12.75 + 20 + 15 \\ 35.2 + 11.7 + 21.25 + 18.4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 90.05 \\ 76.55 \\ 86.55 \end{bmatrix}$$

∵ Alexandra has grade 90,

Megan has 77,

Brigida has 87.

3. In a competitive Examination one mark is awarded for every correct answer while ¼ mark is deducted for every wrong answer. A student answered 100 questions and got 80 marks.

∵ How many questions did he answer correctly?

Solution:

x = No. of questions answered correctly

y = No. of questions answered wrongly

$\because$ x + y = 100------ (1)

$x - \frac{1}{4}y = 80$

4x - y = 320 ------ (2)

By Cramer's Rule,

$$\Delta = \begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} = -1 - 4 \Rightarrow -5$$

$$\Delta_x = \begin{vmatrix} 100 & 1 \\ 320 & -1 \end{vmatrix} = -100 - 320 \Rightarrow -420$$

$$\Delta_y = \begin{vmatrix} 1 & 100 \\ 4 & 320 \end{vmatrix} = 320 - 400 \Rightarrow -80$$

$$x = \frac{\Delta x}{\Delta} = \frac{-420}{-5} \Rightarrow 84$$

$$y = \frac{\Delta y}{\Delta} = \frac{-80}{-5} \Rightarrow 16$$

$\because$ He answered 84 questions correctly.

4. An electronics warehouse ships televisions and DVD players in certain combinations to retailers throughout the country. The weight of 2 televisions and 5 DVD players is 62.5 pounds, and the weight of 3 televisions and 2 DVD players is 52 pounds. Then solve it to find out each television and DVD player weights (use matrix inversion method to solve the problem).

Solution:

Televisions	DVD Player	Weight in Pounds
2	5	62.5
3	2	52

Let the weight of Television & DVD player be x and y respectively.

2x + 5y = 62.5

3x + 2y = 52

Writing the above equations in matrix form

$$\begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 62.5 \\ 52 \end{bmatrix}$$

AX = B

$\Rightarrow x = A^{-1}B$

To find A^{-1} : $A^{-1} = \frac{1}{|A|}$ adj A

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix} = 4 - 15 = -11 \neq 0$$

$$\text{adj } A = \begin{bmatrix} +2 & -5 \\ -3 & +2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{-11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

$$\text{Now } x = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B$$

$$= \frac{1}{-11} \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 62.5 \\ 52 \end{bmatrix}$$

$$= \frac{1}{-11} \begin{bmatrix} 125 & -260 \\ -187.5 & 104 \end{bmatrix}$$

$$= \frac{1}{-11} \begin{bmatrix} -135 \\ -83.5 \end{bmatrix}$$

$$X = \begin{bmatrix} 12.27 \\ 7.59 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$\therefore$ Each Television weight is 12.27 pounds and each DVD player weight is 7.59 pounds

5) You are in IT organization that wants to equip some new cabinets into your office. You contact a furniture company, and they informed you that cabinet A costs Rs. 10 per unit, requires 5 square feet of floor space. The cost of cabinet B is 20 per unit, requires 15 square feet of floor space. You have space only for 125 square feet of cabinets and a budget of Rs. 200 to make purchase. How many of models A & B should you be purchased ?

Solution:

	A	B	Total
Cost	10	20	200
Space	5	15	125

Let x = No. of model A cabinets purchased

y = No. of model B cabinets purchased.

$$10x + 20y = 200 \Rightarrow x + 2y = 20$$

$$5x + 15y = 125 \Rightarrow x + 3y = 25$$

The matrix form of the above equations are

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 20 \\ 25 \end{bmatrix}$$

$$AX = B$$

The augment Matrix (A, B) is

$$[A, B] = \begin{bmatrix} 1 & 2 & 20 \\ 1 & 3 & 25 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 20 \\ 0 & 1 & 5 \end{bmatrix} \quad R_2 \to R_2 - R_1$$

The above matrix is in echelon form. Now writing the equivalent equations.

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 20 \\ 5 \end{bmatrix}$$

$x + 2y = 20$

$\boxed{y = 5}$

$x + 2(5) = 20$

$x = 20 - 10$

$\boxed{x = 10}$

No. of model A cabinets purchased = 10

No. of model B cabinets purchased = 5

6) A Company Manufacturers two products A and B. Both products are processed on two machines M_1& M_2

Machine / Products	M_1	M_2
A	6 hrs/unit	2 hrs/unit
B	4 hrs/unit	4 hrs/unit
Availability	7200 hrs/month	4000 hrs/month

Profit per unit for A is Rs. 100 and for B is Rs. 80. Find out the monthly production of A and B is to maximize profit by Graphical Method (or) and also in Matrix Method.

Solution:

By Graphical Method

Formulation of LPP

x_1 = No. of units of A/Month

x_2 = No. of units of B/Month

Objective Function:

Max $z = 100\,x_1 + 80\,x_2$

Subject to the constraints,

$6x_1 + 4x_2 \leq 7200$

$2x_1 + 4x_2 \leq 4000$

$x_1, x_2 \geq 0$

Constraint (1)

$$6x_1 + 4x_2 \le 7200$$

Converting into equality,

$$6x_1 + 4x_2 = 7200$$

Let $x_1 = 0$ $\quad \therefore 4x_2 = 7200$

$$x_2 = 1800$$

$$\boxed{\therefore P_1\ (0, 1800)}$$

Let $x_2 = 0$ $\quad \because 6x_1 = 7200$

$$x_1 = 1200$$

$$\boxed{\therefore P_2\ (1200, 0)}$$

Constraint (2)

$$2x_1 + 4x_2 \le 4000$$

Convert into equality,

$$2x_1 + 4x_2 = 4000$$

Let $x_1 = 0$ $\quad \therefore 4x_2 = 4000$

$$x_2 = 1000$$

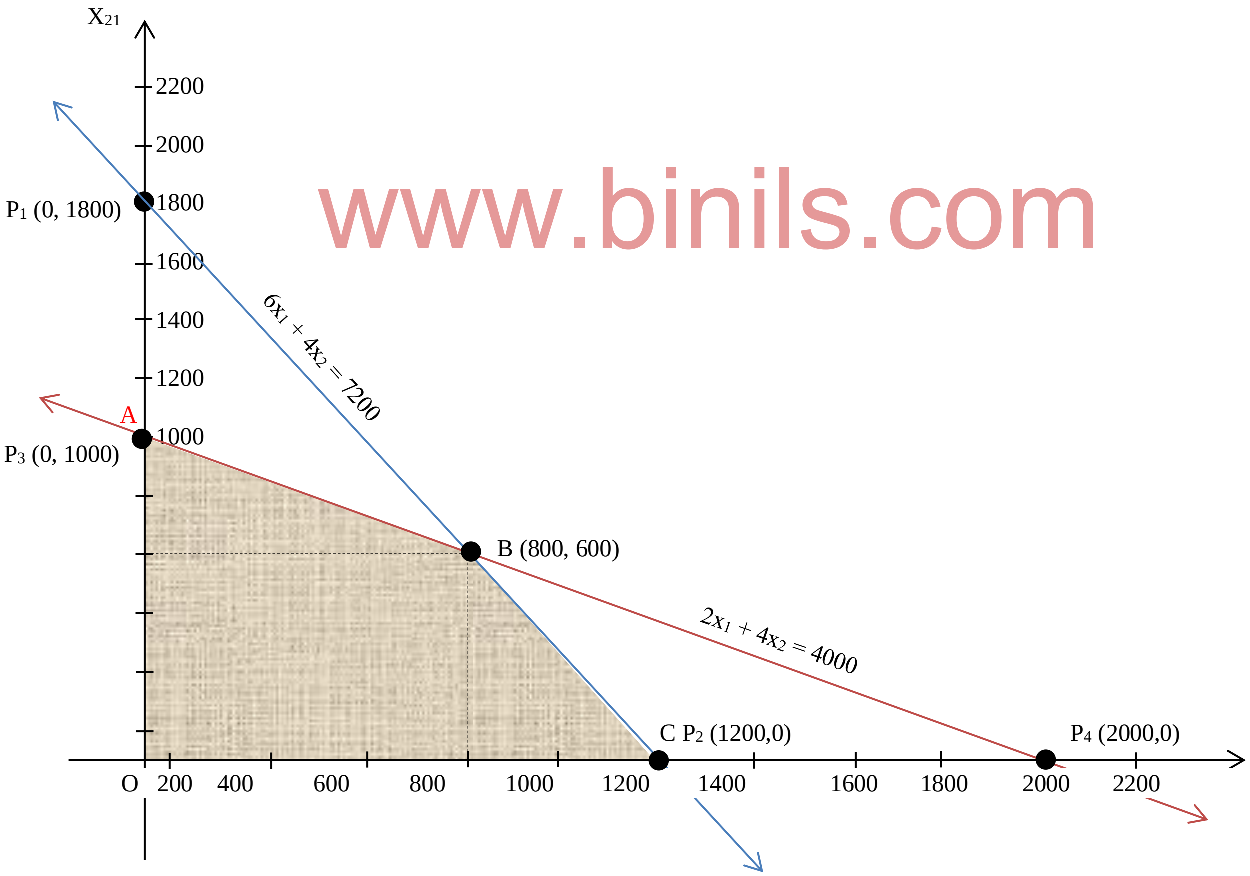
$$\boxed{\therefore P_3\ (0, 1000)}$$

Let $x_2 = 0$ $\quad 2x_1 = 4000$

$$x_1 = 2000$$

$$\boxed{\therefore P_4\ (2000, 0)}$$

Draw a Graph to the points P_1, P_2, P_3, P_4



From the Graph,

We can identify the feasible Region OABC then the points are 0 (0, 0) .A (0, 1000), B (800, 600) C (1200, 0).

Then we have to find the optimal solution, O (0, 0) $Z = 100\, x_1 + 80\, x_2$

$= 100\,(0) + 80\,(0)$

$Z = 0$

A (0, 1000) $Z = 100\,(0) + 80\,(1000)$

$Z = 80{,}000$

B (800, 600) $Z = 100\,(800) + 80\,(600)$

$= 80{,}000 + 48{,}000$

$Z = 1{,}28{,}000$

C (1200, 0) $Z = 100\,(1200) + 90\,(0)$

$Z = 1{,}20{,}000$

We want maximise of Z

$\because$ Max Z = 1,28,000 at B (800, 600)

$\because$ The optimal solution is $X_1 = 800$

$X_2 = 600$

Matrix Method:

Max $Z = 100\, X_1 + 80\, X_2$

$6\, X_1 + 4X_2 = 7200$ ----- (1)

$2\, X_1 + 4X_2 = 4000$ ----- (2)

$AX = B$

$$\begin{bmatrix} 6 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7200 \\ 4000 \end{bmatrix}$$

By Matrix Inversion Method, $\boxed{X = A^{-1} B}$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A$$

$$|A| = \begin{vmatrix} 6 & 4 \\ 2 & 4 \end{vmatrix} = 24 - 8 = 16$$

$$\text{adj } A = \begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix}$$

$$\because A^{-1} = \frac{1}{16} \begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix}$$

$$X = \frac{1}{16} \begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} 7200 \\ 4000 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -4 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} 450 \\ 250 \end{bmatrix}$$

$$= \begin{bmatrix} 1800 - 1000 \\ -900 + 1500 \end{bmatrix}$$

$$X = \begin{bmatrix} 800 \\ 600 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 800 \\ 600 \end{bmatrix}$$

$\because X_1 = 800 \quad X_2 = 600$

$\because$ Max Z $= 100\, x_1 + 80\, x_2$

$= 100\,(800) + 80\,(600)$

$= 80{,}000 + 48{,}000$

Max Z $= 1{,}28{,}000$

From above these two methods (i) Graphical Method (or) (ii) Matrix Method. We have same optimal solution and maximize profit.

7) A house wife wishes to mix two types of food F_1 and F_2 in such a way that the vitamin contents of the mixture contain at least 8 units of Vitamin A and 11 units of Vitamin B. Food F_1 costs 60/kg and Food F_2 costs 80/kg. Food F_1 contains 3 units/kg of Vitamin A and 5 units/kg of Vitamin B while Food F_2 contains 4 units/kg of Vitamin A and 2 units / kg of Vitamin B. Formulate this problem as a LPP to minimize the cost of the mixture. Also find the optimal solution.

Solution: (i) By Graphical Method:

Vitamin Content	Food (in Kg)		Requirement (in units)
	F_1	F_2	
Vitamin A (Units/Kg)	3	4	8
Vitamin B (Units/Kg)	5	2	11
Cost (E/Kg)	60	80	

Decision Variables:

x = Mixture of Food F_1 in Kg

y = Mixture of Food F_2 in Kg

Objective Function:

Min C = 60X + 80Y

Subject to the Constraints:

$3x + 4y \geq 8$ ---------- (1)

$5x + 2y \geq 11$ ---------- (2)

$x \geq 0,\ y \geq 0$

Constraint: 1

$$3x + 4y \geq 8$$

$$3x + 4y = 8$$

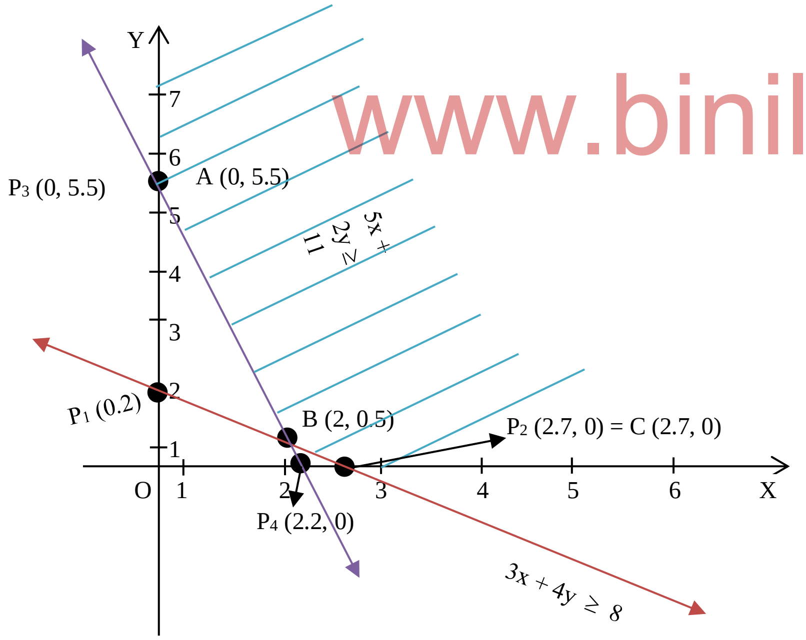
If $x = 0$	$4y = 8$	If $y = 0$	$3x = 8$
	$y = 2$		$x = 8/3$
			$x = 2.7$
	$\therefore P_1\ (0, 2)$		$P_2\ (2.7, 0)$

Constraint: 2

$$5x + 2y \geq 11$$

$$5x + 2y = 11$$

If $x = 0$	$2y = 11$	If $y = 0$	$5x = 11$
	$y = 11/2$		$x = 11/5$
	$y = 5.5$		$x = 2.2$
	$P_3\ (0, 5.5)$		$P_4\ (2.2, 0)$



All the constraints are greater than or equal to the type.

$\because$ Feasible Region should be above (to the Right of) all constraints.

$\because$ The vertices of the feasible region are A, B, C.

By Corner Point Method:

$$\text{Min } C = 60\,x + 80\,y$$

A (0, 5.5) $\quad \text{Min } C = 60\,(0) + 80\,(5.5)$

$= 0 + 440$

Min C $= 440$

B (2, 0.5) Min C $= 60 (2) + 80 (0.5)$

$= 120 + 40$

Min C $= 160$

C (2.7, 0) Min C $= 60 (2.7) + 80 (0)$

Min C $= 162$

We want Minimize C

$\because$ Min C = 160 at B (2, 0.5)

$\because$ The optimal solution is $x = 2$

$Y = 0.5 = \frac{1}{2}$

$\because$ 2 Kg Mixture of food F_1 and

½ kg Mixture of food F_2

Matrix Method:

Min C $= 60x + 80y$

$3x + 4y = 8$ ------ (1)

$5x + 2y = 11$ ------ (2)

$AX = B$

$$\begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

By Matrix Inversion Method

$X = A^{-1} B$

$A^{-1} = \frac{1}{|A|} \text{ adj } A$

$$|A| = \begin{vmatrix} 3 & 4 \\ 5 & 2 \end{vmatrix} = 6 - 20 = -14$$

$|A| = -14$

$$\text{Adj } A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix} \because A^{-1} = \frac{1}{-14} \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$$

$X = A^{-1} B$

$$X = \frac{1}{-14} \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 11 \end{bmatrix}$$

$$= \frac{1}{-14} \begin{bmatrix} 16 - 44 \\ -40 + 33 \end{bmatrix}$$

$$X = \frac{1}{-14} \begin{bmatrix} -28 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$$

$\because$ x= 2, y = 0.5

Min C = 60 x + 80 y

= 60 (2) + 80 (0.5)

= 120 + 40

Min C = 160

From above these two methods (i) Graphical Method, (ii) Matrix Method.

We have same optimal solution and minimize cost of mixture.

8) A small – scale industry manufactures electrical regulators the assembly of which being accomplished by a small group of 11 skilled workers both men and women. Due to limitations of space and finance, the number of workers employed cannot exceed and their salary bill not more than Rs. 60,000 per month.

The male members of the skilled workers are paid Rs. 6,000 per month, while the female worker doing the same work as the male member gets Rs. 5,000. Data collected on the performance of these workers indicate that a male member contributes Rs. 10,000 per month to total return of the industry, while the female worker contribute Rs. 8500 per month. Determine the member of male and female workers to be employed in order to maximize the monthly total return.

Solution: By Graphical Method:

Let X_1 = No. of Male workers to be employed

X_2 = No. of Female workers to be employed

Objective function Max Z = 10000 X_1 + 8500 X_2

Subject to the Constraints

$x_1 + x_2 \leq 11$ ------ (1)

$6000\,x_1 + 5000\,x_2 \leq 60{,}000$

$6\,x_1 + 5\,x_2 \leq 60$ ------ (2)

$x_1, x_2 \geq 0$

Constraint : 1

$x_1 + x_2 \leq 11$

$x_1 + x_2 = 11$

If $x_1 = 0$ $x_2 = 11$

P_1 (0, 11)

$x_2 = 0$, $x_1 = 11$

P_2 (11, 0)

Constraint : 2

$6x_1 + 5x_2 \leq 60$

$6x_1 + 5x_2 = 60$

If $x_1 = 0$ $5x_2 = 60$

$x_2 = 12$

$\because P_3$ (0, 12)

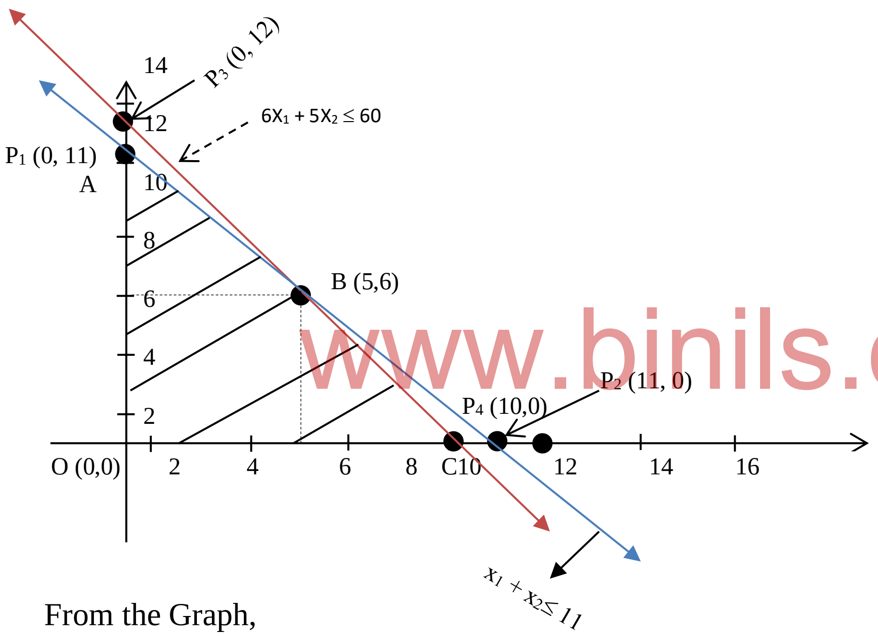
If $x_2 = 0$ $6x_1 = 60$

$x_1 = 60/6$

$x_1 = 10$

$\because P_4$ (10, 0)

Draw a Graph to the points P_1(0, 11), P_2(11, 0), P_3(0, 12), P_4(10, 0)



From the Graph,

We can identify the feasible Region OABC then the points are 0 (0, 0), A (0, 11), B (5, 6) C (10, 0).

At O (0, 0) MaxZ = $10000\, x_1 + 8500\, x_2$

= 10000 (0) + 8500 (0)

Max Z = 0

At A (0, 11) Max Z = 10000 (0) + 8500 (11)

= 93,500

At B (5, 6) Max Z = 10000 (5) + 8500 (6)

= 50,000 + 51,000

= 1,01,000

At C (10, 0) Max Z = 10000 (10) + 8500 (0)

= 1,00,000

But we want maximise of Z

$\because$ Max Z = 1,01,000 at B (5, 6)

Then the optimal solution is

$X_1 = 5, \quad X_2 = 6$

(ii) Matrix Method:

Max Z = 10000 X_1 + 85000 X_2

$X_1 + X_2 = 11$ ----- (1)

$6X_1 + 5X_2 = 60$ ----- (2)

AX = B

$$\begin{bmatrix}1 & 1\\6 & 5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}11\\60\end{bmatrix}$$

By Matrix Inversion Method, $\boxed{X = A^{-1} B}$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A$$

$$|A| = \begin{vmatrix}1 & 1\\6 & 5\end{vmatrix}$$

$$= 5 - 6$$

$$|A| = -1$$

$$\text{Adj } A = \begin{bmatrix}5 & -1\\-6 & 1\end{bmatrix}$$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix}5 & -1\\-6 & 1\end{bmatrix} = \begin{bmatrix}-5 & 1\\6 & -1\end{bmatrix}$$

$$X = A^{-1} B$$

$$= \begin{bmatrix}-5 & 1\\6 & -1\end{bmatrix}\begin{bmatrix}11\\60\end{bmatrix}$$

$$= \begin{bmatrix}-55+60\\66-60\end{bmatrix}$$

$$\begin{bmatrix}X_1\\X_2\end{bmatrix} = \begin{bmatrix}5\\6\end{bmatrix}$$

$\because X_1 = 5 \qquad X_2 = 6$

$\because$ Max Z = 10000 x_1 + 8500 (x_2)

= 10000 (5) + 8500 (6)

= 50,000 +51,000

Max Z = 1,01,000

Hence We have same optimal solution in both way.

Chapter 1.3 BINOMIAL THEOREM

Introduction:

The word ‘Binomial’ stands for expressions having <u>two terms</u>”.

i.e. Bi-nomial $\Rightarrow$ Bi = two nomial = terms

eg. 1) Bicycle
2) Binocular
3) Binary

} which all are represents two things

Factorial Notation:

The product of first ‘n’ natural numbers.

- Denoted by ∟n $\Rightarrow$ (read as factorial n)
  n! $\Rightarrow$ (read as ‘n’ factorial)
- Defined by

$$n! = 1 \times 2 \times 3 \times \ldots\ldots\ldots \times n.$$

For a positive integer ‘n’

$$n! = n \times (n-1) \times (n-2) \times \ldots\ldots\ldots \times 3 \times 2 \times 1.$$
$$= n\,(n-1)\,! \text{ for } n > 1$$
$$= n\,(n-1)\,(n-2)\,! \text{ for } n > 2 \text{ \& so on.}$$

Eg. 1! = 1

2! = 1 x 2 = 2

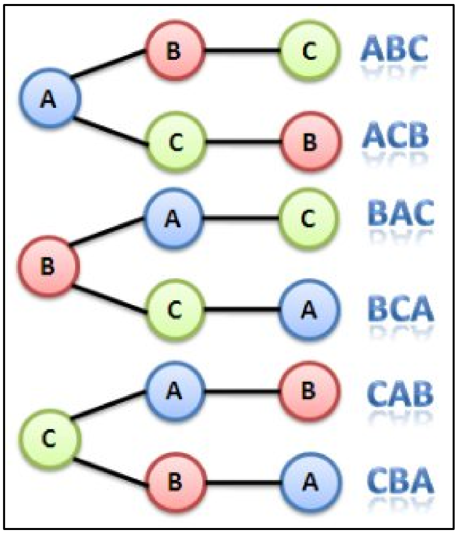
3! = 1 x 2 x 3 = 6

> The notation n! was introduced by Christian Kramp French Mathematician

Permutation:

The number of arrangements that can be made out of ‘n’ things taken ‘r’ at a time is called the Permutation of n things taken r at a time.

Consider any three object,



Thus above example shows six possible ways in which they can arrange themselves.

Thus if 3 objects have to be arranged in a row there are 3 x 2 x 1 = 3! Possible permutations (taken at a time).

Like that 4 objects (taken at a time) = 4!

5 objects (taken at a time) = 5!

Suppose you have 7 objects, we want to make group of four objects. Then we get the result npr

npr i.e = $7p_4$

Generally, the number of distinct permutations of r objects which can be made from n distinct objects

$$\Rightarrow \frac{n!}{(n-r)!}$$

Which is denoted by nP_r

Combinations: (Selections)

Combination is the selection of n things taken r at a time.

Consider the 'Badminton' team with 4 members named as 'A, B, C, D'. But the first match will allow only two members.

The possible choices are : AB : AC : AD : BC, BD : CD.

Thus the number of combinations of 4 different players taken 2 at a time is 6.

The number of combinations of n different objects taken r at a time is represented by nCr.

Note: $\boxed{nPr = nC_r \times r!}$

The above result is relationship between permutation & combination.

Value of permutation & combination

Permutation	Combination
1) nPr	nCr
2) number of permutation of n different things taken r at a time	The number of combination of 'n' things taken 'r' at a time
3) n, r (non-negative integers) $r \leq n$ $nP_r = \frac{\lfloor n}{\lfloor n-r} = \frac{n!}{(n-r)!}$ (1) $nP_n = n!$ (2) $nP_1 = n$	n, r (non-negative integers) $r \leq n$ $nC_r = \frac{n!}{(n-r)!\, r!}$ (1) $nC_n = 1$ (2) $nC_o = 1$ (3) $nC_1 = n$ (4) $nC_r = nC_{n-r}$ (5) $\frac{nCr}{nC_{(r-1)}} = \frac{n-r+1}{r}$ (6) If $nC_r = nC_2 \Rightarrow$ either r = 2 $r + q = n$ [$\because nC_r = nC_{n-r}$

Examples:

1. Find the numbers of permutations of 6 objects taken 4 at a time.

Solution:

$$\boxed{nP_r = \frac{n!}{(n-r)!}}$$

$$6P_4 = \frac{\lfloor 6}{\lfloor 6-4}$$

$$= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 6 \times 5 \times 4 \times 3 = 360.$$

2. In how many ways can 7 books be arranged in a shelf?

Solution:

7 books can be arranged in 7! ways = 5040.

3. In how many ways 4 boys can be seated in a long bench?

4 boys can be seated in 4! Ways = 24

4. Find the value of (i) $7C_5$ (ii) $10C_4$ (iii) $6C_2$ (iv) $8C_8$ (v) $5C_o$ (vi) $10C_7$

Solution:

(i) $7C_5 = 7C_5 = 7C_{7-5} = 7C_2$ [$\because nC_r = n\,C_{n-r}$]

$= \frac{7 \times 6}{1 \times 2} = 21$

(ii) $10C_4 = \frac{10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4} = 210$

(iii) $6C_2 = \frac{6 \times 5}{1 \times 2} = 15$

(iv) $8C_8 = 1$ $\{\because nC_n = 1\}$

(v) $5C_o = 1$ $\{\because nC_o = 1\}$

(vi) $10C_7 = 10C_3$

$= \frac{10 \times 9 \times 8}{3 \times 2 \times 1} = 120$

Binomial Expressions:

This chapter deals with the expansion of the various powers of binomial expression.

Before we state the General theorem, we can recall some known results for known powers of Binomial expressions.

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

The above results may easily be verified by direct multiplication. But for any positive integer n, we define the general theorem known as Binomial theorem as given below.

Binomial Theorem for Positive Integral Index:

For any positive integer n,

$$(x + a)^n = x^n + nC_1\, x^{n-1} a^1 + nC_2\, x^{n-2} a^2 + nC_3\, x^{n-3} a^3 + \ldots\ldots + nC_r\, x^{n-r} a^r + \ldots + a^n$$

Note:

1. Number of terms in the expansion is n + 1
2. General term of Binomial expansion is $nC_r\, x^{n-r}\, a^r$

   i.e. $(r + 1)^{th}$ term = $nC_r\, x^{n-r}\, a^r$

   i.e. $T_{r+1} = nC_r\, x^{n-r}\, a^r$

   Every term is of n^{th} degree in x and a.
3. nC_1, nC_2 ……. nC_r etc. are called the Binomial coefficients. The Binomial co-efficients are all integers since nC_r is an integer.
4. Co-efficients of terms equidistant from the beginning and the end of the expansion are equal since $nC_r = nC_{n-r}$.

Example Problems:

1) Find the general term in the expansion of $\left[X + \frac{2}{x}\right]^{12}$

Solution:

$$\left[X + \frac{2}{x}\right]^{12}$$

Here X = x, a = $\frac{2}{x}$, n = 12

We know that the general term is

$$\begin{aligned} T_r + 1 &= nC_r\, x^{n-r}\, a^r \\ &= 12C_r\, (x)^{12-r} \left[\frac{2}{x}\right]^r \\ &= 12C_r\, (x)^{12-r} \left(\frac{2^r}{x^r}\right) \\ &= 12C_r 2^r\ x^{12-r-r} \\ &= 12C_r 2^r\ x^{12-2r} \end{aligned}$$

$\therefore$ The general term is $12C_r\, 2^r\, x^{12-2r}$

2) Find the general term in the expansion of $\left[x^2 - \frac{2}{x}\right]^9$

Solution:

The general term of $(x + a)^n$ is $T_{r+1} = nC_r\, x^{n-r} a^r$

$$\begin{aligned} T_{r+1} &= 9C_r\, (x^2)^{9-r} \left(\frac{-2}{x}\right)^r \\ &= 9C_r\, x^{18-2r} \frac{(-2)^r}{(x)^r} \\ &= 9C_r\, x^{18-2r} (-2)^r.\, x^{-r} \\ &= 9C_r\, (-2)^r\, x^{18-2r-r} \\ T_{r+1} &= 9C_r\, (-2)^r\, x^{18-3r} \end{aligned}$$

Here X = x^2

a = $\frac{-2}{x}$

n = 9

3) Find the 7^{th} term in the expansion of $\left[x^2 + \frac{1}{x}\right]^{10}$

Solution:

Here $X = x^2$, $a = \frac{1}{x}$, $n = 10$

The general term is $T_{r+1} = nC_r\, x^{n-r}\, a^r$

To find the 7^{th} term. Put r = 6.

$\therefore\ T_{6+1} = 10C_6\, (x^2)^{10-6} \left(\frac{1}{x}\right)^6$

i.e. $T_7 = 10C_6\, (x^2)^4 \cdot \frac{1}{x^6}$

$= 10C_6\, x^8 \cdot \frac{1}{x^6}$ $\quad [\because r + 1 = 7$

$= 10C_6\, x^{8-6}$ $\quad r = 7 - 1$

$T_7 = 10C_6\, x^2$ $\quad r = 6$

$\therefore$ The 7^{th} term is $10\,C_6\, x^2$

Note: The middle term / terms in the expansion of $(x + a)^n$ with n+1 terms.

Cases:

1. n is even. There exists only one middle term that is $\left[\frac{n+2}{2}\right]^{th}$ term.
2. n is odd. There exists two middle term. They are $\left[\frac{n+1}{2}\right]^{th}$, $\left[\frac{n+3}{2}\right]^{th}$ terms.

4. Expand $[5x - y]^4$ using Binomial theorem.

Solution:

Given $[5x - y]^4$

In the expansion of $[5x - y]^4$; there are 5 terms. Each term can be found out using the formula.

$T_{r+1} = nC_r\, x^{n-r}\, a^r$

Here $x = 5x$

$a = -y$

$n = 4$

$r = 0, 1, 2, 3, 4$

Put $\boxed{r = 0}$

$T_1 = 4C_0\, (5x)^{4-0}\, (-y)^0$

$= 1(5^4 x^4) \cdot 1$

$T_1 = 625x^4$

Put $\boxed{r = 1}$

$T_2 = 4C_1 \cdot (5x)^{4-1}\, (-y)^1$

$= 4 . (5x)^3 (-y)$

$= 4.(5)^3 x^3 (-y)$

$= -500 x^3 y$

Put $\boxed{r = 2}$

$T_3 = 4C_2 . (5x)^{4-2} (-y)^2$

$= \frac{4 \times 3}{1 \times 2} . (5x)^2 y^2$

$= 6.(5)^2 x^2 y^2$

$= 150 x^2 y^2$

Put $\boxed{r = 3}$

$T_4 = 4C_3 (5x)^{4-3} (-y)^3$

$= 4C_1 (5x)^1 (-)y^3$

$= - 4 (5x) y^3$

$= -20 xy^3$

Put $\boxed{r = 4}$

$T_5 = 4C_4 (5x)^{4-4} (-y)^4$

$= 1 . (1) . y^4$

$(5x - y)^4 = T_1 + T_2 + T_3 + T_4 + T_5$

$= 625 x^4 - 500 x^3y + 150 x^2y^2 - 20xy^3 + y^4$

5. Find the middle term in the expansion of $\left[2x^4 + \frac{3}{x}\right]^{16}$

Solution:

$(2x^4 + \frac{3}{x})^{16}$, Here n is even. So, we have only one middle term.

Middle term $= \left(\frac{n+2}{2}\right)^{th}$ term

$= \left(\frac{16+2}{2}\right)^{th}$ term

$= 9^{th}$ term

$\therefore$ Middle term = 9^{th} term.

$T_9 = ?$	$n = 16$
$r + 1 = 9$	$x \sim 2x^4$
$\Rightarrow r = 8$	$a = \frac{3}{x}$

$T_{r+1} = nC_r x^{n-r} a^r$

$\therefore T_9 = 16C_8 (2x^4)^{16-8} \left(\frac{3}{x}\right)^r$

$= 16C_8\, (2x^4)^8 \left(\frac{3}{x}\right)^8$

$= 16C_8\, 2^8\, 3^8\, \frac{x^{32}}{x^8}$

$T_9 = 16C_8\, 6^8\, x^{24}$

6) Find the co-efficient of x^{32} in the expansion of $\left(x^4 + \frac{1}{x^3}\right)^{15}$.

Solution: Here $n = 15$

$x \sim x^4$, $a = \frac{1}{x^3}$

General term

$T_{r+1} = nC_r\, x^{n-r}\, a^r$

$= 15Cr\, (x^4)^{15-r} \left[\frac{1}{x^3}\right]^r$

$= 15Cr\, (x)^{60-4r} \left[\frac{1}{x^{3r}}\right]$

$= 15Cr \left[\frac{x^{60-4r}}{x^{3r}}\right]$

$T_{r+1} = 15Cr\, x^{60-7r}$ ------------------------- (1)

We assume that x^{32} occurs in $15C_r\, x^{60-7r}$

$\therefore x^{60-7r} = x^{32}$

Equating the indices $60 - 7r = 32$

$-7r = 32 - 60$

i.e. $-7r = -28$

i.e. Put $\boxed{r = 4}$ in (1)

Co-efficient of x^{32} = coefficient of x^{60-7r}

$T_{4+1} = 15C_4$

$T_5 = 15C_4$

Independent Term

When we find co-efficient of x^o, we get independent term of x.

7) Find the term independent of 'x' in the expansion of $\left(\sqrt{x} - \frac{1}{x^2}\right)^{20}$

Solution:

Here $n = 20$

$x = \sqrt{x}$

$a = -\frac{1}{x^2}$

General Term $= T_{r+1} = n\, Cr\, x^{n-r}\, a^r$

$= 20\, C_r \left(\sqrt{x}\right)^{20-r} \left(-\frac{1}{x^2}\right)^r$

$$= 20\,C_r x^{\frac{20-r}{2}}\left(-\frac{1}{x^2}\right)^r$$

$$= 20\,C_r x^{\frac{20-r}{2}}(-1)^r x^{-2r}$$

$$= 20\,C_r x^{\frac{20-r}{2}\,-2r}(-1)^r$$

$$= 20\,C_r x^{\frac{20-r-4r}{2}}(-1)^r$$

$$= 20\,C_r x^{\frac{20-5r}{2}}(-1)^r$$

To find the independent term of x, find the co-efficient of x^o.

$$x^{\frac{20-5r}{2}} = x^0$$

$$\frac{20-5r}{2} = 0$$

$$20-5r = 0$$

$$-5r = -20$$

$$r = 4$$

Put r = 4 in (1)

$\therefore$ Independent term of x $= 20\,C_r\,(-1)^r$

$= 20\,C_4\,(-1)^4$

$= 20\,C_4$.

Binomial Theorem for Rational Index:

In the previous chapter, we have seen the expansion of Binomial expression only for positive integer. Now, we will see the expansion of Binomial expression for any rational number.

Statements of Binomial Theorem for a Rational Index:

If 'n' is a rational number and $-1 < x < 1$,

then $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots.$

$+ \frac{n(n-1)\ldots\ldots(n-r+1)}{r!}x^r + \ldots\ldots. + \infty.$

Note: (1) The binomial theorem for rational index is true only when the first term is 1 and $|x| < 1$.

(2) Number of terms in this expansion is infinite.

Expansion using Binomial theorem for rational index upto 3

1) $(1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \ldots$

$= 1 + x + x^2 + x^3 + \ldots$

2) $(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{1.2}x^2 + \frac{(-1)(-2)(-3)}{1.2.3}(-x)^3 + \ldots.$

$= 1 - x + x^2 - x^3 + x^4 + \ldots$

3) $(1-x)^{-2} = 1 + (-2)(-x) + \frac{(-2)(-3)}{1.2}(-x)^2 + \ldots.$

$= 1 + 2x + 3x^2 + 4x^3 + \ldots$

4) $(1+x)^{-2} = 1 + (-2)(x) + \frac{(-2)(-3)}{1.2}(-x)^2 + \ldots.$

$= 1 - 2x + 3x^2 - 4x^3 + \ldots$

5) $(1+x)^{-3} = 1 + (-3)x + \frac{(-3)(-4)}{1.2}x^2 + \frac{(-3)(-4)(-5)}{1.2.3}(-x)^3 + \ldots.$

$= 1 - 3x + 6x^2 - 10x^3 + \ldots\ldots \ldots$

6) $(1-x)^{-3} = 1 + (-3)(-x) + \frac{(-3)(-4)}{2}(-x)^2 + \frac{(-3)(-4)(-5)}{1.2.3}(-x)^3 + \ldots.$

$= 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots \ldots.$

7) Expand $(1 – 2x)^{-3}$ using Binomial theorem.

Solution:

$(1 – 2x)^{-3} = 1 + (-3)(-2x) + \frac{(-3)(-4)}{2}(-2x)^2 + \frac{(-3)(-4)(-5)}{1.2.3}(-2x) + \ldots\ldots..$

$= 1 + 6x + 6(4x^2) + 10(8x^3) + \ldots\ldots.$

$= 1 + 6x + 24x^2 + 80x^3 + \ldots\ldots$

8) Expand $(3 – 4x)^{-3}$ using Binomial theorem.

$(3 – 4x)^{-3} = 3^{-3}(1 - \frac{4}{3}x)^{-3}$

$$= \frac{1}{3^3}\left[1 + (-3)\left(\frac{-4}{3}x\right) + \frac{(-3)(-4)}{1.2}\left(\frac{-4}{3}x\right)^2 + \frac{(-3)(-4)(-5)}{1.2.3}\left(\frac{-4}{3}x^3\right) + \ldots\ldots\ldots\right]$$

$$= \frac{1}{27}\left[1 + 4x + \frac{32}{3}x^2 + \cdots\right]$$

Exercises

1. Find the value of $11C_4$
2. How many middle terms are there in the expansion of $(3x^2 – 5)^{10}$
3. Find 7^{th} term in $\left(x^2 - \frac{1}{x}\right)^{10}$
4. Find 11^{th} term in $\left(x + \frac{3}{x}\right)^{20}$
5. Find the general term in $\left(2x - \frac{1}{2y^2}\right)^{17}$
6. Find the general term in $(3x + y)^8$
7. Find the middle term in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{14}$
8. Find the middle term in the expansion of $\left(x^4 + \frac{1}{x^3}\right)^{11}$
9. Find the coefficients of x^{32} in the expansion of $\left(x^4 - \frac{2}{x^3}\right)^{15}$
10. Find the coefficients of x^{-17} in the expansion of $\left(3x^4 + \frac{1}{x^3}\right)^{15}$
11. Find the term independent of x in the expansion of $\left(2x^2 - \frac{1}{x}\right)^{12}$

12. Find the term independent of x in the expansion $\left(4x^3 + \frac{3}{x^2}\right)^{20}$

13. Expand the following using binomial theorem

a) $(4 + 3x)^{-1}$ b) $(1 + 3x)^{-1}$

b) $(5 - 2x)^{-2}$ d) $(1 + 5x)^{-2}$

c) $(1 - 3x)^{-3}$ f) $(1 + 2x)^{-3}$

14. Find the first three terms in the expansion of the following

a) $(1 - 4x)^{-2}$ b) $(1 - 2x)^{-3}$

b) $(1 - 5x^2)^{-2}$ d) $(1 - 3x^2)^{-2}$

c) $\left[1 - \frac{x}{2}\right]^{-2}$ f) $\left[1 - \frac{4x}{5}\right]^{-3}$

Application of Binomial theorem

Binomial theorem has a wide range of application in mathematics like

1. Finding the remainder
2. Finding digits of a number
3. Finding greatest term

Worked Examples

1. Finding digits of a number using binomial theorem

(i) $(99)^5$ (ii) $(102)^6$ (iii) $(10.1)^5$

$(99)^5 = (100 - 1)^5 = 5C_o\,(100)^5 - 5C_1\,(100)^4 + 5C_2\,(100)^3 - 5C_3\,(100)^2 + 5C_4\,(100) - 5C_5$

$= (100)^5 - 5\,(100)^4 + 10\,(100)^3 - 10\,(100)^2 + 5\,(100) - 1$

$= 10^{10} - 5 \times 10^8 + 10^7 - 10^5 + 5 \times 10^2 - 1$

$= (10^{10} + 10^7 + 5 \times 10^2) - (5 \times 10^8 + 10^5 + 1)$

$= 10010000500 - 500100001$

$= 9509900499.$

(ii) $(102)^6 = (100 + 2)^6$

$= 6C_o\,(100)^6 + 6C_1\,(100)^5\,2 + 6C_2\,(100)^4\,2^2 + 6C_3\,(100)^3 2^3 + 6C_4(100)^2\,2^4 + 6C_5(100)^1 2^5 + 6C_6(100)^o 2^6.$

$= (100)^6 + 6 \times (100)^5 \times 2 + 15 \times (100)^4 \times 2^2 + 20 \times (100)^3 \times 2^3 + 15 \times (100)^2 \times 2^4 + 6 \times (100)^1\,2^5 + 2^6.$

$= 10^{12} + 12 \times 10^{10} + 6 \times 10^9 + 16 \times 10^7 + 24 \times 10^5 + 192 \times 10^2 + 64$

$= 1126162419264.$

(iii) $(10.1)^5 = (10 + 0.1)^5$

$= 5C_o\,(10)^5(0.1)^o + 5C_1\,(10)^4\,(0.1) + 5C_2\,(10)^3\,(0.1)^2 + 5C_3\,(10)^2\,(0.1)^3 + 5C_4\,(10)^1\,(0.1)^4 + 5C_5\,(10)^o\,(0.1)^5$

$= (10)^5 + 5 \times 10^4 \times 0.1 + 10 \times 10^3 \times (0.1)^2 + 10 \times (10)^2 \times (0.1)^3 + 5 \times 10 \times (0.1)^4 + (0.1)^5$

$= 10^5 + 5 \times 10^3 + 10^2 + 1 + 5 \times 0.001 + 0.00001$

$= 100000 + 5000 + 100 + 1 + 0.005 + 0.00001$

$= 105101.00501$

Finding the Remainder

1. Find the remainder when 7^{103} is divided by 25.

$(7^{103} \div 25) = [7(7^2)^{51}/25]$

$= [7(49)^{51}/25]$

$= 7\,(50-1)^{51} / 25$

$= 7\,[\,51C_0\,(50)^{51} - 51C_1\,(50)^{50} \ldots\ldots - 51C_{51}\,] / 25$

$= 7\,[\,51C_0\,(50)^{51} - 51C_1\,(50)^{50} \ldots\ldots - 1\,] / 25$

$= [\,7\,(51C_0\,(50)^{51} - 51C_1\,(50)^{50} \ldots\ldots) - 7\,] \div 25$

$= [\,7\,(51C_0\,(50)^{51} - 51C_1\,(50)^{50} \ldots\ldots) - 7 + 18 - 18\,] \div 25$

$= [\,7\,(51C_0\,(50)^{51} - 51C_1\,(50)^{50} \ldots\ldots) - 25\,] + 18$

$= K + 18$ ($\because$ K is divisible of 25)

$\therefore$ remainder $= 18$

2. Show that $11^9 + 9^{11}$ is divisible by 10.

$11^9 + 9^{11} = (10 + 1)^9 + (10 - 1)^{11}$

$= (9c_0\, 10^9 + 9C_1\, 10^8 + \ldots 9c_9) + (11c_0\, 10^{11} - 11C_1\, 10^{10} + \ldots - 11c_{11})$

$= 9c_0\, 10^9 + 9C_1 10^8 + \ldots + 9c_8\, 10 + 1 + 11c_0\, 10^{11} - 11c_1\, 10^{10} + \ldots - 11c_{10}\, 10 - 1$

$= 10\,[\,9c_0\, 10^8 + 9C_1\, 10^7 + \ldots + 9c_8 + 11c_0\, 10^{10} - 11c_1\, 10^9 + \ldots + 11c_{10}\,]$

$= 10k$, which is divisible by 10.

Find the digits of the number

1. Find the last two digits of the number $(13)^{10}$

Solution:

$(13)^{10} = (13^2)^5$

$= (169)^5$

$= (170 - 1)^5$

$(13)^{10} = 5C_0\,(170)^5 - 5C_1\,(170)^4 + 5C_2\,(170)^3 - 5C_3\,(170)^2 + 5C_4\,(170) - 5C_5$

$= [\,5C_0\,(170)^5 - 5C_1\,(170)^4 + 5C_2\,(170)^3 - 5C_3\,(170)^2\,] + 5\,(170) - 1$

$=$ [A multiple of 100] $+ 850 - 1$

$= 100\,K + 849$

$\therefore$ The last two digits are 49.

2. Find the last two digits of the number 7^{400}.

Solution:

$7^{400} = (7^2)^{200} = 49^{200} = (50 - 1)^{200}$

$= 200C_o\, 50^{200} – 200C_1\, 50^{199} + … + 200C_{199}\, 50\, (-1)^{199} + 200\, C_{200}\, (-1)^{200}$

$= 50^2\, [\, 200C_o\, 50^{198} – 200C_1\, 50^{197} + …. + 200\, C_{198}\, (-1)^{198}] – 200 \times 50 + 1$

As 50^2 and 200 are divisible by 100, the last two digits of 7^{400} is 01.

Finding the Greatest Term:

- If $\{ (n + 1) | x | / | x | + 1 \} = p$ is a positive integer then P^{th} term and $(p + 1)^{th}$ terms are numerically greatest terms in the expansion of $(1 + x)^n$.
- If $[(n + 1) | x |] / [| x | + 1]$. $P + F$, where P is a positive integer $o < F < 1$ then $(P + 1)^{th}$ term is greatest term in expansion of $(1 + x)^n$.

1. Find the numerically greatest term in $(1 – 3x)^{10}$ when $x = (1/2)$

Solution:

$[(n + 1) | a |] / [| a | + 1] = [(11 \times 3/2)] / [(3/2) + 1]$

$= \frac{(33/2)}{(5/2)}$

$= \frac{33}{\cancel{2}} \times \frac{\cancel{2}}{5} = \frac{33}{5}$

2. Find the greatest term in $(1 + 5x)^9$ when $x = ¼$

$[(n + 1) | a |] / [| a | + 1]$

$= [(10 \times \frac{5}{4})] / [5/4 + 1]$

$= \frac{50/4}{9/4} \Rightarrow \frac{50}{\cancel{4}} \times \frac{\cancel{4}}{9} = \frac{50}{9}$

Exercise Problems:

1) Finding the digits of a number using Binomial expansion

(i)	$(52)^9$	(vi)	$(13.2)^5$
(ii)	$(65)^8$	(vii)	$(109)^5$
(iii)	$(112)^5$	(viii)	$(120)^4$
(iv)	$(125)^7$	(ix)	$(158)^6$
(v)	$(12.1)^6$	(x)	$(50.1)^5$

2) Finding the remainder using the binomial theorem

(i) $6^{102} \div 20$ (ii) $5^{98} \div 30$

3) Find the remainder when $9^{50} \div 13$

4) Show that $10^7 + 8^2$ is divisible by 8

5) Show that $19^2 + 6^3$ is divisible by 5.

Determinants of order 2 and 3

Properties of Determinants 1 to 5

Property 6

Introduction to Matrix and Cramer's Rule

Operations on Matrices, Adjoint and Inverse of Matrix

Factorial, Permutation and Exercise

Combinations, Binomial Theorem

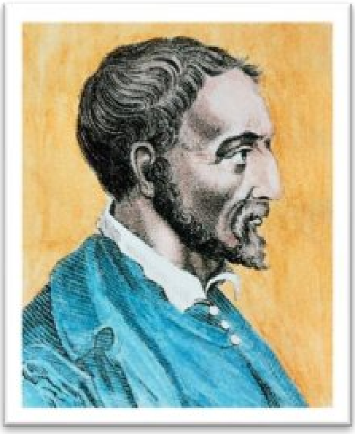
Binomial Theorem – Exercise

Expansion of Binomials using Binomial Theorem, Coefficient of X^n and Term Independent of x

UNIT - II

COMPLEX NUMBERS

Chapter 2.1 ALGEBRA OF COMPLEX NUMBERS



Many mathematicians contributed to the full development of complex numbers.

The 16th century Italian mathematician **Gerolamo Cardano** is credited with introducing complex numbers in his attempts to find solutions to cubic equations.

The set of all complex numbers denoted by C and the nomenclature of a complex number was introduced by a German mathematician C.F. Gauss.

The famous Swiss mathematician, Leonhard Euler was the first to denote the letter 'i' as the imaginary unit in the complex number system in 1748. It is the only imaginary number. However, when you square it, it becomes real. Of course, it wasn't instantly created. It took several centuries to convince certain mathematicians to accept this new number.

In 1833, William Rowan Hamilton expressed complex numbers as pair of real numbers

Introduction to complex numbers

The introduction of complex numbers plays a very important role in the theory of numbers. The number system that we are aware of today is the gradual development from natural numbers to integers, from integers to rational numbers and from rational numbers to real numbers.

Consider the equations

$$x^2 - 1 = 0 \text{ and}$$

$$x^2 - 9 = 0$$

We see that the equations have solutions in the real number system, whereas the equations

$$x^2 + 1 = 0 \text{ and}$$

$$x^2 + 9 = 0 \text{ have no real solutions.}$$

So, we need to extend the real number system to a larger system, so that we can find the solution of the above equations.

Example 1 : Let us consider the equation

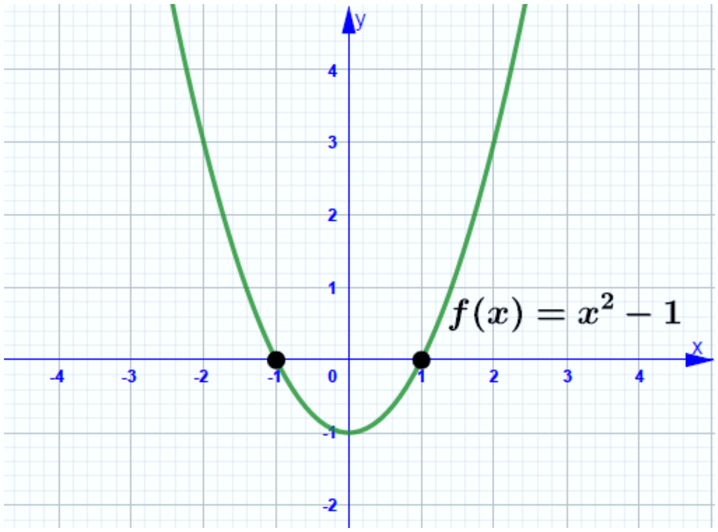
$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm\sqrt{1}$$

$$x = \pm 1$$

It has two real solutions x = -1 and x = 1



The graph $f(x) = x^2 - 1$, crosses the x-axis at (-1, 0) and (1, 0).

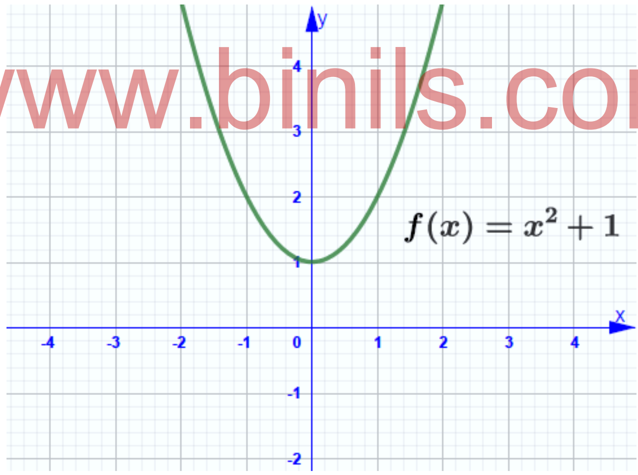
Example 2 : Let us consider the equation

$$x^2 + 1 = 0$$

$$x^2 = -1$$

$$x = \pm\sqrt{-1} \quad (\sqrt{-1} = i)$$

This equation has no real solution



The graph $f(x) = x^2 + 1$ does not cross the x – axis

When we square a real number, it is always positive. It cannot be negative. The solution of the equation $x^2 + 1=0$ extends the real number system to a new kind of number system that allows the square root of negative numbers.

The square root of -1 denoted by the symbol 'i' ($\sqrt{-1}$ = i), called the imaginary unit.

Therefore, +i and –i are the solutions of the equation $x^2 + 1 = 0$

DEFINITION : COMPLEX NUMBER

A number which is of the form a + ib, where a and b are real numbers is called a complex number. It is denoted by z and i = $\sqrt{-1}$ is called as Imaginary unit.

consider the complex number z = a + ib

- Real part of z = a + ib is 'a' and it is denoted by Re(z)

- Imaginary part of z = a + ib is 'b' and it is denoted by Im(z)

For example: z = 2 + 3i, then Re(z) = 2

Im (z) = 3

NOTE:

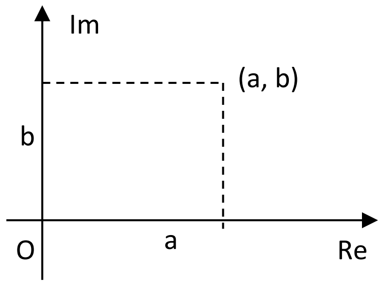
1. A complex number z is said to be real if Im(z) = 0
2. A complex number z is said to be purely imaginary if Re(z) = 0

> **MATH FACT**
>
> Zero is the only number which is at once real and purely imaginary

Ordered pair

A complex number z = a +i b can be written in ordered pair of real numbers as (a,b)

For example : 2 + i can be written as (2,1)



Equality of complex numbers

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

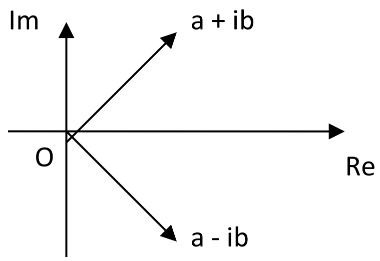
For the complex numbers

z_1 = a + ib and

z_2 = c + id

$z_1 = z_2 \iff$ a= c and b = d

DEFINITION : CONJUGATE



The Conjugate of the complex number z = a + ib is defined as the complex number $a - ib$ and it is denoted by

$\bar{z}$ = a – ib.

i.e., conjugate of a complex number is obtained by changing the sign of imaginary part of z.

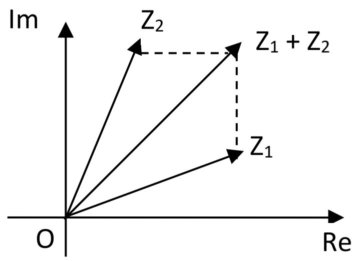
For example : z=1 + 4i, then $\bar{z}$ = 1 – 4i

z = -2 -3i, then $\bar{z}$ = -2 + 3i

z = i, then $\bar{z}$ = -i

> **MATH FACT**
>
> Two complex numbers $a + ib$ and $a - ib$ are conjugate to each other.

Basic Algebraic operations of complex numbers:



- ### Addition of two complex numbers

If $z_1 = a + ib$ and $z_2 = c + id$, then

$z_1 + z_2 = (a + ib) + (c + id)$

$$\boxed{z_1 + z_2 = (a + c) + i(b+d)}$$

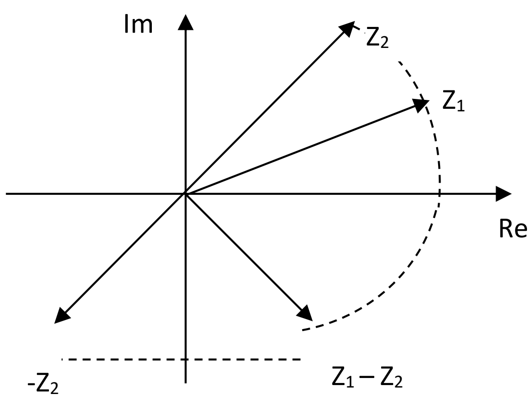
Properties of complex numbers under addition

Closure Property	For any two complex numbers Z_1 and Z_2, $Z_1 + Z_2$ is also a complex number.
Commutative Property	For any Z_1 and Z_2, $Z_1 + Z_2 = Z_2 + Z_1$
Associative Property	For any complex number Z_1, Z_2 and Z_3, $(Z_1 + Z_2) + Z_3 = Z_1 + (Z_2 + Z_3)$
Additive Identity	There exist a complex number $0 = 0 + 0i$ such that $Z + 0 = 0 + Z = Z \;\; \forall \;\; Z$.
Additive Inverse	For any complex number Z, there exist a complex number $-Z$ such that $Z + (-Z) = (-Z) + Z = 0$

MATH FACT

$0 = 0 + 0i$ is known as additive identity and

$-Z$ is called as the additive inverse of Z.

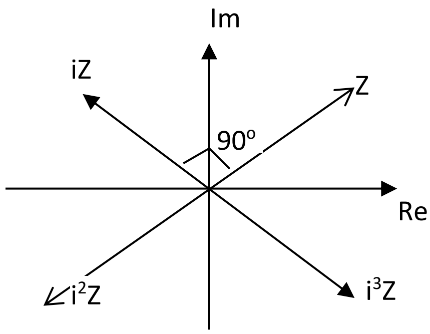


- ### Subtraction of two complex numbers

If $z_1 = a + ib$and $z_2 = c + id$, then

$z_1 - z_2 = (a + ib) - (c + id)$

$$\boxed{z_1 - z_2 = (a - c) + i(b - d)}$$



Multiplication of complex number z by 'i'

Multiplication of a complex number z by 'i' is a rotation of z by 90° or $\frac{\pi}{2}$ radians in the counter clockwise direction about the origin.

- ### Multiplication of two complex numbers

If $z_1 = a + ib$ and $z_2 = c + id$, then

$$\begin{aligned} z_1 z_2 &= (a + ib)(c + id) \\ &= ac + iad + ibc + i^2bd \end{aligned}$$

$= \quad ac + i(ad + bc) - bd \qquad (\because i^2 = -1)$

$$\boxed{z_1 z_2 = (ac - bd) + i(ad + bc)}$$

Properties of complex numbers under multiplication

Closure Property	**For any complex numbers Z_1 and Z_2, $Z_1 Z_2$ is also a complex number.**
Commutative Property	For any Z_1 and Z_2, $Z_1 Z_2 = Z_2 Z_1$
Associative Property	For any complex numbers Z_1, Z_2 and Z_3, $(Z_1 Z_2) Z_3 = Z_1 (Z_2 Z_3)$
Multiplicative Identity	There exist a complex number $1 = 1 + 0i$ such that $Z \cdot 1 = 1 \cdot Z = Z \ \forall \ Z$.
Multiplicative Inverse	For any complex number $Z \neq 0$, there exist a complex number $\frac{1}{z}$ such that $Z \frac{1}{z} = \frac{1}{z} Z = 1$
Distributive Property	For any three complex numbers Z_1, Z_2 and Z_3. $Z_1 (Z_2 + Z_3) = Z_1 Z_2 + Z_1 Z_3$

MATH FACT

$1 = 1 + 0i$ is known as Multiplicative Identity.

$\frac{1}{z}$ is known as multiplicative inverse of Z and denoted by Z^{-1}

- **Division of two complex numbers**

If $z_1 = a + ib$ and $z_2 = c + id$, then

$$\frac{z_1}{z_2} = \frac{(a + ib)}{(c + id)}$$

$$= \frac{(a + ib)}{(c + id)} \times \frac{(c - id)}{(c - id)}$$

$$= \frac{ac - iad + ibc - i^2bd}{c^2 + d^2} \qquad [\because (a+ib)(a-ib) = (a)^2 + (b)^2]$$

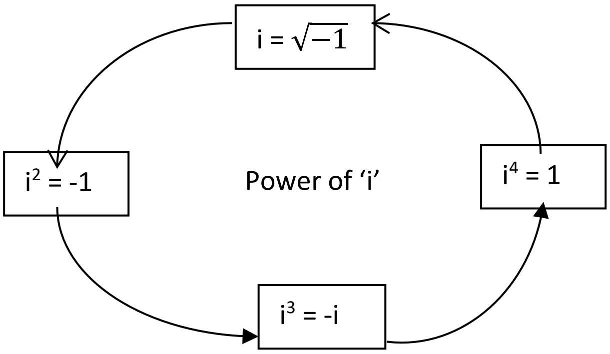
$$= \frac{ac + i(bc - ad) - (-1)bd}{c^2 + d^2} \quad [\because i^2 = -1]$$

$$= \frac{ac + i(bc - ad) + bd}{c^2 + d^2}$$

$$= \frac{ac + bd + i(bc - ad)}{c^2 + d^2}$$

$$\boxed{\frac{z_1}{z_2} = \frac{ac + bd}{c^2 + d^2} + i\frac{(bc - ad)}{c^2 + d^2}}$$

Various powers of Imaginary unit 'i':



1) $i^0 = 1$
2) $i^1 = i$
3) $i^2 = -1$ $\quad [\because i = \sqrt{-1}]$
4) $i^3 = i^2 \times i = (-1) \times i = -i$
5) $i^4 = i^2 \times i^2 = (-1) \times (-1) = 1$
6) $i^5 = i^4 \times i = 1 \times i = i$
7) $i^6 = i^4 \times i^2 = 1 \times (-1) = -1$
8) $(i)^{-1} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$
9) $(i)^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1$
10) $(i)^{-3} = \frac{1}{i^3} = \frac{1}{i^3} \times \frac{i}{i} = \frac{i}{i^4} = \frac{i}{1} = i$
11) $(i)^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1$

Properties of complex conjugates :

1) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
2) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
3) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
4) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0$

5) $Re(z) = \frac{z + \bar{z}}{2}$
6) $Im(z) = \frac{z - \bar{z}}{2i}$
7) $\overline{(z^n)} = (\bar{z})^n$, Where n is an integer
8) z is real if and only if $z = \bar{z}$
9) z is purely imaginary if and only if $z = -\bar{z}$
10) $\bar{\bar{z}} = z$

We shall verify some of the above properties

Property : $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Let $z_1 = a_1 + ib_1$

$z_2 = a_2 + ib_2$

$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$

$\overline{z_1 + z_2} = \overline{(a_1 + ib_1) + (a_2 + ib_2)}$

$= \overline{(a_1 + a_2) + i(b_1 + b_2)}$

$= (a_1 + a_2) - i(b_1 + b_2)$

$= (a_1 - ib_1) + (a_2 - ib_2)$

$$\boxed{\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}}$$

Property : $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$

Let $z_1 = a_1 + ib_1$

$z_2 = a_2 + ib_2$

$z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2)$

$= a_1a_2 + ia_2b_1 + ia_1b_2 + i^2b_1b_2$

$z_1z_2 = (a_1a_2 - b_1b_2) + i(a_2b_1 + a_1b_2)$

$\overline{z_1z_2} = \overline{(a_1a_2 - b_1b_2) + i(a_2b_1 + a_1b_2)}$

$\overline{z_1z_2} = (a_1a_2 - b_1b_2) - i(a_1b_2 + a_2b_1) \quad \rightarrow ①$

$\overline{z_1}\ \overline{z_2} = (a_1 - ib_1)(a_2 - ib_2)$

$= a_1a_2 - ia_2b_1 - ia_1b_2 + i^2b_1b_2$

$= (a_1a_2 - b_1b_2) - i(a_1b_2 + a_2b_1) \quad \rightarrow ②$

By ① and ②, $\boxed{\overline{z_1z_2} = \overline{z_1}\ \overline{z_2}}$

Worked Examples

1) If $z_1 = 1 + 3i$, $z_2 = 6 - 5i$ find $z_1 + z_2$

Solution :

Given $z_1 = 1 + 3i$, and $z_2 = 6 - 5i$

$z_1 + z_2 = (1 + 3i) + (6 - 5i)$

$= 1 + 3i + 6 - 5i$

$\boxed{z_1 + z_2 = 7 - 2i}$

2) If $z_1 = 2 - i$, $z_2 = 3+4i$ find $z_1 - z_2$

Solution :

Given $z_1=2 - i$, and $z_2 = 3+4i$

$z_1 - z_2 = (2 - i) - (3 + 4i)$

$= 2 - i - 3 - 4i$

$\boxed{z_1 - z_2 = -1 - 5i}$

3) If $z_1 = 1 + 4i$, $z_2 = -3 + 6i$ find $3z_1 + 2z_2$

Solution :

Given $z_1= 1 + 4i$ and $z_2 = -3 +6i$

$3z_1 + 2z_2 = 3(1 + 4i) + 2(-3 +6i)$

$= 3 + 12i - 6 + 12i$

$\boxed{3z_1 + 2z_2 = -3 + 24i}$

4) If $z_1 = (3,-1)$ and $z_2 = (4,2)$ find $4z_1 - z_2$

Solution :

Given $z_1 = 3 - i$

$z_2 = 4 + 2i$

$4z_1 - z_2 = 4(3-i) - (4 + 2i)$

$= 12 - 4i - 4 - 2i$

$\boxed{4z_1 - z_2 = 8 - 6i}$

5) Find the value of z_1z_2 if $z_1 = 4 + i$ and $z_2 = 1 - i$

Solution :

Given $z_1 = 4 + i$ and $z_2 = 1 - i$

$z_1z_2 = (4 + i)(1 - i)$

$= 4 - 4i + i - i^2$ [$\because i^2 = -1$]

$= 4 - 3i - (-1)$

$= 4 - 3i + 1$

$\boxed{z_1z_2 = 5 - 3i}$

6) Find the real and imaginary parts of $\frac{1}{5-2i}$

Solution :

Let $z = \frac{1}{5-2i}$

$= \frac{1}{5-2i} \times \frac{(5+2i)}{(5+2i)}$

$= \frac{(5+2i)}{(5)^2+(2)^2}$ [$\because (a + ib)(a - ib) = (a)^2 + (b)^2$]

$= \frac{5+2i}{25+4}$

$z = \frac{5+2i}{29}$

$\boxed{z = \frac{5}{29} + \frac{2i}{29}}$

$Re(z) = \frac{5}{29}$

$Im(z) = \frac{2}{29}$

7) Find the conjugate of $\frac{5}{3+2i}$

Solution :

Let $z = \frac{5}{3+2i}$

$= \frac{5}{3+2i} \times \frac{(3-2i)}{(3-2i)}$

$= \frac{(15-10i)}{(3)^2+(2)^2}$ [$\because (a + ib)(a - ib) = (a)^2 + (b)^2$]

$= \frac{15-10i}{9+4}$

$= \frac{15-10i}{13}$

$\boxed{z = \frac{15}{13} - \frac{10i}{13}}$

$\boxed{\text{Conjugate of } z\ (\bar{z}) = \frac{15}{13} + \frac{10i}{13}}$

8) Find the values of $i^2 + i^3 + i^4$

Solution :

$$i^2 + i^3 + i^4 = i^2 + (i^2)(i) + (i^2)^2$$
$$= -1 + (-1)(i) + (-1)^2$$
$$= -1 - i + 1 \quad [\because i^2 = -1]$$
$$= -i$$

9) Find the real and imaginary parts of $\frac{7 + 2i}{2 - 3i}$

Solution :

Let $z = \frac{7 + 2i}{2-3i}$

$$= \frac{7 + 2i}{2-3i} \times \frac{(2 + 3i)}{(2 + 3i)}$$
$$= \frac{14 + 21i + 4i + 6i^2}{(2)^2 + (3)^2} \qquad [\because (a + ib)(a - ib) = (a)^2 + (b)^2]$$
$$= \frac{14 + 25i + 6(-1)}{4 + 9} \quad [\because i^2 = -1]$$
$$= \frac{14 + 25i - 6}{13}$$
$$= \frac{8 + 25i}{13}$$
$$\boxed{z = \frac{8}{13} + \frac{25}{13} i}$$

$\therefore$ Re(z) $= \frac{8}{13}$

Im(z) $= \frac{25}{13}$

10) Express $\frac{2}{4 + 3i} + \frac{i}{3 - 4i}$ in a + i b form

Solution :

Let $z = \frac{2}{4 + 3i} + \frac{i}{3 - 4i}$

$$= \frac{2}{4 + 3i} \times \frac{(4 - 3i)}{(4 - 3i)} + \frac{i}{3 - 4i} \times \frac{(3 + 4i)}{(3 + 4i)}$$
$$= \frac{8 - 6i}{(4)^2 + (3)^2} + \frac{3i + 4i^2}{(3)^2 + (4)^2} \qquad [\because (a + ib)(a - ib) = (a)^2 + (b)^2]$$
$$= \frac{8 - 6i}{16 + 9} + \frac{3i + 4(-1)}{9 + 16} \qquad [\because i^2 = -1]$$
$$= \frac{8 - 6i}{25} + \frac{3i - 4}{25}$$
$$= \frac{8 - 6i + 3i - 4}{25}$$
$$= \frac{4 - 3i}{25}$$
$$\boxed{z = \frac{4}{25} - \frac{3i}{25}} \quad \text{a + ib form}$$

11) Find the real and imaginary parts of $\frac{(1 + i)(2 + i)}{(1 + 4i)}$

Solution :

$$\text{Let } z = \frac{(1+i)(2+i)}{(1+4i)}$$

$$= \frac{2+i+2i+i^2}{(1+4i)}$$

$$= \frac{2+3i-1}{(1+4i)}$$

$$= \frac{1+3i}{1+4i}$$

$$= \frac{1+3i}{1+4i} x \frac{(1-4i)}{(1-4i)} \qquad [\because (a+ib)(a-ib) = (a)^2 + (b)^2]$$

$$= \frac{1-4i+3i-12i^2}{(1)^2+(4)^2} \qquad [\because i^2 = -1]$$

$$= \frac{1-i+12}{1+16}$$

$$z = \frac{13-i}{17}$$

$$\boxed{z = \frac{13}{17} - \frac{i}{17}}$$

$$\therefore \text{Re}(z) = \frac{13}{17}$$

$$\text{Im}(z) = \frac{-1}{17}$$

12) Express : $\frac{2i+1}{1+3i} + \frac{i-6}{3-i}$ in a + ib form

Solution :

$$\text{Let } z = \frac{2i+1}{1+3i} + \frac{i-6}{3-i}$$

$$= \frac{2i+1}{1+3i} x \frac{(1-3i)}{(1-3i)} + \frac{i-6}{3-i} x \frac{(3+i)}{(3+i)}$$

$$= \frac{2i-6(-1)+1-3i}{1+9} + \frac{3i-1-18-6i}{1+9} \qquad [\because (a+ib)(a-ib) = (a)^2 + (b)^2]$$

$$= \left(\frac{-i+7}{10}\right) + \left(\frac{-3i-19}{10}\right) \qquad [\because i^2 = -1]$$

$$= \frac{-4i-12}{10}$$

$$= \frac{-12-4i}{10}$$

$$\boxed{z = \frac{-12}{10} - \frac{4i}{10}} \text{ hence it is a + ib form}$$

13) Find the smallest positive integer n for which $\left[\frac{1+i}{1-i}\right]^n = 1$

Solution:

$$\text{L.H.S.} = \left[\frac{1+i}{1-i}\right]^n$$

$$= \left[\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right]^n$$

$$= \left[\frac{1+i+i+i^2}{(1)^2+(1)^2}\right]^n$$

$=\left[\frac{1+2i-1}{1+1}\right]^n$

$=\left[\frac{2i}{2}\right]^n$

$=(i)^n$

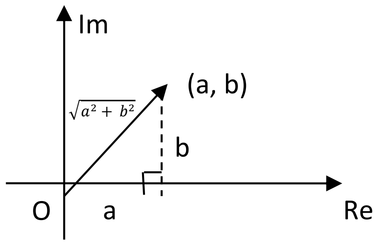
$=(i)^4$ when n = 4, the smallest integer.

= 1 = R.H.S

∴The smallest positive integer is 4.

Modulus of a Complex Number

The Modulus of a complex number is always a real number. It will never be negative.



If z = a + ib, then the modulus (or) absolute value of z, denoted by |z|, is defined by $|z|=\sqrt{a^2+b^2}$

Example :

1) $|2-3i| = \sqrt{(2)^2+(-3)^2} = \sqrt{4+9} = \sqrt{13}$

2) $|i| = \sqrt{(0)^2+(1)^2} = \sqrt{1} = 1$

3) $|-10i| = \sqrt{(0)^2+(-10)^2} = \sqrt{100} = 10$

Properties :

i) $|z| = |\bar{z}|$

ii) $z\bar{z} = |z|^2$

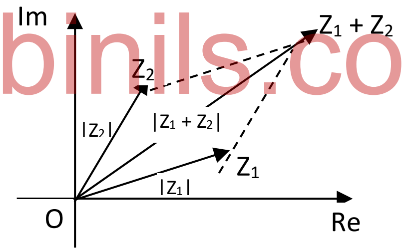
iii) $\mathrm{Re}(z) \le |z|$

iv) $\mathrm{Im}(z) \le |z|$

v) $|z_1+z_2| \le |z_1| + |z_2|$

vi) $|z_1-z_2| \le |z_1| + |-z_2|$

$|z_1-z_2| \le |z_1| + |z_2|$



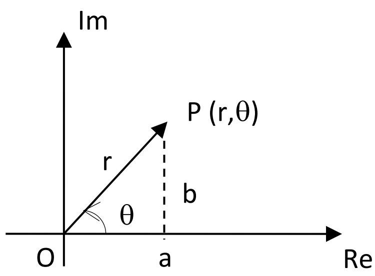
POLAR FORM AND EULER FORM OF A COMPLEX NUMBER

We are familiar with complex numbers in the form z = a + ib. When performing addition and subtraction of complex numbers, we use rectangular form. However, there are some alternate forms namely Polar and Euler form that are useful to perform multiplication or finding powers or roots of complex numbers.

Polar form of a Complex Number

A complex number z in complex plane can be represented by Cartesian co-ordinates, but the representation of z by polar co-ordinates is equally useful.

Let (r, θ) be the Polar co-ordinates of the point P (a, b) representing the complex number z = a + ib.



where r- distance from O to the point P

θ- angle of inclination measured from the initial line in the counterclockwise direction to the line OP

In the above diagram,

$\cos\theta = \frac{OM}{OP} = \frac{a}{r}$ | $\sin\theta = \frac{PM}{OP} = \frac{b}{r}$

$\boxed{a = r\cos\theta} \rightarrow ①$ | $\boxed{b = r\sin\theta} \rightarrow ②$

$$①+② \Rightarrow a^2 + b^2 = (r\cos\theta)^2 + (r\sin\theta)^2$$
$$= r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2(\cos^2\theta + \sin^2\theta)$$
$$a^2 + b^2 = r^2 \qquad [\because \cos^2\theta+\sin^2\theta = 1]$$

$\boxed{\therefore r = \sqrt{a^2+b^2}} = |a + ib|$ is called the modulus of z

Now, $\frac{②}{①} \Rightarrow \frac{r\sin\theta}{r\cos\theta} = \frac{b}{a}$

$$\tan\theta = \frac{b}{a}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{b}{a}\right)}$$

Here, θ is called the amplitude (or) argument of z and it is denoted by amp(z) (or) arg(z)

Polar form:

If r and θ are the polar coordinates of the point P(a,b) that corresponds to a non zero complex number z = a + ib then the polar form or modulus-amplitude form of z is

$$z = r(\cos\theta + i\sin\theta)$$

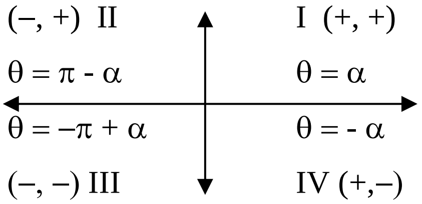
Note :-

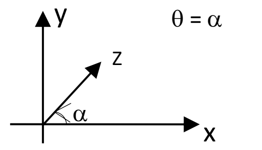
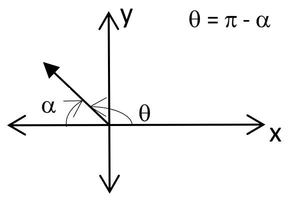
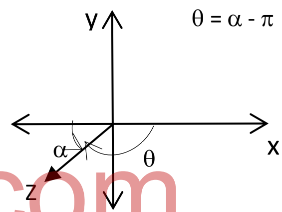
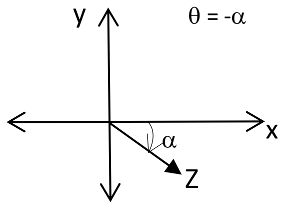
1) If z = 0, the argument θis undefined, so whenever Polar coordinates are used, $z \neq 0$
2) 'r' is single valued, while 'θ' is many valued.
3) The value of 'θ' lies between $-\pi$ andπ is called the principal value of the argument. It is denoted by Arg (z).

Rules to determine the amplitude (argument) of a complex number (θ)

Let $z = a + ib$

Take $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$, where a and b are real numbers



(i)	$z = a + ib$	If Real part (+) Imaginary part (+) It lies in I Quadrant	$\theta = \alpha$	
(ii)	$z = -a + ib$	If Real part (–) Imaginary part (+) It lies in II Quadrant	$\theta = \pi - \alpha$	
(iii)	$z = -a - ib$	If Real part (–) Imaginary part (–) It lies in III Quadrant	$\theta = -\pi + \alpha$	
(iv)	$z = a - ib$	If Real part (+) Imaginary part (–) It lies in IV Quadrant	$\theta = -\alpha$	

Properties of Modulus of Complex numbers:

1. The modulus of the product of two complex numbers is the product of their moduli, and the argument of the product is the sum of their arguments.

For any complex number z_1 and z_2

i) $|z_1 z_2| = |z_1||z_2|$

ii) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

Note :

The above result can be extended to any finite number of complex numbers.

i) $|z_1 z_2 \ldots\ldots\ldots z_n| = |z_1|\,|z_2| \ldots\ldots\ldots\ldots |z_n|$

ii) $\arg(z_1 z_2 \ldots\ldots\ldots z_n) = \arg z_1 + \arg z_2 + \ldots\ldots\ldots + \arg z_n$

2. The modulus of the quotient of two complex numbers is the quotient of their moduli, and the argument of the quotient is the difference of their arguments.

For any complex number z_1 and z_2

i) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ where $z_2 \neq 0$

ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

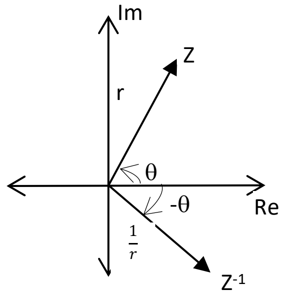
Properties of Polar form:

If $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$

$z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$

then $z_1 z_2 = r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$

$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)]$ where $z_2 \neq 0$



MATH FACT

If $z = r (\cos \theta + i \sin \theta)$ then $z^{-1} = \frac{1}{r} (\cos \theta - i \sin \theta)$

Euler's form of a complex number :

The symbol $e^{i\theta}$ (or) $\exp (i\theta)$ is defined by $e^{i\theta} = \cos \theta + i \sin \theta$. This identity is known as Euler's Formula.

Using Euler's formula, we can rewrite the polar form of a complex number into the exponential form

$\boxed{z = r (\cos \theta + i \sin \theta) = r e^{i\theta}}$ when $z \neq 0$

Note :

- $e^{i\alpha} e^{i\beta} = e^{i(\alpha + \beta)}$
- $\frac{e^{i\alpha}}{e^{i\beta}} = e^{i\alpha} e^{-i\beta} = e^{i(\alpha - \beta)}$
- If $z = re^{i\theta}$ then $\bar{z} = re^{-i\theta}$

MATH FACT

Two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are equal $\Leftrightarrow r_1 = r_2$ and $\theta_1 = \theta_2 + 2n\pi,\ n \epsilon Z$

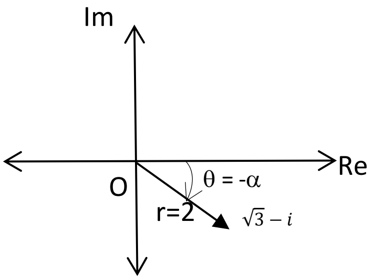
1) Find the modulus and amplitude of $\sqrt{3} - i$

Solution :

Let $z = \sqrt{3} - i = a + i\,b$

$a = \sqrt{3}$, $b = -1$

To find modulus



$$|z| = \sqrt{(a)^2 + (b)^2}$$

$$= \sqrt{(\sqrt{3})^2 + (-1)^2}$$

$$= \sqrt{3 + 1} = \sqrt{4}$$

$$\boxed{|z| = 2}$$

To find amplitude

$$\boxed{\alpha = \tan^{-1}\left(\frac{b}{a}\right)}$$

$$\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\boxed{\alpha = 30^o}$$

∴ The complex number is in the form (+, –)

It lies in IV Quadrant

$$\theta = -\alpha$$

$$\theta = -30^o$$

$$\boxed{\theta = \left(\frac{-\pi}{6}\right)}$$

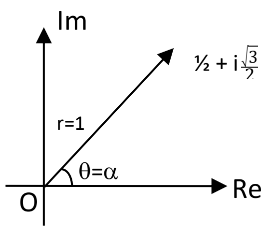
2) Find the modulus and amplitude of $\frac{1}{2} + i\,\frac{\sqrt{3}}{2}$

Solution :

Let $z = \frac{1}{2} + i\,\frac{\sqrt{3}}{2} = a + ib$

Here, $a = \frac{1}{2}$ and $b = \frac{\sqrt{3}}{2}$

To find modulus



$$|z| = \sqrt{(a)^2 + (b)^2}$$

$$= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1}$$

$$\boxed{|z| = 1}$$

To find amplitude

$$\boxed{\alpha = \tan^{-1}\left(\frac{b}{a}\right)}$$

$$= \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right)$$

$$= \tan^{-1}\left(\sqrt{3}\right)$$

$$\boxed{\alpha = 60^o}$$

∴The complex number is in the form (+, +)

It lies in I Quadrant

$$\theta = \alpha$$

$$\theta = 60^o$$

$$\theta = \left(\frac{\pi}{3}\right)$$

3) Write down the Euler form of the complex number 1+i

Solution :

Let $z = 1 + i = a + i\,b$

Here, $a = 1$, $b = 1$

To find modulus

$$r = \sqrt{(a)^2 + (b)^2}$$

$$= \sqrt{(1)^2 + (1)^2}$$

$$= \sqrt{1 + 1}$$

$$\boxed{r = \sqrt{2}}$$

To find amplitude

$$\boxed{\alpha = \tan^{-1}\left(\frac{b}{a}\right)}$$

$$\alpha = \tan^{-1}\left(\frac{1}{1}\right)$$

$$= \tan^{-1}(1)$$

$$\alpha = 45^o$$

$$\boxed{\alpha = \frac{\pi}{4}}$$

∴The complex number is in the form (+, +).

It lies in I Quadrant

$$\boxed{\theta = \alpha}$$

$$\boxed{\theta = \frac{\pi}{4}}$$

Euler form of complex number is

$$\boxed{z = re^{i\theta}}$$

$$\boxed{z = \sqrt{2}e^{i\frac{\pi}{4}}}$$

4) If $z_1 = 5\,(\cos 30^o + i \sin 30^o)$ and $z_2 = 2\,(\cos 10^o + i \sin 10^o)$.

Find (i) $z_1 z_2$ (ii) $\frac{z_1}{z_2}$

Solution :

Given $z_1 = 5\,(\cos 30^o + i \sin 30^o)$

$z_2 = 2\,(\cos 10^o + i \sin 10^o)$

i) $z_1 z_2 = 5(\cos 30^o + i \sin 30^o) \times 2(\cos 10^o + i \sin 10^o)$

$= 10[\cos(30^o + 10^o) + i \sin(30^o + 10^o)]$

$$\boxed{z_1 z_2 = 10[\cos 40^o + i \sin 40^o]}$$

ii) $\frac{z_1}{z_2} = \frac{5(\cos 30^o + i \sin 30^o)}{2(\cos 10^o + i \sin 10^o)}$

$= \frac{5}{2}(\cos 30^o - 10^o) + i \sin(30^o - 10^o)$

$$\boxed{\frac{z_1}{z_2} = \frac{5}{2}[\cos 20^o + i \sin 20^o]}$$

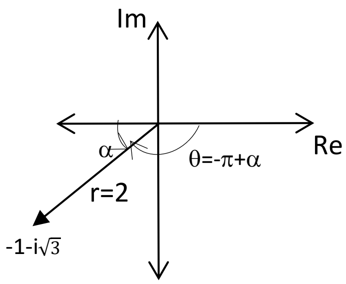
5) Express $-1 - i\sqrt{3}$ in polar form

Solution :

Let $z = -1 - i\sqrt{3} = a + ib$

Here, $a = -1$ and $b = -\sqrt{3}$

To find Modulus



$r = |z| = \sqrt{(a)^2 + (b)^2}$

$= \sqrt{(-1)^2 + (-\sqrt{3})^2}$

$= \sqrt{1 + 3} = \sqrt{4}$

$r = \boxed{|z| = 2}$

Tofind amplitude

$\boxed{\alpha = \tan^{-1}\left(\frac{b}{a}\right)}$ [∵ without taking –ve sign]

$= \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$

$\boxed{\alpha = 60^o} \Rightarrow \boxed{\alpha = \frac{\pi}{3}}$

The complex number is in the form (– , –). It lies in III Quadrant.

$\boxed{\theta = -\pi + \alpha}$

$= -\pi + \frac{\pi}{3}$

$\boxed{\theta = \frac{-2\pi}{3}}$

Polar form $\boxed{z = r(\cos\theta + i \sin\theta)}$

$-1 - i\sqrt{3} = 2\left[\cos\left(\frac{-2\pi}{3}\right) + i \sin\left(\frac{-2\pi}{3}\right)\right]$

$$\boxed{-1 - i\sqrt{3} = 2\left[\cos\frac{2\pi}{3} - i \sin\frac{2\pi}{3}\right]}$$

6) If $z_1 = 2\left(\cos\frac{\pi}{8} + i \sin\frac{\pi}{8}\right)$, $z_2 = 4\left(\cos\frac{3\pi}{8} + i \sin\frac{3\pi}{8}\right)$

What is the value of arg $(z_1 z_2)$

Solution :

Given $\quad z_1 = 2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$

$$z_2 = 4\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$$

$$z_1 z_2 = 2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right) \text{ x } 4\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$$

$$= 8\left[\cos\left(\frac{\pi}{8} + \frac{3\pi}{8}\right) + i\sin\left(\frac{\pi}{8} + \frac{3\pi}{8}\right)\right]$$

$$= 8\left[\cos\frac{4\pi}{8} + i\sin\frac{4\pi}{8}\right]$$

$$z_1 z_2 = 8\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

$$\boxed{\arg(z_1 z_2) = \theta = \frac{\pi}{2}}$$

7) Find the modulus and amplitude of $\frac{5-i}{2-3i}$

Solution :

$$\text{Let } z = \frac{5-i}{2-3i}$$

$$= \frac{5-i}{2-3i} \text{ x } \frac{(2+3i)}{(2+3i)}$$

$$= \frac{10+15i-2i-3i^2}{(2)^2+(3)^2}$$

$$= \frac{10+13i+3}{4+9}$$

$$= \frac{13+13i}{13}$$

$$= \frac{13}{13} + \frac{13i}{13}$$

$$z = 1 + i \quad = a + i\,b$$

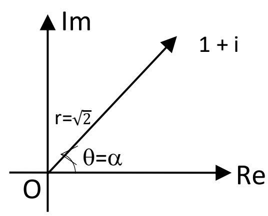
$\therefore a = 1, \quad b = 1$

To find modulus

$$|z| = \sqrt{(a)^2 + (b)^2}$$

$$= \sqrt{(1)^2 + (1)^2} = \sqrt{1+1}$$

$$\boxed{|z| = \sqrt{2}}$$



To find amplitude

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right) \quad [\because \text{ without taking –ve sign}]$$

$$= \tan^{-1}\left(\frac{1}{1}\right)$$

$$\alpha = \tan^{-1}(1)$$

$$\boxed{\alpha = 45^\circ} \Rightarrow \boxed{\alpha = \frac{\pi}{4}}$$

$\therefore$The complex number is in the form (+, +). It lies in I Quadrant.

$$\boxed{\theta = \alpha}$$

$$\boxed{\theta = \frac{\pi}{4}}$$

8) Express the complex number $\frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$ in polar form.

Solution :

Let $z = \frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$

$= \frac{1+3\sqrt{3}i}{\sqrt{3}+2i} \times \frac{\sqrt{3}-2i}{\sqrt{3}-2i}$

$= \frac{\sqrt{3}-2i+9i-6\sqrt{3}i^2}{(\sqrt{3})^2+(2)^2}$

$= \frac{\sqrt{3}+7i+6\sqrt{3}}{3+4}$

$= \frac{7\sqrt{3}+7i}{7}$

$= \frac{7\sqrt{3}}{7} + \frac{7i}{7}$

$z = \sqrt{3} + i \quad = a + ib$

$\therefore a = \sqrt{3}$ and $b = 1$

To find modulus

$r = |z| = \sqrt{(a)^2 + (b)^2}$

$= \sqrt{(\sqrt{3})^2 + (1)^2}$

$= \sqrt{3+1} = \sqrt{4}$

$\boxed{r = 2}$

To find amplitude

$\boxed{\alpha = \tan^{-1}\left(\frac{b}{a}\right)}$

$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\boxed{\alpha = 30^\circ} \Rightarrow \boxed{\alpha = \frac{\pi}{6}}$

$\therefore$ The complex number is in the form (+, +). It lies in I Quadrant

$\boxed{\theta = \alpha}$

$\boxed{\theta = \frac{\pi}{6}}$

Polar form

$\boxed{z = r(\cos\theta + i\sin\theta)}$

$\boxed{\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}$

9) If $(a_1 + i\,b_1)(a_2 + i\,b_2) \ldots (a_n + i\,b_n) = A + i\,B$. Prove that

i) $(a_1^2 + b_1^2)(a_2^2 + b_2^2) \ldots (a_n^2 + b_n^2) = A^2 + B^2$

ii) $\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \ldots \tan^{-1}\left(\frac{b_n}{a_n}\right) = n\pi + \tan^{-1}\left(\frac{B}{A}\right)$

Solution :

Given $\boxed{(a_1 + ib_1)(a_2 + ib_2) \dots (a_n + ib_n) = A + iB} \longrightarrow$ ①

$|(a_1 + ib_1)(a_2 + ib_2) \dots (a_n + ib_n)| = |A + iB|$

$|(a_1 + ib_1)||(a_2 + ib_2)| \dots |(a_n + ib_n)| = |A + iB|$ $\quad [\because |(z_1 z_2)| = |z_1||z_2|]$

$\left(\sqrt{a_1^2 + b_1^2}\right)\left(\sqrt{a_2^2 + b_2^2}\right) \dots \left(\sqrt{a_n^2 + b_n^2}\right) = \left(\sqrt{A^2 + B^2}\right)$

Squaring on both sides, we get

$$\boxed{(a_1^2 + b_1^2)(a_2^2 + b_2^2) \dots (a_n^2 + b_n^2) = A^2 + B^2}$$

Taking argument on both sides in ①

$\arg[(a_1 + ib_1)(a_2 + ib_2) \dots (a_n + ib_n)] = \arg(A+iB)$

$\arg(a_1 + ib_1) + \arg(a_2 + ib_2) + \dots + \arg(a_n + ib_n) = \arg(A+iB)$

$[\because \arg(z_1 z_2) = \arg z_1 + \arg z_2]$

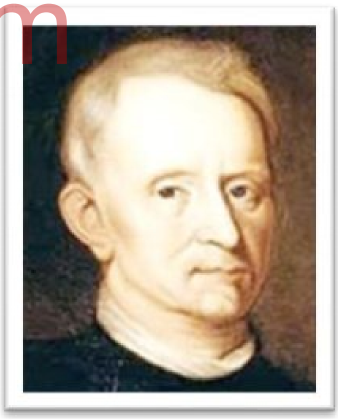
$\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \dots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = \tan^{-1}\left(\frac{B}{A}\right)$

Taking general argument, we get

$$\boxed{\tan^{-1}\left(\frac{b_1}{a_1}\right) + \tan^{-1}\left(\frac{b_2}{a_2}\right) + \dots + \tan^{-1}\left(\frac{b_n}{a_n}\right) = n\pi + \tan^{-1}\left(\frac{B}{A}\right)}$$

Argand Diagram

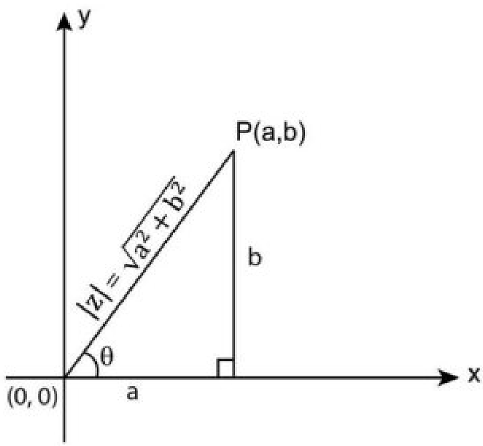
Jean – Robert Argand (1768 – 1822), a Swiss mathematician, is credited with the discovery of Argand diagram, although it was first described by Norwegian – Danish land surveyor and mathematician Caspar Wessel (1745 – 1818).



Argand diagram refers to a geometric plot of complex numbers as points.

It is important because it made the whole idea of a complex number more acceptable. In particular, this visualization helped "imaginary" and "complex" numbers become accepted in mainstream mathematics as a natural extension to negative numbers along the real line.

The complex number z = a+ib is plotted as the point (a, b), where the real part 'a' in the x-axis and imaginary part 'b' in the y-axis. The intersecting lines represent the location of the complex number.

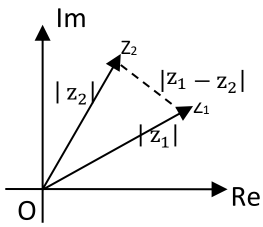


The plane having a complex number assigned to each of its point is called the Argand plane or complex plane.

The system of representing complex numbers in the argand plane is known as Argand diagram

Any complex number z can be represented by an ordered pair (a,b) in an Argand diagram.

Distance between two complex numbers:



Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$|z_1 - z_2| = |(x_1 + iy_1) - (x_2 + iy_2)|$$

$$= |(x_1 - x_2) + i(y_1 - y_2)|$$

$$\boxed{|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}$$

The distance between two complex numbers is given by

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Condition for collinear points

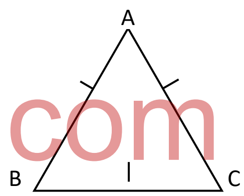
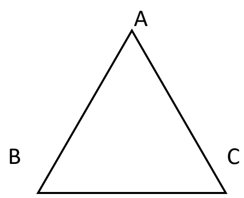
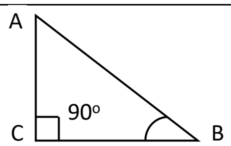
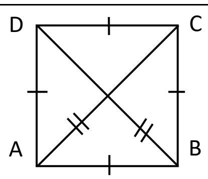
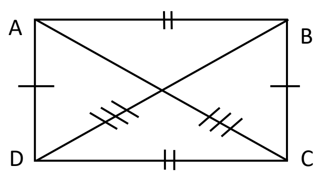
If the three complex numbers x_1+iy_1, x_2+iy_2 and x_3+iy_3 say A, B, and C respectively are collinear, then the area of $\Delta ABC = 0$.

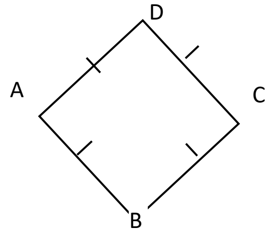
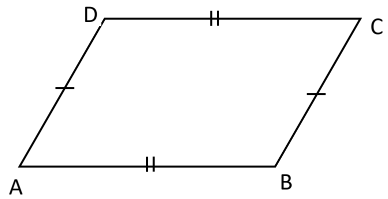
i.e., $\frac{1}{2}[x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)] = 0$

i.e., $x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2) = 0$

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Conditions to check

The triangle ABC is equilateral if AB = BC = AC	
The triangle ABC is an isosceles triangle if AB = BC or BC = AC or AC = AB	
The Δ ABC is right angled triangle if $AB^2 = AC^2 + CB^2$ where AB is hypotenuse.	
If ABCD is a square then (i) AB = BC = CD = AD (ii) Dia AC = Dia BD	
If ABCD is a rectangle then (i) AB = CD and BC = AD (ii) Dia AC = Dia BD	

If ABCD is Rhombus then (i) AB = BC = CD = AD (ii) Dia AC ≠ Dia BD	
If ABCD is a Parallalogram then (i) AB = CD and BC = AD (ii) Dia AC ≠ Dia BD	

Note:

Midpoint of two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is $\left[\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right]$

Worked Examples

1) Find the distance between the complex numbers 3-2i and 1+4i.

Solution :

Let $z_1 = 3\text{-}2i = (3, -2)$

$z_2 = 1+4i = (1, 4)$

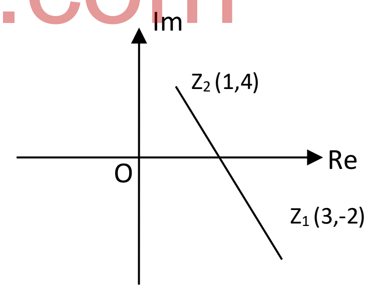
Distance between z_1 and z_2 $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(3-1)^2 + (-2-4)^2}$

$= \sqrt{(2)^2 + (-6)^2}$

$= \sqrt{4 + 36}$

$= \sqrt{40}$



2) Prove that the points in the argand plane represented by the complex numbers 1+4i, 2+7i and 3 + 10i are collinear.

Solution :

Let A = 1+4i = (1, 4)

B = 2+7i = (2, 7)

C = 3+10i = (3, 10)

Condition for collinear

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\text{LHS} = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 7 & 1 \\ 3 & 10 & 1 \end{vmatrix}$$

$$= 1\begin{vmatrix} 7 & 1 \\ 10 & 1 \end{vmatrix} - 4\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 7 \\ 3 & 10 \end{vmatrix}$$

$= 1(7 – 10) – 4(2 – 3) +1(20 – 21)$

$= 1(–3) – 4(–1) +1(–1)$

$= –3 + 4 –1$

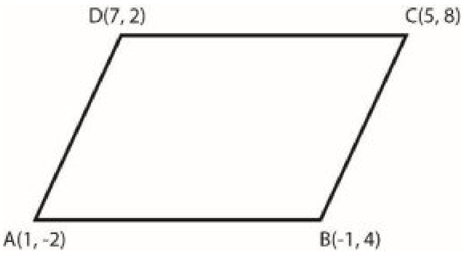
$= –4 + 4$

$= 0$

∴ The given complex numbers are collinear.

3) Show that the complex numbers 1-2i, -1+4i, 5+8i and 7+2i form a parallelogram.

Solution :



Let A = 1-2i = (1, -2)

B = -1+4i = (-1, 4)

C = 5+8i = (5, 8)

D = 7+2i = (7, 2)

$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(1 + 1)^2 + (-2 - 4)^2}$

$= \sqrt{(2)^2 + (-6)^2}$

$= \sqrt{4 + 36}$

$\boxed{AB = \sqrt{40}}$

$BC = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(-1 - 5)^2 + (4 - 8)^2}$

$= \sqrt{(-6)^2 + (-4)^2}$

$= \sqrt{36 + 16}$

$\boxed{BC = \sqrt{52}}$

$CD = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(5 - 7)^2 + (8 - 2)^2}$

$= \sqrt{(-2)^2 + (6)^2}$

$= \sqrt{4 + 36}$

$\boxed{CD = \sqrt{40}}$

$DA = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(7 - 1)^2 + (2 + 2)^2}$

$= \sqrt{(6)^2 + (4)^2}$

$= \sqrt{36 + 16}$

$\boxed{DA = \sqrt{52}}$

Dia AC = $\sqrt{(1-5)^2 + (-2-8)^2}$

$= \sqrt{(-4)^2 + (-10)^2}$

$= \sqrt{16+100}$

$= \sqrt{116}$

Dia BD = $\sqrt{(-1-7)^2 + (4-2)^2}$

$= \sqrt{(-8)^2 + 2^2}$

$= \sqrt{64+4}$

$= \sqrt{68}$

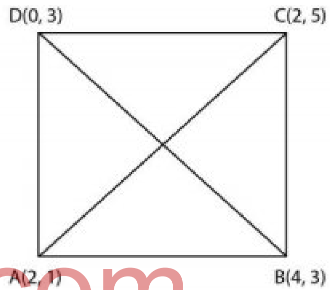
$\therefore$ AB = CD = $\sqrt{40}$

BC = DA = $\sqrt{52}$ & Dia AC $\neq$ Dia BD

$\therefore$ The given points form a parallelogram.

4) Show that the complex numbers 2 + i, 4 + 3i, 2 + 5i and 3i form a square.

Solution :



Let A = 2+i = (2, 1)

B = 4+3i = (4, 3)

C = 2+5i = (2, 5)

D = 3i = (0, 3)

AB $= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

$= \sqrt{(2-4)^2 + (1-3)^2}$

$= \sqrt{(-2)^2 + (-2)^2}$

$= \sqrt{4+4}$

$\boxed{AB = \sqrt{8}}$

BC $= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

$= \sqrt{(4-2)^2 + (3-5)^2}$

$= \sqrt{(2)^2 + (-2)^2}$

$= \sqrt{4+4}$

$\boxed{BC = \sqrt{8}}$

CD $= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

$= \sqrt{(2-0)^2 + (5-3)^2}$

$= \sqrt{(2)^2 + (2)^2}$

$= \sqrt{4+4}$

$\boxed{CD = \sqrt{8}}$

DA $= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

$= \sqrt{(0-2)^2 + (3-1)^2}$

$= \sqrt{(-2)^2 + (2)^2}$

$= \sqrt{4+4}$

$\boxed{DA = \sqrt{8}}$

Dia AC $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(2-2)^2 + (1-5)^2}$

$= \sqrt{(0)^2 + (-4)^2}$

$= \sqrt{0+16}$

$\boxed{AC = \sqrt{16} = 4}$

Dia BD $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(4-0)^2 + (3-3)^2}$

$= \sqrt{(4)^2 + (0)^2}$

$= \sqrt{16+0}$

$\boxed{BD = \sqrt{16} = 4}$

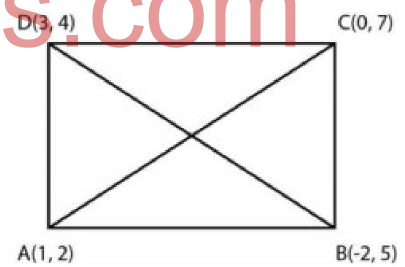
∴ AB = BC = CD = DA = $\sqrt{8}$

Diagonal AC = Diagonal BD = 4

∴ The given complex numbers form a square.

5) Prove that the points representing the complex numbers 1 + 2i, -2 + 5i, 7i and 3 + 4i form a rectangle.

Solution :



Let A = 1+2i = (1, 2)

B = -2+5i = (-2, 5)

C = 7i = (0, 7)

D = 3+4i = (3, 4)

AB $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(1+2)^2 + (2-5)^2}$

$= \sqrt{(3)^2 + (-3)^2}$

$= \sqrt{9+9}$

$\boxed{AB = \sqrt{18}}$

BC $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(-2-0)^2 + (5-7)^2}$

$= \sqrt{(-2)^2 + (-2)^2}$

$= \sqrt{4+4}$

$\boxed{BC = \sqrt{8}}$

$CD = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(0-3)^2 + (7-4)^2}$

$= \sqrt{(-3)^2 + (3)^2}$

$= \sqrt{9+9}$

$\boxed{CD = \sqrt{18}}$

$DA = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(3-1)^2 + (4-2)^2}$

$= \sqrt{(2)^2 + (2)^2}$

$= \sqrt{4+4}$

$\boxed{DA = \sqrt{8}}$

Diagonal $AC = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(1-0)^2 + (2-7)^2}$

$= \sqrt{(1)^2 + (-5)^2}$

$= \sqrt{1+25}$

$= \sqrt{26}$

$\boxed{\text{Diagonal } AC = \sqrt{26}}$

Diagonal $BD = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(-2-3)^2 + (5-4)^2}$

$= \sqrt{(-5)^2 + (1)^2}$

$= \sqrt{25+1}$

$\boxed{\text{Diagonal } BD = \sqrt{26}}$

$\therefore \quad AB = CD = \sqrt{18}$

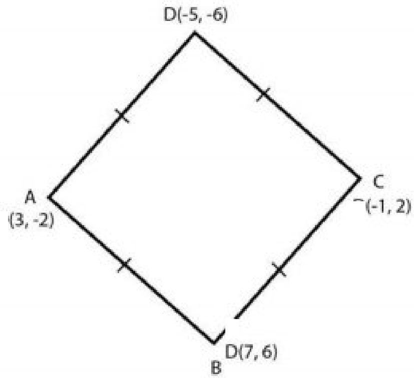
$BC = DA = \sqrt{8}$

Diagonal AC = Diagonal $BD = \sqrt{26}$

$\therefore$ The given complex numbers form a rectangle.

6) Show that the points represented by the complex numbers 3 - 2i, 7 + 6i, -1 + 2i and -5 - 6i form a rhombus.

Solution :



Let A = 3-2i = (3, -2)

B = 7+6i = (7, 6)

C = -1+2i = (-1, 2)

D = -5-6i = (-5, -6)

$AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(3-7)^2 + (-2-6)^2}$

$= \sqrt{(-4)^2 + (-8)^2}$

$= \sqrt{16 + 64}$

$\boxed{AB = \sqrt{80}}$

$BC = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(7 + 1)^2 + (6 - 2)^2}$

$= \sqrt{(8)^2 + (4)^2}$

$= \sqrt{64 + 16}$

$\boxed{BC = \sqrt{80}}$

$CD = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(-1 + 5)^2 + (2 + 6)^2}$

$= \sqrt{(4)^2 + (8)^2}$

$= \sqrt{16 + 64}$

$\boxed{CD = \sqrt{80}}$

$DA = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(-5 - 3)^2 + (-6 + 2)^2}$

$= \sqrt{(-8)^2 + (-4)^2}$

$= \sqrt{64 + 16}$

$\boxed{DA = \sqrt{80}}$

Dia AC $= \sqrt{(3 + 1)^2 + (-2 - 2)^2}$

$= \sqrt{(4)^2 + (-4)^2}$

$= \sqrt{16 + 16}$

$= \sqrt{32}$

Dia BD $= \sqrt{(7 + 5)^2 + (6 + 6)^2}$

$= \sqrt{(12)^2 + (12)^2}$

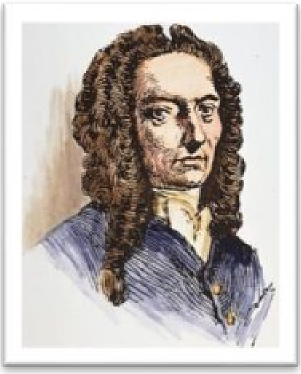
$= \sqrt{144 + 144}$

$= \sqrt{288}$

$\therefore$ AB = BC = CD = DA = $\sqrt{80}$ & Dia AC $\neq$ Dia BD

$\therefore$ The given points form a rhombus.

Chapter 2.2 DE MOIVRE'S THEOREM - APPLICATION



Abraham De Moivre (1667-1754) was a French-born Mathematician proposed the De Moivre's formula.

He was involved in transforming trigonometry, from a branch of geometry into a branch of analysis.

Its importance lies in the relationship, the formula establishes between complex numbers and trigonometry. It entails or suggests a great many valuable identities and thus became one of the most useful steps in the early development of complex number theory.

He discovered a version of his formula in 1707 and proposed the more usual version in 1722.

De Moivre's formula can be used to obtain roots of complex numbers.

De Moivre's Theorem – Statement

i) If 'n' is an integer (positive or negative) then

$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

ii) If 'n' is a fraction, then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$

Example:

i) $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

ii) $(\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$

iii) $(\cos\theta + i\sin\theta)^{1/2} = \cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta$

iv) $(\cos\theta + i\sin\theta)^{-5/2} = \cos\left(\frac{-5}{2}\theta\right) + i\sin\left(\frac{-5}{2}\theta\right)$

$= \cos\frac{5}{2}\theta - i\sin\frac{5}{2}\theta$

Results:

1) $(\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$

2) $\frac{1}{\cos\theta + i\sin\theta} = (\cos\theta + i\sin\theta)^{-1}$

$= \cos(-\theta) + i\sin(-\theta)$

$= \cos\theta - i\sin\theta$

3) $\frac{1}{\cos\theta - i\sin\theta} = (\cos\theta - i\sin\theta)^{-1}$

$= \cos(-\theta) - i\sin(-\theta)$

$= \cos\theta + i\sin\theta$

4) $\sin\theta + i\cos\theta = \cos(90-\theta) + i\sin(90-\theta)$

$= \cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)$

MATH FACT

(i) $\sin\theta + i\cos\theta = i(\cos\theta - i\sin\theta)$

(ii) $\sin\theta - i\cos\theta = -i(\cos\theta + i\sin\theta)$

(iii) $\cos(-\theta) = \cos\theta$

(iv) $\sin(-\theta) = -\sin\theta$

Note :

1. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ then

(i) $ab = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$$\boxed{ab = \cos(\alpha+\beta) + i\sin(\alpha+\beta)}$$

(ii) $\frac{a}{b} = \frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta}$

$$\boxed{\frac{a}{b} = \cos(\alpha-\beta) + i\sin(\alpha-\beta)}$$

2. If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ and $c = \cos\gamma + i\sin\gamma$ then

$abc = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

$$\boxed{abc = \cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma)}$$

Worked Examples

1) If $z = \cos 15^o + i\sin 15^o$ What is the value of z^6.

Solution :

Given $z = \cos 15^o + i\sin 15^o$

$$\begin{aligned} z^6 &= (\cos 15^o + i\sin 15^o)^6 \\ &= \cos(6 \times 15^o) + i\sin(6 \times 15^o) \\ &= \cos 90^o + i\sin 90^o \qquad [\because \cos 90^o = 0, \sin 90^o = 1] \\ &= 0 + i(1) \\ &= 1i \end{aligned}$$

2) If $z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ What is the value of z^4?

Solution :

Given $z = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$$\begin{aligned} z^4 &= \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^4 \\ &= \cos 4\left(\frac{\pi}{2}\right) + i\sin 4\left(\frac{\pi}{2}\right) \\ &= \cos 2\pi + i\sin 2\pi \\ &= \cos 2(180^o) + i\sin 2(180^o) \\ &= \cos 360^o + i\sin 360^o \end{aligned}$$

$= \quad 1 + i(0) \qquad [\because \pi = 180^o]$

$= \quad 1 \qquad [\because \cos 360^o = 1, \sin 360^o = 0]$

3) If z = cos 60° + i sin 60° then what is the value of $\frac{1}{z}$.

Solution :

Given $z = \cos 60^o + i \sin 60^o$

$$\frac{1}{z} = \frac{1}{(\cos 60^o + i \sin 60^o)}$$

$$= (\cos 60^o + i \sin 60^o)^{-1}$$

$$= \cos 60^o - i \sin 60^o$$

$$= \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

4) Find the value of $\frac{\cos 10\,\theta + i \sin 10\,\theta}{\cos 3\,\theta + i \sin 3\,\theta}$

Solution :

$$\frac{\cos 10\,\theta + i \sin 10\,\theta}{\cos 3\,\theta + i \sin 3\,\theta} = \frac{(\cos\theta + i\sin\theta)^{10}}{(\cos\theta + i\sin\theta)^{3}}$$

$$= (\cos\theta + i\sin\theta)^{10-3}$$

$$= (\cos\theta + i\sin\theta)^{7}$$

$$= \cos 7\theta + i \sin 7\theta$$

5) Simplify $\frac{\cos 3\,\theta + i \sin 3\,\theta}{\cos\theta - i \sin\theta}$

Solution :

$$\frac{\cos 3\,\theta + i \sin 3\,\theta}{\cos\theta - i \sin\theta} = \frac{(\cos\theta + i\sin\theta)^{3}}{(\cos\theta + i\sin\theta)^{-1}}$$

$$= (\cos\theta + i\sin\theta)^{3+1}$$

$$= (\cos\theta + i\sin\theta)^{4}$$

$$= \cos 4\theta + i \sin 4\theta$$

6) Simplify $\frac{\cos 5\,\theta - i \sin 5\,\theta}{\cos 3\,\theta + i \sin 3\,\theta}$

Solution :

$$\frac{\cos 5\,\theta - i \sin 5\,\theta}{\cos 3\,\theta + i \sin 3\,\theta} = \frac{(\cos\theta + i\sin\theta)^{-5}}{(\cos\theta + i\sin\theta)^{3}}$$

$$= (\cos\theta + i\sin\theta)^{-5-3}$$

$$= (\cos\theta + i\sin\theta)^{-8}$$

$$= \cos 8\theta - i \sin 8\theta$$

7) Simplify $(\cos\theta + i\sin\theta)^3(\cos\theta + i\sin\theta)^{-2}$

Solution :

$$(\cos\theta + i\sin\theta)^3(\cos\theta + i\sin\theta)^{-2} = (\cos\theta + i\sin\theta)^{3-2}$$

$$= (\cos\theta + i\sin\theta)^{1}$$

$$= \cos\theta + i\sin\theta$$

8) If $x = \cos\theta + i\sin\theta$, find $x + \frac{1}{x}$

Solution :

Given $x = \cos\theta + i\sin\theta$

$$\frac{1}{x} = \cos\theta - i\sin\theta$$

$$x + \frac{1}{x} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$$

$$\boxed{x + \frac{1}{x} = 2\cos\theta}$$

9) If $x = \cos\theta + i\sin\theta$, find $x - \frac{1}{x}$

Solution :

Given : $x = \cos\theta + i\sin\theta$

$$\frac{1}{x} = \cos\theta - i\sin\theta$$

$$x - \frac{1}{x} = (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)$$

$$\boxed{x - \frac{1}{x} = 2i\sin\theta}$$

10) If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, find ab

Solution :

Given : $a = \cos\alpha + i\sin\alpha$

$$b = \cos\beta + i\sin\beta$$

$$ab = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$\boxed{ab = \cos(\alpha + \beta) + i\sin(\alpha + \beta)}$$

11) If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ find $\frac{a}{b}$

Solution :

Given : $a = \cos\alpha + i\sin\alpha$

$$b = \cos\beta + i\sin\beta$$

$$\frac{a}{b} = \frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta}$$

$$\boxed{\frac{a}{b} = \cos(\alpha-\beta) + i\sin(\alpha-\beta)}$$

12) Simplify $(\cos 20^o + i\sin 20^o)(\cos 40^o + i\sin 40^o)(\cos 30^o + i\sin 30^o)$

Solution :

Given : $(\cos 20^o + i\sin 20^o)(\cos 40^o + i\sin 40^o)(\cos 30^o + i\sin 30^o)$

$= \cos(20^o + 40^o + 30^o) + i\sin(20^o + 40^o + 30^o)$

$= \cos 90^o + i\sin 90^o$

$= 0 + i(1)$ $\quad [\because \cos 90^o = 0, \ \sin 90^o = 1]$

$= 1i$

13) Prove that $(\sin\theta + i\cos\theta)^n = \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right)$

Solution :

$$\text{LHS} = (\sin\theta + i\cos\theta)^n = \left[\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right]^n \quad \left[\because \sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)\right.$$

$$= \cos n\left(\frac{\pi}{2}-\theta\right) + i\sin n\left(\frac{\pi}{2}-\theta\right) \qquad \left.\cos\theta = \sin\left(\frac{\pi}{2}-\theta\right)\right]$$

14) Find the product of $3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

Solution :

$$3\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \times 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$= 12\left[\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)\right]$$

$$= 12\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right] \qquad \left[\because \frac{\pi}{2} = \frac{180^o}{2} = 90^o\right]$$

$$= 12[\cos 90^o + i\sin 90^o] \qquad [\because \cos 90^o = 0, \sin 90^o = 1]$$

$$= 12[0 + i(1)]$$

$$= 12(1i)$$

$$= 12i$$

15) If $x = \cos\theta + i\sin\theta$ find $x^m + \frac{1}{x^m}$ and $x^m - \frac{1}{x^m}$

Solution :

Given $x = \cos\theta + i\sin\theta$

$$x^m = (\cos\theta + i\sin\theta)^m$$

$$\boxed{x^m = \cos m\theta + i\sin m\theta} \rightarrow (1)$$

$$\boxed{\frac{1}{x^m} = \cos m\theta - i\sin m\theta} \rightarrow (2)$$

①+ ② $\Rightarrow$

$$x^m + \frac{1}{x^m} = (\cos m\theta + i\sin m\theta) + (\cos m\theta - i\sin m\theta)$$

$$\boxed{x^m + \frac{1}{x^m} = 2\cos m\theta}$$

① −② $\Rightarrow$

$$x^m - \frac{1}{x^m} = (\cos m\theta + i\sin m\theta) - (\cos m\theta - i\sin m\theta)$$

$$= \cos m\theta + i\sin m\theta - \cos m\theta + i\sin m\theta$$

$$\boxed{x^m - \frac{1}{x^m} = 2i\sin m\theta}$$

16) If $a = \cos\theta + i\sin\theta$, $b = \cos\emptyset + i\sin\emptyset$ find $ab + \frac{1}{ab}$

Solution :

Given $a = \cos\theta + i\sin\theta$

$b = \cos\emptyset + i\sin\emptyset$

$$ab = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$$

$$\boxed{ab = \cos(\theta+\phi) + i\sin(\theta+\phi)} \quad \rightarrow \quad ①$$

$$\boxed{\frac{1}{ab} = \cos(\theta+\phi) - i\sin(\theta+\phi)} \quad \rightarrow \quad ②$$

① + ② $\Rightarrow$

$$ab + \frac{1}{ab} = [\cos(\theta+\phi) + i\sin(\theta+\phi)] + [\cos(\theta+\phi) - i\sin(\theta+\phi)]$$

$$\boxed{ab + \frac{1}{ab} = 2\cos(\theta+\phi)}$$

17) If $a = \cos 2\alpha + i\sin 2\alpha$, $b = \cos 2\beta + i\sin 2\beta$ find $\sqrt{ab}$

Solution :

Given $a = \cos 2\alpha + i\sin 2\alpha$

$$b = \cos 2\beta + i\sin 2\beta$$

$$ab = (\cos 2\alpha + i\sin 2\alpha)(\cos 2\beta + i\sin 2\beta)$$

$$= \cos(2\alpha + 2\beta) + i\sin(2\alpha + 2\beta)$$

$$\boxed{ab = \cos 2(\alpha+\beta) + i\sin 2(\alpha+\beta)}$$

$$\sqrt{ab} = (ab)^{1/2}$$

$$= [\cos 2(\alpha+\beta) + i\sin 2(\alpha+\beta)]^{1/2}$$

$$= \cos 2\left(\frac{(\alpha+\beta)}{2}\right) + i\sin 2\left(\frac{(\alpha+\beta)}{2}\right)$$

$$\boxed{\sqrt{ab} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)}$$

18) If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$ and $z = \cos\gamma + i\sin\gamma$. Find the value of $\frac{xy}{z}$

Solution :

Given $x = \cos\alpha + i\sin\alpha$

$$y = \cos\beta + i\sin\beta$$

$$z = \cos\gamma + i\sin\gamma$$

$$xy = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

$$\boxed{xy = \cos(\alpha+\beta) + i\sin(\alpha+\beta)}$$

$$\frac{xy}{z} = \frac{\cos(\alpha+\beta) + i\sin(\alpha+\beta)}{\cos\gamma + i\sin\gamma}$$

$$\boxed{\frac{xy}{z} = \cos(\alpha+\beta-\gamma) + i\sin(\alpha+\beta-\gamma)}$$

19) If $z_1 = 3(\cos 70^o + i\sin 70^o)$, $z_2 = 6(\cos 20^o + i\sin 20^o)$ Find $\frac{z_1}{z_2}$

Solution :

Given $z_1 = 3(\cos 70^o + i\sin 70^o)$, $z_2 = 6(\cos 20^o + i\sin 20^o)$

$$\frac{z_1}{z_2} = \frac{3(\cos 70^o + i\sin 70^o)}{6(\cos 20^o + i\sin 20^o)}$$

$$= \frac{3}{6}[\cos(70^o - 20^o) + i\sin(70^o - 20^o)]$$

$$= \frac{1}{2}[\cos 50^o + i\sin 50^o]$$

20) Simplify : $\frac{(\cos 3\theta + i \sin 3\theta)^4 (\cos 4\theta + i \sin 4\theta)^2}{(\cos 2\theta + i \sin 2\theta)^5 (\cos 5\theta + i \sin 5\theta)^3}$

Solution :

Given $\frac{(\cos 3\theta + i \sin 3\theta)^4 (\cos 4\theta + i \sin 4\theta)^2}{(\cos 2\theta + i \sin 2\theta)^5 (\cos 5\theta + i \sin 5\theta)^3}$

$= \frac{(\cos \theta + i \sin \theta)^{4x3} (\cos \theta + i \sin \theta)^{2x4}}{(\cos \theta + i \sin \theta)^{5x2} (\cos \theta + i \sin \theta)^{3x5}}$

$= \frac{(\cos \theta + i \sin \theta)^{12} (\cos \theta + i \sin \theta)^{8}}{(\cos \theta + i \sin \theta)^{10} (\cos \theta + i \sin \theta)^{15}}$

$= (\cos \theta + i \sin \theta)^{12+8-10-15}$

$= (\cos \theta + i \sin \theta)^{-5}$

$= \cos 5\theta - i \sin 5\theta$

21) Simplify : $\frac{(\cos 2\theta - i \sin 2\theta)^4 (\cos 4\theta + i \sin 4\theta)^{-5}}{(\cos 3\theta + i \sin 3\theta)^2 (\cos 5\theta - i \sin 5\theta)^{-3}}$

Solution :

Given $\frac{(\cos 2\theta - i \sin 2\theta)^4 (\cos 4\theta + i \sin 4\theta)^{-5}}{(\cos 3\theta + i \sin 3\theta)^2 (\cos 5\theta - i \sin 5\theta)^{-3}}$

$= \frac{(\cos \theta + i \sin \theta)^{4x-2} (\cos \theta + i \sin \theta)^{-5x4}}{(\cos \theta + i \sin \theta)^{2x3} (\cos \theta + i \sin \theta)^{-3x-5}}$

$= \frac{(\cos \theta + i \sin \theta)^{-8} (\cos \theta + i \sin \theta)^{-20}}{(\cos \theta + i \sin \theta)^{6} (\cos \theta + i \sin \theta)^{15}}$

$= (\cos \theta + i \sin \theta)^{-8-20-6-15}$

$= (\cos \theta + i \sin \theta)^{-49}$

$= \cos 49\theta - i \sin 49\theta$

22) Simplify : $\frac{(\cos 3\theta + i \sin 3\theta)^{-5} (\cos 2\theta + i \sin 2\theta)^4}{(\cos 4\theta - i \sin 4\theta)^{-2} (\cos 5\theta - i \sin 5\theta)^3}$

Solution :

Given $\frac{(\cos 3\theta + i \sin 3\theta)^{-5} (\cos 2\theta + i \sin 2\theta)^4}{(\cos 4\theta - i \sin 4\theta)^{-2} (\cos 5\theta - i \sin 5\theta)^3}$

$= \frac{(\cos \theta + i \sin \theta)^{-5x3} (\cos \theta + i \sin \theta)^{4x2}}{(\cos \theta + i \sin \theta)^{-2x-4} (\cos \theta + i \sin \theta)^{3x-5}}$

$= \frac{(\cos \theta + i \sin \theta)^{-15} (\cos \theta + i \sin \theta)^{8}}{(\cos \theta + i \sin \theta)^{8} (\cos \theta + i \sin \theta)^{-15}}$

$= (\cos \theta + i \sin \theta)^{-15+8-8+15}$

$= (\cos \theta + i \sin \theta)^{o}$

$= \cos 0 + i \sin 0$

$= 1 + i(o)$ [∵ cos 0=1, sin 0 = 0]

$= 1$

23) Simplify: $\frac{(\cos \theta - i \sin \theta)^3 (\cos 3\theta + i \sin 3\theta)^5}{(\cos 2\theta - i \sin 2\theta)^5 (\cos 5\theta + i \sin 5\theta)^7}$ when $\theta = \frac{2\pi}{13}$

Solution :

Given $\frac{(\cos \theta - i \sin \theta)^3 (\cos 3\theta + i \sin 3\theta)^5}{(\cos 2\theta - i \sin 2\theta)^5 (\cos 5\theta + i \sin 5\theta)^7}$

$$= \frac{(\cos\theta + i\sin\theta)^{3x-1}(\cos\theta + i\sin\theta)^{5x3}}{(\cos\theta + i\sin\theta)^{5x-2}(\cos\theta + i\sin\theta)^{7x5}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{-3}(\cos\theta + i\sin\theta)^{15}}{(\cos\theta + i\sin\theta)^{-10}(\cos\theta + i\sin\theta)^{35}}$$

$$= (\cos\theta + i\sin\theta)^{-3+15+10-35}$$

$$= (\cos\theta + i\sin\theta)^{-13}$$

$$= \cos 13\,\theta - i\sin 13\,\theta$$

when $\boxed{\theta = \frac{2\pi}{13}}$

$$= \cos 13\left(\frac{2\pi}{13}\right) - i\sin 13\left(\frac{2\pi}{13}\right)$$

$$= \cos 2\pi - i\sin 2\pi$$

$$= 1 - i(0) \qquad [\because \cos 2\pi = 1, \sin 2\pi = 0]$$

$$= 1$$

24) Prove that : $\left[\frac{\cos\theta + i\sin\theta}{\sin\theta - i\cos\theta}\right]^4 = 1$

Solution :

$$\text{LHS} = \left[\frac{\cos\theta + i\sin\theta}{\sin\theta - i\cos\theta}\right]^4$$

$$= \left[\frac{\cos\theta + i\sin\theta}{-i\,(\cos\theta + i\sin\theta)}\right]^4$$

$$\boxed{\because \sin\theta - i\cos\theta = -i(\cos\theta + i\sin\theta)}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{(-i)^4(\cos\theta + i\sin\theta)^4} \qquad [\because (-i)^4 = 1]$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{(1)\,(\cos\theta + i\sin\theta)^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^4}{(\cos\theta + i\sin\theta)^4}$$

$$= (\cos\theta + i\sin\theta)^{4-4}$$

$$= (\cos\theta + i\sin\theta)^0$$

$$= \cos 0 + i\sin 0$$

$$= 1 + i(0) \qquad [\because \cos 0 = 1, \sin 0 = 0]$$

$$= 1 \text{ R.H.S.}$$

25) Show that $\left(\frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right)^2 = \cos 2\theta + i\sin 2\theta$

Solution :

$$\text{LHS} = \left(\frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right)^2$$

Let $z = \cos\theta + i\sin\theta$

$$\frac{1}{z} = \cos\theta - i\sin\theta$$

$$= \left(\frac{1+z}{1+\frac{1}{z}}\right)^2$$

$$= \left(\frac{1+z}{\frac{z+1}{z}}\right)^2$$

$$= \left[(1 + z) \times \frac{z}{(1+z)}\right]^2$$

$$= [z]^2$$

$$= (\cos\theta + i\sin\theta)^2$$

$$= \cos 2\theta + i\sin 2\theta$$

$$= \text{RHS}$$

26) Show that $\left[\frac{1+\sin A + i\cos A}{1+\sin A - i\cos A}\right]^n = \cos n\left(\frac{\pi}{2} - A\right) + i\sin n\left(\frac{\pi}{2} - A\right)$

Solution :

$$\text{LHS} = \left[\frac{1+\sin A + i\cos A}{1+\sin A - i\cos A}\right]^n$$

Let $z = \sin A + i\cos A \Rightarrow \boxed{z = \cos\left(\frac{\pi}{2} - A\right) + i\sin\left(\frac{\pi}{2} - A\right)}$

$\frac{1}{z} = \sin A - i\cos A \Rightarrow \boxed{\frac{1}{z} = \cos\left(\frac{\pi}{2} - A\right) - i\sin\left(\frac{\pi}{2} - A\right)}$

$$\text{LHS} = \left(\frac{1+z}{1+\frac{1}{z}}\right)^n$$

$$= \left[\frac{1+z}{\left(\frac{z+1}{z}\right)}\right]^n$$

$$= \left[(1+z)\left(\frac{z}{(z+1)}\right)\right]^n$$

$$= [z]^n$$

$$= \left[\cos\left(\frac{\pi}{2} - A\right) + i\sin\left(\frac{\pi}{2} - A\right)\right]^n$$

$$= \cos n\left(\frac{\pi}{2} - A\right) + i\sin n\left(\frac{\pi}{2} - A\right)$$

$$= \text{RHS}$$

27) If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$ Show that

(i) $\cos(\alpha - \beta) = \frac{1}{2}\left[\frac{a}{b} + \frac{b}{a}\right]$

(ii) $\sin(\alpha + \beta) = \frac{1}{2i}\left[ab - \frac{1}{ab}\right]$

Solution :

Given $a = \cos\alpha + i\sin\alpha$

$b = \cos\beta + i\sin\beta$

(i) $\frac{a}{b} = \frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta}$

$$\boxed{\frac{a}{b} = \cos(\alpha - \beta) + i\sin(\alpha - \beta)} \rightarrow ①$$

$$\frac{b}{a} = \frac{1}{\left(\frac{a}{b}\right)} = \frac{1}{\cos(\alpha-\beta) + i\sin(\alpha-\beta)}$$

$$\boxed{\frac{b}{a} = \cos(\alpha-\beta) - i\sin(\alpha-\beta)} \rightarrow ②$$

① + ② $\Rightarrow$

$$\frac{a}{b} + \frac{b}{a} = [\cos(\alpha-\beta) + i\sin(\alpha-\beta)] + [\cos(\alpha-\beta) - i\sin(\alpha-\beta)]$$

$$\frac{a}{b} + \frac{b}{a} = 2\cos(\alpha-\beta)$$

$$\boxed{\cos(\alpha-\beta) = \frac{1}{2}\left[\frac{a}{b} + \frac{b}{a}\right]}$$

(ii) $ab = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$$\boxed{ab = \cos(\alpha+\beta) + i\sin(\alpha+\beta)} \rightarrow ③$$

$$\frac{1}{ab} = \frac{1}{\cos(\alpha+\beta) + i\sin(\alpha+\beta)}$$

$$\boxed{\frac{1}{ab} = \cos(\alpha+\beta) - i\sin(\alpha+\beta)} \rightarrow ④$$

③ − ④ $\Rightarrow$

$$ab - \frac{1}{ab} = [\cos(\alpha+\beta) + i\sin(\alpha+\beta)] - [\cos(\alpha+\beta) - i\sin(\alpha+\beta)]$$

$$ab - \frac{1}{ab} = 2i\sin(\alpha+\beta)$$

$$\boxed{\sin(\alpha+\beta) = \frac{1}{2i}\left[ab - \frac{1}{ab}\right]}$$

28) If $a = \cos x + i\sin x$, $b = \cos y + i\sin y$, $c = \cos z + i\sin z$, Find the value of

(i) $\frac{ab}{c} + \frac{c}{ab}$ (ii) $\frac{ab}{c} - \frac{c}{ab}$

Solution :

Given $a = \cos x + i\sin x$

$b = \cos y + i\sin y$

$c = \cos z + i\sin z$

(i) $$\frac{ab}{c} = \frac{(\cos x + i\sin x)(\cos y + i\sin y)}{\cos z + i\sin z}$$

$$= \frac{\cos(x+y) + i\sin(x+y)}{\cos z + i\sin z}$$

$$\boxed{\frac{ab}{c} = \cos(x+y-z) + i\sin(x+y-z)} \rightarrow ①$$

$$\frac{c}{ab} = \frac{1}{\left(\frac{ab}{c}\right)} = \frac{1}{\cos(x+y-z) + i\sin(x+y-z)}$$

$$\boxed{\frac{c}{ab} = \cos(x+y-z) - i\sin(x+y-z)} \rightarrow ②$$

① + ② $\Rightarrow$

$$\frac{ab}{c} + \frac{c}{ab} = [\cos(x+y-z) + i\sin(x+y-z)] + [\cos(x+y-z) - i\sin(x+y-z)]$$

$$\boxed{\frac{ab}{c}+\frac{c}{ab} = 2\cos(x+y-z)}$$

① − ② $\Rightarrow$

(ii) $\frac{ab}{c}-\frac{c}{ab} = [\cos(x+y-z)+i\sin(x+y-z)]-[\cos(x+y-z)-i\sin(x+y-z)]$

$$\boxed{\frac{ab}{c}-\frac{c}{ab} = 2i\sin(x+y-z)}$$

29) If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$.

Prove that (i) $\sqrt{ab} - \frac{1}{\sqrt{ab}} = 2\,i\sin\left(\frac{\alpha+\beta}{2}\right)$

(ii) $\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}} = 2\cos\left(\frac{\alpha-\beta}{2}\right)$

Solution :

Given $a = \cos\alpha + i\sin\alpha$

$b = \cos\beta + i\sin\beta$

$ab = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$ab = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

$\sqrt{ab} = (ab)^{1/2} = [\cos(\alpha+\beta)+i\sin(\alpha+\beta)]^{1/2}$

(i) $\boxed{\sqrt{ab} = \cos\left(\frac{\alpha+\beta}{2}\right)+i\sin\left(\frac{\alpha+\beta}{2}\right)} \to ①$

$\frac{1}{\sqrt{ab}} = \frac{1}{\cos\left(\frac{\alpha+\beta}{2}\right)+i\sin\left(\frac{\alpha+\beta}{2}\right)}$

$\boxed{\frac{1}{\sqrt{ab}} = \cos\left(\frac{\alpha+\beta}{2}\right)-i\sin\left(\frac{\alpha+\beta}{2}\right)} \to ②$

① − ② $\Rightarrow$

$\sqrt{ab}-\frac{1}{\sqrt{ab}} = \left[\cos\left(\frac{\alpha+\beta}{2}\right)+i\sin\left(\frac{\alpha+\beta}{2}\right)\right] - \left[\cos\left(\frac{\alpha+\beta}{2}\right)-i\sin\left(\frac{\alpha+\beta}{2}\right)\right]$

$= 2\,i\sin\left(\frac{\alpha+\beta}{2}\right)$

$$\boxed{\sqrt{ab}-\frac{1}{\sqrt{ab}} = 2\,i\sin\left(\frac{\alpha+\beta}{2}\right)}$$

(ii) $\frac{a}{b} = \frac{\cos\alpha+i\sin\alpha}{\cos\beta+i\sin\beta} = \cos(\alpha-\beta)+i\sin(\alpha-\beta)$

$\sqrt{\frac{a}{b}} = \left(\frac{a}{b}\right)^{1/2} = [\cos(\alpha-\beta)+i\sin(\alpha-\beta)]^{1/2}$

$\boxed{\sqrt{\frac{a}{b}} = \cos\left(\frac{\alpha-\beta}{2}\right)+i\sin\left(\frac{\alpha-\beta}{2}\right)} \to ③$

$\sqrt{\frac{b}{a}} = \frac{1}{\sqrt{a/b}} = \frac{1}{\cos\left(\frac{\alpha-\beta}{2}\right)+i\sin\left(\frac{\alpha-\beta}{2}\right)}$

$\boxed{\sqrt{\frac{b}{a}} = \cos\left(\frac{\alpha-\beta}{2}\right)-i\sin\left(\frac{\alpha-\beta}{2}\right)} \to ④$

③ + ④ $\Rightarrow$

$$\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}} = \cos\left(\frac{\alpha-\beta}{2}\right)+i\sin\left(\frac{\alpha-\beta}{2}\right)+\left[\cos\left(\frac{\alpha-\beta}{2}\right)-i\sin\left(\frac{\alpha-\beta}{2}\right)\right]$$

$$\boxed{\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}} = 2\cos\left(\frac{\alpha-\beta}{2}\right)}$$

30) If $x+\frac{1}{x} = 2\cos\theta$ Prove that (i) $x^n+\frac{1}{x^n} = 2\cos n\theta$

(ii) $x^n-\frac{1}{x^n} = 2i\sin n\theta$

Solution :

Given $\boxed{x+\frac{1}{x} = 2\cos\theta}$

$$\frac{x^2+1}{x} = 2\cos\theta$$

$$x^2+1 = 2x\cos\theta$$

$$\boxed{x^2-2x\cos\theta+1=0}$$

$\therefore a = 1, b = -2\cos\theta, c = 1$

Formula

$$\boxed{x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}}$$

$$= \frac{2\cos\theta\pm\sqrt{(-2\cos\theta)^2-4(1)(1)}}{2(1)}$$

$$= \frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}$$

$$= \frac{2\cos\theta\pm\sqrt{4(\cos^2\theta-1)}}{2}$$

$$= \frac{2\cos\theta\pm\sqrt{4(-\sin^2\theta)}}{2} \qquad [\because \sin^2\theta+\cos^2\theta = 1$$

$$= \frac{2\cos\theta\pm i\,2\sin\theta}{2} \qquad \Rightarrow \sin^2\theta = 1-\cos^2\theta$$

$$= \frac{2[\cos\theta\pm i\sin\theta]}{2} \qquad \Rightarrow -\sin^2\theta = -1+\cos^2\theta]$$

$$x = \cos\theta\pm i\sin\theta$$

Let $x = \cos\theta+i\sin\theta$

$$x^n = (\cos\theta+i\sin\theta)^n$$

$$x^n = \cos n\theta+i\sin n\theta \rightarrow ①$$

$$\frac{1}{x^n} = \cos n\theta-i\sin n\theta \rightarrow ②$$

① + ② $\Rightarrow$

$$x^n+\frac{1}{x^n} = (\cos n\theta+i\sin n\theta)+(\cos n\theta-i\sin n\theta)$$

$$\boxed{x^n+\frac{1}{x^n} = 2\cos n\theta}$$

① − ② $\Rightarrow$ $x^n-\frac{1}{x^n} = (\cos n\theta+i\sin n\theta)-(\cos n\theta-i\sin n\theta)$

$$= \cos n\theta+i\sin n\theta-\cos n\theta+i\sin n\theta$$

$$\boxed{x^n-\frac{1}{x^n} = 2i\sin n\theta}$$

31) Prove that $(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^{n+1}\cos\frac{n\pi}{3}$

Solution :

Let $z = 1 + i\sqrt{3}$

$z = r(\cos\theta + i\sin\theta)$

$1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$

To find modulus (r)

$$r = |z| = \sqrt{(a)^2 + (b)^2}$$

$$= \sqrt{(1)^2 + (\sqrt{3})^2}$$

$$= \sqrt{1+3} = \sqrt{4} = 2$$

$$\boxed{r = 2}$$

To find amplitude

$$\boxed{\alpha = \tan^{-1}\left(\frac{b}{a}\right)}$$ Here $a = 1$, $b = \sqrt{3}$

$$\alpha = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$\alpha = \tan^{-1}(\sqrt{3})$$

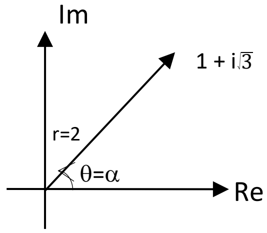
$$= 60^o$$

$$\boxed{\alpha = \frac{\pi}{3}}$$

The complex number is in the form (+, +). It lies in I Quadrant

$$\boxed{\theta = \alpha}$$

$$\Rightarrow \boxed{\theta = \frac{\pi}{3}}$$



Let $1 + i\sqrt{3} = r(\cos\theta + i\sin\theta)$

$$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$(1+i\sqrt{3})^n = \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^n$$

$$\boxed{(1+i\sqrt{3})^n = 2^n\left[\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right]} \rightarrow ①$$

Similarly we can prove that

$$\boxed{(1-i\sqrt{3})^n = 2^n\left[\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right]} \rightarrow ②$$

① + ② $\Rightarrow$

$$(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^n\left[\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3} + \cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right]$$

$$= 2^n\left[2\cos\frac{n\pi}{3}\right]$$

$$\boxed{(1+i\sqrt{3})^n + (1-i\sqrt{3})^n = 2^{n+1}\cos\frac{n\pi}{3}}$$

Chapter 2.3 ROOTS OF COMPLEX NUMBERS

Definition:

A number ω is called the n^{th} root of a complex number z, if $\omega^n = z$ and we write

$\omega = z^{\frac{1}{n}}$

Procedure to find the n^{th} roots of complex numbers

Step 1:

Write the given complex number in polar form

Step 2:

Add 2kπ to the argument

Step 3:

Apply DeMoivre's theorem (bring the power to inside)

Step 4:

Put k = 0, 1, 2, … (n-1)

Illustration:

Step 1: Let $z = r(\cos\theta + i\sin\theta)$

Step 2: $z = r[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]$

$z^{\frac{1}{n}} = r[\cos(2k\pi + \theta) + i\sin(2k\pi + \theta)]^{\frac{1}{n}}$

Step 3: $z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right]$

Step 4: Put k = 0, 1, 2, … n-1 for getting n roots

Only these values of k will give 'n' different values of $z^{\frac{1}{n}}$ provided $z \neq 0$.

To find n^{th} roots of unity

Let $1 = \cos 0 + i\sin 0$

$1 = \cos 2k\pi + i\sin 2k\pi$

∴ n^{th} roots of unity

$(1)^{\frac{1}{n}} = [\cos 2k\pi + i\sin 2k\pi]^{\frac{1}{n}}$

$(1)^{\frac{1}{n}} = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$,

Where k = 0, 1, 2, …. (n-1)

∴ The n^{th} roots of unity are

When $k = 0$, $R_1 = \cos 0 + i\sin 0 = 1 + i0 = 1 = e^o$

$k = 1$, $R_2 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = e^{\frac{i2\pi}{n}} = \omega$(say)

$k = 2$, $R_3 = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n} = e^{\frac{i4\pi}{n}}$

$$= \left[e^{\frac{i\,2\pi}{n}}\right]^2$$

$$= \omega^2$$

$k = n-1$, $R_n = \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}$

$$= e^{\frac{i\,2(n-1)\pi}{n}}$$

$$= \omega^{n-1}$$

The n^{th} roots of unity are $1, \omega, \omega^2, \dots\dots\dots \omega^{n-1}$

MATH FACT

n^{th} roots are equally spread along the circle because successive n^{th} roots have arguments that differ by $\frac{2\pi}{4}$

Exponential form n^{th} roots of unity

The n^{th} roots of unity are in exponential form is given by

$e^{o}, e^{\frac{i\,2\pi}{n}}, e^{\frac{i\,4\pi}{n}}, \dots\dots\dots, e^{\frac{i\,2(n-1)\pi}{n}}$

Result:

If ω is the n^{th} roots of unity then (i) $\omega^n = 1$, (ii) $1 + \omega + \omega^2 + \dots\dots + \omega^{n-1} = 0$

(i) $\omega^n = 1$

$$\omega^n = \left(\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}\right)^n$$

$$= \cos 2\pi + i\sin 2\pi$$

$$= 1 + 0\,i$$

$$\omega^n = 1$$

(ii) Sum of the roots is zero

$1 + \omega + \omega^2 + \dots\dots + \omega^{n-1} = 0$

LHS $= 1 + \omega, \omega^2, \dots\dots\dots \omega^{n-1}$ is a G.P with n terms

$$= \frac{1-\omega^n}{1-\omega} \quad \left[\because 1 + r + r^2 + \dots + r^{n-1} = \frac{1-r^n}{1-r}\right]$$

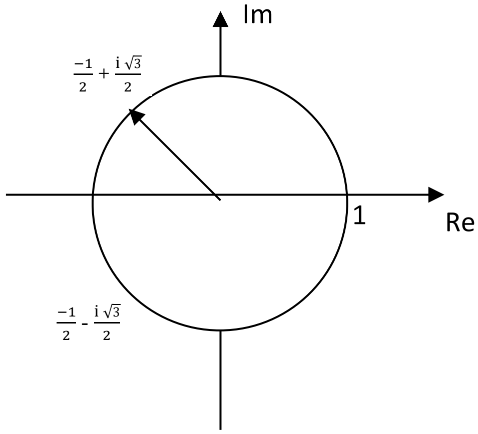
$$= \frac{1-1}{1-\omega} = \frac{0}{1-\omega} = 0 \qquad [\because \omega^n = 1]$$

$$\therefore \quad 1 + \omega + \omega^2 + \dots + \omega^{n-1} = 0$$

Note: In nth roots of unity,

- The arguments are in A.P with common difference $\frac{2\pi}{n}$
- The roots are in geometric progression with common ratio ω.
- The product of the roots is $(-1)^{n-1}$.

Cube roots of unity



Let x be a cube roots of unity

$$x = (1)^{\frac{1}{3}}$$

$$= (\cos 0 + i \sin 0)^{\frac{1}{3}}$$

$$= (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{3}}$$

$$= \cos\left(\frac{2k\pi}{3}\right) + i \sin\left(\frac{2k\pi}{3}\right)$$

Where k = 0, 1, 2

∴ The roots are

When $k = 0$, $\cos 0 + i \sin 0 = 1 + i(0) = 1$

$k = 1$, $\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$

$k = 2$, $\cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3} = \frac{-1}{2} - \frac{i\sqrt{3}}{2}$

∴ The cube roots of unity are 1, $\frac{-1}{2}+\frac{i\sqrt{3}}{2}$ and $\frac{-1}{2} - \frac{i\sqrt{3}}{2}$

$\Rightarrow 1, \omega$ and ω^2 where $\omega = e^{\frac{i\,2\pi}{3}}$

Note:

If ω is cube roots of unity then

(i) $\omega^3 = 1$

(ii) $1 + \omega + \omega^2 = 0$ [∵ The sum of the cube root of unity is zero]

Fourth roots of unity

Let x be a fourth roots of unity

$$x = (1)^{\frac{1}{4}}$$

$$= (\cos 0 + i \sin 0)^{\frac{1}{4}}$$

$$= (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{4}}$$

$$= \cos\left(\frac{2k\pi}{4}\right) + i \sin\left(\frac{2k\pi}{4}\right)$$

Where k = 0, 1, 2, 3

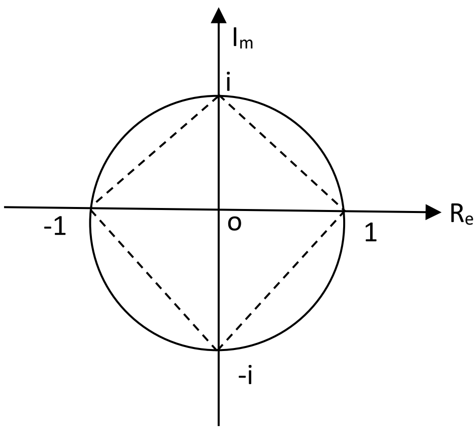
∴ The roots are

When $k = 0$, $\cos 0 + i \sin 0 = 1 + i\,0 = 1$

$k = 1$, $\cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = 0 + i\,(1) = i = \omega$

$k = 2$, $\cos \pi + i \sin\pi = -1 + i\,(0) = -1 = \omega^2$

$k = 3$, $\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2} = 0 + i\,(-1) = -i = \omega^3$



The fourth roots of unity are 1, i, -1 and -i

$\Rightarrow 1, \omega, \omega^2$ and ω^3 where $\omega = e^{\frac{i\pi}{2}}$

Note:

If ω is the fourth roots of unity, then

(i) $\omega^4 = 1$

(ii) $1 + \omega + \omega^2 + \omega^3 = 0$ [∵ The sum of the fourth root of unity is zero]

MATH FACT

The value of ω used in cube roots of unity and in fourth roots of unity are different.

Standard polar form

(i) $1 = \cos 0 + i \sin 0$

(ii) $-1 = \cos \pi + i \sin \pi$

(iii) $i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$

(iv) $-i = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$

Note:

If ω is the sixth roots of unity, then

(i) $\omega^6 = 1$

(ii) $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 = 0$

Worked Examples

1) If ω is the cube roots of unity what is the value of $1+\omega+\omega^2$?

Solution :

$1+\omega+\omega^2 = 0$ [∵ Sum of cube roots of unity is zero]

2) If ω is the cube roots of unity what is the value of $\omega(\omega+1)$?

Solution :

$\omega(\omega+1) = \omega^2 + \omega$ [∵ $1 + \omega + \omega^2 = 0 \Rightarrow \omega + \omega^2 = -1$]

$= -1$

3) If ω is a cube roots of unity, find the value of $\omega^5+\omega^6+\omega^7$

Solution :

If ω is a cube roots of unity [∵ $\omega^3 = 1$]

$\omega^5 + \omega^6 + \omega^7 = \omega^3.\omega^2 + \omega^3.\omega^3 + \omega^3.\omega^3.\omega$

$= (1)\omega^2 + (1)(1) + (1)(1)\omega$

$= \omega^2 + 1 + \omega$

$= 1 + \omega + \omega^2$ [∵ $1 + \omega + \omega^2 = 0$]

$= 0$

4) Find the value of $(-1)^{\frac{1}{2}}$

Solution :

Let $x = (-1)^{\frac{1}{2}}$

$x = [\cos\pi + i\sin\pi]^{\frac{1}{2}}$

$= [\cos(2k\pi+\pi) + i\sin(2k\pi+\pi)]^{\frac{1}{2}}$

$$\boxed{x = \cos\left(\frac{2k\pi+\pi}{2}\right) + i\sin\left(\frac{2k\pi+\pi}{2}\right)}$$

Where k = 0, 1

5) Find the value of $\left(\frac{1+i\sqrt{3}}{2}\right)^3$

Solution :

$\left(\frac{1+i\sqrt{3}}{2}\right)^3 = \left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^3$

$= \left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]^3$

$$\boxed{\because \cos\frac{\pi}{3} = \frac{1}{2} \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}}$$

$= \cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}$

$= \cos\pi + i\sin\pi$

$= -1 + i(0) \quad [\because \cos\pi = -1, \sin\pi = 0]$

$= -1$

6) Find the value of $(i)^{\frac{1}{3}}$

Solution :

Let $x = (i)^{\frac{1}{3}}$

$= \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^{\frac{1}{3}}$

$= \left[\cos\left(2k\pi+\frac{\pi}{2}\right) + i\sin\left(2k\pi+\frac{\pi}{2}\right)\right]^{\frac{1}{3}}$

$x = \left[\cos\left(\frac{4k\pi+\pi}{2}\right) + i\sin\left(\frac{4k\pi+\pi}{2}\right)\right]^{\frac{1}{3}}$

$$\boxed{x = \cos\left(\frac{4k\pi+\pi}{6}\right) + i\sin\left(\frac{4k\pi+\pi}{6}\right)}$$

Where k = 0, 1, 2

∴ The roots are

When k = 0, $x = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

k = 1, $x = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

k = 2, $x = \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}$

7) Solve $x^3 - 1 = 0$ (or) Find the cube roots of unity.

Solution :

Given $x^3 - 1 = 0$

$$x^3 = 1$$

$$x = (1)^{\frac{1}{3}}$$

$$x = [\cos 0 + i \sin 0]^{\frac{1}{3}}$$

$$= [\cos 2k\pi + i \sin 2k\pi]^{\frac{1}{3}}$$

$$\boxed{x = \cos\left(\frac{2k\pi}{3}\right) + i \sin\left(\frac{2k\pi}{3}\right)}$$

Where k = 0, 1, 2

∴ The roots are

When $k = 0, \quad x = \cos 0 + i \sin 0$

$k = 1, \quad x = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}$

$k = 2, \quad x = \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3}$

8) Solve $x^2 + 16 = 0$

Solution :

Given $x^2 + 16 = 0$

$$x^2 = -16$$

$$x^2 = 16(-1)$$

$$x = (16)^{\frac{1}{2}}(-1)^{\frac{1}{2}}$$

$$x = 4\,[\cos\pi + i\sin\pi]^{\frac{1}{2}}$$

$$= 4\,[\cos(2k\pi + \pi) + i \sin(2k\pi + \pi)]^{\frac{1}{2}}$$

$$\boxed{x = 4\left[\cos\left(\frac{2k\pi+\pi}{2}\right) + i \sin\left(\frac{2k\pi+\pi}{2}\right)\right]}$$

where k = 0, 1

∴ The roots are

When $k = 0, \quad x = 4\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$

$k = 1, \quad x = 4\left[\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right]$

9) Solve $x^5 - 1 = 0$

Solution :

Given $x^5 - 1 = 0$

$$x^5 = 1$$

$$x = (1)^{\frac{1}{5}}$$

$$= (\cos 0 + i \sin 0)^{\frac{1}{5}}$$

$$= (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{5}}$$

$$\boxed{x = \cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)}$$

Where k = 0, 1, 2, 3, 4

∴ The roots are

When $k = 0, \quad x = \cos 0 + i \sin 0$

$k = 1, \quad x = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$k = 2, \quad x = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$k = 3, \quad x = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$

$k = 4, \quad x = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$

10) Find all the values of $(1)^{\frac{1}{6}}$

Solution :

Let $\quad x = (1)^{\frac{1}{6}}$

$$= (\cos 0 + i \sin 0)^{\frac{1}{6}}$$

$$= (\cos 2k\pi + i \sin 2k\pi)^{\frac{1}{6}}$$

$$\boxed{x = \cos\left(\frac{2k\pi}{6}\right) + i\sin\left(\frac{2k\pi}{6}\right)}$$

Where k = 0, 1, 2, 3, 4, 5

∴ The roots are

When $k = 0, \quad x = \cos 0 + i \sin 0$

$k = 1, \quad x = \cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6}$

$k = 2, \quad x = \cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6}$

$k = 3, \quad x = \cos\frac{6\pi}{6} + i\sin\frac{6\pi}{6}$

$k = 4, \quad x = \cos\frac{8\pi}{6} + i\sin\frac{8\pi}{6}$

$k = 5, \quad x = \cos\frac{10\pi}{6} + i\sin\frac{10\pi}{6}$

11) Solve $x^7 + 1 = 0$ (or) Find all the values of $(-1)^{\frac{1}{7}}$

Solution :

Given $x^7 + 1 = 0$

$x^7 = -1$

$x = (-1)^{\frac{1}{7}}$

$x = (\cos\pi + i\sin\pi)^{\frac{1}{7}}$

$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{7}}$

$$\boxed{x = \cos\left(\frac{2k\pi+\pi}{7}\right) + i\sin\left(\frac{2k\pi+\pi}{7}\right)}$$

where k = 0, 1, 2, 3, 4, 5, 6

∴ The roots are

When $k = 0$, $x = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$

$k = 1$, $x = \cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}$

$k = 2$, $x = \cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}$

$k = 3$, $x = \cos\frac{7\pi}{7} + i\sin\frac{7\pi}{7}$

$k = 4$, $x = \cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7}$

$k = 5$, $x = \cos\frac{11\pi}{7} + i\sin\frac{11\pi}{7}$

$k = 6$, $x = \cos\frac{13\pi}{7} + i\sin\frac{13\pi}{7}$

12) Solve $x^7 + x^4 + x^3 + 1 = 0$

Solution :

Given $x^7 + x^4 + x^3 + 1 = 0$

$x^4(x^3 + 1) + 1(x^3 + 1) = 0$

$(x^4 + 1)(x^3 + 1) = 0$

$x^4 + 1 = 0;\quad x^3 + 1 = 0$

Case (i)

$x^4 + 1 = 0$

$x^4 = -1$

$x = (-1)^{\frac{1}{4}}$

$= (\cos\pi + i\sin\pi)^{\frac{1}{4}}$

$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{4}}$

$$\boxed{x = \cos\left(\frac{2k\pi+\pi}{4}\right) + i\sin\left(\frac{2k\pi+\pi}{4}\right)}$$

where k = 0, 1, 2, 3

∴ The roots are

When $k = 0, \quad x = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

$k = 1, \quad x = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}$

$k = 2, \quad x = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$

$k = 3, \quad x = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$

Case (ii)

$$x^3 + 1 = 0$$

$$x^3 = -1$$

$$x = (-1)^{\frac{1}{3}}$$

$$= (\cos\pi + i\sin\pi)^{\frac{1}{3}}$$

$$= [\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)]^{\frac{1}{3}}$$

$$\boxed{x = \cos\left(\frac{2k\pi+\pi}{3}\right) + i\sin\left(\frac{2k\pi+\pi}{3}\right)}$$

where k = 0, 1, 2

∴ The roots are

When $k = 0, \quad x = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

$k = 1, \quad x = \cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3}$

$k = 2, \quad x = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$

13) Solve $x^9 + x^5 - x^4 - 1 = 0$

Solution :

Given $x^9 + x^5 - x^4 - 1 = 0$

$x^5(x^4 + 1) - 1(x^4 + 1) = 0$

$(x^5 - 1)(x^4 + 1) = 0$

$x^5 - 1 = 0; \; x^4 + 1 = 0$

Case (i)

$$x^5 - 1 = 0$$

$$x^5 = 1$$

$$x = (1)^{\frac{1}{5}}$$

$$= (\cos 0 + i\sin 0)^{\frac{1}{5}}$$

$$= (\cos 2k\pi + i\sin 2k\pi)^{\frac{1}{5}}$$

$$\boxed{x = \cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)}$$

where k = 0, 1, 2, 3, 4

∴ The roots are

When $k = 0, \quad x = \cos 0 + i \sin 0$

$k = 1, \quad x = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$

$k = 2, \quad x = \cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}$

$k = 3, \quad x = \cos \frac{6\pi}{5} + i \sin \frac{6\pi}{5}$

$k = 4, \quad x = \cos \frac{8\pi}{5} + i \sin \frac{8\pi}{5}$

Case (ii)

Given $x^4 + 1 = 0$

$x^4 = -1$

$x = (-1)^{\frac{1}{4}}$

$x = (\cos \pi + i \sin \pi)^{\frac{1}{4}}$

$= [\cos(2k\pi + \pi) + i \sin(2k\pi + \pi)]^{\frac{1}{4}}$

$$\boxed{x = \cos\left(\frac{2k\pi+\pi}{4}\right) + i \sin\left(\frac{2k\pi+\pi}{4}\right)}$$

where k = 0, 1, 2, 3

∴ The roots are

When $k = 0, \quad x = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$

$k = 1, \quad x = \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}$

$k = 2, \quad x = \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}$

$k = 3, \quad x = \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4}$

14) If ω, ω^2 are the cube roots of unity, prove that $(1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5 = 32$

Solution :

LHS $= (1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5$

$= (1+\omega^2-\omega)^5 + (1+\omega-\omega^2)^5$

$[\because 1+\omega+\omega^2 = 0$

$\Rightarrow 1+\omega^2 = -\omega$

$\Rightarrow 1+\omega = -\omega^2]$

$= (-\omega-\omega)^5 + (-\omega^2-\omega^2)^5$

$= (-2\omega)^5 + (-2\omega^2)^5$

$= (-2)^5 (\omega)^5 + (-2)^5 (\omega^2)^5$

$= -32\omega^5 - 32\omega^{10}$

$$[\because \omega^5 = \omega^3 \times \omega^2 = 1 \times \omega^2$$

$$\boxed{\omega^5 = \omega^2}$$

$$\omega^{10} = \omega^9 \times \omega = 1 \times \omega]$$

$$\boxed{\omega^{10} = \omega}$$

$= -32\omega^2 - 32\omega$

$= -32(\omega^2 + \omega)$ $\quad [1 + \omega + \omega^2 = 0$

$= -32(-1)$ $\quad \Rightarrow \omega + \omega^2 = -1]$

$= 32$

$= \text{RHS}$

15) Find the value of $\left(\sqrt{3} + i\right)^{\frac{2}{3}}$

Solution :

Let $z = a + ib$

$z = \sqrt{3} + i$

Here, $a = \sqrt{3}$, $b = 1$

To find modulus (r)

$$r = |z| = \sqrt{(a)^2 + (b)^2}$$

$$= \sqrt{(\sqrt{3})^2 + (1)^2}$$

$$= \sqrt{3+1}$$

$$= \sqrt{4}$$

$$\boxed{r = 2}$$

To find amplitude

$\boxed{\alpha = \tan^{-1}\left(\frac{b}{a}\right)}$ Here $a = \sqrt{3}, b = 1$

$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\alpha = 30^{o}$$

$\because$ The complex number is in the form $(+,+)$. It lies in I Quadrant. $\boxed{\theta = \alpha}$

$$\boxed{\theta = \frac{\pi}{6}}$$

Let $\boxed{z = r(\cos\theta + i\sin\theta)}$

$$\sqrt{3} + i = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$\left(\sqrt{3} + i\right)^{\frac{2}{3}} = \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^{\frac{2}{3}}$$

$$= 2^{\frac{2}{3}}\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]^{\frac{2}{3}}$$

$$= 2^{\frac{2}{3}}\left[\cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6}\right]^{\frac{1}{3}}$$

$$= 2^{\frac{2}{3}}\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]^{\frac{1}{3}}$$

$$= 2^{\frac{2}{3}}\left[\cos\left(2k\pi + \frac{\pi}{3}\right) + i\sin\left(2k\pi + \frac{\pi}{3}\right)\right]^{\frac{1}{3}}$$

$$= 2^{\frac{2}{3}}\left[\cos\left(\frac{6k\pi+\pi}{3}\right) + i\sin\left(\frac{6k\pi+\pi}{3}\right)\right]^{\frac{1}{3}}$$

$$= 2^{\frac{2}{3}}\left[\cos\frac{1}{3}\left(\frac{6k\pi+\pi}{3}\right) + i\sin\frac{1}{3}\left(\frac{6k\pi+\pi}{3}\right)\right]$$

$$= 2^{\frac{2}{3}}\left[\cos\left(\frac{6k\pi+\pi}{9}\right) + i\sin\left(\frac{6k\pi+\pi}{9}\right)\right]$$

Where k = 0, 1, 2

When $k = 0$, $x = 2^{\frac{2}{3}}\left[\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right]$

$k = 1$, $x = 2^{\frac{2}{3}}\left[\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}\right]$

$k = 2$, $x = 2^{\frac{2}{3}}\left[\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right]$

Applications of Complex Numbers

Complex numbers are useful in representing a phenomenon that has two parts varying at the same time, for instance an alternating current. Numbers in the physical world are often represented by their real number component, such as in measurement, money and time.

For example, a mile is a unit of measurement that is equivalent to 5,280 feet. As a complex number, this measurement would be 5280 + 0i feet.

However, the expression in complex form does not produce any additional meaning if the imaginary number component is equal to zero. Therefore, complex numbers are useful when the imaginary number component is non-zero.

There are several instances in which complex numbers are important in the physical world.

1) Complex numbers have essential concrete applications in signal processing, control theory, electromagnetism, fluid dynamics, quantum mechanics, cartography and vibration analysis.
2) In electronics, the two real numbers R(Resistance) and X(Reactance) can be described as a single complex number Z = R + jX, where Z is the impedance.
3) In electromagnetism, instead of describing an electromagnetic field by two real quantities (electric field strength and magnetic field strength), it is best to describe as a single complex number, of which electric and magnetic components are simply the real and imaginary parts.
4) The laws of electricity can be expressed using complex addition and multiplication.
5) Engineers, doctors, scientists, vehicle designers and others who use electromagnetic signals need complex numbers for strong signal to reach its destination.

An application of Complex numbers : AC CIRCUITS

Complex numbers help to analyse and design AC circuits

DEFINITIONS:

AC : (ALTERNATING CURRENT)

Alternating current (AC) is an electric current which periodically reverses direction and changes its magnitude continuously with time in contrast to direct current (DC) which flows only in one direction.

CURRENT :

Flow of electrons in any conductor is called current. It is represented by the letter 'I'. The unit of the current is 'ampere'.

$$\text{Current, } I = \frac{\text{Charge}}{\text{Time}}$$

$$\boxed{I = \frac{Q}{T}}$$

Where 'Q' is in coulomb and 't' is in second.

VOLTAGE :

The difference of potentials between two points is called voltage or potential difference. It is denoted by 'V'. The unit of the voltage is 'volt'.

RESISTANCE :

The opposition offered by a substance to the flow of current is called resistance. It is denoted by 'R'. The unit of the resistance is ohm (Ω).

IMPEDANCE :

Impedance is the measure of the resistance that a circuit offers to a current when a voltage is applied.

It is ratio of the applied voltage to the resulting current. It is denoted by 'Z'. Its unit is ohm (Ω).

$$\boxed{Z = \frac{V}{I} \text{ ohm}}$$

ADMITTANCE :

Admittance is defined as a measure of how easily a circuit or device will allow current to flow through it.

It is also defined as the reciprocal of impedance. It is denoted by 'Y'. Its unit is 'mho'.

$$Y = \frac{1}{Z}$$

$$Y = \frac{1}{\left(\frac{V}{I}\right)}$$

$$\boxed{Y = \frac{I}{V}} \text{ mho}$$

Note :

In electrical engineering, the letter 'j' is placed in front of a real number to signify its imaginary number operation. Normally, the letter 'i' is used, but in electrical engineering 'j' is used instead to avoid conflict with the symbol for current (i).

FORMULAS :

1) Impedance $= \frac{\text{voltage}}{\text{current}}$ ohm

2) Admittance $= \frac{\text{current}}{\text{voltage}}$ mho

Worked Examples

1) The Admittance and Voltage of a circuit are given by the complex numbers 1+2j and 2–3j respectively. Find the current of the circuit.

Solution :

Given

Admittance = 1+2j

Voltage = 2–3j

To Find the current

$$\boxed{\text{Admittance} = \frac{\text{current}}{\text{voltage}}}$$

$$1 + 2j = \frac{\text{current}}{2-3j}$$

$$\Rightarrow \text{Current} = (1 + 2j) \text{ x } (2 - 3j)$$

$$= 2 - 3j + 4j - 6j^2$$

$$= 2 + j - 6(-1) \quad [\because j^2 = -1]$$

$$= 2 + j + 6$$

$$\boxed{\text{Current} = 8 + j} \text{ amperes}$$

2) The impedance and current of a circuit are given by the complex numbers 6 + 2j and 3 + j respectively. Find the voltage of the circuit.

Solution :

Given

Impedance = 6 + 2j

Current = 3 + j

To Find the voltage

$$\boxed{\text{Impedance} = \frac{\text{Voltage}}{\text{Current}}}$$

$$6 + 2j = \frac{\text{Voltage}}{3+j}$$

$$\Rightarrow \text{Voltage} = (6 + 2j) \text{ x } (3 + j)$$

$$= 18 + 6j + 6j + 2j^2$$

$$= 18 + 12j + 2(-1) \quad [\because j^2 = -1]$$

$$= 18 + 12j - 2$$

$$\boxed{\text{Voltage} = 16 + 12j} \text{ volts}$$

3) The voltage and current of a circuit are given by the complex numbers 2 + j and 3 – 2j respectively. Find the impedance of the circuit.

Solution :

Given

Voltage = 2 + j

Current = 3 – 2j

To Find

Impedance = ?

$$\boxed{\text{Impedance} = \frac{\text{Voltage}}{\text{Current}}}$$

$$= \frac{2+j}{3-2j}$$

$$= \frac{(2+j)}{(3-2j)} \times \frac{(3+2j)}{(3+2j)}$$

$$= \frac{6+4j+3j+2j^2}{(3)^2+(2)^2} \qquad [\because (a+jb)(a-jb) = (a)^2+(b)^2]$$

$$= \frac{6+7j+2(-1)}{9+4} \qquad [\because j^2 = -1]$$

$$= \frac{6+7j-2}{13}$$

$$= \frac{4+7j}{13}$$

$$\boxed{\text{Impedance} = \frac{4}{13} + \frac{7}{13}j} \text{ ohm}$$

4) The current in a circuit is 9 – 2j amperes when the voltage across the circuit is 50 + 10j volts. Find the admittance of the circuit. Also find the magnitude of the current of the circuit.

Solution :

Given

Current = 9 – 2j amperes

Voltage = 50 + 10j volts

To Find

Admittance = ?

Magnitude of the current |I|= ?

Step 1 :

$$\boxed{\text{Admittance} = \frac{\text{current}}{\text{voltage}}}$$

$$= \frac{9-2j}{50+10j}$$

$$= \frac{(9-2j)}{(50+10j)} \times \frac{(50-10j)}{(50-10j)}$$

$$= \frac{450-90j-100j+20j^2}{(50)^2+(10)^2} \qquad [\because (a+jb)(a-jb) = (a)^2+(b)^2]$$

$$= \frac{450-190j+20(-1)}{2500+100} \qquad [\because j^2 = -1]$$

$$= \frac{450 - 190j - 20}{2600}$$

$$= \frac{430 - 190j}{2600}$$

$$= \frac{430}{2600} - \frac{190}{2600}j$$

$$\boxed{\text{Admittance} = 0.1653 - 0.073j} \text{ mho}$$

Step 2 : Magnitude of the current

Given

Current = 9 – 2j amperes

RP = 9, IP = –2

$$|I| = \sqrt{(RP)^2 + (IP)^2}$$

$$= \sqrt{(9)^2 + (-2)^2}$$

$$= \sqrt{81 + 4}$$

$$|I| = \sqrt{85}$$

$$\boxed{\text{Magnitude of the current} = \sqrt{85} \text{ amperes}}$$

5) The Admittance and current of a circuit are given by the complex numbers 7 + 5j and 17 – 6j respectively. Find the voltage of the circuit.

Solution :

Given

Admittance = 7 + 5j

Current = 17 – 6j

To Find

Voltage =?

$$\boxed{\text{Admittance} = \frac{\text{Current}}{\text{Voltage}}}$$

$$7 + 5j = \frac{17 - 6j}{\text{Voltage}}$$

$$\text{Voltage} = \frac{17 - 6j}{7 + 5j}$$

$$= \frac{(17 - 6j)}{(7 + 5j)} \times \frac{(7 - 5j)}{(7 - 5j)}$$

$$= \frac{119 - 85j - 42j + 30j^2}{(7)^2 + (5)^2} \quad [\because (a + jb)(a - jb) = (a)^2 + (b)^2]$$

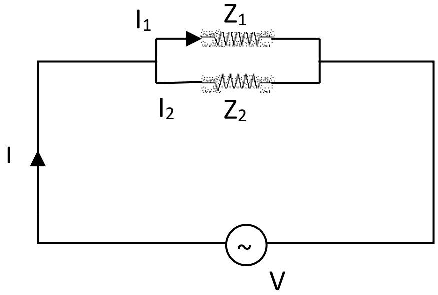
$$= \frac{119 - 127j + 30(-1)}{49 + 25} \quad [\because j^2 = -1]$$

$$= \frac{119 - 127j - 30}{74}$$

$$\boxed{\text{Voltage} = \frac{89}{74} - \frac{127}{74}j} \text{ volts}$$

6) Two circuits of impedances 2 + 4j ohms and 3 + 4j ohms are connected in parallel and a.c voltage of 100 volts is applied across the parallel combination. Calculate the magnitude of the current for each circuit.

Solution :



Given two impedances

$z_1 = 2 + 4j$ ohms

$z_2 = 3 + 4j$ ohms

Voltage = 100 volts

To Find

Magnitude of the current for each circuit

$|I_1| = ?$

$|I_2| = ?$

Step 1:

Let $z_1 = 2 + 4j$ ohms

Voltage = 100 volts

$$\boxed{\text{Impedance} = \frac{\text{Voltage}}{\text{Current}}}$$

$$2 + 4j = \frac{100}{\text{Current}}$$

$$\text{Current} = \frac{100}{2+4j}$$

$$= \frac{(100)}{(2+4j)} \times \frac{(2-4j)}{(2-4j)}$$

$$= \frac{200-400j}{(2)^2 + (4)^2} \qquad [\because (a + jb)\,(a - jb) = (a)^2 + (b)^2]$$

$$= \frac{200-400j}{4+16}$$

$$= \frac{200-400j}{20}$$

$$= \frac{200}{20} - \frac{400}{20}j$$

$$\boxed{I = 10 - 20j}$$

R.P = 10, I.P = –20

$$|I_1| = \sqrt{(RP)^2 + (IP)^2}$$

$$= \sqrt{(10)^2 + (-20)^2}$$

$$= \sqrt{100 + 400}$$

$$= \sqrt{500}$$

$$\boxed{|I_1| = \sqrt{500}} \text{ amperes}$$

Step 2:

Let $z_2 = 3 + 4j$ ohms

Voltage = 100 volts

$$\boxed{\text{Impedance} = \frac{\text{Voltage}}{\text{Current}}}$$

$$3 + 4j = \frac{100}{\text{Current}}$$

$$\text{Current} = \frac{100}{3+4j}$$

$$= \frac{(100)}{(3+4j)} \times \frac{(3-4j)}{(3-4j)}$$

$$= \frac{300-400j}{(3)^2+(4)^2} \qquad [\because (a+jb)(a-jb) = (a)^2 + (b)^2]$$

$$= \frac{300-400j}{9+16}$$

$$= \frac{300-400j}{25}$$

$$= \frac{300}{25} - \frac{400}{25}j$$

$$\boxed{I_2 = 12 - 16j}$$

R.P = 12, I.P = –16

$$|I_2| = \sqrt{(RP)^2 + (IP)^2}$$

$$= \sqrt{(12)^2 + (-16)^2}$$

$$= \sqrt{144 + 256}$$

$$= \sqrt{400}$$

$$\boxed{|I_2| = 20} \text{ amperes}$$

7) Two impedances $z_1 = 10 + 6j$ and $z_2 = 8 - 12j$ are connected in parallel across 200 volts, 50 cycles per sec. A.C. mains. Calculate the magnitude of the current in each branch and magnitude of the total current in the circuit.

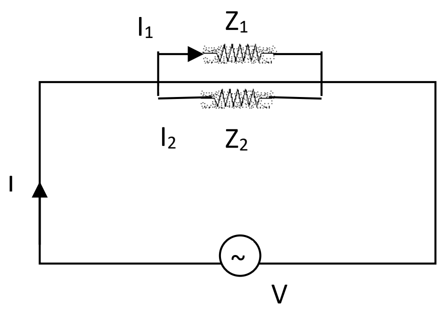
Solution :

Given two impedances

$z_1 = 10 + 6j$

$z_2 = 8 - 12j$

Voltage = 200 volts



To Find

i) Magnitude of the current in each branch

$|I_1| = ?$

$|I_2| = ?$

ii) Magnitude of the total current

$|I| = ?$

Step 1:

Let $z_1 = 10 + 6j$

Voltage = 200 volts

$$\boxed{\text{Impedance} = \frac{\text{Voltage}}{\text{Current}}}$$

$$10 + 6j = \frac{200}{\text{Current}}$$

$$\text{Current} = \frac{200}{10 + 6j}$$

$$I_1 = \frac{(200)}{(10 + 6j)} \times \frac{(10 - 6j)}{(10 - 6j)}$$

$$= \frac{2000 - 1200j}{(10)^2 + (6)^2} \qquad [\because (a + jb)(a - jb) = (a)^2 + (b)^2]$$

$$= \frac{2000 - 1200j}{100 + 36}$$

$$= \frac{2000 - 1200j}{136}$$

$$= \frac{2000}{136} - \frac{1200}{136} j$$

$$\boxed{I_1 = 14.705 - 8.823j}$$

RP = 14.705

I.P = –8.823

$$|I_1| = \sqrt{(RP)^2 + (IP)^2}$$

$$= \sqrt{(14.705)^2 + (-8.823)^2}$$

$$= \sqrt{294.082354}$$

$$\boxed{|I_1| = 17.148} \text{ amperes}$$

Step 2:

Let $z_2 = 8 - 12j$

Voltage = 200 volts

$$\boxed{\text{Impedance} = \frac{\text{Voltage}}{\text{Current}}}$$

$$8 - 12j = \frac{200}{\text{Current}}$$

$$\text{Current} = \frac{200}{8 - 12j}$$

$$I_2 = \frac{(200)}{(8 - 12j)} \times \frac{(8 + 12j)}{(8 + 12j)}$$

$$= \frac{1600 + 2400j}{(8)^2 + (12)^2} \qquad [\because (a + jb)(a - jb) = (a)^2 + (b)^2]$$

$$= \frac{1600 + 2400j}{64 + 144}$$

$$= \frac{1600 + 2400j}{208}$$

$$= \frac{1600}{208} + \frac{2400}{208} j$$

$$\boxed{I_2 = 7.692 + 11.538j}$$

R.P = 7.692

$$\text{I.P} = 11.538$$

$$|I_2| = \sqrt{(RP)^2 + (IP)^2}$$

$$= \sqrt{(7.692)^2 + (11.538)^2}$$

$$= \sqrt{192.292308}$$

$$\boxed{|I_2| = 13.866} \text{ amperes}$$

Step 3:

Total current $\boxed{I = I_1 + I_2}$

$$I = (14.705 - 8.823j) + (7.692 + 11.538j)$$

$$= (14.705 + 7.692) + (-8.823j + 11.538j)$$

$$\boxed{I = 22.397 + 2.715j}$$

$$\text{R.P} = 22.397$$

$$\text{I.P} = 2.715$$

$$|I| = \sqrt{(RP)^2 + (IP)^2}$$

$$= \sqrt{(22.397)^2 + (2.715)^2}$$

$$= \sqrt{508.996834}$$

$$\boxed{|I| = 22.560} \text{ amperes}$$

Exercise

1) The admittance and voltage of a circuit are given by the complex numbers 5+3j and $-3+4j$ respectively. Find the current of the circuit.

[Ans: current $= -27 + 11j$ amp]

2) The admittance and voltage of a circuit are given by the complex numbers 2+j and $2-3j$ respectively. Find the current of the circuit.

[Ans: current $= 7 - 4j$ amp]

3) The impedance and current of a circuit are given by the complex numbers 7+ 3j and 4+j respectively. Find the voltage of the circuit.

[Ans: voltage $= 25 + 19j$ volts]

4) The impedance and current of a circuit are given by the complex numbers 4+5j and $1-j$ respectively. Find the voltage of the circuit.

[Ans: voltage $= 9 + j$ volts]

5) The voltage and current of a circuit are given by the complex numbers 2+j and $3-4j$ respectively. Find the complex number of the impedance of the circuit.

$\left[\text{Ans: impedance } = \frac{2}{25} + \frac{11}{25}j\right]$

6) The admittance and current of a circuit are given by the complex numbers 5+3j and $15-4j$ respectively. Find the voltage of the circuit.

$\left[\text{Ans: voltage } = \frac{63}{34} - \frac{65}{34}j\right]$

7) The voltage and current of a circuit are given by the complex numbers 3+4j and 2−5j respectively. Find the complex number of the impedance of the circuit.

$$\left[\text{Ans: impedance } = \frac{-14}{29} + \frac{23}{29}\,j\right]$$

8) The voltage and current of a circuit are given by the complex numbers 70+20j and 20-6j respectively. Find the complex numbers representing

(i)Admittance (ii) Impedance of the circuit.

$$\begin{bmatrix}\text{Ans: (i) } 0.2415 - 0.1547\,j \\ \text{(ii) } 2.9358 + 1.8807\,j\end{bmatrix}$$

9) The current in a circuit is $10 - 2j$ amperes when the voltage across the circuit is $60 + 20j$ volts. Find the admittance. Also, find the magnitude of the current of the circuit.

$$\begin{bmatrix}\text{Ans: Admittance } = \frac{7}{50} - \frac{2}{25}\,j \\ \text{Magnitude } = \sqrt{104}\text{ amperes}\end{bmatrix}$$

10) Two impedances $z_1 = 8+6j$ and $z_2 = 6 - 8j$ are connected in parallel across 200 volts, 50 cycles per sec. A.C. mains. Calculate the magnitude of the current in each branch and magnitude of total current.

[Ans : $|I_1| = 20$ amperes

$|I_2| = 20$ amperes

$|I| = 28.28$ amperes]

11) The voltage and current of a circuit are given as $60 - 20j$ volts and $12 - 3j$ amperes respectively. Find the magnitude of current, admittance and impedance.

[Ans : Magnitude = 12.3693 amperes

Admittance = 0.195+0.015 j mho

Impedance = 5.098 – 0.392 j]

12) Two impedances $z_1 = 4+3j$ ohms and $z_2 = 12-5j$ ohms are connected in parallel across 100 volts, 50 cycles per sec. A.C. mains. Calculate of the magnitude of current in each branch and total current.

[Ans: 20 amperes, 7.7 amperes, 24.8 amperes]

13) Two impedances $z_1 = 7 - 15j$ and $z_2 = 12 + 10j$ are connected in parallel across 200 volts, 50 cycles per sec. A.C. main. Calculate the magnitude of the current in each branch and magnitude of total current.

[Ans : $|I_1| = 12.081$ amperes

$|I_2| = 12.80$ amperes

$|I| = 15.196$ amperes]

14) Two impedances $z_1 = -1 + 4j$ and $z_2 = 5 + 8j$ are connected in parallel across 100 volts, 50 cycles per sec. A.C. main. Calculate the magnitude of the current in each branch and magnitude of the total current.

[Ans : $|I_1| = 24.253$ amperes

$|I_2| = 10.599$ amperes

$|I| = 32.518$ amperes]

Exercise

1) Find the real and imaginary parts of the following.

i) $1 + 2i$ ii) $\frac{1}{2} + \frac{i\sqrt{3}}{2}$ iii) i

iv) $1 - i$ v) 1 vi) $a + ib$

2) If $z_1 = 4 + 7i$ and $z_2 = 9 - 2i$, find $z_1 + z_2$.

3) If $z_1 = 1 + i$ and $z_2 = 3 - i$, find $\overline{z_1 + z_2}$

4) If $z_1 = (-1, 2)$ and $z_2 = (-3, 4)$, find $2z_1 - 3z_2$.

5) If $z_1 = 4 + 3i$ and $z_2 = 3 - 2i$, find $4z_1 - 5z_2$.

6) If $z_1 = 1 + i$ and $z_2 = 2 + i$, find z_1z_2.

7) If $z_1 = 2 + i$ and $z_2 = 3 - 2i$, find $\frac{z_2}{z_1}$

8) Find the value of $i^3 + i^5$

9) Prove that $i^5 + 2i^3 + i^9 = 0$

10) Find the real and imaginary parts of the following complex numbers.

i) $\frac{1}{2+3i}$ ii) $\frac{3}{2+i}$

iii) $\frac{1}{1+i}$ iv) $(4 + 3i)(1 - 2i)$

11) Express the following complex numbers in a + ib form.

i) $\frac{1+i}{1-i}$ ii) $\frac{1}{4-3i}$

iii) $\frac{1}{i-2}$ iv) $\frac{1}{3+i}$

12) Find the complex conjugate of the following

i) $(1 - i)(1 - 2i)$ ii) $\frac{2}{5-2i}$ iii) $\frac{7}{5+2i}$

13) Find the modulus and amplitude (or) argument of the following

i) $1+i$ ii) $1-i\sqrt{3}$ iii) $1+\sqrt{-3}$

iv) $\frac{1}{2} + \frac{i\sqrt{3}}{2}$ v) i

14) Find the distance between the following two complex numbers

i) $2 + i$ and $1 - 2i$ ii) $1 + i$ and $3 - 2i$ iii) $7 + 9i$ and $-3+7i$

15) State DeMoivre's theorem

16) Simplify the following

i) $(\cos\theta + i\sin\theta)^4 (\cos\theta + i\sin\theta)^{-3}$ ii) $(\cos\alpha + i\sin\alpha)^{-9} (\cos\alpha + i\sin\alpha)^5$

iii) $(\cos x + i\sin x)^2 (\cos x - i\sin x)^3$

17) Simplify the following

i) $\frac{\cos 9\theta + i\sin 9\theta}{\cos 2\theta + i\sin 2\theta}$ ii) $\frac{\cos 5\theta + i\sin 5\theta}{\cos\theta - i\sin\theta}$ iii) $\frac{\cos 7\theta - i\sin 7\theta}{\cos 3\theta - i\sin 3\theta}$

iv) $\frac{(\cos\theta + i\sin\theta)^{12}}{(\cos\theta + i\sin\theta)^{-5}}$ v) $\frac{\cos 4\theta + i\sin 4\theta}{(\cos\theta - i\sin\theta)^3}$

18) If z = cos 30° + i sin 30°, what is the value of z^3.

19) Find the product of $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

20) Find the product of $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

21) If $z_1 = 5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $z_2 = 3\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

22) If x = cos θ+ isin θ, Find $x + \frac{1}{x}$.

23) If a = cos α+ i sin α, b = cos β+ i sin β, Find ab.

24) If a = cos θ+ isin θ, b = cos ∅+ i sin ∅. Find $\frac{b}{a}$

25) If ω is the n^{th} roots of unity, what is the value of $1+\omega+\omega^2+\ldots+\omega^{n-1}$?

26) If ω is the cube roots of unity $\omega^4+\omega^5+\omega^6$.

27) If ω is the cube roots of unity what is the value of ω^3?

28) If ω is 4^{th} roots of unity, what is the value of $1+\omega+\omega^2$?

29) Find the value of $(-1)^{1/3}$

30) Find the value of $\left(\frac{-1+i\sqrt{3}}{2}\right)^3$

31) Solve $x^3 - 1 = 0$

32) If ω is the sixth roots of unity what is the value of $1+\omega+\omega^2+\omega^3$?

33) Find the value of $(i)^{1/2}$

1) Find the real and imaginary parts of the following complex numbers.

i) $\frac{5-i}{2-3i}$ ii) $\frac{2+3i}{4+5i}$ iii) $\frac{1}{3-2i} + \frac{1}{2-3i}$

iv) $\frac{4i+1}{2-3i}$ v) $\frac{1+\sqrt{3}i}{1+i}$

2) Express the following complex numbers in a + ib form

i) $\frac{i-4}{3-2i}$ ii) $\frac{(2-i)^2}{1+i}$ iii) $\frac{-1+3i}{1+3i}$

3) Find the complex conjugate of the following.

i) $\frac{2}{4+i}$ ii) $\frac{i+1}{2-i}$ iii) $\frac{1}{4+i} + \frac{1}{4-i}$

4) Find the modulus and amplitude of the following complex numbers.

i) $1 + \sqrt{3}i$ ii) $\frac{5-i}{2-3i}$ iii) $\frac{\sqrt{3}}{2} + i\frac{\sqrt{3}}{2}$

5) Express in modulus – amplitude form (polar form) the following complex numbers.

i) $-1 + \sqrt{3}i$ ii) $\frac{-1}{2} - \frac{\sqrt{3}}{2}i$

6) If $z_1 = 3(\cos 40^o + i\sin 40^o)$, $z_2 = 2(\cos 10^o + i\sin 10^o)$ Find $\frac{z_1}{z_2}$

7) Prove that the following complex numbers are collinear.

i) 5 + 8i, 13 + 20i and 19 + 29i

ii) $1 + 3i,\ 5 + i$ and $3 + 2i$

iii) $1 - i,\ 6 + 3i$ and $-4 - 5i$

iv) $3 + 7i,\ 6 + 5i$ and $15 - i$

8) Prove that the following complex numbers form a parallelogram

i) $1 - 2i,\ -1 + 4i,\ 5 + 8i,\ 7 + 2i$

ii) $1,\ 4 + 3i,\ -2 - i,\ -5 - 4i$

iii) $-3 + 3i,\ -2i,\ 2 + 6i,\ 5 + i$

9) Simplify the following

i) $\frac{(\cos 2\theta - i \sin 2\theta)^3}{(\cos 4\theta + i \sin 4\theta)^2}$ ii) $\frac{(\cos 3\theta - i \sin 3\theta)^5}{(\cos 5\theta + i \sin 5\theta)^4}$

iii) $\frac{(\cos 7\theta + i \sin 7\theta)^3}{(\cos 6\theta - i \sin 6\theta)^5}$ iv) $\frac{(\cos \theta + i \sin \theta)^9}{(\cos 15\theta + i \sin 15\theta)^2}$

10) If $x = \cos\theta + i \sin\theta$. Find the value of

i) $x^2 + \frac{1}{x^2}$ ii) $x^2 - \frac{1}{x^2}$ iii) $x^3 + \frac{1}{x^3}$

iv) $x^3 - \frac{1}{x^3}$ v) $x^7 - \frac{1}{x^7}$ vi) $x^m + \frac{1}{x^m}$

vii) $x^9 + \frac{1}{x^9}$ viii) $x^n + \frac{1}{x^n}$

11) If $a = \cos\alpha + i \sin\alpha$, $b = \cos\beta + i \sin\beta$ then find $\sqrt{ab}$

12) If $a = \cos\alpha + i \sin\alpha$, $b = \cos\beta + i \sin\beta$, Find $\frac{a^2}{b^2}$

13) If $a = \cos\theta + i \sin\theta$, $b = \cos\emptyset + i \sin\emptyset$, Find $ab - \frac{1}{ab}$.

14) If $x = \cos\alpha + i \sin\alpha$, $y = \cos\beta + i \sin\beta$, Find $x^m y^n$

15) Solve $x^2 - 1 = 0$

16) Solve $x^3 + 1 = 0$ (or) Find all the values of $(-1)^{1/3}$

17) Solve $x^4 - 1 = 0$ (or) Find the fourth roots of unity.

18) Find all the values of $(i)^{2/3}$

19) Find the cube roots of 8.

20) Solve $x^2 + 16 = 0$.

1) Find the real and imaginary parts of the following complex numbers.

i) $\frac{(2+i)^2\,(2+i)}{1+4i}$ ii) $\frac{(3+i)^2\,(1+3i)}{(3+i)}$ iii) $\frac{(1+i)\,(1+2i)}{(1+3i)}$

iv) $\frac{(1+i)\,(2-i)}{(2+i)^2}$ v) $\frac{5-i}{2+3i} + \frac{3-i}{4+5i}$ vi) $\frac{3}{3+4i} + \frac{1}{5-2i}$

vii) $\frac{1}{3-2i} + \frac{1}{2-3i}$ viii) $\frac{2}{4+3i} + \frac{i}{3-4i}$ ix) $\left(\frac{1-i}{1+i}\right)^2$

x) $\frac{(2-3i)\,(2-i)^2}{3+i}$ xi) $\frac{1}{1+\cos\theta + i \sin\theta}$

2) Express the following complex members in a + ib form

i) $\frac{i-4}{3-2i}+\frac{4i+1}{2-3i}$ ii) $\frac{(1+i)\,(1+2i)}{(1+3i)^2}$ iii) $\frac{3+i}{(1+2i)\,(1-4i)}$

iv) $\frac{1+i}{(2-i)^2}$ v) $\frac{(2+3i)^2}{(1-2i)^2}$ vi) $\frac{(2+3i)}{1-i}$

3) Find the conjugate of the following complex number

i) $\frac{2+i}{1-4i}$ ii) $\frac{1+i}{1-i}$ iii) $\left(\frac{2-3i}{4+7i}\right)^2$

iv) $\frac{(1+i)\,(1+2i)^2}{(1+3i)}$

4) Find the modulus and amplitude (or) argument of the complex numbers

i) $\frac{5-i}{2-3i}$ ii) $\frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$ iii) $\frac{2+i\sqrt{3}}{1+i\sqrt{3}}$

iv) $\frac{1+2i}{3+7i}$ v) $\frac{4+i}{3-2i}$

5) Express in modulus – amplitude form (polar form) the following:-

i) $\left(\frac{2+i}{3-i}\right)^2$ ii) $\frac{5+i}{2+3i}$ iii) $\frac{1+3\sqrt{3}i}{\sqrt{3}+2i}$

iv) $\frac{(1+i)\,(2+i)}{(3-i)}$

6) Prove that the following complex numbers form a square.

i) 3 + 2i, 5 + 4i, 3 + 6i and 1 + 4i

ii) -i, 2 + i, 3i and -2 + i

iii) 9 + i, 4 + 13i, -8 + 8i and -3 -4i

iv) 4 + 5i, 1 + 2i, 4 -i and 7 + 2i

v) 1 + i, 2 + i, 2 + 2i and 1 + 2i

7) Prove that the following complex numbers form a rectangle.

i) 2 – 2i, 8 + 4i, 5 + 7i and -1 + i

ii) -3, 1 - 2i, 5 + 6i and 1 + 8i

iii) 2i, 1 + i, 4 + 4i and 3 + 5i

iv) 4 + 3i, 12 + 9i, 15 + 5i and 7 -i

8) Prove that the following complex numbers form a rhombus.

i) 3 + 4i, 9 + 8i, 5 + 2i and -1 -2i

ii) 4 -2i, 6 + 8i, -4 + 6i and -6 -4i

iii) 6 + 4i, 4 + 5i, 6 + 3i and 8 + i

iv) 1 + i, 2 + i, 2 + 2i and 1 + 2i

9) Simplify the following using DeMoivre's Theorem.

i) $\frac{(\cos 3\theta - i\sin 3\theta)^5(\cos 4\theta + i\sin 4\theta)^4}{(\cos 2\theta + i\sin 2\theta)^7(\cos 3\theta - i\sin 3\theta)^6}$

ii) $\frac{(\cos 3\theta + i\sin 3\theta)^4(\cos 4\theta + i\sin 4\theta)^2}{(\cos 2\theta + i\sin 2\theta)^4(\cos 5\theta + i\sin 5\theta)^3}$

iii) $\dfrac{(\cos 3\theta+ i\sin 3\theta)^4(\cos 4\theta- i\sin 4\theta)^5}{(\cos 4\theta + i\sin 4\theta)^{-2}(\cos 5\theta+ i\sin 5\theta)^4}$

iv) $\dfrac{(\cos 5\theta- i\sin 5\theta)^2(\cos 7\theta + i\sin 7\theta)^{-3}}{(\cos 4\theta- i\sin 4\theta)^9(\cos\theta+ i\sin\theta)^6}$

v) $\dfrac{(\cos 3\theta + i\sin 3\theta)^{-5}(\cos 2\theta + i\sin 2\theta)^4}{(\cos 4\theta- i\sin 4\theta)^{-2}(\cos 5\theta- i\sin 5\theta)^3}$

vi) $\dfrac{(\cos 2\theta + i\sin 2\theta)^3(\cos 4\theta- i\sin 4\theta)^3}{\cos 3\theta + i\sin 3\theta}$ when $\theta = \frac{\pi}{9}$

vii) $\dfrac{(\cos 5x- i\sin 5x)(\cos 2x- i\sin 2x)^{-3}}{(\cos x + i\sin x)^5(\cos 3x + i\sin 3x)^{-5}}$ when $x = \frac{2\pi}{11}$

10) Prove that $\left[\dfrac{\cos\theta + i\sin\theta}{\sin\theta + i\cos\theta}\right]^4 = \cos 8\theta + i\sin 8\theta$

11) Prove that $\left[\dfrac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right]^n = \cos n\theta + i\sin n\theta$

12) Prove that $\left[\dfrac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right]^6 = \cos 6\theta + i\sin 6\theta$

13) $\left[\dfrac{1+\sin\frac{\pi}{8} + i\cos\frac{\pi}{8}}{1+\sin\frac{\pi}{8} - i\cos\frac{\pi}{8}}\right]^8 = -1$

14) If $a = \cos A + i\sin A$, $b = \cos B + i\sin B$. Prove that

i) $\dfrac{a^2+b^2}{2ab} = \cos(A - B)$ ii) $\dfrac{a^2-b^2}{2ab} = i\sin(A - B)$

15) If $a = \cos 2\alpha + i\sin 2\alpha$, $b = \cos 2\beta + i\sin 2\beta$, $c = \cos 2\gamma + i\sin 2\gamma$ then prove that

i) $\sqrt{abc} + \dfrac{1}{\sqrt{abc}} = 2\cos(\alpha+\beta+\gamma)$ ii) $\dfrac{a^2b^2+c^2}{abc} = 2\cos 2(\alpha+\beta-\gamma)$

16) If $x = \cos 3\alpha + i\sin 3\alpha$, $y = \cos 3\beta + i\sin 3\beta$, prove that

i) $\sqrt[3]{xy} + \dfrac{1}{\sqrt[3]{xy}} = 2\cos(\alpha+\beta)$ ii) $\sqrt[3]{xy} - \dfrac{1}{\sqrt[3]{xy}} = 2i\sin(\alpha+\beta)$

17) If $a = \cos\alpha + i\sin\alpha$, $b = \cos\beta + i\sin\beta$, $c = \cos\gamma + i\sin\gamma$ and if $a + b + c = 0$, show that

i) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha+\beta+\gamma)$

ii) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha+\beta+\gamma)$

18) If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$ and $z = \cos\gamma + i\sin\gamma$ and if $x + y + z=0$, then prove that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 0$.

19) If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$ prove that

$x^my^n + \dfrac{1}{x^my^n} = 2\cos(m\alpha + n\beta)$

20) Prove that $(1 + i)^n + (1-i)^n = 2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}$

21) If 'n' is a positive integer, prove that $\left(\sqrt{3}+i\right)^n + \left(\sqrt{3}-i\right)^n = 2^{n+1}\cos\frac{n\pi}{6}$

22) Prove that $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cos\left(\frac{n\theta}{2}\right)$

23) Solve $x^4-1 = 0$

24) Solve $x^5-1 = 0$

25) Solve $x^6 - 1 = 0$

26) Solve $x^7 - 1 = 0$

27) Solve $x^4 + 1 = 0$

28) Solve $x^5 + 1 = 0$

29) Solve $x^6 + 1 = 0$

30) Solve $x^7 + 1 = 0$

31) Solve $x^7 - x^4 + x^3 - 1 = 0$

32) Solve $x^8 - x^5 + x^3 - 1 = 0$

33) Solve $x^9 + x^5 - x^4 - 1 = 0$

34) Solve $x^5 + x^3 + x^2 + 1 = 0$

35) Find all the values of $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{3/4}$ Hence prove that the product of four values is 1.

Introduction to Complex Numbers

Modulus and Amplitude of Complex Numbers

Modulus and Amplitude-Exercise

De Moivre's Theorem

De Moivre's Theorem - Related Problems

Roots of Complex Numbers

UNIT - III TRIGONOMETRY

Chapter 3.1 TRIGONOMETRIC FUNCTIONS & ALLIED ANGLES

Trigonometry is primarily a branch of Mathematics that studies relationship involving sides and angles of triangles.

The word trigonometry stems from the Greek word trigonon which means triangle and metron which means to measure. So, literally trigonometry is the study of measuring triangles.

The General principles of trigonometry were formulated by the Greek astronomer Hipparchus and he is credited as the founder of Trigonometry.

Indian Mathematician and Astronomer Aryabhata defined sine, cosine, inverse sine, inverse cosine and he gave mathematical results in the form of 108 verses, which included a formula for the Area of a triangle. Mohammed ibn Musaalkhwarizimi discovered the tangent Abu al-wafa Buzjani discovered the secant, cotangent and cosecant.

Trigonometry has applications in both pure mathematics and in applied mathematics, where it is essential in many branches of science and technology.

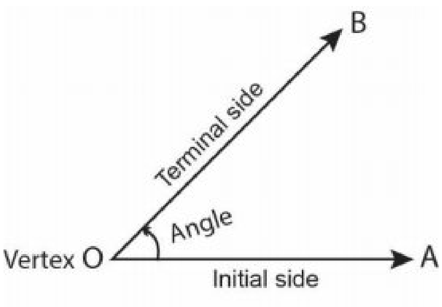
Trigonometry in Real life :

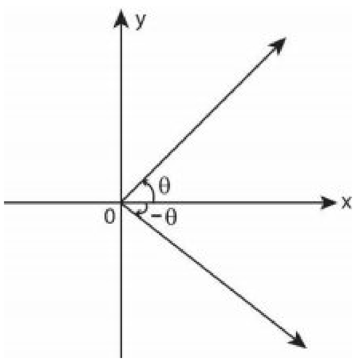
Our GPS system in cars and mobile phones is based on trigonometric calculations. Advanced medical scanning procedures such as CT and MRI, used in detecting tumours, involving sine and cosine functions.

Angles:

The angle is a measure defined by the two rays OA and OB sharing the common point O as shown in figure. The common point O is called the vertex of the angle.

Note: An anticlockwise rotation generates a positive angle while clockwise rotation generates a negative angle.

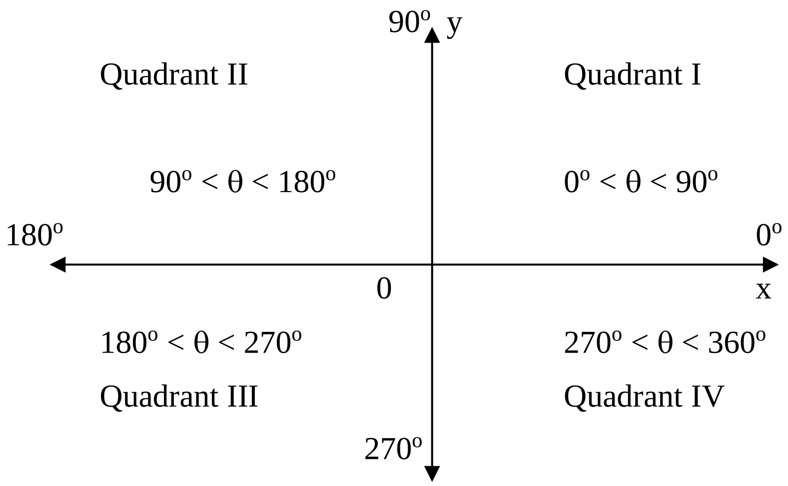




MATH FACT :

* Two angles that have the exact same measure are called congruent angles.
* Two angles that have their measures adding to 90^o are called complementary angles.
* Two angles that have their measures adding to 180^o are called supplementary angles.
* Two angles between 0^o and 360^o are conjugate if their sum equals 360^o

Angles in Standard Position :



MATH FACT :

The degree measurement of a quadrantal angle is a multiple of 90^o.

Coterminal Angles :

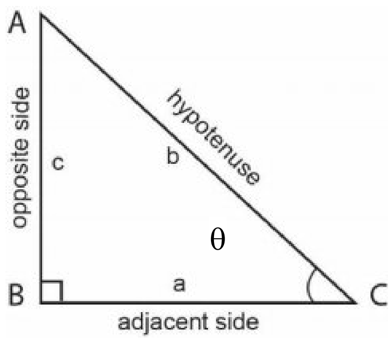
If the difference of two angles is $K(360^o)$ then they are coterminal angles. K is an integer. The measurements of coterminal angles differ by an integral multiple of 360^o.

MATH FACT :

The pairs of angles $(30^o, 390^o)$, $(280^o, 1000^o)$ and $(-85^o, 275^o)$ are coterminal angles.

Basic Trigonometric Ratios using Right Triangle :

We know that, six ratios can be formed using the three lengths a, b, c of sides of a right triangle ABC.



The trigonometric ratios which are defined with reference to a right angle are

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} ;$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad ; \quad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} \quad ; \quad \sec\theta = \frac{1}{\cos\theta}$$

Exact values of trigonometric functions of standard angles.

θ	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined
cosec	undefined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined
cot	undefined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

Example 1 :

Identify the quadrant in which an angle of each given measure lies?

(i) 25^o

(ii) -55^o

(iii) 328^o

Solution :

(i) 25^o – I Quadrant

(ii) -55^o – IV Quadrant

(iii) 328^o – IV Quadrant

Example 2 :

For each given angle, find the coterminal angle with measure of θ, $0^o \le \theta \le 360^o$.

(i) 395^o (ii) -450^o

Solution :

(i) $35^o - 395^o = -360^o$

$= -1(360^o)$ [∵-1 is an integer]

35^o is the coterminal angle

(ii) $(-450^o, 270^o) = 270^o - (-450^o)$

$= 270^o + 450^o$

$= 720^o$

$= 2(360^o)$ 270^o is the coterminal angle

Example 3 :

Eliminate θ from a cos θ = b and c sin θ = d, where a, b, c, d are constants.

Solution :

Given $a \cos \theta = b \rightarrow$ ①

$c \sin \theta = d \rightarrow$ ②

Multiply Equation ① by 'c' and Equation ② by 'a' on both side

$ac \cos \theta = bc \rightarrow$ ③

$ac \sin \theta = ad \rightarrow$ ④

Squaring and adding ③ and ④, we get,

$a^2c^2 \cos^2 \theta + a^2c^2 \sin^2 \theta = b^2c^2 + a^2d^2$

$a^2c^2 [\cos^2 \theta + \sin^2 \theta] = b^2c^2 + a^2d^2$

$a^2c^2 (1) = b^2c^2 + a^2d^2$ $[\because \cos^2 \theta + \sin^2 \theta = 1]$

$a^2c^2 = b^2c^2 + a^2d^2$

Example 4 :

If $a \cos x + b \sin x = m$ and $a \sin x - b \cos x = n$ then prove that $a^2 + b^2 = m^2 + n^2$.

Solution :

$m = a \cos x + b \sin x$

$n = a \sin x - b \cos x$

$m^2 = (a \cos x + b \sin x)^2$

$n^2 = (a \sin x - b \cos x)^2$

$m^2 = a^2 \cos^2 x + b^2 \sin^2 x + 2ab \cos x \sin x$

$n^2 = a^2 \sin^2 x + b^2 \cos^2 x - 2ab \cos x \sin x$

To prove : $m^2 + n^2 = a^2 + b^2$

$m^2 + n^2 = a^2 \cos^2 x + b^2 \sin^2 x + 2ab \cos x \sin x + a^2 \sin^2 x + b^2 \cos^2 x - 2ab \cos x \sin x$

$= a^2[\cos^2 x + \sin^2 x] + b^2 [\sin^2 x + \cos^2 x]$ $[\because \sin^2 x + \cos^2 x = 1]$

$m^2 + n^2 = a^2 + b^2$ Hence Proved

Example 5 :

If $m = \tan x + \sin x$, $n = \tan x - \sin x$ show that $m^2 - n^2 = 4\sqrt{mn}$

Solution :

Given $m = \tan x + \sin x$

$n = \tan x - \sin x$

To prove : $m^2 - n^2 = 4\sqrt{mn}$

$m^2 - n^2 = (\tan x + \sin x)^2 - (\tan x - \sin x)^2$

$m^2 - n^2 = 4 \tan x \sin x \rightarrow$ ① $[\because (a+b)^2 - (a-b)^2 = 4ab]$

$4\sqrt{mn} = 4\sqrt{(\tan x + \sin x)(\tan x - \sin x)}$

$= 4\sqrt{\tan^2 x - \sin^2 x}$

$$= \quad 4\sqrt{\frac{\sin^2 x}{\cos^2 x} - \sin^2 x}$$

$$= \quad 4 \sin x \sqrt{\frac{1}{\cos^2 x} - 1}$$

$$= \quad 4 \sin x \sqrt{\sec^2 x - 1} \qquad [\sec^2 x - 1 = \tan^2 x]$$

$$= \quad 4 \sin x \sqrt{\tan^2 x}$$

$$4\sqrt{mn} \quad = \quad 4 \sin x \tan x \quad \rightarrow ②$$

∴ From ① & ②, we have,

$$m^2 - n^2 \quad = \quad 4\sqrt{mn}. \qquad \text{Hence Proved.}$$

Example 6 :

A ball is thrown off the edge of a building at an angle of 70° and with an initial velocity of 5 meters per second. The equation that represents the horizontal distance of the ball x is $x = v_o (\cos \theta)\, t$, where v_o is the initial velocity, θ is the angle at which it is thrown and t is the time in seconds. About how far will the ball travel in 10 seconds?

Solution :

Given $v_o = 5$m/s

$\theta = 70^\circ$

$t = 10$ sec

$x = v_o (\cos \theta)\, t$

$= 5 (\cos 70^\circ)\, 10$

$= 50 (\cos 70^\circ)$

$x \simeq 17.1$ meters. The ball travels 17.1 meters in 10 sec.

EXERCISE

1) Identify the quadrants in which an angle of each given measure lies.

   i) 825° ii) -230° iii) 380° iv) -150°

2) Find the coterminal angle for each of the given angle with measure of θ, $0^\circ \leq \theta \leq 360^\circ$

   i) 525° ii) 1150° iii) -270° iv) 390°

3) If $\cos x + \sin x = \sqrt{2} \cos x$, prove that $\cos x - \sin x = \sqrt{2} \sin x$.

4) If $a = \sec x - \tan x$, $b = \operatorname{cosec} x + \cot x$, show that $ab + a - b + 1 = 0$.

5) If $\cot \theta (1 + \sin \theta) = 4m$ and $\cot \theta (1 - \sin \theta) = 4n$ then prove that $(m^2 - n^2)^2 = mn$.

6) Prove that $\frac{\tan \theta + \sec \theta - 1}{\tan \theta - \sec \theta + 1} = \frac{1 + \sin \theta}{\cos \theta}$

7) Prove that $(\sec A - \operatorname{cosec} A)(1 + \tan A + \cot A) = \tan A \sec A - \cot A \operatorname{cosec} A$.

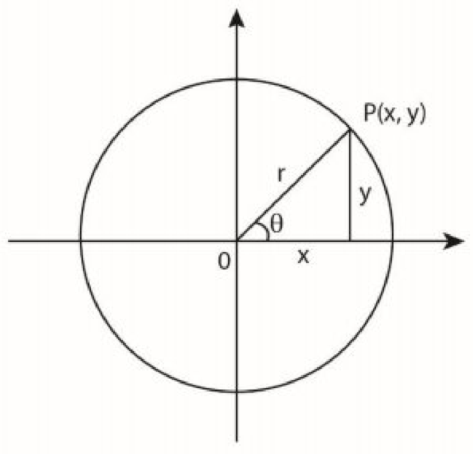
Properties of Trigonometric Function

Trigonometric functions of any angle in terms of Cartesian coordinates.

We have studied about the principles of trigonometric ratios using acute angle. We shall extend the concept and define trigonometric functions for any angle.

Trigonometric functions:

The trigonometric ratios to any angle in terms of radian measure are called trigonometric functions.



Let p(x, y) be any point on the terminal side of an angle θ in standard position.

Let OP = r

$r = \sqrt{x^2 + y^2}$ [using Pythagoras theorem]

The six trigonometric functions of θ are defined as

$\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$

using this,

$\tan\theta = \frac{y}{x}$; $x \neq 0$

$\cot\theta = \frac{x}{y}$; $y \neq 0$

$\operatorname{cosec}\theta = \frac{r}{y}$; $y \neq 0$

$\sec\theta = \frac{r}{x}$; $x \neq 0$

MATH FACT:

- Since $|x| \leq r$, $|y| \leq r$

  We have $|\sin\theta| \leq 1$ and

  $|\cos\theta| \leq 1$

- The trigonometric functions have positive or negative values depending on the quadrant in which the point p(x, y) on the terminal side of θ lies.

Characteristics of Trigonometric functions

- sine and cosine functions are complementary to each other.

  i.e. $\sin(90^\circ - \theta) = \cos\theta$

  $\cos(90^\circ - \theta) = \sin\theta$

- Trigonometric functions repeats its value in regular intervals.

Periodicity of Trigonometric functions:

A function f is said to be a periodic function with period p, if there exists a smallest positive number 'p' such that f(x+p) = f(x) for all x in the domains.

$$\boxed{\sin(x + 2n\pi) = \sin x}, \; n \in Z$$

i.e.) $\sin(x + 2\pi) = \sin(x + 4\pi) = \sin(x + 6\pi) = \ldots = \sin x$.

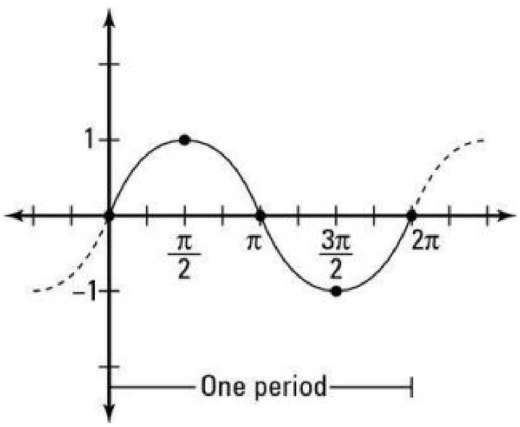
Thus sin x is a periodic function with period 2π.

cos x, cosec x and sec x are periodic functions with period 2π.

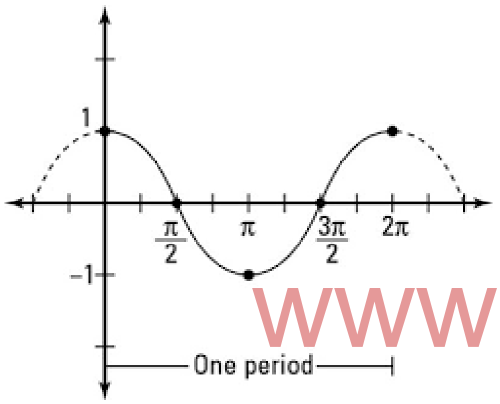
tan x and cot x are periodic functions with period π.

The graph of sine function:

Here 'x' represents a variable angle. Graph of the function y = sin x, shown in figure is periodic of period 2π.



The graph of cosine function:



The graph of the function y = cos x resembles like the graph of y = sin x except it is being translated to the left by $\frac{\pi}{2}$. This is because of the identity $\cos x = \sin\left(\frac{\pi}{2} + x\right)$

Note: sine and cosine functions have the property that $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$

Importance of sine and cosine functions:

The sine and cosine functions are useful for an important reason, since they repeat in a regular pattern. (i.e. Periodic)

There are a vast array of things in and around us that repeat periodically.

- ✷ The rising and setting of the sun, the motion of a spring up and down, the tides of the ocean and so on, are repeating at regular intervals of time.
- ✷ All periodic behaviour can be studied in the combination of the sine and cosine functions.
- ✷ Periodic functions are used in science to describe oscillations, waves and other phenomena that occur periodically.

Odd and Even Trigonometric functions:

A real valued function f(x) is an even function if it satisfies, f(-x) = f(x) for all real number x.

Even function – symmetric about the y – axis.

A real valued function f(x) is an odd function if it satisfies,

f(-x) = -f(x) for all real number x.

Odd function – Symmetric about the origin.

MATH FACT:

- The only function that is even and odd is f(x) = 0
- Even and odd functions are useful in analyzing trigonometric functions of sum and difference formula.
- It is also useful in evaluating some definite integrals, which comes in calculus.

Example 1

Determine whether the function cos x + sin x even, odd or neither

Solution

$f(x) = \cos x + \sin x$

$f(-x) = \cos(-x) + \sin(-x)$

$= \cos x - \sin x$

$f(-x) \neq f(x)$; $f(-x) \neq -f(x)$

∴ It is neither even function nor odd function.

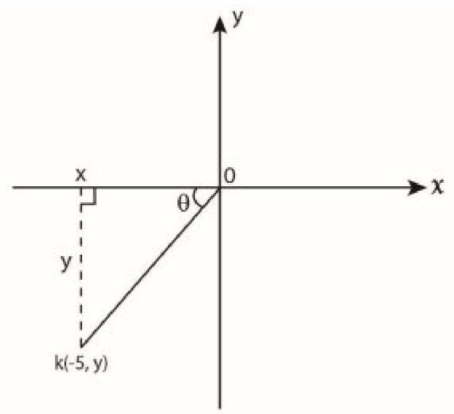
Example 2

XOK = θ is an angle in the third quadrant where x is a point on the x-axis and k is the point (-5, y) and OK is 13 units.

i) Determine the value of y.

ii) Prove that $\tan^2\theta + 1 = \sec^2\theta$

Solution:



i) Using Pythagoras theorem

$r^2 = x^2 + y^2$

$y^2 = r^2 - x^2$

$= (13)^2 - (-5)^2$

$= 169 - 25$

$y^2 = 144$

$y = \pm 12$

Given θ lies in III quadrant, y must be negative.

$\therefore y = -12$

ii) We have, x = -5, y = -12, r = 13

LHS, $\tan^2\theta + 1 = \left(\frac{y}{x}\right)^2 + 1$ $\quad \left[\because \tan\theta = \frac{y}{x}\right]$

$= \left(\frac{-12}{-5}\right)^2 + 1$

$= \left(\frac{144}{25}\right) + 1$

$= \frac{144+25}{25}$

$$\tan^2\theta + 1 = \frac{169}{25} \quad \rightarrow ①$$

RHS $\quad \sec^2\theta = \left(\frac{r}{x}\right)^2$

$$= \left(\frac{13}{-5}\right)^2$$

$$\sec^2\theta = \frac{169}{25} \quad \rightarrow ②$$

From ① and ②, we have,

$$\tan^2\theta + 1 = \sec^2\theta$$

Hence Proved.

Example 3

Find the value of sin A + cos A, given that 13 sin A – 12 = 0, where cos A < 0.

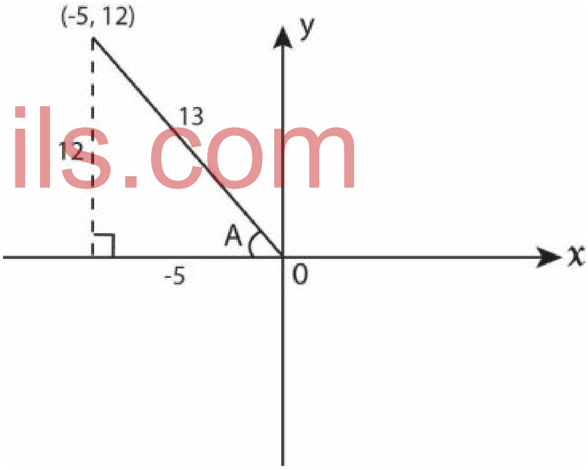
Solution:

Given 13 sin A – 12 = 0

$$13 \sin A = 12$$

$$\sin A = \frac{12}{13}$$

$\therefore y = 12 \; ; \; r = 13 \quad \left[\because \sin A = \frac{y}{r}\right]$



Using Pythagoras theorem

$$x^2 = r^2 - y^2$$

$$= (13)^2 - (12)^2$$

$$= 169 - 144$$

$$x^2 = 25$$

$$x = \pm 5$$

Given that cos A < 0 and ‘r’ is positive.

$\therefore x = -5$

$$\sin A + \cos A = \frac{y}{r} + \frac{x}{r}$$

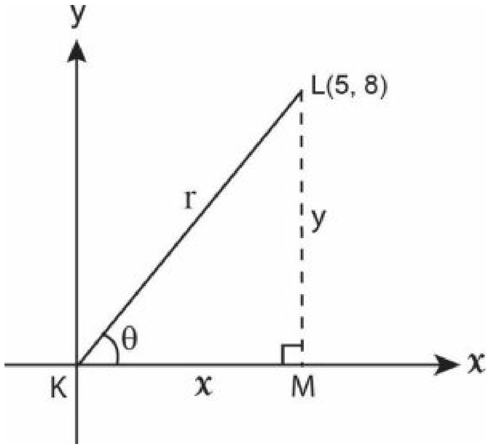
$$= \frac{y + x}{r}$$

$$= \frac{12 - 5}{13}$$

$$\sin A + \cos A = \frac{7}{13}$$

Example 4

L is a point with coordinates (5, 8) on a Cartesian plane. LK forms an angle θ with the positive x – axis. Set up a diagram and use it to find the distance LK.



Solution:

Given L (5, 8)

The angle lies in I quadrant

$\therefore x = 5, \quad y = 8$

Using Pythagoras theorem,

$$\begin{aligned} LK &= r \\ LK^2 &= KM^2 + ML^2 \\ &= 5^2 + 8^2 \\ &= 25 + 64 \\ LK^2 &= 89 \\ LK &= \sqrt{89} \end{aligned}$$

EXERCISE

1) Describe whether the functions

i) $\sin^2 x - 2\cos^2 x$ (ii) $\sin(\cos x)$ (iii) $\sin x - \cos x$ are even, odd or neither

2) If $\sin\theta = -\frac{15}{17}$ and $\cos\theta < 0$. Find the values of following.

i) $\cos\theta$ ii) $\tan\theta$

3) Given that $\sin\theta = \frac{\sqrt{5}}{5}$ and $\cos\theta = \frac{2\sqrt{5}}{5}$. Find the exact values of four remaining trigonometric functions.

Relation between Degree and Radian measure

Two types of units of measurement of an angle which are most commonly used namely Degree measure and Radian measure. One measuring unit is better than another if it can be defined in a simpler and more intuitive way.

Radian measure is more convenient for analysis whereas degree measure of an angle is more convenient to communicate the concept between people.

Degree Measure :

If a rotation from the initial side to terminal side is $\left(\frac{1}{360}\right)^{th}$ of a revolution, the angle is said to have a measure of one degree, written as 1^o.

A degree is divided into 60 minutes, and a minute is divided into 60 seconds.

$1^o = 60$ minutes $(60')$

$1' = 60$ seconds $(60'')$

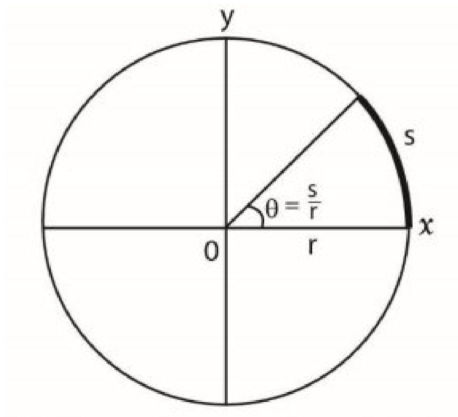
One sixtieth of a degree is called a minute, written as $1'$ and one sixtieth of a minute is called a second, written as $1''$.

$$1' = \frac{1}{60} \text{ of a degree}$$

$$1'' = \frac{1}{60} \text{ of a minute}$$

Radian Measure:

The Radian Measure of an angle is the ratio of the arc length it subtends, to the radius of the circle in which it is the central angle.



Consider a circle of a radius ‘r’. Let ‘s’ be the arc length subtending an angle θ at the centre.

$$\theta = \frac{\text{arc length}}{\text{radius}}$$

$$\theta = \frac{s}{r} \text{ radians}$$

$$S = r\,\theta$$

Relation between Degree and Radians:

In unit circle, a full rotation corresponds to 360^o, whereas a full rotation is related to 2π radians, the circumference of the unit circle.

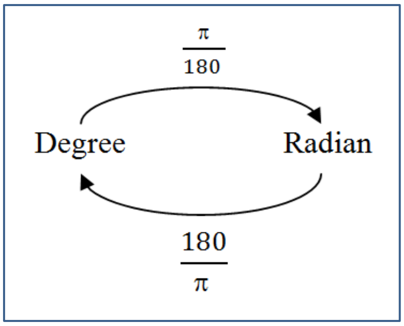
$$2\pi \text{ radians} = 360^o$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^o$$

$$x \text{ radian} = \left(\frac{180}{\pi} x\right)^o$$

$$1^o = \frac{\pi}{180} \text{ radians}$$

$$x^o = \frac{\pi x}{180} \text{ radians}$$



Value of π :

The number π is a mathematical constant. It plays a crucial role in radian measure. It is defined as the ratio of a circles circumference to its diameter. It is approximately equal to 3.14159. Whether the circle is big or small. The value of π remains same. It is pronounced as ‘pie’.

$$\pi \text{ value} = 3.14 \text{ (in decimal)}$$

$$\pi \text{ value} = \frac{22}{7} \text{ (in fraction)}$$

MATH FACT:

- ✷ The ratio of the circumference of any circle to its diameter is always a constant. This constant is denoted by the irrational number π.
- ✷ If the unit of angle measure is not specified, then the angle is taken to be in radian.

Commonly used angles in Degree and Radian Measures:

Degrees	0^o	30^o	45^o	60^o	90^o	180^o	270^o	360^o
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

MATH FACT:

Sin 90^o = 1 but sin 90 $\neq$ 1 (in radian measure)

Example 1

Convert each of the following angles in radian measure.

i) 18^o ii) 30^o

Solution:

i) $1^o = \frac{\pi}{180}$ radians

$18^o = \frac{\pi}{180} \times 18$ radians

$18^o = \frac{\pi}{10}$ radians

ii) $1^o = \frac{\pi}{180}$ radians

$30^o = \frac{\pi}{180} \times 30$ radians

$30^o = \frac{\pi}{6}$ radians

Example 2

Convert each of the following radians to degrees.

i) $\frac{\pi}{3}$ ii) $\frac{7\pi}{3}$

Solution:

i) 1 radian $= \left(\frac{180}{\pi}\right)^o$

$\frac{\pi}{3}$ radians $= \left(\frac{180}{\pi} \times \frac{\pi}{3}\right)^o$

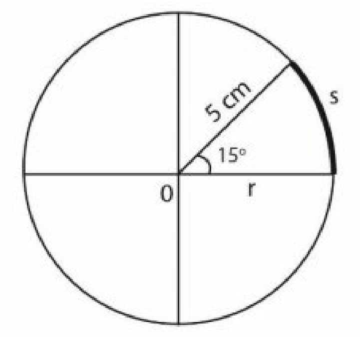
$\frac{\pi}{3}$ radians $= 60^o$

ii) $\frac{7\pi}{3}$ radians $= \left(\frac{180}{\pi} \times \frac{7\pi}{3}\right)$

$\frac{7\pi}{3}$ radians $= 420^o$

Example 3

Find the length of an arc of a circle of radius 5 cm subtending a central angle measuring 15^0.



Solution:

Let ‘s’ be the arc length of a circle of radius ‘r’ subtending a central angle θ.

We have, $\theta = 15^o$ and $r = 5$ cm.

Arc length, $s = r\,\theta$

$15^o = \frac{\pi}{180} \times 15$ radians

$15^o = \frac{\pi}{12}$ radians

Arc length, $S = 5 \times \frac{\pi}{12}$

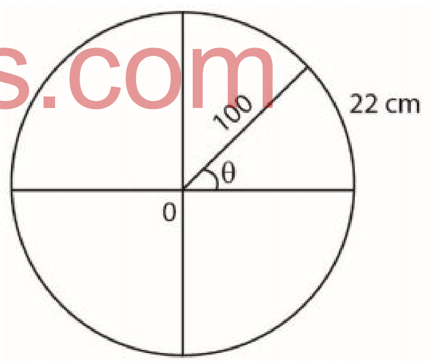
$S = \frac{5\pi}{12}$ cm

MATH FACT:

In the product r θ, θ must always be in radians.

Example 4

Find the degree measure of the angle subtended at the centre of a circle of radius 100 cm by an arc of length 22 cm. $\left(\pi = \frac{22}{7}\right)$



We have, r = 100 cm, S = 22 cm

Angle $\theta = \frac{s}{r}$

$= \frac{22}{100}$

$\theta = 0.22$

$= \left(0.22 \times \frac{180}{\pi}\right)^o$

$= \left(0.22 \times \frac{180}{22} \times 7\right)^o$

$= 12.60$

$\theta = 12^o36'$

Example 5

If the arc of the same lengths in the two circles subtend angles 60^o and 75^o at the centre find the ratio of their radii.

Solution:

Let θ_1 and θ_2 be the angles corresponding to the radii r_1 and r_2.

$\theta_1 = 60^o$

$= \frac{\pi}{180} \times 60$ radians

$\theta_1 = \frac{\pi}{3}$ radians

$\theta_2 = 75^o$

$= \frac{\pi}{180} \text{ x } 75$ radians

$\theta_2 = \frac{5\pi}{12}$ radians

We have, $\theta_1 = \frac{S}{r_1}$

$S = r_1\,\theta_1$

Similarly, $S = r_2\,\theta_2$

$\therefore\ r_1\,\theta_1 = r_2\,\theta_2$

$\frac{r_1}{r_2} = \frac{\theta_2}{\theta_1}$

$= \frac{5\pi}{12} \text{ x } \frac{3}{\pi}$

$\frac{r_1}{r_2} = \frac{5}{4}$

$r_1 : r_2 = 5:4$

Example 6:

A train is moving on a circular track of 1500 m radius at the rate of 66 km / hr. What angle will it turn in 20 seconds?

Solution:

We have, radius r = 1500 m

66 km = 66000 m

Length in 20 seconds $= \frac{66000}{60 \text{ x } 60} \text{ x } 20$

$= \frac{3300}{9}$

Angle, $\theta = \frac{S}{r}$

$= \frac{3300}{9 \text{ x } 1500}$

$\theta = \frac{11}{45}$ radians

$\frac{11}{45}$ radians $= \frac{11}{45} \text{ x } \frac{180}{\pi}$

$= \frac{11}{45} \text{ x } \frac{180}{22} \text{ x } 7$

$\theta = 14^o$

Example 7:

In a circle of diameter 40 cm, a chord is of length 20 cm. Find the length of the minor arc of the chord.

Solution:

Diameter of circle = 40 cm

radius of circle = 20 cm

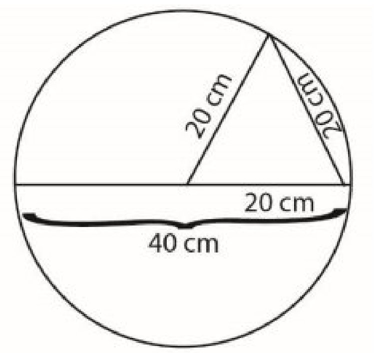
length of the chord = 20 cm

∴ The triangle is an equilateral triangle

Hence angle $= 60^\circ$

$\theta = 60 \times \frac{\pi}{180}$

$\theta = \frac{\pi}{3}$ radians



Length of the minor arc $= r\,\theta$

$= 20 \times \frac{\pi}{3}$

$= 20 \times \frac{22}{7 \times 3}$

$\therefore l = \frac{440}{21}$ cm

EXERCISE

1) Express each of the following angles in radian measure

i) 150° ii) -205° iii) 330° iv) 135°

2) Find the degree measure for the following radian measures.

i) $\frac{2\pi}{5}$ ii) 6 radians iii) $\frac{10\pi}{9}$ iv) $\frac{3\pi}{2}$

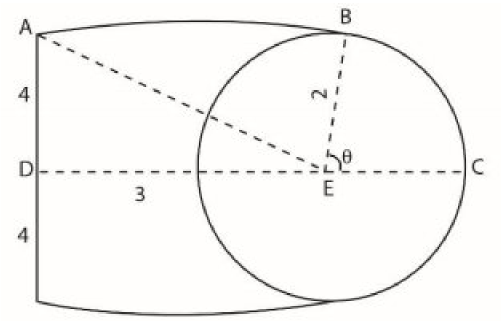
3) Find the length of an arc of a circle of radius 5 cm subtending a central angle of 15°.

4) If the arc of the same length in two circles subtended a central angles of 30° and 80°. Find the ratio of their radii.

5) What must be the radius of a circular running path around which an athlete must run 5 times in order to describe 1 km?

6) A bicycle tire makes 8 revolutions in one minute. The tire has a radius of 15 inches. Find the angle θ in radians through which the tire rotates in one second.

7) A rope is fastened to a wall in two places 8 ft apart at the same height. A cylindrical container with a radius of 2 ft is pushed away from the wall as far as it can go while being held by the rope, as in figure which shows top view. If the center of the container is 3 feet away from the point on the wall midway between the ends of the rope, what is the length L of the rope?

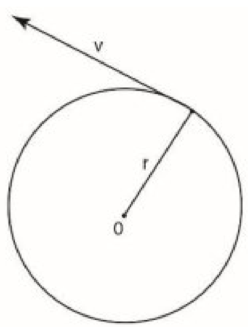


8) The centers of two belt pulleys, with radii of 3 inches and 6 inches respectively are 13 inches apart. Find the total length L of the belt around the pulleys.

9) A central angle in a circle of radius 2 cm cuts off an arc of length 4.6 cm. What is the measure of the angle in radians? What is the measure of the angle in degrees?

Application of Radian Measure

Circular Motion :



Circular motion is described as a movement of an object while rotating along a circular path.

Circular motion can be either uniform or non-uniform.

Uniform Circular Motion:

The angular rate of rotation and speed will be constant.

Non-Uniform Motion:

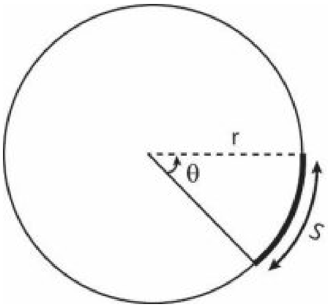
The rate of rotation keeps changing.

Common Examples of Circular Motion:

Man made satellite that revolves around the earth, a rotating ceiling fan, a moving car's wheel, the blades in wind mill and gears in gas turbines.

A particle is said to execute circular motion when it moves along the circumference of a circular path.

Angular Displacement:



It is defined as the angle turned by a rotating particle per unit time. It is measured in radians.

Angular displacement, $\theta = \frac{s}{r}$

Arc length :

On a circle of radius r, the arc length 's' intercepted by a central angle θ with radian measure is given by

$s = r\,\theta$

MATH FACT:

To calculate arc length, the measure of central angle must be in radians.

1 revolution = 2π radians = 360^o

Linear and Angular Speed :

Linear speed is the distance of an object travelled in a given time interval whereas angular speed is the object's angular rotation during a given time interval.

Anything that moves or turns in the circular direction has both linear velocity and angular velocity.

Linear Velocity:

The rate of change of displacement with respect to time when the object moves along a straight path. It is measured in m/s.

Angular Velocity:

Angular velocity is defined as the rate of change of angular displacement of a particle in circular motion with respect to time.

It is measured in rad/sec.

Angular velocity	Linear velocity
$\omega = \theta/t$ **Where a rotation θ takes place in a time 't'**	$v = s/t$ It depends the distance travelled S by an object in a time 't'.

MATH FACT:

- ✷ Linear velocity is a measure of distance per unit time.
- ✷ Angular velocity is a measure of turning per unit time.

Relation between linear velocity and Angular velocity :

For an object moving in a circular path of radius ‘r’ with constant angular velocity ω the linear velocity of the object is given by

$$v = \frac{s}{t}$$

We have $s = r\,\theta$

$$v = \frac{r\theta}{t}$$

$$\boxed{v = r\omega} \qquad \left[\because\ \omega = \frac{\theta}{t}\right]$$

Linear Velocity = Radius x Angular velocity

Angular Acceleration:

The rate of change of angular velocity of the rotating particle with respect to time is known as angular acceleration.

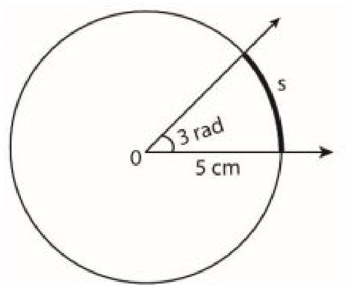
It is measured in rad/sec^2.

Circular motion can also be described in terms of angular variables.

Example 1:

Find the length (s) of the arc intercepted by a central angle of size 3 radians, if the radius of the circle is 5 cm.

Solution:



Given r = 5cm and θ = 3 radians.

$$\text{Arc length } s = r\,\theta$$

$$= 5 \times 3$$

$$\boxed{s = 15 \text{ cm}}$$

Example 2:

Find the length of the arc intercepted by a central angle of size–100^o if the radius of the circle is 7cm.

Solution :

Given $\theta = -100^o$

$$= (-100)\left(\frac{\pi}{180}\right)$$

$$\theta = -1.75 \text{ radians}$$

$$r = 7 \text{ cm}$$

$$\text{Arc length, } s = r\,\theta$$

$$= 7(-1.75)$$

$$s = -12.25 \text{ cm}$$

The negative value of the angle and the arc length refers to negative direction.

Example 3:

If the earth has a radius of 4050 miles and rotates one complete revolution (2π radians) earth 24 hrs. what is the linear velocity of an object located on the equator?

Solution :

Given $r = 4050$ miles

$\theta = 2\pi$ rad (1 revolution)

$t = 24$ hrs.

$$\text{linear velocity, } v = \frac{r\theta}{t}$$

$$= \frac{(4050)\,(2\pi)}{24}$$

$$= \frac{(4050)\,(6.28)}{24}$$

$v \cong 1060$ miles per hr.

Example 4:

A particle in uniform circular motion makes an angular displacement of 1.57 rad in a half second. Find its angular speed and linear speed if the radius of circular path is $\frac{1}{\pi}$ m.

Solution :

Given angular displacement, $\theta = 1.57$ rad.

$t = 0.5$ sec.

$r = \frac{1}{\pi}$ m

$$\text{Angular speed, } \omega = \frac{\theta}{t}$$

$$= \frac{1.57}{0.5}$$

$\omega = 3.14$ rad/sec.

linear speed, $v = r\,\omega$

$$= \frac{1}{\pi} \times 3.14$$

$v = 1$ m/s.

Example 5:

What is the angular velocity of a spinning top if it travels π radians in one third of a second?

Solution :

Given $\theta = \pi$ radians

$t = \frac{1}{3}$ sec.

$$\text{Angular velocity, } \omega = \frac{\theta}{t}$$

$$= \frac{\pi}{1/3}$$

$\omega = 3\pi$ rad / sec.

Example 6:

Suppose a circular disk is rotating at a rate of 40 revolutions per minute. Determine the angular velocity ω of the point in radians per minute and also determine the linear velocity v (in feet per second) of a point that is 3 feet from the center of the disk.

Solution :

Given θ = 40 revolutions

= $40 \times 2\pi$ radians

$\theta = 8\,0\pi$ rad

$r = 3$ ft.

$t = 1$ minute.

Angular velocity $\omega = \frac{\theta}{t}$

$= \frac{80\pi}{1}$ rad

$\omega = 80\pi$ rad/min.

$\omega = \frac{8\pi}{6}$ rad/sec.

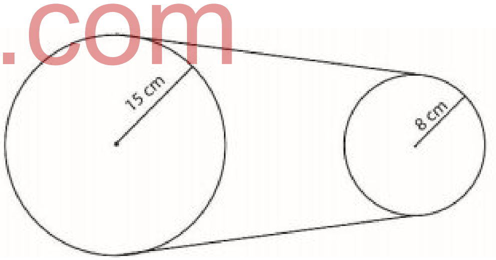
Linear velocity, $v = r\,\omega$

$= (3 \text{ ft}) \times \left(\frac{8\pi}{6} \text{ rad/sec}\right)$

$v = 4\pi$ ft / sec.

Example 7:

The two pulleys in the figure have radii of 15cm and 8 cm respectively. The larger pulley rotates 25 times in 36 seconds. Find the angular velocity of each pulley in rad/sec.



Solution :

Given radius of big pulley $r_B = 15$ cm

$\theta = 25$ rotations

$= 25\,(2\pi)$

$\theta = 50\,\pi$ radians

$t = 36$ sec

Angular Velocity of big pulley, $\omega_B = \frac{\theta}{t}$

$\omega_B = \frac{50\pi}{36}$ rad/sec

To find angular velocity of small pulley, let us find the linear velocities.

The linear velocities of two pulleys are same, because the two pulleys are connected.

$v_B = v_s$

$r_B\omega_B = r_s\omega_s$

$(15)\left(\frac{50\pi}{36}\right) = (8)\,\omega_s$

$\omega_s = \frac{125\pi}{48}$ rad/sec

Example 8:

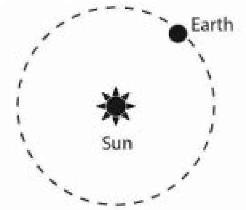
Earth travels about the sun in a circular orbit. Assume the orbit is circular with radius 93,000,000 miles. Its angular and linear velocities are used to design solar power facilities.

a) Assume a year is 365 days, find θ, the angle formed by earth's movement in one day.

b) Find the angular velocity in rad/hr.

c) Find the linear velocity of earth in miles/hour.

Solution :



Given $\theta = \frac{1 \text{ revolution}}{\text{year}}$

$\theta = \frac{2\pi}{365}$

b) $t = \frac{1 \text{ day}}{24 \text{ hrs}}$

Angular velocity, $\omega = \frac{\theta}{t}$

$= \frac{2\pi}{365} \text{ x } \frac{1}{24}$

$= \frac{2\pi}{(365)\,(24)}$ rad/hr.

$\omega = \frac{\pi}{4380}$ rad/hr.

c) r = 93,000,000 miles.

v = rω

$= (93,000,000) \left(\frac{\pi}{4380}\right)$

v = 66,732 mph.

Example 9:

Earth takes 365 days to complete a revolution around the sun. Calculate its angular speed.

Solution :

Given θ = 2π (1 revolution)

t = 365 days

= 365 x 24 x 60 x 60

t = 31536000 sec.

Angular speed, $\omega = \frac{\theta}{t}$

$= \frac{2\pi}{31536000}$

$\omega = 1.99 \text{ x } 10^{-7}$ rad/sec.

Example 10:

The wheel of a wagon of radius 1m is travelling with the speed of 5m per sec. calculate its angular speed.

Solution :

Given Linear speed, v = 5 m/s

Radius, r = 1m

Angular speed, $\omega = \frac{v}{r}$ $\left[\because v = r\omega\,;\ \omega = \frac{v}{r}\right]$

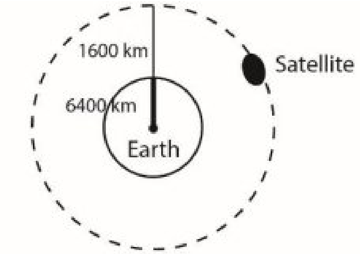
$= \frac{5}{1}$

$\omega = 5$ rad/sec.

EXERCISE

1) A child is spinning a rock at the end of a 3-foot rope at a rate of 100 revolutions per minute. Find the angular speed of the rock and the linear speed of the rock when it is released.

2) A satellite travelling in a circular orbit 1600 km above the surface of earth takes 2 hr to make an orbit. The radius of earth is 6400 km. See Figure.



   a) Find the linear speed of the satellite.

   b) Find the distance the satellite travels in 4.5 hr.

3) Suppose that point p is on a circle with radius 10 cm and ray OP is rotating with angular speed $\frac{\pi}{18}$ radian per sec.

   a) Find the angle generated by P in 6 sec.

   b) Find the distance travelled by P along the circle in 6 sec.

   c) Find the linear speed of P.

4) A vehicle travels at a steady speed on a straight road. Each of its wheel rotates 5 times per second. If each wheel has a diameter of 40 cm, then find the angular speed of the wheel and the speed of the car (approximately).

5) Calculate the angular velocity of 0.300 m radius of a car tire when the car travels at 15.0 m/s.

6) If a ball is travelling in a circle of diameter 10 m with velocity 20 m/s. Find the angular velocity of the ball.

7) A car wheel of radius 20 inches rotates at 8 revolutions per second on the highway. What is the angular speed of the tire?

Trigonometric Ratios of Allied Angles

Allied Angles

Two angles are said to be allied if their sum or difference is a multiple of $\frac{\pi}{2}$ radians.

Thus any two angles of θ such as $-\theta, \frac{\pi}{2} \pm \theta, \pi \pm \theta, \frac{3\pi}{2} \pm \theta, \ldots$ are all allied angles.

Using trigonometric ratios of the allied angles, we can find the trigonometric ratios of any angle.

Generation of Allied angles

Consider an acute angle θ, to generate a group of allied angles, keep on adding or subtracting integral multiples of 90^o or $\frac{\pi}{2}$. In this way, we can generate as many allied angle as we desire.

$n(90^o) \pm \theta$	or	$n\left(\frac{\pi}{2}\right) \pm \theta$ and
$\theta \pm n(90^o)$	or	$\theta \pm n\left(\frac{\pi}{2}\right)$
are generators of allied angles		

$0^o < \theta < 90^o$

$n \in Z$

Consider the angle 30^o

30^o	30^o	$30^o + 90^o = 120^o$	$30^o + 2(90^o) = 210^o$	$30^o + 3(90^o) = 300^o$	$30^o + 4(90^o) = 390^o$
$1(90^o) - 30^o = 60^o$	$30^o - 90^o = -60^o$				
$2(90^o) - 30^o = 150^o$	$30^o - 2(90^o) = -150^o$				
$3(90^o) - 30^o = 240^o$	$30^o - 3(90^o) = -240^o$				
$4(90^o) - 30^o = 330^o$	$30^o - 4(90^o) = -330^o$				

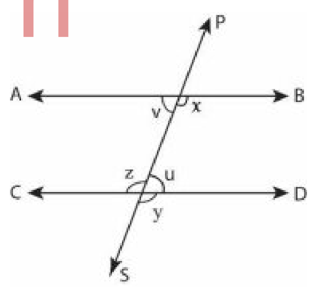
$n(90^o) \pm 30^o$ or $n\left(\frac{\pi}{2}\right) \pm 30^o$

and

$30^o \pm n(90^o)$ or $30^o \pm n\left(\frac{\pi}{2}\right)$

Note:

If two parallel lines are cut by a transversal as shown in figure, the angles u and x (Z and v) are Allied (or co-interior) angles.



Trigonometric ratios of $-\theta$ in terms of θ.

$\sin(-\theta) = -\sin\theta$

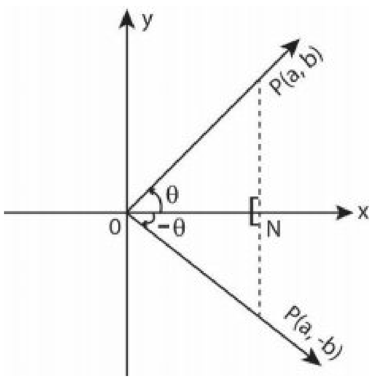
$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$\cot(-\theta) = -\cot\theta$

$\text{cosec}(-\theta) = -\text{cosec}\,\theta$

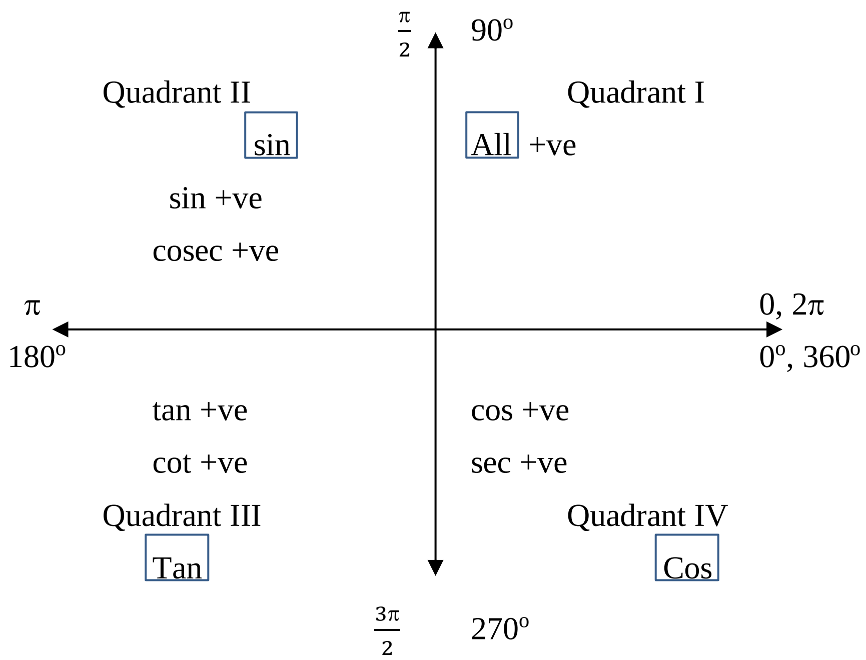
$\sec(-\theta) = \sec\theta$



MATH FACT:

Negative – angle identities are used to determine whether trigonometric function is an odd function or an even function.

Signs of Trigonometric Ratios of an angle θ as it varies from 0º to 360º



Sign of other Trigonometric functions in different quadrant

	I	II	III	IV
sin θ	+	+	−	−
cos θ	+	−	−	+
tan θ	+	−	+	−
cosec θ	+	+	−	−
sec θ	+	−	−	+
cot θ	+	−	+	−

Trigonometric ratios of allied angles $\frac{\pi}{2} \pm \theta$, $\pi \pm \theta$, $\frac{3\pi}{2} \pm \theta$, $2\pi \pm \theta$.

θ	**−θ**	**90º −θ or $\frac{\pi}{2} - \theta$**	**90º +θ or $\frac{\pi}{2} + \theta$**	**180º −θ or $\pi - \theta$**	**180º +θ or $\pi + \theta$**	**270º −θ or $\frac{3\pi}{2} - \theta$**	**270º +θ or $\frac{3\pi}{2} + \theta$**	**360º −θ or $2\pi - \theta$**	**360º +θ or $2\pi + \theta$**
sin	−sin θ	cos θ	cos θ	sin θ	−sin θ	−cos θ	−cos θ	−sin θ	sin θ
cos	cos θ	sin θ	−sin θ	−cos θ	−cos θ	−sin θ	sin θ	cos θ	cos θ
tan	−tan θ	cot θ	−cot θ	−tan θ	tan θ	cot θ	−cot θ	−tan θ	tan θ

Uses of Allied Angles :

To find the trigonometric values of numerically large angles.

Note:

i) If the allied angles of the form $\pm\theta$, $\pi\pm\theta$, $2\pi\pm\theta$ …. Then the form of trigonometric ratio is unaltered [(i.e.) sine remains sine. Cosine remains cosine etc.]

ii) If the allied angles of the form $\frac{\pi}{2}\pm\theta$, $\frac{3\pi}{2}\pm\theta$, … then the form of trigonometric ratio is altered to its complementary ratio.

Example 1:

Evaluate the values of (i) $\sin(-45^o)$ ii) $\cos 135^o$ iii) $\tan 120^o$.

Solution :

(i) $\sin(-45^o)$

$$\boxed{\sin(-\theta) = -\sin\theta}$$

$$\sin(-45^o) = -\sin 45^o$$

$$= -\frac{1}{\sqrt{2}}$$

(ii) $\cos(135^o) = \cos(90^o + 45^o)$

$$\boxed{\cos(90^o + \theta) = -\sin\theta}$$

$$\cos(90^o + 45^o) = -\sin(45^o)$$

$$= -\frac{1}{\sqrt{2}}$$

(iii) $\tan(120^o) = \tan(180^o - 60^o)$

$$\boxed{\tan(180^o - \theta) = -\tan\theta}$$

$$\tan(180^o - 60^o) = -\tan 60^o$$

$$= -\sqrt{3}$$

Example 2:

Prove that the ratio of the product of $\sin(180^o + \theta)$ and $\cot(90^o - \theta)$ to $\sec(-\theta)$ is $\sin^2\theta$.

Solution :

To prove : $\frac{\sin(180^o+\theta)\cot(90^o-\theta)}{\sec(-\theta)} = \sin^2\theta$

Using Allied Angles,

$$\boxed{\begin{aligned} \sin(180^o + \theta) &= -\sin\theta \\ \cot(90^o - \theta) &= -\tan\theta \\ \sec(-\theta) &= \sec\theta \end{aligned}}$$

consider,

$$\text{LHS} = \frac{\sin(180^o+\theta)\cot(90^o-\theta)}{\sec(-\theta)}$$

$$= \frac{(-\sin\theta)(-\tan\theta)}{\sec\theta}$$

$$= \frac{(-\sin\theta)\left(-\frac{\sin\theta}{\cos\theta}\right)}{\left(\frac{1}{\cos\theta}\right)} \qquad \{\text{using quotient identity}\}$$

$$= \frac{(-\sin\theta)\,(-\sin\theta)}{\cos\theta} \text{ x } \cos\theta$$

$$= \sin^2\theta$$

$$\text{LHS} = \text{RHS}$$

Hence the statement is proved.

Example 3:

Prove that $\cos\left(\frac{\pi}{2}+\theta\right)\sec(-\theta)\tan\left(\frac{3\pi}{2}-\theta\right) = -1$

Solution :

Using Allied Angles,

$$\cos\left(\frac{\pi}{2}+\theta\right) = -\sin\theta$$
$$\sec(-\theta) = \sec\theta$$
$$\tan\left(\frac{3\pi}{2}-\theta\right) = \cot\theta$$

consider,

$$\text{LHS} = \cos\left(\frac{\pi}{2}+\theta\right)\sec(-\theta)\tan\left(\frac{3\pi}{2}-\theta\right)$$

$$= (-\sin\theta)(\sec\theta)(\cot\theta)$$

$$= (-\sin\theta)\left(\frac{1}{\cos\theta}\right)\left(\frac{\cos\theta}{\sin\theta}\right) \qquad \{\text{using reciprocal and quotient identities}\}$$

$$= -1$$

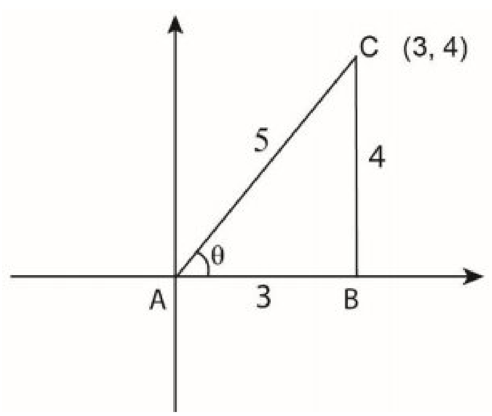
$$\text{LHS} = \text{RHS}$$

Hence $\cos\left(\frac{\pi}{2}+\theta\right)\sec(-\theta)\tan\left(\frac{3\pi}{2}-\theta\right) = -1$

Example 4:

(3, 4) is a point on the terminal side of an angle θ in standard position. Determine the six trigonometric function values of angle θ.

Solution :



Using Pythagoras theorem,

$$AC^2 = AB^2 + BC^2$$
$$= 3^2 + 4^2$$
$$AC = \sqrt{9+16}$$
$$= \sqrt{25}$$
$$AC = 5$$

$$\sin\theta = \frac{4}{5}, \quad \cos\theta = \frac{3}{5}$$

$$\tan\theta = \frac{4}{3}, \quad \cot\theta = \frac{3}{4}$$

$$\sec\theta = \frac{5}{3}, \quad \text{cosec}\,\theta = \frac{5}{4}$$

Example 5:

Find the values of all trigonometric functions of $\cos\theta = -\frac{1}{2}$, θ lies in III quadrant.

Solution :

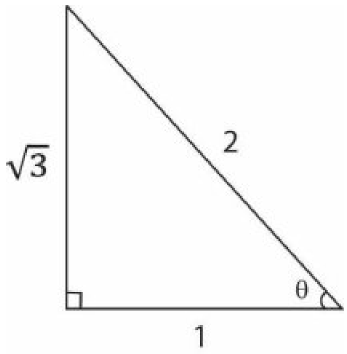
$$\cos\theta = -\frac{1}{2}$$

$$\sin\theta = \sqrt{1-\cos^2\theta}$$

$$= \sqrt{1-\left(\frac{-1}{2}\right)^2}$$

$$= \sqrt{1-\frac{1}{4}}$$

$$\sin\theta = \frac{\sqrt{3}}{2}$$



Given θ lies in III quadrant

$$\sin\theta = -\frac{\sqrt{3}}{2}$$

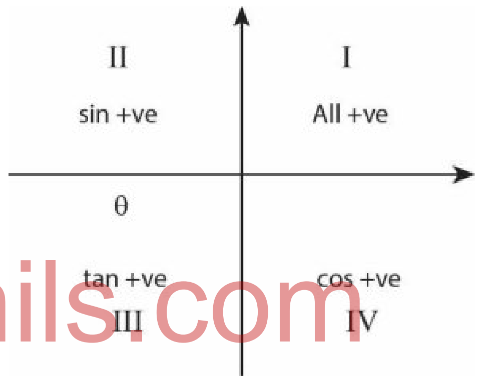
$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$= -\frac{\sqrt{3}}{2} \times \frac{-2}{1} = \sqrt{3}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{-2}{\sqrt{3}}$$

$$\sec\theta = \frac{1}{\cos\theta} = -2$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = -\frac{1}{2} \times \frac{-2}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$



Example 6:

Find the exact values of all trigonometric functions of 120°, without using calculator.

Solution :

i) $\sin 120^\circ = \sin(90^\circ + 30^\circ)$

$$\boxed{\sin(90^\circ+\theta) = \cos\theta}$$

$$\sin 120^\circ = \cos 30^\circ$$

$$\sin 120^\circ = \frac{\sqrt{3}}{2}$$

ii) $\cos 120^\circ = \cos(90^\circ + 30^\circ)$

$$\boxed{\cos(90^\circ+\theta) = -\sin\theta}$$

$$\cos(90^\circ+30^\circ) = -\sin 30^\circ$$

$$\cos 120^\circ = -\frac{1}{2}$$

iii) $\tan 120^\circ = \tan(90^\circ + 30^\circ)$

$$\boxed{\tan(90^\circ+\theta) = -\cot\theta}$$

$$\tan(90^\circ+30^\circ) = -\cot 30^\circ$$

$$\tan 120^\circ = -\sqrt{3}$$

iv) $\cot 120^{o} = \cot(90^{o} + 30^{o})$

$\boxed{\cot(90^{o} + \theta) = -\tan\theta}$

$\cot(90^{o} + 30^{o}) = -\tan 30^{o}$

$\cot 120^{o} = -\frac{1}{\sqrt{3}}$

v) $\operatorname{cosec} 120^{o} = \operatorname{cosec}(90^{o} + 30^{o})$

$\boxed{\operatorname{cosec}(90^{o} + \theta) = \sec\theta}$

$\operatorname{cosec}(90^{o} + 30^{o}) = \sec 30^{o}$

$\operatorname{cosec} 120^{o} = \frac{2}{\sqrt{3}}$

vi) $\sec 120^{o} = \sec(90^{o} + 30^{o})$

$\sec(90^{o} + 30^{o}) = -\operatorname{cosec} 30^{o}$

$\sec 120^{o} = -2$

Example 7:

Prove that $\sec\left(\frac{3\pi}{2} - \theta\right)\sec\left(\theta - \frac{5\pi}{2}\right) + \tan\left(\frac{5\pi}{2} + \theta\right)\tan\left(\theta - \frac{5\pi}{2}\right) = 1$

Solution :

$$\begin{aligned}
\sec\left(\frac{3\pi}{2} - \theta\right) &= \sec(270^{o} - \theta) \\
&= -\operatorname{cosec}\theta \\
\sec\left(\theta - \frac{5\pi}{2}\right) &= \sec\left[-\left(\frac{5\pi}{2} - \theta\right)\right] \\
&= \sec\left(\frac{5\pi}{2} - \theta\right) \qquad [\sec(-\theta) = \sec\theta] \\
&= \sec(450^{o} - \theta) \\
&= \sec(360^{o} + (90^{o} - \theta)) \\
&= \sec(90^{o} - \theta) \\
&= \operatorname{cosec}\theta \\
\tan\left(\frac{5\pi}{2} + \theta\right) &= \tan(450^{o} + \theta) \\
&= \tan(360^{o} + (90^{o} + \theta)) \\
&= \tan(90^{o} + \theta) \\
&= -\cot\theta \\
\tan\left(\theta - \frac{5\pi}{2}\right) &= \tan\left[-\left(\frac{5\pi}{2} - \theta\right)\right] \\
&= -\tan\left(\frac{5\pi}{2} - \theta\right) \qquad [\tan(-\theta) = -\tan\theta] \\
&= -\tan(450^{o} - \theta) \\
&= -\tan(360^{o} + (90^{o} - \theta)) \\
&= -\tan(90^{o} - \theta) \\
&= -\cot\theta
\end{aligned}$$

Consider, LHS $= (-\operatorname{cosec}\theta)(\operatorname{cosec}\theta) - \cot\theta(-\cot\theta)$

$= -\operatorname{cosec}^2\theta + \cot^2\theta$

$= -[\operatorname{cosec}^2\theta - \cot^2\theta]$

$= -1$ $[\because \operatorname{cosec}^2\theta - \cot^2\theta = 1]$

LHS = RHS

$$\therefore \sec\left(\frac{3\pi}{2} - \theta\right)\sec\left(\theta - \frac{5\pi}{2}\right) + \tan\left(\frac{5\pi}{2} + \theta\right)\tan\left(\theta - \frac{5\pi}{2}\right) = -1$$

Example 8:

Prove that $2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3} = 10$

Solution :

Consider, LHS $= 2\sin^2 135^o + 2\cos^2 45^o + 2\sec^2 60^o$

$= 2\sin^2(90^o + 45^o) + 2\cos^2 45^o + 2\sec^2 60^o$

$= 2\cos^2 45^o + 2\cos^2 45^o + 2\sec^2 60^o$

$= 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2(2)^2$

$= 2 \text{ x } \frac{1}{2} + 2 \text{ x } \frac{1}{2} + 8$

$= 1 + 1 + 8$

$= 10$

LHS = RHS

$$\therefore 2\sin^2\frac{3\pi}{4} + 2\cos^2\frac{\pi}{4} + 2\sec^2\frac{\pi}{3} = 10$$

Hence proved.

Example 9:

Prove $\cot\left(\frac{3\pi}{2} - \theta\right) + \cot(2\pi + \theta) + \cos\left(\frac{3\pi}{2} + \theta\right)\sec(\pi - \theta) = \cot\theta$.

Solution :

Consider, LHS $= \cot\left(\frac{3\pi}{2} - \theta\right) + \cot(2\pi + \theta) + \cos\left(\frac{3\pi}{2} + \theta\right)\sec(\pi - \theta)$

$= \cot(270^o - \theta) + \cot(360^o + \theta) + \cos(270^o + \theta)\sec(180^o - \theta)$

$= \tan\theta + \cot\theta + \sin\theta(-\sec\theta)$ [using Allied Angles]

$= \tan\theta + \cot\theta + \sin\theta\left(-\frac{1}{\cos\theta}\right)$

$= \tan\theta + \cot\theta - \tan\theta$

$= \cot\theta$

LHS = RHS

Hence proved.

EXERCISE

1) Find the values of each of the following trigonometric ratios.

   (i) sin 150^o ii) cosec 390^o iii) tan 120^o

2) Determine the values of all trigonometric functions for sec $\theta = \frac{13}{5}$, θ lies in IV quadrant.

3) Show that $\sin^2\frac{\pi}{18} + \sin^2\frac{\pi}{9} + \sin^2\frac{7\pi}{18} + \sin^2\frac{4\pi}{9} = 2$

4) Find all the angles between 0^o and 360^o. which satisfy the equation $\sin^2\theta = \frac{3}{4}$

5) If $\left(\frac{5}{7}, \frac{2\sqrt{6}}{7}\right)$ is a point on the terminal side of an angle θ in standard position, then determine the six trigonometric function values of angle θ.

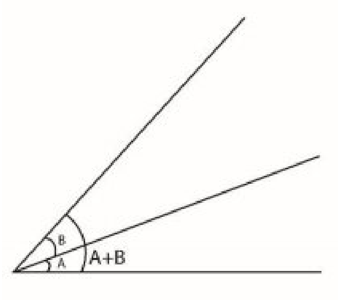
6) Prove that $\tan(\pi - x)\cot(x - \pi) - \cos(2\pi - x)\cos(2\pi + x) = \sin^2 x$

7) Prove that $2\sin^2\frac{\pi}{6} + \text{cosec}^2\frac{7\pi}{6}\ \cos^2\frac{\pi}{3} = \frac{3}{2}$

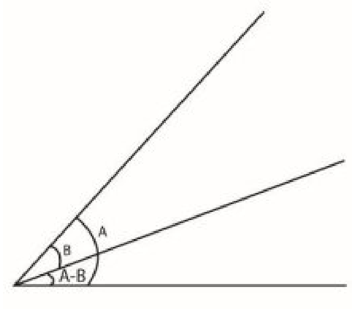
Chapter 3.2 TRIGONOMETRIC IDENTITIES

Compound Angles :

A compound angle is an algebraic sum or difference of two or more angles.



Compound angle A+B



Compound angle A–B

A Compound angle formula is basically a trigonometric identity that express a trigonometric function of (A+B) or (A–B) in terms of trigonometric function of A and B.

Trigonometric function of Sum and Difference Angles

1. $\sin(A+B) = \sin A \cos B + \cos A \sin B$
2. $\sin(A-B) = \sin A \cos B - \cos A \sin B$
3. $\cos(A+B) = \cos A \cos B - \sin A \sin B$
4. $\cos(A-B) = \cos A \cos B + \sin A \sin B$
5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

MATH FACT:

i) Compound Angle Identities used to compute the trigonometric functions of sum and difference of angles.

ii) Trigonometric function do not satisfy the functions relation like $f(x + y) = f(x) + f(y)$ $\boxed{\text{(i.e) } \sin(A \pm B) \neq \sin A \pm \sin B,}$ for other trigonometric functions also.

iii) If $A = B$ then $\sin(A - B) = \sin 0 = 0$

iv) If $A = \frac{\pi}{2}$ and $B = \theta$ then $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ (using Allied Angles)

The trigonometric functions of sum and difference formulas as single equation

Sine	$\mathbf{\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B}$
Cosine	$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
Tangent	$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Example 1:

Find the value of (i) sin 75° (ii) cos 15° (iii) tan 15°

Solution :

(i) $\sin 75^o = \sin(45^o + 30^o)$

$= \sin 45^o \cos 30^o + \cos 45^o \sin 30^o$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

(ii) $\cos 15^o = \cos(45^o - 30^o)$

$= \cos 45^o \cos 30^o + \sin 45^o \sin 30^o$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

(iii) $\tan 15^o = \tan(45^o - 30^o)$

$= \frac{\tan 45^o - \tan 30^o}{1 + \tan 45^o \tan 30^o}$

$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$

$= \frac{\sqrt{3}-1}{\sqrt{3}+1}$

Example 2:

Find the value of

(i) cos 50° cos 40° - sin 50° sin 40°

(ii) sin 72° cos 18° + cos 72° sin 18°

(iii) $\frac{\tan 20^o + \tan 25^o}{1 - \tan 25^o \tan 20^o}$

Solution :

(i) $\cos 50^o \cos 40^o - \sin 50^o \sin 40^o = \cos(50^o + 40^o)$

$= \cos 90^o$

$= 0$

(ii) $\sin 72^o \cos 18^o + \cos 72^o \sin 18^o = \sin(72^o + 18^o)$

$= \sin 90^o$

$= 1$

(iii) $\frac{\tan 20^o + \tan 25^o}{1 - \tan 20^o \tan 25^o} = \tan(20^o + 25^o)$

$= \tan 45^o = 1$

Example 3:

If $\tan A = \frac{5}{6}$ and $\tan B = \frac{1}{11}$ show that $A+B = \pi/4$

Solution :

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{5}{6} + \frac{1}{11}}{1 - \frac{5}{6} \cdot \frac{1}{11}}$$

$$= \frac{61/66}{61/66}$$

$$\tan(A+B) = 1$$

$$A+B = \pi/4$$

Hence proved.

Example 4:

The force F (in pounds) on a person's back when he or she bends over at an angle θ is

$$F = \frac{0.6\, w \sin(\theta + 90^o)}{\sin 12^o}$$

Where w is the person's weight (in pounds). Simplify the formula.

Solution :

Given

$$F = \frac{0.6\, w \sin(\theta + 90^o)}{\sin 12^o}$$

Expand $\sin(\theta + 90^o)$ using $\sin(A + B)$

$$= \frac{0.6\, w (\sin\theta \cos 90^o + \cos\theta \sin 90^o)}{\sin 12^o}$$

$$= \frac{0.6\, w (\sin\theta\,(0) + \cos\theta\,(1))}{\sin 12^o}$$

$$= \frac{0.6\, w (\cos\theta)}{0.208}$$

$$F = 2.88\, w \cos\theta$$

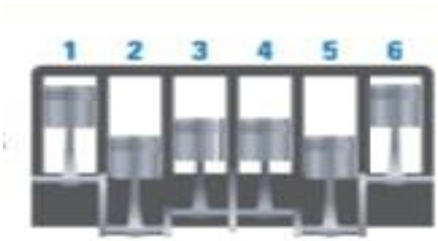
Example 5:

The heights h (in inches of pistons 1 and 2 of an automobile engine can be modeled by $h_1 = 3.75 \sin 733t + 7.5$ and $h_2 = 3.75 \sin 733\left(t + \frac{4\pi}{3}\right) + 7.5$ where t is measured in seconds. How often are these two pistons at the same height?

Solution :

Let $h_1 = h_2$ and solve for t.

$$3.75 \sin 733\,t + 7.5 = 3.75 \sin 733\left(t + \frac{4\pi}{3}\right) + 7.5$$



$$3.75 \sin 733\,t = 3.75 \sin\left(733\,t + \frac{2932\pi}{3}\right)$$

$$\sin 733\,t = \sin\left(733\,t + \frac{2932\pi}{3}\right)$$

$$= \sin 733\,t \cos \frac{2932\pi}{3} + \cos 733\,t \sin \frac{2932\pi}{3}$$

$$= -\frac{\sin 733\,t}{2} - \cos 733\,t \left(\frac{\sqrt{3}}{2}\right)$$

$$\sin 733\,t + \frac{\sin 733\,t}{2} = \frac{-\sqrt{3}}{2} \cos 733\,t$$

$$\frac{3 \sin 733\,t}{2} = \frac{-\sqrt{3}}{2} \cos 733\,t \qquad \left[\because \cos \frac{2932\pi}{3} = -\frac{1}{2},\ \sin \frac{2932\pi}{3} = \frac{\sqrt{3}}{2}\right]$$

$$\frac{\sin 733\,t}{\cos 733\,t} = \frac{-\sqrt{3}}{2} \times \frac{2}{3}$$

$$\tan 733\,t = \frac{-\sqrt{3}}{3}$$

$$\tan 733\,t = -\frac{1}{\sqrt{3}}$$

$$733\,t = \frac{-\pi}{6} + n\pi$$

$$t = \frac{-\pi}{4398} + \frac{n\pi}{733}$$

The heights are equal once every $\frac{\pi}{733}$ secs. So in 1 sec, the heights are equal, the following number of times,

$$= \frac{1}{\pi / 733}$$

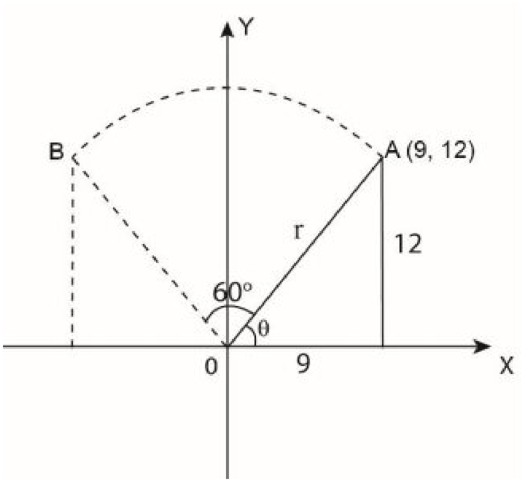
$$= \frac{733}{\pi} = \frac{733}{3.141}$$

$$= 233.4 \text{ times}$$

Example 6:

The point A(9, 12) rotates around the origin O in a plane through 60° in the anti-clockwise direction to a new position B. Find the coordinates of the point B.

Solution:



Let A(9, 12) = A $(r \cos \theta,\ r \sin \theta)$

Where r = OA

$$r = \sqrt{9^2 + 12^2}$$

Thus, $r = \sqrt{81 + 144} = \sqrt{225}$

$$r = 15$$

Hence, the point A is given by

A$(15 \cos \theta,\ 15 \sin \theta)$

Now, the point B is given by

B$(15 \cos (\theta + 60^\circ),\ 15 \sin (\theta + 60^\circ))$

$$15 \cos (\theta + 60^\circ) = 15 (\cos \theta \cos 60^\circ - \sin \theta \sin 60^\circ)$$

$$= 15 \cos \theta \cos 60^\circ - 15 \sin \theta \sin 60^\circ$$

$$= 9 \times \frac{1}{2} - 12 \times \frac{\sqrt{3}}{2}$$

$$= \frac{3}{2}(3 - 4\sqrt{3})$$

Similarly,

$$15 \sin(\theta + 60^o) = \frac{3}{2}(4 + 3\sqrt{3})$$

Hence the point B is

$$B\left(\frac{3}{2}(3 - 4\sqrt{3}),\ \frac{3}{2}(4 + 3\sqrt{3})\right)$$

Example 7:

If $\cos A = \frac{1}{7}$ and $\cos B = \frac{13}{14}$ Prove that $A–B = \pi/3$

Solution:

$\cos A = \frac{1}{7}$

$\sin A = \sqrt{1 - \cos^2 A}$

$= \sqrt{1 - \frac{1}{49}}$

$= \frac{\sqrt{48}}{7}$

$\cos B = \frac{13}{14}$

$\sin B = \sqrt{1 - \cos^2 B}$

$= \sqrt{1 - \frac{169}{196}}$

$= \frac{\sqrt{27}}{14}$

$$\text{Cos}(A\text{-}B) = \cos A \cos B + \sin A \sin B$$

$$= \frac{1}{7}\frac{13}{14} + \frac{\sqrt{48}}{7}\frac{\sqrt{27}}{14}$$

$$= \frac{13}{98} + \frac{36}{98}$$

$$= \frac{49}{98}$$

$$= \frac{1}{2}$$

$$A–B = \pi/3$$

Hence proved.

Example 8:

If $A+B = 45^o$. Prove that $(1+\tan A)(1 + \tan B) = 2$. Hence deduce the value of $\tan 22\frac{1}{2}^o$.

Solution:

$A+B = 45^o$

$\tan(A+B) = \tan 45^o$

$\frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$

$\tan A + \tan B = 1 - \tan A \tan B \quad \rightarrow \quad ①$

LHS $(1+\tan A)(1+\tan B) = 1 + \tan B + \tan A + \tan A \tan B$

$= 1 + 1 - \tan A \tan B + \tan A \tan B$ [by (1)]

$= 2$

$\Rightarrow$ RHS

$\therefore (1 + \tan A)(1 + \tan B) = 2$

Put $A = B = 22\frac{1}{2}^{o}$

$$\left(1 + \tan 22\frac{1}{2}^{o}\right)\left(1 + \tan 22\frac{1}{2}^{o}\right) = 2$$

$$\left(1 + \tan 22\frac{1}{2}^{o}\right)^2 = 2$$

$$1 + \tan 22\frac{1}{2}^{o} = \sqrt{2}$$

$$\tan 22\frac{1}{2}^{o} = \sqrt{2} - 1$$

Hence proved.

Example 9:

If $A+B+C = 180^o$ then prove that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

Solution:

$$A + B + C = 180^o$$

$$A+B = 180^o - C$$

$$\tan(A+B) = \tan(180^o - C)$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$$\tan A + \tan B = -\tan C\,[1 - \tan A \tan B]$$

$$= -\tan C + \tan A \tan B \tan C$$

$$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C$$

Hence proved.

Example 10:

In the study of the propagation of electromagnetic waves, Snell's law gives the relation.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Show that the P - Polarization transmission Fresnel Coefficient defined by

$$t_{12p} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2} \quad \text{can be written as} \quad t_{12p} = \frac{2\cos\theta_1\sin\theta_2}{\sin(\theta_1 + \theta_2)}$$

Solution:

$$\therefore \text{ Given that, } t_{12p} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2}$$

Multiply and divide by $\sin\theta_1 \sin\theta_2$,

$$t_{12p} = \frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2} \text{ x } \frac{\sin\theta_1\sin\theta_2}{\sin\theta_1\sin\theta_2}$$

$$= \frac{2n_1\sin\theta_1\cos\theta_1\sin\theta_2}{n_1\cos\theta_1\sin\theta_1\sin\theta_2 + n_2\cos\theta_2\sin\theta_1\sin\theta_2}$$

$$= \frac{2(n_1\sin\theta_1)\cos\theta_1\sin\theta_2}{(n_1\sin\theta_1)\cos\theta_1\sin\theta_2 + (n_2\sin\theta_2)\cos\theta_2\sin\theta_1}$$

$$= \frac{2(n_1\sin\theta_1)\cos\theta_1\sin\theta_2}{(n_1\sin\theta_1)\cos\theta_1\sin\theta_2 + (n_1\sin\theta_1)\cos\theta_2\sin\theta_1} \quad \text{[By Snell's law]}$$

$$= \frac{n_1 \sin\theta_1\,(2\cos\theta_1 \sin\theta_2)}{n_1 \sin\theta_1\,[\cos\theta_1\ \sin\theta_2 + \cos\theta_2 \sin\theta_1]}$$

$$t_{12p} = \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1 + \theta_2)}$$

Hence proved.

Example 11:

Prove that sin A + sin (120° + A) – sin (120° – A) = 0

Solution:

LHS, sin A + sin (120° + A) – sin (120° – A)

= sin A + sin 120° cos A + cos 120° sin A – sin 120° cos A + cos 120° sin A

= sin A + 2 cos 120° sin A

$= \sin A + 2\left(-\frac{1}{2}\right)\sin A$

= sin A – sin A

= 0 = RHS

Hence proved.

Example 12:

Prove that (i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B$

Solution:

(i) LHS,

$\sin(A+B)\sin(A-B) = [\sin A\cos B + \cos A\sin B][\sin A\cos B - \cos A\sin B]$

$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$

$= \sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B$

$= \sin^2 A - \sin^2 B$

$\Rightarrow$ RHS

(ii) LHS,

$\cos(A+B)\cos(A-B) = [\cos A\cos B - \sin A\sin B][\cos A\cos B + \sin A\sin B]$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$

$= \cos^2 A(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B$

$= \cos^2 A - \cos^2 A\sin^2 B - \sin^2 B + \cos^2 A\sin^2 B$

$= \cos^2 A - \sin^2 B$

$\Rightarrow$ RHS

Hence proved.

EXERCISE

1) Find the value of (i) $\sin 15^\circ$ (ii) $\cos 75^\circ$ (iii) $\tan 105^\circ$

2) Simplify : (i) $\sin 42^\circ \cos 18^\circ + \cos 42^\circ \sin 18^\circ$

(ii) $\cos 18^\circ \cos 12^\circ - \sin 18^\circ \sin 12^\circ$

(iii) $\frac{\tan 50^\circ + \tan 10^\circ}{1 - \tan 50^\circ \tan 10^\circ}$

3) If $\sin A = \frac{1}{\sqrt{10}}$ and $\sin B = \frac{1}{\sqrt{5}}$ show that $A + B = \frac{\pi}{4}$

4) The point A(3,4) rotates around the origin O in a plane through 30° in the anticlockwise direction to a new position B. Find the coordinates of the point B.

5) The heights h (in feet) of two people in different seats on a Ferris wheel can be modeled by $h_1 = 28 \cos 10t + 38$ and $h_2 = 28 \cos 10\left(t - \frac{\pi}{6}\right) + 38$. Where t is the time (in mins) when are the two people at the same height?

6) If $A+B = 45^\circ$ prove that $(\cot A - 1)(\cot B - 1) = 2$. Hence deduce the value of $\cot 22\frac{1^\circ}{2}$

7) If $\sin P = 0.8142$ and $\cos Q = 0.4432$

Evaluate the following and correct to 3 decimal places.

(i) $\sin (P - Q)$

(ii) $\cos (P + Q)$

8) Prove that $\tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A$

9) If $A + B + C = 180^\circ$ then prove that $\Sigma \cot A \cot B = 1$

10) Prove that $\frac{\sin (A-B)}{\cos A \cos B} + \frac{\sin (B-C)}{\cos B \cos C} + \frac{\sin (C-A)}{\cos C \cos A} = 0$

11) If $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$. Find the value of $\tan (A+B)$

12) If angle θ is divided into two parts such that the tangent of one part is k times the tangent of other and $\emptyset$ is their difference, then show that,

$$\sin \theta = \frac{k+1}{k-1} \sin \emptyset$$

13) In the study of the propagation of electromagnetic waves, Snell's law gives the relation $n_1 \sin \theta_1 = n_2 \sin \theta_2$.

where θ_1 is the angle of incidence at which a wave strikes the planar boundary between two mediums, θ_2 is the angle of transmission of the wave through the new medium and n_1 and n_2 are the indexes of refraction of the two mediums. The quantity

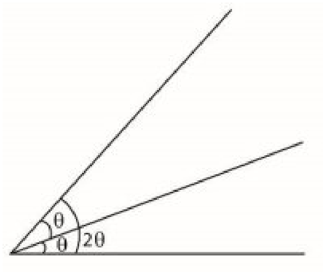
$$r_{12s} = \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_2 \cos \theta_1 + n_2 \cos \theta_2}$$

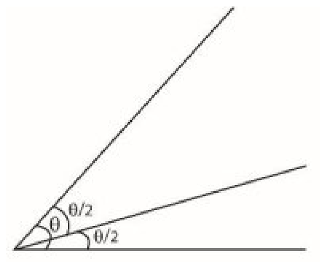
is called the Fresnel coefficient for normal incidence reflection of the wave for s-polarization show that this can be written as $r_{12s} = \frac{\sin (\theta_2 - \theta_1)}{\sin (\theta_2 + \theta_1)}$

Multiple and Submultiple Angles

If A is an angle then 2A, 3A, . . . are called multiple angle of A and the angle A/2, A/3, . . . are called sub multiple of angles.

Multiple angle is associated with all common trigonometric ratios. It is useful in solving trigonometric equation.





Double angle:

Double angle identities are a special case of sum identities when two angles are equal, the sum identities are reduced to double angle identities.

We know that, $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Taking $B = A$

$\sin(A+A) = \sin A \cos A + \cos A \sin A$

We get $\sin 2A = 2 \sin A \cos A$

Further Double angle identities can be used to derive the reduction identities.

Submultiple Angle :

Submultiple angle are closely related to the double angle identities. We use half angle identities when we have an angle that is half the size of a special angle.

If we put $2A = \theta$ or $A = \frac{\theta}{2}$ in the double angle identities, we get new identities in terms of angle $\frac{\theta}{2}$.

Trigonometric function of Multiple and Submultiple angles

	Double Angle Identity	Submultiple Angle Identity
sine function	$\sin 2A = 2 \sin A \cos A$ $\sin 2A = \frac{2 \tan A}{1+\tan^2 A}$	$\sin A = 2 \sin A/2 \cos A/2$ $\sin A = \frac{2 \tan A/2}{1+\tan^2 A/2}$
cosine function	$\cos 2A = \cos^2 A - \sin^2 A$ $\cos 2A = 1 - 2 \sin^2 A$ $\cos 2A = 2 \cos^2 A - 1$ $\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$	$\cos A = \cos^2 A/2 - \sin^2 A/2$ $\cos A = 1 - 2 \sin^2 A/2$ $\cos A = 2 \cos^2 A/2 - 1$ $\cos A = \frac{1-\tan^2 A/2}{1+\tan^2 A/2}$
tangent function	$\tan 2A = \frac{2 \tan A}{1-\tan^2 A}$	$\tan A = \frac{2 \tan A/2}{1-\tan^2 A/2}$

Reduction Identities

Power Reducing Identities or Reduction Identities can be derived from double angle Identities.

We know that, $\cos 2A = 2 \cos^2 A - 1$

We get, $\cos^2 A = \frac{1+\cos 2A}{2}$

Power Reducing Identities helps to write the squared trigonometric function in terms of smaller power.

sine function	$\sin^2 A = \frac{1-\cos 2A}{2}$	$\sin^2 A/2 = \frac{1-\cos A}{2}$
cosine function	$\cos^2 A = \frac{1+\cos 2A}{2}$	$\cos^2 A/2 = \frac{1+\cos A}{2}$

Example 1:

If $\sin A = \frac{1}{2}$ find the value of $\cos 2A$

Solution:

$$\begin{aligned}\cos 2A &= 1 - 2\sin^2 A\\ &= 1 - 2 \times \frac{1}{4}\\ &= 1 - \frac{1}{2}\\ &= \frac{1}{2}\end{aligned}$$

Example 2:

Simplify : (i) $2 \sin 30^o \cos 30^o$

(ii) $\frac{2\tan 30^o}{1-\tan^2 30^o}$

Solution:

(i) $2\sin 30^o \cos 30^o = \sin 2(30^o)$

$$\begin{aligned}&= \sin 60^o\\ &= \frac{\sqrt{3}}{2}\end{aligned}$$

(ii) $\frac{2\tan 30^o}{1-\tan^2 30^o} = \tan 2(30^o)$

$$\begin{aligned}&= \tan 60^o\\ &= \sqrt{3}\end{aligned}$$

Example 3:

Prove that $\frac{\sin A}{1+\cos A} = \tan A/2$

Solution:

LHS, $\frac{\sin A}{1+\cos A} = \frac{2\sin A/2 \cos A/2}{2\cos^2 A/2}$

$$\begin{aligned}&= \frac{\sin A/2}{\cos A/2}\\ &= \tan A/2\end{aligned}$$

Example 4:

If $\sin\theta = x$ for $\frac{-\pi}{2} \leq 0 \leq \frac{\pi}{2}$ Find an expression for $\sin 2\theta$ in terms of x.

Solution:

Given that,

$$\sin\theta = x$$

$$\cos\theta = \sqrt{1-\sin^2\theta}$$

$$= \sqrt{1-x^2}$$

$$\sin 2\theta = 2\cos\theta\sin\theta$$

$$= 2x\sqrt{1-x^2}$$

Example 5:

The horizontal distance travelled by a Football is given by $x = \frac{1}{32}\upsilon^2 \sin 2\theta$. The football kicked from the ground with an initial speed of 80 feet per seconds. In what angle, will the ball kick, to make 200 feet travel.

Solution:

Given that, $x = \frac{1}{32}\upsilon^2 \sin 2\theta$

Here, $x = 200$, $\upsilon = 80$

$$200 = \frac{1}{32}(80)^2 \sin 2\theta$$

$$1 = \sin 2\theta$$

$$90^\circ = 2\theta$$

$$\theta = 45^\circ$$

To make 200 feet travel, ball will be kicked at an angle of 45°.

Example 6:

Prove that $\frac{1-\cos A+\sin A}{1+\cos A+\sin A} = \tan A/2$

Solution:

LHS, $$\frac{1-\cos A+\sin A}{1+\cos A+\sin A} = \frac{2\sin^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$= \frac{2\sin\frac{A}{2}\left[\sin\frac{A}{2} + \cos\frac{A}{2}\right]}{2\cos\frac{A}{2}\left[\cos\frac{A}{2} + \sin\frac{A}{2}\right]}$$

$$= \tan\frac{A}{2} = \text{RHS}$$

Hence proved

Example 7:

If $\tan\alpha = \frac{1}{3}$, $\tan\beta = \frac{1}{7}$ show that $2\alpha + \beta = \frac{\pi}{4}$

Solution:

$$\tan(2\alpha+\beta) = \frac{\tan 2\alpha + \tan\beta}{1-\tan 2\alpha\tan\beta}$$

Now, $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

$$= \frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}$$

$$= \frac{\frac{2}{3}}{1-\frac{1}{9}}$$

$$\tan 2\alpha = \frac{3}{4}$$

$$\tan(2\alpha+\beta) = \frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}$$

$$= \frac{\frac{21+4}{28}}{1-\frac{3}{28}}$$

$$= \frac{25/28}{25/28} = 1$$

$$\therefore\ 2\alpha+\beta = \frac{\pi}{4}$$

Hence proved.

Example 8:

Prove that $\frac{\sin 2A}{1-\cos 2A} = \cot A$. Hence deduce the value of $\cot 22\frac{1^o}{2}$

Solution:

LHS $\frac{\sin 2A}{1-\cos 2A} = \frac{2\sin A\cos A}{2\sin^2 A}$

$$= \frac{\cos A}{\sin A}$$

$$= \cot A = \text{RHS}$$

We have $\cot A = \frac{\sin 2A}{1-\cos 2A}$

Put $A = 22\frac{1^o}{2}$,

$$\cot 22\frac{1^o}{2} = \frac{\sin 2\left(22\frac{1^o}{2}\right)}{1-\cos 2\left(22\frac{1^o}{2}\right)}$$

$$= \frac{\sin 45^o}{1-\cos 45^o}$$

$$= \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}$$

$$= \frac{1}{\sqrt{2}-1} = \sqrt{2}+1$$

Example 9:

Prove that $\sin^2 A + \sin^2(60^o+A) + \sin^2(60^o-A) = \frac{3}{2}$

Solution:

LHS $\sin^2 A + \sin^2(60^o+A) + \sin^2(60^o-A)$

$$= \frac{1-\cos 2A}{2} + \frac{1-\cos(120^o - 2A)}{2} + \frac{1-\cos(120^o + 2A)}{2}$$

$$= \frac{3}{2} - \frac{1}{2}\left[\cos 2A + \cos(120^o - 2A) + \cos(120^o + 2A)\right]$$

$$= \frac{3}{2} - \frac{1}{2}\left[\cos 2A + 2\cos 120^o \cos 2A\right]$$

$$= \frac{3}{2} - \frac{1}{2}\left[\cos 2A + 2\left(-\frac{1}{2}\right)\cos 2A\right]$$

$$= \frac{3}{2} - \frac{1}{2}(\cos 2A - \cos 2A)$$

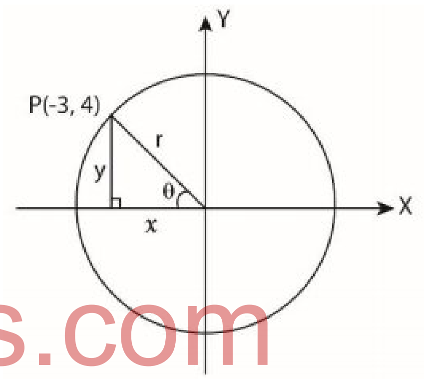
$$= \frac{3}{2} = \text{RHS}$$

∴ Hence proved.

Example 10:

Suppose P (−3, 4) lies on the terminal side of θ when θ is plotted in standard position find cos 2 θ and sin 2 θ.

Solution:



Using $x^2 + y^2 = r^2$ with (x, y) = (-3, 4)

We find $r = \sqrt{x^2 + y^2}$

$= \sqrt{(-3)^2 + (+4)^2} = 5$

Now, $\cos\theta = \frac{x}{r} = \frac{-3}{5}$

$\sin\theta = \frac{y}{r} = \frac{4}{5}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= \left(\frac{-3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$

$= \frac{-7}{25}$

$\sin 2\theta = 2\cos\theta \sin\theta$

$= 2\left(\frac{-3}{5}\right)\left(\frac{4}{5}\right)$

$= \frac{-24}{25}$

Example 11:

A spring is being moved up and down. An object is attached to the end of the spring that undergoes a vertical displacement. The displacement is given by the equation y = 3.50 sin t + 1.20 sin 2t. Find the first two values of t (in seconds) for which y = 0.

Solution:

Let y = 0

3.50 sin t + 1.20 sin2 t = 0

3.50 sin t + 2.40 sin t cos t = 0

sin t (3.50 + 2.40 cos t) = 0

sin t = 0 (or) 3.50 + 2.40 cos t = 0

$$2.40 \cos t = -3.50$$

$$\cos t = -1.46$$

(no solution because $-1 \leq \cos t \leq 1$

$\therefore \sin t = 0 \Rightarrow t = 0, \pi$

EXERCISE

1) If $\cos A = \frac{1}{2}$ find the value of sin 2A.

2) Find the value of tan 2A if $\tan A = \frac{1}{2}$

3) Simplify : (i) $\frac{2 \tan 15^o}{1+\tan^2 15^o}$

(ii) $\cos^2 15^o - \sin^2 15^o$

(iii) $2 \sin 75^o \cos 75^o$

4) Prove that $\frac{1+\sin 2A - \cos 2A}{1+\sin 2A + \cos 2A} = \tan A$

5) Prove that $\frac{\sin 2A}{1+\cos 2A} = \tan A$ and hence deduce the value of $\tan 22\frac{1^o}{2}$

6) Prove that $\frac{\sin\frac{A}{2} + \sin A}{1 + \cos\frac{A}{2} + \cos A} = \tan\frac{A}{2}$

7) The horizontal distance travelled by a football is given by $x = \frac{1}{32} v^2 \sin 2\theta$. The football kicked from the ground with an initial speed of 50 feet per second. Find the distance travelled by the football at an angle of 40^o.

8) (i) If $\sin\theta = \frac{x}{2}$ for $\frac{-\pi}{2} < 0 < \frac{\pi}{2}$ Find an expression for $\cos 2\theta$ in terms of x.

(ii) If $\tan\theta = \frac{x}{7}$ for $\frac{-\pi}{2} < 0 < \frac{\pi}{2}$ Find an expression for $\sin 2\theta$ in terms of x.

9) Prove that $\cos^6 A + \sin^6 A = 1 - \frac{3}{4}\sin^2 A$

10) Prove that (i) $(\sin A + \cos A)^2 = 1 + \sin 2A$

(ii) $\cot A - \tan A = 2 \cot 2A$

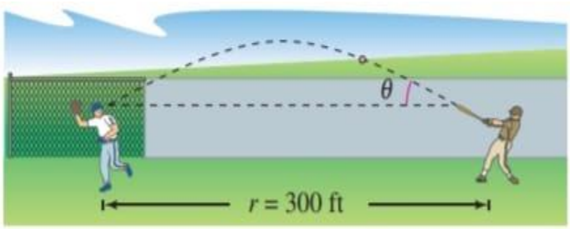
11) Prove that $\cos^2 A + \cos^2(60^o - A) + \cos^2(60^o + A) = \frac{3}{2}$

12) If $\tan\theta = \frac{a}{b}$ find the value of $a \sin 2\theta + b \cos 2\theta$.

13) Prove that $\cos 4A = 8 \sin^4 A - 8 \sin^2 A + 1$

14) Prove that $\frac{\sin 2\theta + \sin\theta}{1+\cos 2\theta + \cos\theta} = \tan\theta$.

15) A Baseball is hit the horizontal (at an angle of θ) with an initial velocity v_o = 100 feet per second. An outfielder catches the ball 300 feet from home plate. Find θ when the range r of a projectile is given by $r = \frac{1}{32} v_0^2 \sin 2\theta$.



Triple Angle Identities

Trigonometric Triple angle identities give a relationship between the basic trigonometric functions applied to three times an angle in terms of trigonometric function of the angle itself.

By combining the sum formula and the double angle formula, formulae for triple angles and more can be found.

Consider $\sin 3A = \sin(2A + A)$

$= \sin 2A \cos A + \sin A \cos 2A$

$= (2 \sin A \cos A) \cos A + \sin A (1 - 2 \sin^2 A)$

$= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A$

$= 3 \sin A - 4 \sin^3 A$

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

Triple angle identities are used, to expand or simplify the triple angle trigonometric functions.

Example 1

If $\sin A = \frac{3}{5}$ Find the value of $\sin 3A$

Solution:

Given $\sin A = \frac{3}{5}$

$\sin 3A = 3 \sin A - 4 \sin^3 A$

$= 3\left(\frac{3}{5}\right) - 4\left(\frac{3}{5}\right)^3$

$= \frac{9}{5} - 4\left(\frac{27}{125}\right)$

$= \frac{9}{5} - \frac{108}{125}$

$= \frac{117}{125}$

Example 2

Simplify : (i) $4 \cos^3 20^o - 3 \cos 20^o$

(ii) $\frac{3 \tan 15^o - \tan^3 15^o}{1 - 3 \tan^2 15^o}$

Solution:

(i) $4 \cos^3 20^o - 3 \cos 20^o = \cos 3(20^o)$

$= \cos 60^o$

$= \frac{1}{2}$

(ii) $\frac{3 \tan 15^o - \tan^3 15^o}{1 - 3 \tan^2 15^o} = \tan 3(15^o)$

$= \tan 45^o$

$= 1$

Example 3

Prove that $\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = 2$

Solution:

LHS $\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A}$

$= \frac{3\sin A - 4\sin^3 A}{\sin A} - \frac{4\cos^3 A - 3\cos A}{\cos A}$

$= 3 - 4\sin^2 A - 4\cos^2 A + 3$

$= 6 - 4$

$= 2 \Rightarrow$ RHS

Hence proved.

Example 4

Show that $\cos 3\alpha = \sin 2\alpha$ if $\sin \alpha = 0.3090$

Solution:

Given, $\sin \alpha = 0.3090$

$\cos \alpha = \sqrt{1 - \sin^2 \alpha}$

$= \sqrt{1 - (0.3090)^2}$

$= 0.9511$

LHS $\cos 3\alpha = 4\cos^3 \alpha - 3\cos \alpha$

$\cos 3\alpha = 4(0.9511)^3 - 3(0.9511)$

$= 0.588 \rightarrow$ ①

RHS, $\sin 2\alpha = 2\sin \alpha \cos \alpha$

$= 2(0.3090)(0.9511)$

$= 0.588 \rightarrow$ ②

From ① & ②

LHS = RHS

Hence proved.

Example 5

Prove that $\frac{\cos^3 A - \cos 3A}{\cos A} + \frac{\sin^3 A + \sin 3A}{\sin A} = 3$

Solution:

LHS, $= \frac{\cos^3 A - \cos 3A}{\cos A} + \frac{\sin^3 A + \sin 3A}{\sin A}$

$= \frac{\cos^3 A - 4\cos^3 A + 3\cos A}{\cos A} + \frac{\sin^3 A + 3\sin A - 4\sin^3 A}{\sin A}$

$= \frac{3\cos A - 3\cos^3 A}{\cos A} + \frac{3\sin A - 3\sin^3 A}{\sin A}$

$$= 3(1 - \cos^2 A) + 3(1 - \sin^2 A)$$

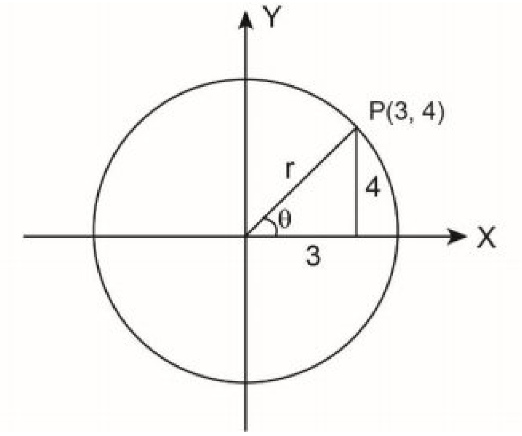
$$= 3\sin^2 A + 3\cos^2 A$$

$$= 3 \Rightarrow \text{RHS}$$

Hence proved.

Example 6

Suppose P(3,4) lies on the terminal side of θ when θ is plotted in standard position. Find the value of sin 3θ.



Solution:

Using,

$x^2 + y^2 = r^2$ with (3,4)

$$r = \sqrt{x^2 + y^2}$$

$$= \sqrt{(3)^2 + (4)^2}$$

$$= 5$$

$$\sin\theta = \frac{y}{r} = \frac{4}{5}$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$= 3\left(\frac{4}{5}\right) - 4\left(\frac{4}{5}\right)^3$$

$$= \frac{12}{5} - \frac{256}{125}$$

$$= \frac{44}{125}$$

Example 7

Prove that $\cos 10^o \cos 30^o \cos 50^o \cos 70^o = \frac{3}{16}$

Solution:

LHS $\cos 10^o \cos 30^o \cos 50^o \cos 70^o$

$$= \frac{\sqrt{3}}{2} \cos 10^o \cos(60^o - 10^o) \cos(60^o + 10^o)$$

$$= \frac{\sqrt{3}}{2} \cos 10^o \left[\cos^2 60^o - \sin^2 10^o\right]$$

$$= \frac{\sqrt{3}}{2} \cos 10^o \left[\frac{1}{4} - 1 + \cos^2 10^o\right]$$

$$= \frac{\sqrt{3}}{2} \cos 10^o \left[\frac{-3 + 4\cos^2 10^o}{4}\right]$$

$$= \frac{\sqrt{3}}{8} \left[4\cos^3 10^o - 3\cos 10^o\right]$$

$$= \frac{\sqrt{3}}{8} \cos 3(10^o)$$

$$= \frac{\sqrt{3}}{8} \cos 30^o$$

$$= \frac{\sqrt{3}}{8} \cdot \frac{\sqrt{3}}{2}$$

$= \frac{3}{16} \Rightarrow$ RHS

Hence proved

Example 8

Prove that $(1 + 2\cos A)^2 = \frac{1-\cos 3A}{1-\cos A}$

Solution:

RHS, $\frac{1-\cos 3A}{1-\cos A} = \frac{1-(4\cos^3 A-3\cos A)}{1-\cos A}$

$= \frac{1-4\cos^3 A+3\cos A)}{1-\cos A}$

$= \frac{1-4\cos^3 A+4\cos A-\cos A}{1-\cos A}$

$= \frac{1-\cos A+4\cos A\,(1-\cos A)\,(1+\cos A)}{1-\cos A}$

$=$ 1+4 cos A (1+cosA)

$=$ 1+4 cos A + 4 $\cos^2$ A

$= (1 + 2\cos A)^2$

$\Rightarrow$ LHS

Hence proved.

Example 9

Prove that $\sin A = \frac{\sin 3A}{1+2\cos 2A}$ and deduce the value of sin 15°.

Solution:

RHS, $\frac{\sin 3A}{1+2\cos 2A} = \frac{3\sin A - 4\sin^3 A}{1+2(1-2\sin^2 A)}$

$= \frac{\sin A\,(3-4\sin^2 A)}{3-4\sin^2 A}$

$=$ sin A

$\Rightarrow$ LHS

Hence proved.

$\sin A = \frac{\sin 3A}{1+2\cos 2A}$

Put A = 15°

$\sin 15^o = \frac{\sin 3\,(15^o)}{1+2\cos 2(15^o)}$

$= \frac{\sin 45^o}{1+2\cos 30^o}$

$= \frac{\frac{1}{\sqrt{2}}}{1+2\frac{\sqrt{3}}{2}}$

$\sin 15^o = \frac{\sqrt{3}-1}{2\sqrt{2}}$

EXERCISE

1) If $\cos A = \frac{4}{5}$ find the value of cos 3A.

2) Simplify : (i) $3 \sin 40^o - 4 \sin^3 40^o$

(ii) $4 \cos^3 10^o - 3 \cos 10^o$

3) Prove that $\frac{\cos 3A}{\cos A} + \frac{\sin 3A}{\sin A} = 4 \cos 2A$

4) Suppose P(−2, 5) lies on the terminal side of θ when θ is plotted in standard position. Find the value of cos 3θ.

5) Prove that $\frac{\cos 3A}{2 \cos 2A - 1} = \cos A$ and deduce the value of $\cos 15^o$.

6) Prove that $\frac{\sin 3A - \sin^3 A}{\cos^3 A - \cos 3A} = \cot A$.

7) Show that $\sin 10^o \sin 30^o \sin 50^o \sin 70^o = \frac{1}{16}$

8) Prove that $\tan 20^o \tan 40^o \tan 80^o = \sqrt{3}$

9) Prove that $\sin 20^o \sin 40^o \sin 80^o = \frac{\sqrt{3}}{8}$

Sum and Product Identities

Some application of trigonometric function demand that a product of trigonometric function to be written as sum or difference of trigonometric function.

The process of converting sum into products (or) products into sum can make an easy solution to a problem. Two sets of identities sum-product & product-sum can be derived from compound angle identities

Product-sum Identities

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta \rightarrow ①$$

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta \rightarrow ②$$

Adding ① & ②

$$\sin(\alpha+\beta) + \sin(\alpha - \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta + \sin\alpha \cos\beta - \cos\alpha \sin\beta$$

$$= 2 \sin\alpha \cos\beta$$

$$\frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)] = \sin\alpha \cos\beta$$

$$\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

$$\cos\alpha \sin\beta = \frac{1}{2}[\sin(\alpha + \beta) - \sin(\alpha - \beta)]$$

$$\sin\alpha \sin\beta = \frac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos\alpha \cos\beta = \frac{1}{2}[\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

Product-sum Identities

- Transform the product of sine and cosine functions into sum whenever need arises.
- These identities are valid for degree or radian measure whenever both sides of the identity are defined

Sum-Product Identity

To get sum to product identities, Let us introduce the substitutions $\alpha+\beta = C$ & $\alpha-\beta = D$ (or) equivalently $\alpha = \frac{C+D}{2}$, $\beta = \frac{C-D}{2}$ in the product to sum identities.

We get,

$$\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$$

$$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

Example 1:

Express each of the following product as sum (or) difference.

(i) $\sin 40^o \cos 30^o$ (ii) $\sin\frac{x}{2}\cos\frac{3x}{2}$

Solution:

(i) Since, $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

Here, $\alpha = 40^o$ & $\beta = 30^o$

Now,

$$2\sin 40^o \cos 30^o = \sin(40^o + 30^o) + \sin(40^o - 30^o)$$

$$= \sin(70^o) + \sin(10^o)$$

$$\sin 40^o \cos 30^o = \frac{1}{2}[\sin 70^o + \sin 10^o]$$

(ii) $\sin\frac{x}{2}\cos\frac{3x}{2}$

Since, $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

Here, $\alpha = \frac{x}{2}$ & $\beta = \frac{3x}{2}$

Now,

$$2\sin\frac{x}{2}\cos\frac{3x}{2} = \sin\left(\frac{x}{2}+\frac{3x}{2}\right) + \sin\left(\frac{x}{2}-\frac{3x}{2}\right)$$

$$= \frac{1}{2}[\sin 2x - \sin x]$$

Example 2:

Express each of the following as product

(i) $\sin 75^o - \sin 35^o$ (ii) $\cos 65^o + \cos 15^o$

Solution:

(i) Since, $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

Here, $C = 75^o$, $D = 35^o$

Now, $\sin 75^o - \sin 35^o = 2\cos\left(\frac{75^o+35^o}{2}\right)\sin\left(\frac{75^o-35^o}{2}\right)$

$= 2\cos 55^o \sin 20^o$

(ii) $\cos 65^o + \cos 15^o$

Since, $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

Here, $C = 65^o$, $D = 15^o$

Now, $\cos 65^o + \cos 15^o = 2\cos\left(\frac{65^o+15^o}{2}\right)\cos\left(\frac{65^o-15^o}{2}\right)$

$= 2\cos 40^o \cos 25^o$

Example 3:

Prove that $\cos 20^o + \cos 70^o = \sqrt{2}\cos 25^o$

Solution:

We know that $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

Here, $C = 20^o$ & $D = 70^o$

$$\cos 20^o + \cos 70^o = 2\cos\left(\frac{20^o+70^o}{2}\right)\cos\left(\frac{20^o-70^o}{2}\right)$$

$$= 2\cos\left(\frac{90^o}{2}\right)\cos\left(\frac{-50^o}{2}\right)$$

$$= 2\cos 45^o \cos(-25^o)$$

$$= \sqrt{2}\ \sqrt{2}\frac{1}{\sqrt{2}}\cos 25^o$$

$$\cos 20^o + \cos 70^o = \sqrt{2}\cos 25^o$$

Hence proved.

Example 4:

Prove that $\sin 10^o + \sin 70^o = \sqrt{3}\sin 40^o$

Solution:

LHS

$$\sin 10^o + \sin 70^o = 2\sin\left(\frac{10^o+70^o}{2}\right)\cos\left(\frac{10^o-70^o}{2}\right)$$

$$= 2\sin\left(\frac{80^o}{2}\right)\cos\left(\frac{-60^o}{2}\right)$$

$\left[\because \sin c + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}\right]$

$$= 2\sin 40^o \cos 30^o$$

$$= 2\sin 40^o \frac{\sqrt{3}}{2}$$

$$\sin 10^o + \sin 70^o = \sqrt{3}\sin 40^o \text{ RHS}$$

Hence proved.

Example 5:

In an AC electrical circuit, the instantaneous power p(t) delivered to the entire circuit in the sinusoidal steady state at time t is gives by p(t) = v(t) i(t) where the voltage v(t) and current i(t) are given by $v(t) = V_m \cos \omega t$, $i(t) = I_m \cos(\omega t + \emptyset)$, for some constants V_m, I_m, ω and $\emptyset$. Show that the instantaneous power can be written as

$$p(t) = \frac{1}{2} V_m I_m \cos \emptyset + \frac{1}{2} V_m I_m \cos(2\omega t + \emptyset)$$

Solution:

Given

$$v(t) = V_m \cos \omega t$$

$$i(t) = I_m \cos(\omega t + \emptyset)$$

$$\begin{aligned} \text{Now,} \quad p(t) &= v(t)\, i(t) \\ &= V_m I_m \cos \omega t \cos(\omega t + \emptyset) \\ &= V_m I_m \frac{1}{2}[\cos(2\omega t + \emptyset) + \cos(-\emptyset)] \\ &= \frac{1}{2} V_m I_m \cos \emptyset + \frac{1}{2} V_m I_m \cos(2\omega t + \emptyset) \end{aligned}$$

Hence proved.

Example 6:

Prove that $\cos 52^o + \cos 68^o + \cos 172^o = 0$

Solution:

$$\begin{aligned} \cos 52^o + \cos 68^o + \cos 172^o &= 2\cos\left(\frac{52^o+68^o}{2}\right)\cos\left(\frac{52^o-68^o}{2}\right) + \cos 172^o \\ &= 2\cos 60^o \cos(-8^o) + \cos 172^o \\ &= \cos 8^o + \cos 172^o \\ &= 2\cos\left(\frac{8^o+172^o}{2}\right)\cos\left(\frac{8^o-172^o}{2}\right) \\ &= 2\cos 90^o \cos(-82^o) \\ &= 2(0)\cos 82^o \\ &= 0 \end{aligned}$$

Hence proved.

Example 7:

Prove that $\sin 40^o + \sin 20^o - \cos 10^o = 0$

Solution:

$$\begin{aligned} \sin 40^o + \sin 20^o - \cos 10^o &= 2\sin\left(\frac{40^o+20^o}{2}\right)\cos\left(\frac{40^o-20^o}{2}\right) - \cos 10^o \\ &= 2\sin 30^o \cos 10^o - \cos 10^o \\ &= 2\left(\frac{1}{2}\right)\cos 10^o - \cos 10^o \\ &= \cos 10^o - \cos 10^o \\ &= 0 \end{aligned}$$

Hence proved.

Example 8:

In the study of electronics, the function f(t) = sin (200t + π) + sin (200t − π) is used to analyze frequency. Simplify this function using the sum-to-product formula.

Solution:

Using the sum-to-product formula.

$$f(t) = \sin(200t + \pi) + \sin(200t - \pi)$$
$$= 2\sin\left(\frac{(200t+\pi)+(200t-\pi)}{2}\right)\cos\left(\frac{(200t+\pi)-(200t-\pi)}{2}\right)$$
$$= 2\sin\left(\frac{400t}{2}\right)\cos\left(\frac{2\pi}{2}\right)$$
$$= 2\sin 200t \cos\pi$$
$$= 2\sin 200t\,(-1)$$
$$f(t) = -2\sin 200t$$

Example 9:

Prove that $(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = 4\sin^2\left(\frac{\alpha-\beta}{2}\right)$

Solution:

$$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 = \left[-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right]^2 + \left[2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right]^2$$
$$= 4\sin^2\left(\frac{\alpha+\beta}{2}\right)\sin^2\left(\frac{\alpha-\beta}{2}\right) + 4\cos^2\left(\frac{\alpha+\beta}{2}\right)\sin^2\left(\frac{\alpha-\beta}{2}\right)$$
$$= 4\sin^2\left(\frac{\alpha-\beta}{2}\right)\left[\sin^2\left(\frac{\alpha+\beta}{2}\right) + \cos^2\left(\frac{\alpha+\beta}{2}\right)\right]$$
$$= 4\sin^2\left(\frac{\alpha-\beta}{2}\right)(1)$$
$$= 4\sin^2\left(\frac{\alpha-\beta}{2}\right)$$

Hence proved.

Example 10:

If a = sin α + sin β; b = cos α + cos β prove that $\sec^2\left(\frac{\alpha-\beta}{2}\right) = \left(\frac{4}{a^2+b^2}\right)$

Solution:

Given, a = sin α + sin β; b = cos α + cos β

$$\text{RHS} = \frac{4}{a^2+b^2}$$
$$= \frac{4}{(\sin\alpha + \sin\beta)^2 + (\cos\alpha + \cos\beta)^2}$$
$$= \frac{4}{\left[2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]^2 + \left[2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right]^2}$$
$$= \frac{4}{4\sin^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right) + 4\cos^2\left(\frac{\alpha+\beta}{2}\right)\cos^2\left(\frac{\alpha-\beta}{2}\right)}$$
$$= \frac{4}{4\cos^2\left(\frac{\alpha-\beta}{2}\right)\left[\sin^2\left(\frac{\alpha+\beta}{2}\right) + \cos^2\left(\frac{\alpha+\beta}{2}\right)\right]}$$

$$= \frac{4}{4\cos^2\left(\frac{\alpha-\beta}{2}\right)}$$

$$= \frac{1}{\cos^2\left(\frac{\alpha-\beta}{2}\right)}$$

$$= \sec^2\left(\frac{\alpha-\beta}{2}\right)$$

$$= \text{LHS}$$

Hence proved.

Example 11:

In the study of the propagation of electromagnetic waves, snell's law gives the relation $n_1 \sin \theta_1 = n_2 \sin \theta_2$. Show that the p-polarization transmission Fresnel coefficient defined by

$t_{12p} = \frac{2n_1 \cos\theta_1}{n_2\cos\theta_1 + n_1\cos\theta_2}$ can be written as

$t_{12p} = \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}$

Solution:

Multiply $\sin\theta_1 \sin\theta_2$ in the numerator and denominator of t_{12p}

$$t_{12p} = \frac{2n_1\cos\theta_1}{n_2\cos\theta_1 + n_1\cos\theta_2} \text{ x } \frac{\sin\theta_1 \sin\theta_2}{\sin\theta_1\sin\theta_2}$$

$$= \frac{2(n_1\sin\theta_1)\cos\theta_1\sin\theta_2}{(n_2\sin\theta_2)\sin\theta_1\cos\theta_1 + (n_1\sin\theta_1)\sin\theta_2\cos\theta_2}$$

$$= \frac{2(n_1\sin\theta_1)\cos\theta_1\sin\theta_2}{(n_1\sin\theta_1)\sin\theta_1\cos\theta_1 + (n_1\sin\theta_1)\sin\theta_2\cos\theta_2}$$

$$= \frac{2(n_1\sin\theta_1)\cos\theta_1\sin\theta_2}{n_1\sin\theta_1[\sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_2]}$$

$$= \frac{2\cos\theta_1\sin\theta_2}{\frac{1}{2}(\sin 2\theta_1 + \sin 2\theta_2)}$$

$$= \frac{2\cos\theta_1\sin\theta_2}{\frac{1}{2}\left[2\sin\left(\frac{2\theta_1+2\theta_2}{2}\right)\cos\left(\frac{2\theta_1-2\theta_2}{2}\right)\right]}$$

$$t_{12p} = \frac{2\cos\theta_1\sin\theta_2}{[\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)]}$$

EXERCISE

1) Express the following in the form of sum (or) difference.

(i) $2\sin 4A\cos 2A$ (ii) $2\sin 3A\sin A$

(iii) $2\cos 8\theta\cos 6\theta$ (iv) $\sin 4A\cos 7A$ (v) $\cos\frac{3A}{2}\sin\frac{A}{2}$

2) Express in the following as product.

(i) $\sin 70^\circ + \sin 50^\circ$ (iii) $\sin 13\theta - \sin 6\theta$

(ii) $\cos 70^\circ + \cos 70^\circ$ (iv) $\cos 13A - \cos 5A$ (v) $\cos 5\theta - \cos\theta$

3) Prove the following

(i) $\cos 10^\circ + \cos 70^\circ = \sqrt{3} \cos 40^\circ$

(ii) $\cos 55^\circ + \sin 55^\circ = \sqrt{2} \cos 10^\circ$

(iii) $\sin 50^\circ + \sin 10^\circ = \cos 20^\circ$

(iv) $\cos 35^\circ + \cos 85^\circ + \cos 155^\circ = 0$

(v) $\sin 50^\circ - \sin 70^\circ + \sin 10^\circ = 0$

(vi) $\sin 72^\circ - \sin 12^\circ - \cos 42^\circ = 0$

4) Prove that $(\sin \alpha + \sin \beta)^2 + (\cos \alpha - \cos \beta)^2 = 4 \sin^2 \left(\frac{\alpha+\beta}{2}\right)$

5) Prove that $(\cos \alpha + \cos \beta)^2 + (\sin \alpha + \sin \beta)^2 = 4 \cos^2 \left(\frac{\alpha-\beta}{2}\right)$

6) Prove that $(\cos \alpha + \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 = 4 \cos^2 \left(\frac{\alpha+\beta}{2}\right)$

7) If $a = \sin \alpha + \sin \beta$, $b = \cos \alpha + \cos \beta$ prove that

$\tan^2 \left(\frac{\alpha-\beta}{2}\right) = \frac{4-a^2-b^2}{a^2+b^2}$

8) prove the following

(i) $\frac{\sin 2A - \sin 2B}{\cos 2A + \cos 2B} = \tan (A-B)$

(ii) $\frac{\sin A + \sin 2A}{\cos A - \cos 2A} = \cot \frac{A}{2}$

9) If $a = \sin \alpha + \sin \beta$; $b = \cos \alpha + \cos \beta$. Prove that $\sin (\alpha + \beta) = \frac{2ab}{a^2+b^2}$

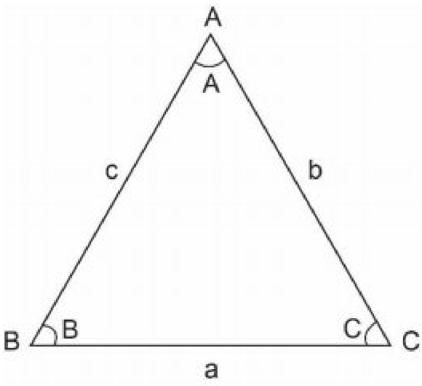
10) Prove the following

(i) $\frac{\sin 3A - \sin A}{\cos A - \cos 3A} = \cot 2A$

(ii) $\frac{\sin A + \sin 2A + \sin 3A}{\cos A + \cos 2A + \cos 3A} = \tan 2A$

Chapter 3.3 PROPERTIES OF TRIANGLE & INVERSE TRIGONOMETRIC FUNCTIONS

One important aspect of trigonometry is to solve practical problems that can be modeled by a triangle. Determination of all sides and angles of a triangles is referred as solving the triangle.



Main objective is to develop theorems which allows us to solve triangle.

The angles corresponding to the vertices A, B, C are denoted by A, B, C themselves. The sides opposite to the angles A, B, C are denoted by a, b, c respectively.

The symbol Δ is used to denote the area of a triangle.

Law of sine

In any Δ ABC, the length of the sides are proportional to the sines of the opposite angles.

i.e. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

Where R is the circumradius of the triangle.

It can be written as

✷ $\frac{a}{b} = \frac{\sin A}{\sin B}, \frac{a}{c} = \frac{\sin A}{\sin C}, \frac{b}{c} = \frac{\sin B}{\sin C}$

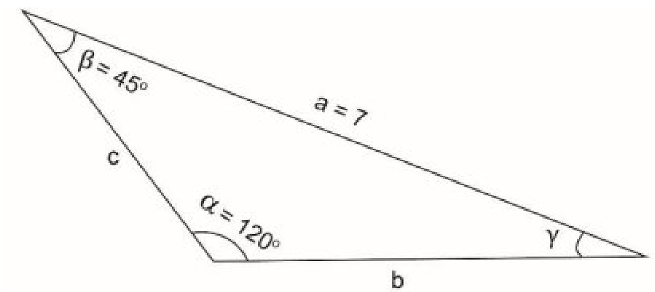
Law of sine used

✷ To find an angle if two sides and one angle which is not included, by them are given.

✷ To find a side if two angles and one side which is opposite to one of the given angles.

Example 1:

Find the area of the triangle in which $\alpha = 120^o$, a = 7 units and $\beta = 45^o$.



Solution:

Given,

$\alpha = 120^o$, a = 7 and $\beta = 45^o$

$\gamma = 180^o - (120^o + 45^o)$

$\gamma = 15^o$

Law of sine,

$$\frac{C}{\sin 15^o} = \frac{7}{\sin 120^o}$$

$$C = \frac{7 \sin 15^o}{\sin 120^o}$$

$$C = 2.09 \text{ units}$$

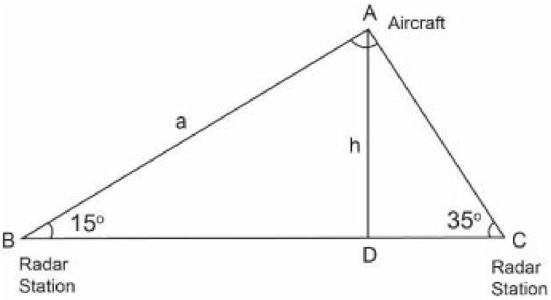
$$A = \frac{1}{2} ac \sin \beta$$

$$= \frac{1}{2} (7)(2.09) \sin 45^o$$

$$A = 5.18 \text{ square units.}$$

Example 2:

Suppose two radar stations located 20 miles apart each detect an aircraft between them. The angle of elevation measured by the first station in 15 degrees, where as the angle of elevation measured by the second station is 35 degrees. Find the altitude of the aircraft.



Solution:

$\angle A = 180^o - (15^o + 35^o)$

$= 130^o$

let a be the distance of aircraft from first station.

Apply,

Law of sines,

$$\frac{a}{\sin 35^o} = \frac{20}{\sin 130^o}$$

$$a \sin 130^o = 20 \sin 35^o$$

$$a = \frac{20 \sin 35^o}{\sin 130^o}$$

$$a = 14.98$$

consider Δ ABD,

$$\sin 15^o = \frac{h}{a}$$

$$h = a \sin 15^o$$

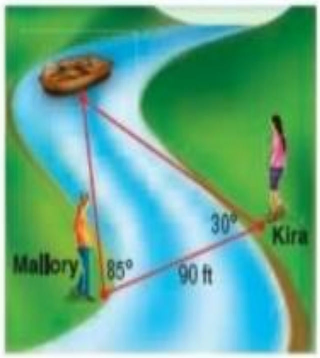
$$= 14.98 \sin 15^o$$

$$h = 3.88$$

The aircraft is at an altitude of 3.88 miles.

Example 3:

Boat kira and Mallory are standing on opposite sides of a river. How far is kira from the boat?



Solution:

Measure of angle at boat, A is

$= 180^o - (85^o + 30^o)$

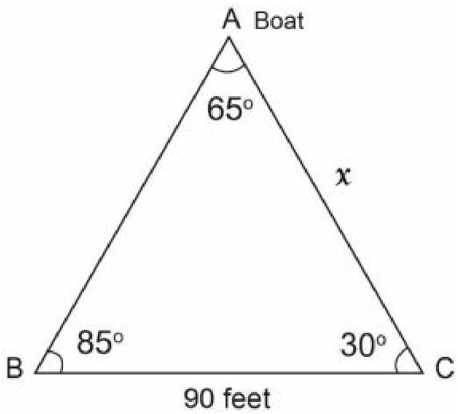
$= 65^o$

Let x be distance between kira and the boat use the law of sines to find x.

$$\frac{\sin 85^o}{x} = \frac{\sin 65^o}{90}$$

$$x = \frac{90 \sin 85^o}{\sin 65^o}$$

$$x = 98.9 \text{ ft.}$$



Law of cosine

In Δ ABC,

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos B = \frac{c^2+a^2-b^2}{2ca}$$

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

Law of cosines used,

When two sides and included angle or the three sides of a triangle are given.

Results :

In Δ ABC,

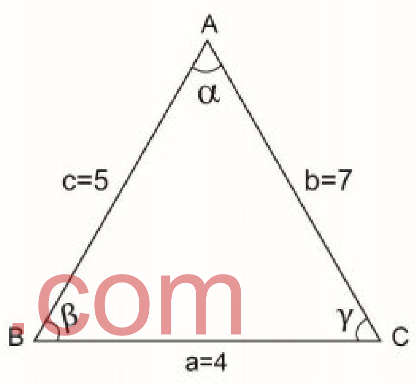
$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = c^2 + a^2 - 2ca \cos B$$

Example 4:

Solve for the angles in the triangle with sides of length a = 4 units, b = 7 units and c = 5 units.



Solution:

Since all three sides and no angles are given. We are forced to use the law of cosines,

We find β first, since its opposite the longest side b.

$$\cos \beta = \frac{a^2+c^2-b^2}{2ac}$$

$$= \frac{(4)^2+(5)^2-(7)^2}{2(4)\,(5)}$$

$$= \frac{-1}{5}$$

$$\beta = \cos^{-1}\left(\frac{-1}{5}\right)$$

$$\beta = 101.54^o$$

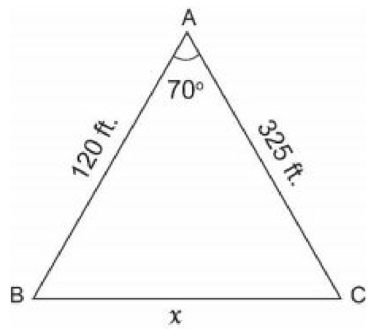
Similarly, we get $\alpha = 34.05^o$

$\gamma = 44.42^o$

Example 5:

A farmer wants to fence a piece of his land. Two sides of the triangular field have length of 120 feet and 325 feet. The measure of the angle between those sides is 70°. How much fencing will be farmer need.

Solution:



The triangle should be solved by the law of cosines

$\angle A = 70^o$, b = 120, c = 325

Let x be the unknown side length

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$x^2 = 120^2 + 325^2 - 2(120)\,(325) \cos 70^o$$

$$x^2 = 93347.429$$
$$x = 305.5 \text{ ft.}$$

The length of the fence needed $= 305.5 + 120 + 325$

$= 750.5$ ft.

Projection formula

In a Δ ABC, a side of triangle is equal to sum of the projection of other two sides on it.

$$a = b \cos C + c \cos B$$
$$b = c \cos A + a \cos C$$
$$c = a \cos B + b \cos A$$

Area of the triangle

Area of an oblique triangle is equal to one half of the product of two sides and the sine of their included angle.

In Δ ABC, Area of the triangle is

i.e. $\Delta = \frac{1}{2} ab \sin C$

$$\Delta = \frac{1}{2} bc \sin A$$
$$\Delta = \frac{1}{2} ca \sin B$$

Half angle formula

In ΔABC,

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$
$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$
$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Where s is the semi perimeter of ΔABC given by $s = \frac{a+b+c)}{2}$

Area of triangle (Heron's formula)

In ΔABC, Area of a triangle is

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

Where s is the semi-perimeter of ΔABC.

This formula is used only when all the three sides are known.

Example 6:

In a ΔABC, prove that

$$b^2 \sin 2C + c^2 \sin 2B = 2 bc \sin A$$

Solution:

The sine formula is $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

Thus a = 2R sin A, b = 2R sin B, C = 2R sin C

LHS,

$$b^2 \sin 2C + c^2 \sin 2B = 4R^2 \sin^2 B \sin 2C + 4R^2 \sin^2 C \sin 2B$$

$$= 4R^2 (2 \sin^2 B \sin C \cos C + 2 \sin^2 C \sin B \cos B)$$

$$= 8R^2 \sin B \sin C (\sin B \cos C + \sin C \cos B)$$

$$= 8R^2 \sin B \sin C \sin (B + C)$$

$$= 8R^2 \sin B \sin C \sin (\pi - A)$$

$$= 8R^2 \sin B \sin C \sin A$$

$$= 8R^2 \left(\frac{b}{2R}\right)\left(\frac{c}{2R}\right) \sin A$$

$$= 2bc \sin A \Rightarrow \text{RHS}$$

Hence proved.

Example 7:

Derive cosine formula using the law of sine in a ΔABC.

Solution:

The law of sine : $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

$$\frac{b^2+c^2-a^2}{2bc} = \frac{(2R \sin B)^2 + (2R \sin C)^2 - (2R \sin A)^2}{2(2R \sin B)(2R \sin C)}$$

$$= \frac{\sin^2 B + \sin(C+A)\sin(C-A)}{2 \sin B \sin C}$$

$$= \frac{\sin B [\sin B + \sin(C-A)]}{2 \sin B \sin C}$$

$$= \frac{\sin(C+A) + \sin(C-A)}{2 \sin C}$$

$$= \frac{2 \sin C \cos A}{2 \sin C} = \cos A$$

Thus, $\cos A = \frac{b^2+c^2-a^2}{2bc}$

Similarly we can derive other two cosine formulas.

Example 8:

In a ΔABC, prove that $\sin\left(\frac{B-C}{2}\right) = \frac{b-c}{2} \cos \frac{A}{2}$

Solution:

The sine formula is $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

RHS,

$$\frac{b-c}{a} \cos \frac{A}{2} = \frac{2R \sin B - 2R \sin C}{2R \sin A} \cos \frac{A}{2}$$

$$= \frac{2 \sin\left(\frac{B-C}{2}\right) \cos\left(\frac{B+C}{2}\right)}{2 \sin \frac{A}{2} \cos \frac{A}{2}} \cos \frac{A}{2}$$

$$= \frac{\sin\left(\frac{B-C}{2}\right)\cos\left(90^o - \frac{A}{2}\right)}{\sin\frac{A}{2}}$$

$$= \frac{\sin\left(\frac{B-C}{2}\right)\sin\frac{A}{2}}{\sin\frac{A}{2}}$$

$$= \sin\left(\frac{B-C}{2}\right)$$

$\Rightarrow$ LHS

Hence proved.

Example 9:

In a ΔABC, prove that $\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$

Solution:

We know the sine formula

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

RHS,

$$\frac{a-b}{a+b}\cot\frac{C}{2} = \frac{2R\sin A - 2R\sin B}{2R\sin A + 2R\sin B}\cot\frac{C}{2}$$

$$= \frac{\sin A - \sin B}{\sin A + \sin B}\cot\frac{C}{2}$$

$$= \frac{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)}{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}\cot\frac{C}{2}$$

$$= \cot\frac{A+B}{2}\tan\frac{A-B}{2}\cot\frac{C}{2}$$

$$= \cot\left(90^o - \frac{C}{2}\right)\tan\frac{A-B}{2}\cot\frac{C}{2}$$

$$= \tan\frac{C}{2}\tan\frac{A-B}{2}\cot\frac{C}{2}$$

$$= \tan\frac{A-B}{2} \Rightarrow \text{LHS}$$

Hence proved.

EXERCISE

1) If the three angles in a triangle are in the ratio 1:2:3 then prove that the corresponding sides are in the ratio 1: $\sqrt{3}$: 2.

2) In a ΔABC, prove that $(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c$

3) In a triangle ABC, prove that

$$\frac{a^2+b^2}{a^2+c^2} = \frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}$$

4) In a ΔABC, prove that $\frac{\sin B}{\sin C} = \frac{c-a\cos B}{b-a\cos C}$

5) In a ΔABC, prove that $a\cos A + b\cos B + c\cos C = 2a\sin B\sin C$

6) In a ΔABC, $\angle A = 60^o$. Prove that

$$b+c = 2a\cos\left(\frac{B-C}{2}\right)$$

7) In a ΔABC, prove that $a(\cos B + \cos C) = 2(b+c)\sin^2\frac{A}{2}$

8) In a ΔABC, prove that $\frac{a^2-c^2}{b^2} = \frac{\sin(A-C)}{\sin(A+C)}$

9) In a ΔABC, prove that $\frac{a+b}{a-b} = \tan\left(\frac{A+B}{2}\right)\cot\left(\frac{A-B}{2}\right)$

10) A rope of length 12m is given. Find the largest area of the triangle formed by this rope and find the dimensions of the triangle so formed.

Application of Triangle

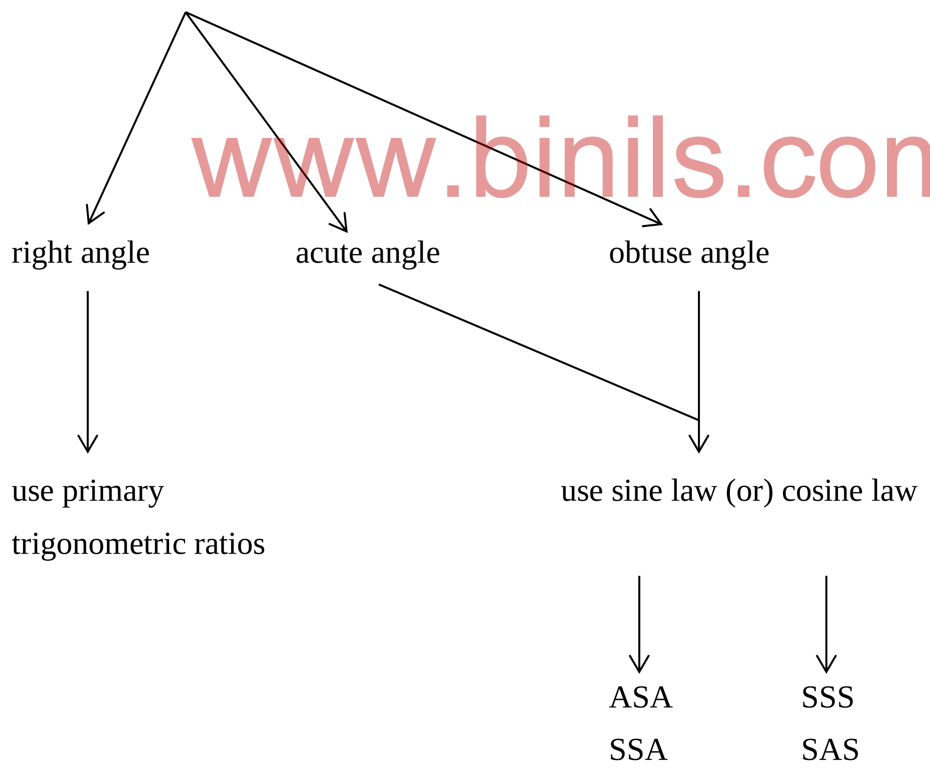
A triangle is a polygon with three edges and three vertices. It is one of the basic shape in Geometry.

Throughout its early development, trigonometry was often used as a means of indirect measurement, eg. determining large distance or length by using measurements of angles and small, known distances. Today trigonometry is widely used in various fields of mathematics and other disciplines.

In this section we will see some of the ways in which trigonometry can be applied

Key ideas

When solving problems involving trigonometry, try to follow a problem – solving model.



* If both the sine law and cosine law can be used to solve a triangle, use the sine law, since it is the easier method.

* When solving a problem in three dimensions. It is helpful to separate the information into horizontal and vertical triangles.

Example 1

In a ΔABC, if a = 12 cm, b = 8 cm, $\angle c = 30^\circ$ then show that its area is 24 sq.cm.

Solution:

Area of $\Delta ABC = \frac{1}{2}ab \sin C$

$= \frac{1}{2} \times 12 \times 8 (\sin 30^o)$

$= 6 \times 8 \times \frac{1}{2}$

$=$ 24.sq.cm. Hence proved.

Example 2

Find the area of the triangle whose sides are 13cm, 14 cm, and 15 cm.

Solution:

Here, a = 13cm, b = 14 cm, c = 15 cm

Since $s = \frac{a+b+c}{2}$

$= \frac{13+14+15}{2}$

$s = 21$ cm.

Area of triangle $= \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{21(21-13)(21-14)(21-15)}$

$= 84$ sq.cm.

Example 3

Mark is a landscaper who is creating a triangular planting garden. The homeowner wants the garden to have two equal sides and contain an angle of 135°. Also the longest side of the garden must be exactly 5m. Determine the area of the garden.

Solution:

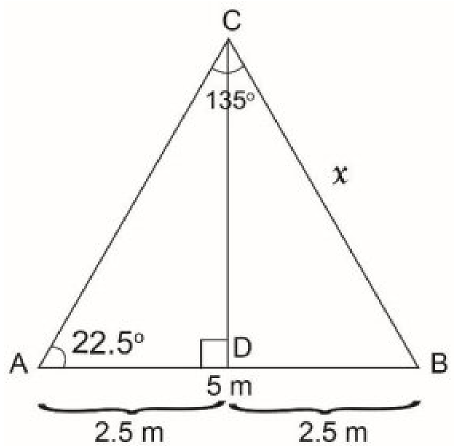
Since two sides must be equal, the triangle is isosceles.

Therefore, AC = CB

$\angle A = \angle B$

$= \frac{(180^o - 135^o)}{2}$

$= 22.5^o$



In ΔACD,

$\tan 22.5^o = \frac{CD}{AD}$

$CD = 2.5 \tan 22.5^o$

$= 2.5 \times 0.4142$

$CD = 1.03$

Area of the garden $= \frac{1}{2}(AB)(CD)$

$= \frac{1}{2} \times 5 \times 1.03$

$= 2.575$ sq.m.

Example 4

If the sides of ΔABC are a = 4, b = 6, c = 8 then show that 4 cos B + 3 cos C = 2.

Solution:

LHS $4\cos B + 3\cos C = 4\left[\frac{a^2+c^2-b^2}{2ac}\right] + 3\left[\frac{a^2+b^2-c^2}{2ab}\right]$

$= 4\left[\frac{16+64-36}{2(4)(8)}\right] + 3\left[\frac{16+36-64}{2(4)(6)}\right]$

$= 4\left[\frac{80-36}{64}\right] + 3\left[\frac{52-64}{48}\right]$

$= \frac{11}{4} - \frac{3}{4} = \frac{8}{4}$

$=$ 2 RHS

Hence proved.

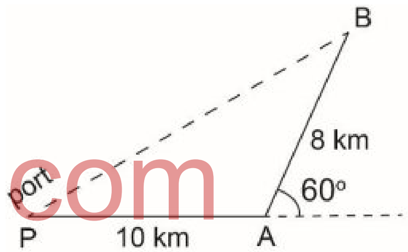
Example 5

Suppose that a boat travels 10 km from the port towards east and then turns 60^o to its left. If the boat travels further 8 km, how far from the port is the boat?

Solution:

Here, PA = 10, AB = 8 km

Let BP be the required distance, by using cosine formula



$c^2 = a^2 + b^2 - 2ab\cos c$

$BP^2 = 10^2 + 8^2 - 2 \times 10 \times 8 \times \cos 120^o$

$= 100 + 64 - 160\left(-\frac{1}{2}\right)$

$= 244$

$BP = \sqrt{244}$

$= \sqrt{61 \times 4}$

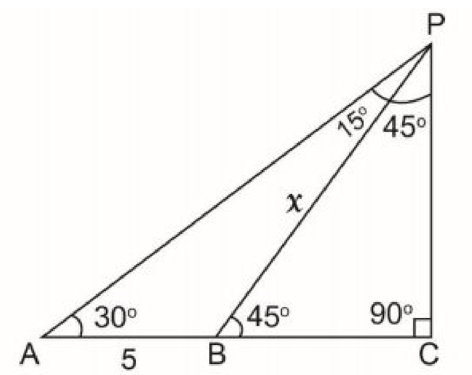
$BP = 2\sqrt{61}$ km.

Example 6

Two soldiers A and B in two different underground bunkers on a straight road, spot an intruder at the top of a hill. The angle of elevation of the intruder from A and B be the ground level in the eastern direction are 30^o and 45^o respectively. If A and B stand 5 km apart, find the distance of the intruder from B.

Solution:

Let P be the position of intruder, A & B are the position of soldiers $\angle PAB = 30^o$,
$\angle PBC = 45^o$, $\angle APB = 15^o$



By using sine formula,

$\frac{5}{\sin 15^o} = \frac{x}{\sin 30^o}$

$x = \frac{5}{\sin 15^o}\sin 30^o$

$$\frac{5}{\sin 15^o} \times \frac{1}{2}$$

$$x = \frac{5}{2 \sin 15^o} \rightarrow ①$$

$$\sin 15^o = \sin (45^o - 30^o)$$

$$= \sin 45^o \cos 30^o - \cos 45^o \sin 30^o$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \rightarrow ②$$

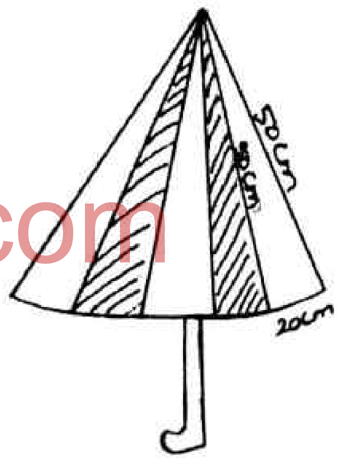
Sub ② in ① we get,

$$x = \frac{5}{\frac{2(\sqrt{3}-1)}{2\sqrt{2}}} = \frac{5\sqrt{2}}{\sqrt{3}-1}$$

Example 7

An umbrella is made by stitching 10 triangular pieces of cloth of two different colours, each piece measuring 20 cm, 50 cm and 50 cm. How much cloth of each colour is required for the umbrella?

Solution:



Given, a = 50 cm, b = 50 cm, c = 20 cm

$$\text{Semi perimeter, } S = \frac{a+b+c}{2}$$

$$= \frac{50+50+20}{2}$$

$$= \frac{120}{2}$$

$$S = 60 \text{ cm}$$

Using Heron's formula,

$$\text{Area of one triangular piece} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{60(60-50)(60-50)(60-20)}$$

$$= \sqrt{60(10)(10)(40)}$$

$$= 200\sqrt{6} \text{ cm}^2$$

$$\text{Piece of cloth of each colour required for the umbrella} = 5 \times 200\sqrt{6}$$

$$= 1000\sqrt{6} \text{ cm}^2$$

Example 8

A researcher wants to determine the width of a pond from east to west, which cannot be done by actual measurement. From a point P, he finds the distance to the eastern most point of a pond to be 8 km, while the distance to the western most point from P to be 6 km. If the angle between the two lines of sight is 60° find the width of the pond.

Solution:

Let A be the eastern point & B be the western point and PA = 8, PB = 6.

Let a = 6, b = 8 & $\angle c = 60^o$

By using cosine formula

$$c^2 = a^2 + b^2 - 2ab \cos c$$

$$= 36 + 64 - 2(6)(8) \cos 60^\circ$$

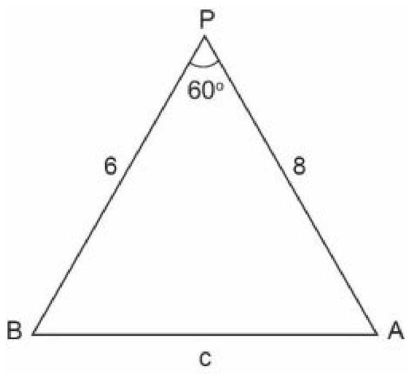
$$= 100 - 12 \times 8 \times \frac{1}{2}$$

$$c^2 = 52$$

$$c = \sqrt{52} = \sqrt{4 \times 13}$$

$$c = 2\sqrt{13} \text{ km}$$

Width of the river is $2\sqrt{13}$ km



Example 9

A straight tunnel is to be made through a mountain. A surveyor observes the two extremities A and B of the tunnel to be built from a point P in front of the mountain. If AP = 3 km, BP = 5km and ∠APB = 120° then find the length of the tunnel to be built.

Solution:

Let AB = C be the length of the tunnel.

Let a = 5 b = 3 and ∠c = 130°

By using cosine formula

$$c^2 = a^2 + b^2 - 2ab \cos c$$

$$= 25 + 9 - 2(5)(3) \cos 120^\circ$$

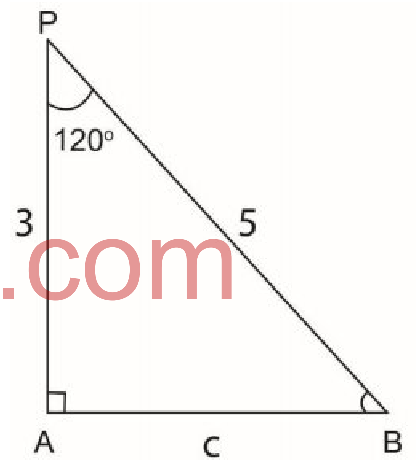
$$= 34 - 30 \cos (180^\circ - 60^\circ)$$

$$= 34 - 30 (-\cos 60^\circ)$$

$$c^2 = 49$$

$$c = 7 \text{ km}$$

∴ The length of the tunnel = 7 km



Example 10

A Rhombus shaped fields has green grass for 18 cows to graze. If each side of the rhombus is 30 m and its longer diagonal is 48m. How much area of grace field will each cow be grazing?

Solution:

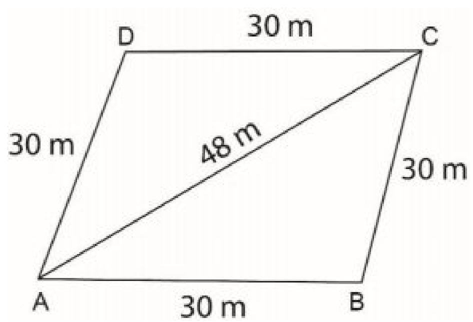
Consider ΔABC & ΔADC,

Given, a = 30 m, b = 48 m, c = 30 m

Semi perimeter, $S = \frac{a+b+c}{2}$

$$= \frac{30+48+30}{2} = \frac{108}{2}$$

$$S = 54 \text{ m}$$



Using Heron's formula,

Area of ΔABC = ΔADC $= \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{54(54-30)(54-48)(54-30)}$$

$= \sqrt{54(24)\,(6)\,(24)}$

Area of $\Delta ABC = \Delta ADC$ $= 432 \text{ m}^2$

$\therefore$ Area of Rhombus $= 432 \times 2$

$= 864 \text{ m}^2$

Area of grass field grazed by each cow $= \frac{864}{18}$

$= 48 \text{ m}^2$

EXERCISE

1) In a ΔABC, if $a = 2\sqrt{2}$, $b = 2\sqrt{3}$ and $c = 75^o$. Find the other side and the angles.
2) In a ΔABC, $a = 3$, $b = 5$, and $c = 7$. Find the values of cos A, cos B and cos C.
3) In a ΔABC if $a = 18$cm, $b = 24$cm, $c = 30$cm then show that its area is 216 sq.cm.
4) In ΔABC, $A = 30^o$, $B = 60^o$ and $C = 10^o$. Find a & b.
5) Suppose two radar stations located 100 km apart each detect a fighter aircraft between them. The angle of elevation measured by the first station is 30^o, where as the angle of elevation measured by the second station is 45^o. Find the altitude of the aircraft at the instant.
6) Two navy helicopters A and B are flying over the Bay of Bengal at same altitude from the sea level to search a missing boat, pilots of both the helicopters sight the boat at the same time while they are apart 10 km from each other. If the distance of the boat from A is 6 km. and if the line segment AB subtends 60^o at the boat. Find the distance of the boat from B.

7) A former wants to purchase a triangular shaped land with sides 120 feet and 60 feet and the angle included between these two sides is 60^o. If the land costs Rs.500 per Sq.ft. Find the amount he needed to purchase the land. Also find the perimeter of the land.

8. A triangular pack ABC has sides 120m, 80m and 50m. A gardener has to put a fence all around it and also plant grass inside. How much area does she needs to plant? Find the cost of fencing it with barbed wire at the rate of Rs.20 per meter leaving a space of 3m wide for a gate on one side.
9. A fighter jet has to hit a small target by flying a horizontal distance. When the target is sighted, the pilot measures the angle of depression to be 30^o. If after 100 km, the target has an angle of depression of 60^o, how far is the target from the fighter jet at that instant?
10. A plane is 1 km from one landmark and 2 km from another. From the planes point of view the land between them subtends an angle of 45^o. How far apart are the landmarks?

Inverse Trigonometric Function

The inverse trigonometric functions are the inverse functions of the trigonometric functions. It is used to obtain an angle from any of the angles of trigonometric ratios.

Inverse trigonometric functions are simply defined as the inverse functions of the basic trigonometric function.

$\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$ and $\text{cosec}^{-1}x$ are the inverse trigonometric functions of sine, cosine, tangent, cotangent, secant and cosecant functions respectively.

$\sin^{-1}x$ is pronounced as sine inverse of x.

- ✷ The inverse function cannot be defined if it fails to be one to one.
- ✷ The inverse trigonometric function is used to determine the angle measure with the use of basic trigonometric ratios.

MATH FACT :

$$\sin^{-1} x \neq \frac{1}{\sin x}$$

$$\cos^{-1} x \neq \frac{1}{\cos x}$$

$$\tan^{-1} x \neq \frac{1}{\tan x}$$

Defining the inverse trigonometric ratios :

$\sin\theta = \frac{y}{r} \rightarrow \sin^{-1}\frac{y}{r} = \theta$

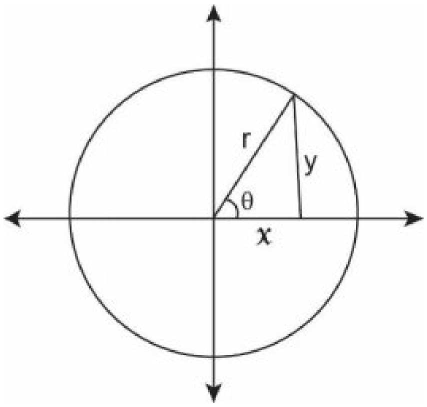
$\cos\theta = \frac{x}{r} \rightarrow \cos^{-1}\frac{x}{r} = \theta$

$\tan\theta = \frac{y}{x} \rightarrow \tan^{-1}\frac{y}{x} = \theta$

$\cot\theta = \frac{x}{y} \rightarrow \cot^{-1}\frac{x}{y} = \theta$

$\sec\theta = \frac{r}{x} \rightarrow \sec^{-1}\frac{r}{x} = \theta$

$\mathrm{cosec}\,\theta = \frac{r}{y} \rightarrow \mathrm{cosec}^{-1}\frac{r}{y} = \theta$



Domain and range of inverse trigonometric functions

S. No.	Function	Domain	Range (Principal value)
1	$y = \sin^{-1}x$	$-1 \leq x \leq 1$	$\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$
2	$y = \cos^{-1}x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
3	$y = \tan^{-1}x$	R	$\frac{-\pi}{2} < y < \frac{\pi}{2}$
4	$y = \mathrm{cosec}^{-1}x$	$x \geq 1$ (or) $x \leq -1$	$\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$
5	$y = \sec^{-1}x$	$x \geq 1$ (or) $x \leq -1$	$0 < y \leq \pi$, $y \neq \frac{\pi}{2}$
6	$y = \cot^{-1}x$	R	$0 < y < \pi$

Principal value

The principal value of an inverse function is that value of the general value, which is numerically least. It may be positive or negative. When there are two values, one is positive and the other is negative such that they are numerically equal, then the principal value is the positive one.

Example 1

Find the measure of the angle θ for the triangle shown.

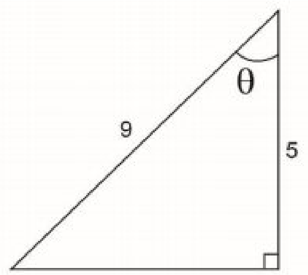
Solution:

In right angle triangle,

$\cos\theta = \frac{5}{9}$

$\theta = \cos^{-1}\frac{5}{9}$

$\theta = 56.3^{o}$



Example 2

Find the principal value of $\sin^{-1}(½)$

Solution:

Let $x = \sin^{-1}(½)$

$\sin x = ½$

$= \sin\frac{\pi}{6}$

$x = \frac{\pi}{6}$

The principal value of $\sin^{-1}(½) = \frac{\pi}{6}$

Example 3

Find the value of $\cos(\sin^{-1}(-¼))$

Solution:

Let $\theta = \sin^{-1}(-¼)$

$\sin\theta = -¼$

Note,

$\cos(\sin^{-1}(-¼)) = \cos\theta$

$= \sqrt{1-\sin^2\theta} = \sqrt{1-(-¼)^2} = \sqrt{1-\frac{1}{16}}$

$= \frac{\sqrt{15}}{4}$

Example 4

Evaluate $\cos(\tan^{-1}(¾))$

Solution:

Let $\theta = (\tan^{-1}(¾))$

$\tan\theta = ¾$

Now,

$\cos(\tan^{-1}(¾)) = \cos\theta$ $\boxed{\sec^2\theta = 1+\tan^2\theta}$

$= \frac{1}{\sec\theta}$

$= \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+(¾)^2}}$

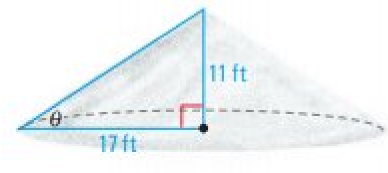
$$= \frac{1}{\sqrt{1+\frac{9}{16}}} = \frac{1}{\sqrt{\frac{25}{16}}} = \frac{1}{5/4}$$

$$= \frac{4}{5}$$

Example 5

When rock salt is stored in a cone-shaped pile 11 feet high, the diameter of the pile's base is about 34 feet. Find the angle of repose for rock salt.

Solution:



Given, Height of the pile, h = 11 feet

Diameter of the pile base = 34 feet

∴ Radius of the pile base, r = 17 feet

Now, $\tan\theta = \frac{h}{r}$

$= \frac{11}{17}$

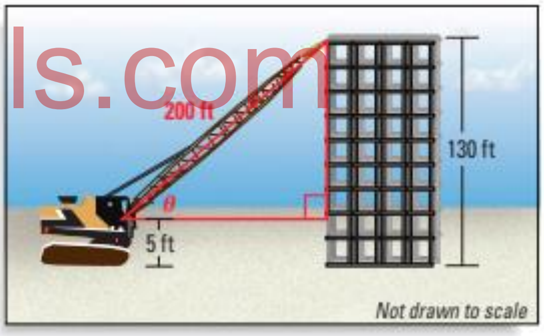
$\theta = \tan^{-1}\left(\frac{11}{17}\right)$

$\theta = 32.9^{o}$

∴ The angle of repose for rock salt is about 32.9^{o}

Example 6

A crane has a 200 foot arm whose lower end is 5 feet of the ground. The arm has to reach the top of a building 130 feet high. At what angle θ should the arm be set?



Solution:

Given,

The arm of the crane, AB = 200

Height of the building BC = 130

Height of the building above the crane level = 130 – 5 = 125

Now,

$\sin\theta = \frac{BD}{AB}$

$= \frac{125}{200} = \frac{5}{8}$

$\theta = \sin^{-1}\left(\frac{5}{8}\right)$

$\theta = 38.7^{o}$

∴ The arm to be set at $\theta = 38.7^{o}$

Example 7

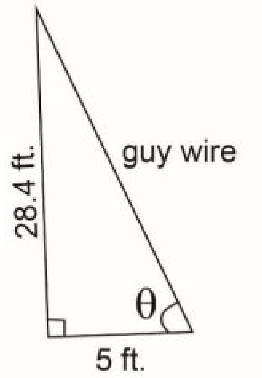
A tower 28.4 feet high, must be secured with a guy wire anchored 5 feet from the base of the tower, what angle will the guy wire make with the ground?

Solution:

Given,

Height of the tower, h = 28.4 feet

Distance from the base of the tower = 5 feet



Now, $\tan\theta = \frac{h}{b}$

$= \frac{28.4}{5}$

$\tan\theta = 5.68$

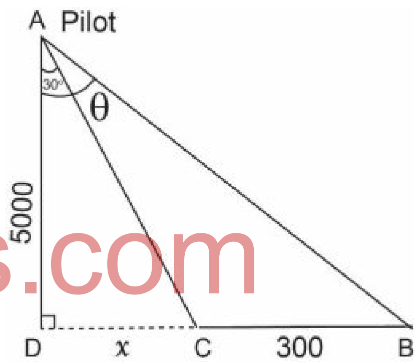
$\theta = \tan^{-1} 5.68$

$\theta = 80.02^o$

Example 8

The pilot of an airplane flying at an elevation of 5000 feet sights two towers that are 300 feet apart. If the angle of depression to the tower close to him is 30°, determine the angle of depression to the second tower.

Solution:



First need to find x in order to find θ.

In ΔACD,

$\tan 30^o = \frac{x}{5000}$

$x = 5000 \tan 30^o$

$x = 2886.75$

In ΔABD,

$\tan\theta = \frac{x + 300}{5000}$

$= \frac{3186.75}{5000} = 0.63735$

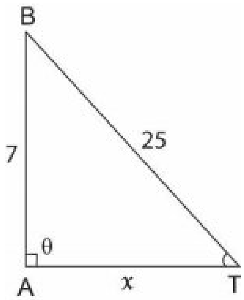
$\theta = \tan^{-1} (0.63735)$

$\theta = 32.51^o$

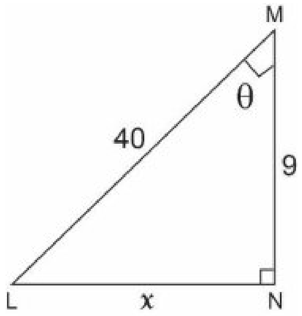
EXERCISE

1) Find the measure of the angle θ for the following triangle shown below.

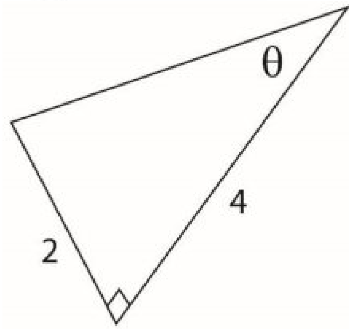
i)

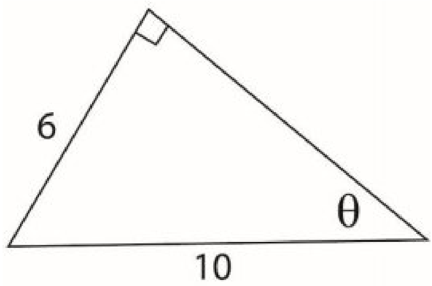


ii)



iii)





2) Find the principal values of the following.

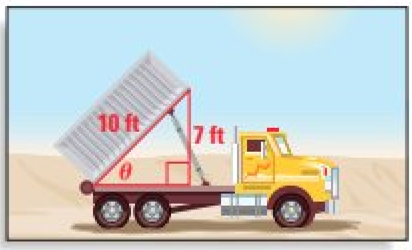
i) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ ii) $\cos^{-1}$ (½) iii) $\tan^{-1}$ (1) iv) cosec^{-1} (-1) v) $\tan^{-1}\left(\sqrt{3}\right)$

3) Find the values of the following.

i) $\sin\left(\frac{1}{2}\cos^{-1}(4/5)\right)$ ii) $\cos\left(\sin^{-1}(-3/5)\right)$ iii) $\tan\left(\cos^{-1}(8/17)\right)$

iv) $\sin\left(\cos^{-1}(\frac{1}{2})\right)$ iv) $\cos\left(\sin^{-1}\left(\frac{5}{13}\right)\right)$

4) A ladder is 15m long and rests against a wall with the bottom 2m from the wall. What is the angle of the ladder to the ground?



5) The dump truck shown has a 10 foot bed. When tilted at its maximum angle, the bed reaches a height of 7 feet above its original position, what is the maximum angle θ that the truck bed can tilt?

6) The intensity of a certain type of polarised light is given by the equation $I = I_o \sin 2\theta \cos\theta$. Solve for θ.

7) The electric current in a certain circuit is given by

$i = I_m [\sin(wt+\alpha)\cos\Psi + \cos(wt+\alpha)\sin\Psi]$ solve for t.

8) A pile of an airplane flying at an elevation of 10,000 feet sights two towers that are 500 feet apart. If the angle of depression to the tower closer to him is 18^o, determine the angle of depression to the second tower.

9) A silo is 40 feet high and 12 feet across. Find the angle of depression from the top edge of the silo to the floor of the opposite edge.

Properties of Inverse Trigonometric Functions

There are a few inverse trigonometric function properties which are crucial to not only solve problem but also to have a deeper understanding of this concept.

Here are a few important properties related to inverse trigonometric functions.

Property 1

i) $\sin^{-1}(\sin x) = x$

Proof

$$\text{Let } \sin x = y$$
$$x = \sin^{-1} y$$
$$x = \sin^{-1}(\sin x)$$

$$\cos^{-1}(\cos x) = x$$
$$\tan^{-1}(\tan x) = x$$
$$\sec^{-1}(\sec x) = x$$
$$\text{cosec}^{-1}(\text{cosec } x) = x$$
$$\cot^{-1}(\cot x) = x$$

Property 2

i) $\sin^{-1}(1/x) = \text{cosec}^{-1} x$

Proof

$$\text{Let } \sin^{-1}(1/x) = \theta$$
$$\sin\theta = 1/x$$
$$\text{cosec } \theta = x$$
$$\theta = \text{cosec}^{-1} x$$
$$\sin^{-1}(1/x) = \text{cosec}^{-1}(x)$$

$$\cos^{-1}(1/x) = \sec^{-1}(x)$$
$$\tan^{-1}(1/x) = \cot^{-1}(x)$$
$$\text{cosec}^{-1}(1/x) = \sin^{-1}(x)$$
$$\sec^{-1}(1/x) = \cos^{-1}(x)$$
$$\cot^{-1}(1/x) = \tan^{-1}(x)$$

Property 3 (Reflection Identities)

$\sin^{-1}(-x) = -\sin^{-1} x$

Proof

$$\text{Let } \sin^{-1}(-x) = \theta$$
$$-x = \sin\theta$$
$$x = -\sin\theta$$
$$x = \sin(-\theta)$$

$$\Rightarrow \quad -\theta = \sin^{-1} x$$

$$\theta = -\sin^{-1} x$$

$$\Rightarrow \quad \sin^{-1}(-x) = -\sin^{-1} x$$

$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$

$$\tan^{-1}(-x) = -\tan^{-1} x$$

$$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$$

$$\sec^{-1}(-x) = \pi - \sec^{-1} x$$

$$\cot^{-1}(-x) = \pi - \cot^{-1}(x)$$

Property 4 (Cofunction Inverse Identities)

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

Proof

$$\text{Let } \sin^{-1} x = \theta$$

$$x = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\cos^{-1} x = \frac{\pi}{2} - \theta$$

$$= \frac{\pi}{2} - \sin^{-1} x$$

$$\Rightarrow \quad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$$

Property 5

If $xy < 1$ then $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Proof

Let $\tan^{-1} x = \theta_1$ and $\tan^{-1} y = \theta_2$

$$x = \tan\theta_1 \quad y = \tan\theta_2$$

$$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

$$= \frac{x+y}{1-xy}$$

$$\theta_1 + \theta_2 = \tan^{-1}\frac{x+y}{1-xy}$$

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$$

$$\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}\frac{x \pm y}{1 \mp xy}$$

Property 6

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

Proof

Let $\theta_1 = \sin^{-1}x \qquad \theta_2 = \sin^{-1}y$

$\sin\theta_1 = x \qquad \sin\theta_2 = y$

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

$$= \sin\theta_1\sqrt{1-\sin^2\theta_2} + \sqrt{1-\sin^2\theta_1}\sin\theta_2$$

$$= x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

$$\theta_1+\theta_2 = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

$$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$$

$$\boxed{\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]}$$

Property 7

$$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$$

Proof

Let $\theta_1 = \cos^{-1}x \qquad \theta_2 = \cos^{-1}y$

$\cos\theta_1 = x \qquad \cos\theta_2 = y$

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

$$= xy - \sqrt{1-\cos^2\theta_1}\sqrt{1-\cos^2\theta_2}$$

$$= xy - \sqrt{1-x^2}\sqrt{1-y^2}$$

$$\theta_1+\theta_2 = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$$

$$\therefore \quad \cos^{-1}x + \cos^{-1}y = \cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$$

$$\boxed{\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left[xy \mp \sqrt{1-x^2}\sqrt{1-y^2}\right]}$$

Example 1

Prove that $\tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right] = 3\tan^{-1}x$

Solution

Let $x = \tan\theta$

$\theta = \tan^{-1}x$

Now,

$$\tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right] = \tan^{-1}\left[\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right]$$

$$= \tan^{-1}(\tan 3\theta)$$

$$= 3\theta$$

$$\tan^{-1}\left[\frac{3x-x^3}{1-3x^2}\right] = 3\tan^{-1}x$$

Hence proved

Example 2

Prove that $\tan^{-1}(1/7) + \tan^{-1}(1/13) = \tan^{-1}(2/9)$

Solution

L.H.S. $\tan^{-1}(1/7) + \tan^{-1}(1/13)$

$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\left(\frac{1}{7}\right)\left(\frac{1}{13}\right)}\right)$$

$$= \tan^{-1}\left(\frac{\frac{13+7}{91}}{1-\frac{1}{91}}\right)$$

$$= \tan^{-1}\left(\frac{20/91}{90/91}\right)$$

$$= \tan^{-1}\left(\frac{2}{9}\right)$$

R.H.S. Hence proved

Example 3

Show that $2\tan^{-1}(2/3) = \tan^{-1}(12/5)$

Solution:

L.H.S. $2\tan^{-1}(2/3) = \tan^{-1}(2/3) + \tan^{-1}(2/3)$

$$= \tan^{-1}\left[\frac{2/3+2/3}{1-(2/3)(2/3)}\right]$$

$$= \tan^{-1}\left[\frac{4/3}{1-4/9}\right]$$

$$= \tan^{-1}\left[\frac{4/3}{5/9}\right]$$

$$= \tan^{-1}\left[\frac{4}{3} \times \frac{9}{5}\right]$$

$$= \tan^{-1}\left[\frac{12}{5}\right] \qquad \text{R.H.S.}$$

Hence Proved.

Example 4

Prove that $\cos^{-1}(4/5) + \tan^{-1}(3/5) = \tan^{-1}(27/11)$

Solution:

Let $\cos^{-1}(4/5) = \theta$

$\cos\theta = 4/5$

$\sec\theta = 5/4$

$\tan^2\theta = \sec^2\theta - 1$

$$\tan\theta = \sqrt{\left(\frac{5}{4}\right)^2 - 1}$$

$$= \sqrt{\frac{25}{16} - 1}$$

$$\tan\theta = \frac{3}{4}$$

$\theta = \tan^{-1}(3/4)$

$\therefore \quad \cos^{-1}\left(\frac{4}{5}\right) = \tan^{-1}(3/4) \ldots.$ ①

L.H.S. $\cos^{-1}(4/5) + \tan^{-1}(3/5)$

$= \tan^{-1}(3/4) + \tan^{-1}(3/5)$

$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\left(\frac{3}{4}\right)\left(\frac{3}{5}\right)}\right)$

$= \tan^{-1}\left[\frac{27/20}{11/20}\right]$

$= \tan^{-1}\left[\frac{27}{11}\right]$ R.H.S.

Hence proved.

Example 5

Prove that $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \sin^{-1}\left(\frac{77}{85}\right)$

Solution

Since $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}$

L.H.S. $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \sin^{-1}\left\{\frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2} + \frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2}\right\}$

$= \sin^{-1}\left\{\frac{3}{5}\sqrt{1-\frac{64}{289}} + \frac{8}{17}\sqrt{1-\frac{9}{25}}\right\}$

$= \sin^{-1}\left\{\frac{3}{5}\sqrt{\frac{289-64}{289}} + \frac{8}{17}\sqrt{\frac{25-9}{25}}\right\}$

$= \sin^{-1}\left\{\frac{3}{5}\sqrt{\frac{225}{289}} + \frac{8}{17}\sqrt{\frac{16}{25}}\right\}$

$= \sin^{-1}\left\{\frac{3}{5}\left(\frac{15}{17}\right) + \frac{8}{17}\left(\frac{4}{5}\right)\right\}$

$= \sin^{-1}\left\{\frac{45}{85} + \frac{32}{85}\right\}$

$= \sin^{-1}\left(\frac{77}{85}\right)$ R.H.S.

Hence proved.

Example 6

Prove that $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-yz-zx}\right]$

Solution:

L.H.S. $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = (\tan^{-1}x + \tan^{-1}y) + \tan^{-1}z$

$= \tan^{-1}\left[\frac{x+y}{1-xy}\right] + \tan^{-1}z$

$= \tan^{-1}\left[\frac{\left(\frac{x+y}{1-xy}\right)+z}{1-\left(\frac{x+y}{1-xy}\right)z}\right]$

$$= \tan^{-1}\left[\frac{\frac{x+y+z\,(1-xy)}{1-xy}}{\frac{(1-xy)-(x+y)\,z}{1-xy}}\right]$$

$$= \tan^{-1}\left[\frac{x+y+z-xyz}{1-xy-xz-yz}\right] \qquad \text{R.H.S.}$$

Hence proved.

EXERCISE

1. Prove that the following:

    i) $2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

    ii) $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2\tan^{-1}x$

    iii) $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

    iv) $\cos^{-1}(2x^2-1) = 2\cos^{-1}x$

    v) $\cos^{-1}(4x^3-3x) = 3\cos^{-1}x$

2. Evaluate $\cos\left[\sin^{-1}(3/5) + \sin^{-1}(5/13)\right]$

3. Prove that $\tan^{-1}(4/3) - \tan^{-1}(1/7) = \pi/4$

4. Show that $2\tan^{-1}(1/3) = \tan^{-1}(3/4)$

5. Prove the following identities.

    i) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left[\frac{x-y}{1+xy}\right]$

    ii) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

    iii) $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left[xy + \sqrt{1-x^2}\,\sqrt{1-y^2}\right]$

6. Prove that $2\tan^{-1}(1/5) + \tan^{-1}(1/7) + 2\tan^{-1}(1/8) = \frac{\pi}{4}$

7. Find the value of $\cos^{-1}(4/5) + \tan^{-1}(3/5)$.

8. Prove that $\sin^{-1}(4/5) + \sin^{-1}(5/13) + \sin^{-1}(16/65) = \frac{\pi}{2}$

9. Show that $\tan^{-1}\left(\frac{m}{n}\right) - \tan^{-1}\left(\frac{m-n}{m+n}\right) = \frac{\pi}{4}$

Introduction to Trigonometry

Identities

Related Problems (Identities)

Multiple Angles

Related Problems(Multiple Angles)

Sum and Product Formulae

Inverse Trigonometric Functions - Introduction

UNIT - IV

DIFFERENTIAL CALCULUS-I

Chapter 4.1: LIMITS

Introduction:

Calculus is that branch of mathematics which mainly deals with the study of change in the value of a function as the points in the domain change.

Calculus was created independently in England by Sir Issac Newton and in German by **Gottfried Wilhelm Leibnitz**. In 1665, Newton began his study of the rates of change of quantities such as distances or temperatures that varied continuously. The result of this study was what we today call Differential Calculus.

In the development of Calculus of the 17th and 18th centuries, the modern idea of the limit of a function goes back to Bolzano who, in 1817, introduced the basics of the epsilon – delta technique to define continuous functions. However his work was not known during his life time.

Augustin – Louis Cauchy gave the first reasonably clear definition of a limit and was the first to define the derivative as the limit of the difference quotient.

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Karl Weierstrass first introduced the epsilon – delta definition of the concepts of limit, continuity and differentiability.

Limits

The limit of a function is a fundamental concept in Calculus and analysis. The notion of a limit has many applications in modern Calculus.

The modern notation of placing the arrow below the limit symbol is due to Hardy, which is introduced in his Book 'A course of pure Mathematics' in 1908.

The notion of limit is very intimately related to the intuitive idea of nearness or closeness. It comes into play in situations where one quantity depends on another varying quantity.

Calculus is a Limit Machine that involves three stages.

Stage - 1 — Pre Calculus mathematics such as the slope of a line or the area of a rectangle.

Stage - 2 — Limit Process

Stage - 3 — New calculus formulation such as Derivative, Integral.

MATH FACT

Pre Calculus mathematics is more static where as Calculus is more dynamic.

For Example

The slope of a line can be analysed with Pre Calculus mathematics. To analyse the slope of a curve, you need Calculus.

Definition:

Let f be a function of a real variable x. Let x_0 and l be the two fixed numbers. If f (x) approaches the value l as x approaches to x_0, we say l is the limit of the function f (x) as x tends to x_0.

This can be written as $\lim\limits_{x \to x_0} f(x) = l$

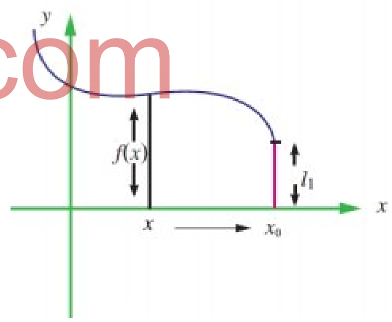
MATH FACT

$x \to 0$ means that x gets nearer and nearer to 0 but never becomes equal to 0.

$x = 0$ means that x takes the value 0.

Left Hand Limit

Left Hand Limit of f(x) as x approaches x_0 is equal to l_1 if we can make the values of f (x) arbitrarily close to l_1, by taking x to be sufficiently close to x_0 and less than x_0.

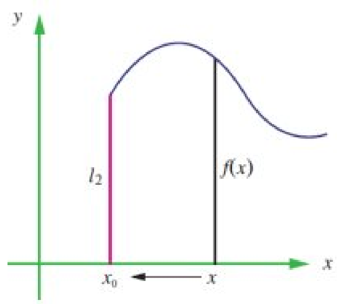


It is written as

$$Lf(x_0) = \lim_{x \to x_0 -} f(x) = l_1$$

Right Hand Limit

Right Hand Limit of f(x) as x approaches x_0 is equal to l_2 if we can make the values of f(x) arbitrarily close to l_2, by taking x to be sufficiently close to x_0 and greater than x_0.



It is written as

$$Rf(x_0) = \lim_{x \to x_0 +} f(x) = l_2$$

Note:

- $f(x_0^-) = Lf(x_0) = \lim\limits_{x \to x_0 -} f(x)$
- $f(x_0^+) = Rf(x_0) = \lim\limits_{x \to x_0 +} f(x)$
- Left and Right hand limits are also known as one sided limits.

For the existence of $\lim\limits_{x \to x_0} f(x)$, it is necessary that

(i) Both $f(x_0^+)$ and $f(x_0^-)$ exists.

(ii) $f(x_0^+) = f(x_0^-) = \lim\limits_{x \to x_0} f(x)$

MATH FACT

From the definition of one sided limits and that of the limit of f (x).

$$\lim_{x \to x_0} f(x) = l \Leftrightarrow f(x_0^-) = f(x_0^+) = l.$$

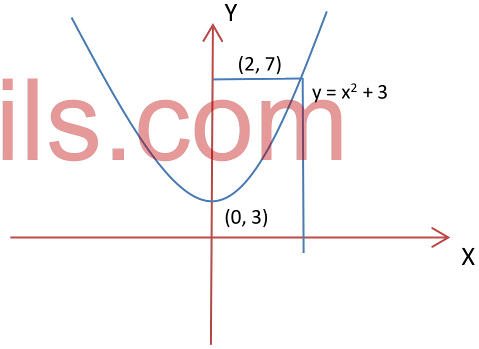
Calculation of Limits

Consider the function $f : R \to R$ given by $f(x) = x^2 + 3$. We can see two sets of x values. One set approaches 2 from left and one set that approaches 2 from the right.

x approaches 2 from left → ← **x approaches 2 from right**

x	1.9	1.99	1.999	2	2.0001	2.001	2.01
f (x)	6.61	6.9601	6.99601	7	7.0040001	7.004001	7.0101

From the table, we can observe that x gets close to 2, $f(x) = x^2 + 3$ gets close to 7.



$\therefore \lim\limits_{x \to 2^-} f(x) = 7$ and

$\therefore \lim\limits_{x \to 2^+} f(x) = 7$

$\Rightarrow \lim\limits_{x \to 2} f(x) = 7.$

Note:

(i) $\lim\limits_{x \to 2^-} f(x)$ and $\lim\limits_{x \to 2^+} f(x)$ are same.

(ii) $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^-} f(x)$ is a unique real number.

Definition:

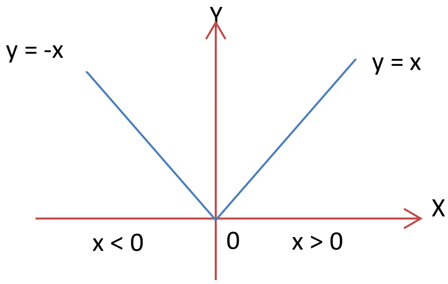
Let I be an open internal containing $x_0 \in R$. Let $f : I \to R$ then we say that the limit of f (x) is L as x approaches x_0 $\left(\lim\limits_{x \to x_0} f(x) = L\right)$, if whenever x becomes sufficiently close to x_0 from either side with $x \neq x_0$, f (x) gets sufficiently close to L.

Example: 1

Calculate $\lim\limits_{x \to 0} |x|$

Solution:

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$$



If x > 0, then | x | = x,

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0.$$

If x < 0, then | x | = -x,

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = 0$$

Thus $\lim_{x \to 0^-} |x| = 0 = \lim_{x \to 0^+} |x|$,

Hence $\lim_{x \to 0} |x| = 0$

Example: 2

Consider the function $f(x) = \sqrt{x}$, $x \geq 0$. Does $\lim_{x \to 0} f(x) = 0$.

Solution:

For $x < 0$, $f(x) = \sqrt{x}$ is not defined.

$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sqrt{x}$ does not exist.

However, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \sqrt{x} = 0.$

$\therefore \lim_{x \to 0} \sqrt{x}$ does not exist.

Example: 3

Evaluate $\lim_{x \to 1^-} \lfloor x \rfloor$ and $\lim_{x \to 1^+} \lfloor x \rfloor$

Solution:

The greatest integer function $f(x) = \lfloor x \rfloor$ is defined as the greatest integer lesser than or equal to x.

For any integer n, (i) $\lim_{x \to n^-} \lfloor x \rfloor = n - 1$

(ii) $\lim_{x \to n^+} \lfloor x \rfloor = n$

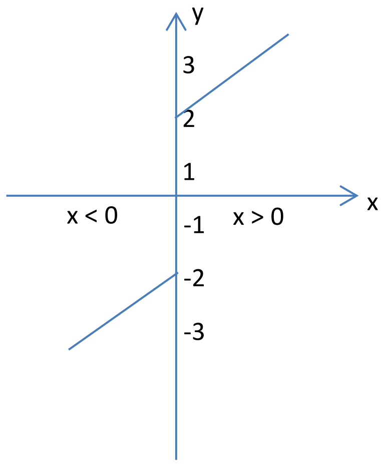
It is clear that $\lim_{x \to 1^-} \lfloor x \rfloor = 0$ and $\lim_{x \to 1^+} \lfloor x \rfloor = 1$

Example: 4

Let $f(x) = \begin{cases} x + 2, & x > 0 \\ x - 2, & x < 0 \end{cases}$

Verify the existence of limit as $x \to 0$

Solution:



For x < 0, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x - 2$

$= -2$

For x > 0, $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x + 2$

$= 2$

Here $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$ exist

But, $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\Rightarrow \lim_{x \to 0} f(x)$ does not exist.

Example: 5

Check if $\lim_{x \to -1} f(x)$ exist or not, where $f(x) = \begin{cases} \frac{|x+1|}{x+1}, & x \neq -1 \\ 0 & x = 1 \end{cases}$

Solution:

For $x < -1$, $|x + 1| = -(x + 1)$

Thus $f(-1^-) = \lim_{x \to -1} \frac{-(x+1)}{x+1} = -1$

For, $x > -1$, $|x + 1| = x + 1$

Thus $f(-1^+) = \lim_{x \to -1^+} \frac{(x+1)}{(x+1)} = 1$

Here Both $f(-1^-)$ & $f(-1^+)$ exist but

$f(-1^-) \neq f(-1^+)$

Hence the limit does not exist.

Example: 6

Test the existence of the limit, $\lim_{x \to 2} \frac{3|x-2|+x-2}{|x-2|}$, $x \neq 2$

Solution:

For $x > 2$, $|x - 2| = x - 2$ and $f(2^+) = \lim_{x \to 2^+} \frac{3(x-2)+x-2}{x-2}$

$f(2^+) = \lim_{x \to 2^+} \frac{4(x-2)}{(x-2)} = 4$

For $x < 2$, $|x - 2| = -(x - 2)$ and $f(2^-) = \lim_{x \to 2^-} \frac{-3(x-2)+x-2}{-(x-2)}$

$f(2^-) = \lim_{x \to 2^-} \frac{2(x-2)}{x-2} = 2$

$f(2^+)$ and $f(2^-)$ are exist but not equal

$\therefore \lim_{x \to 2}$ does not exist.

Limits of Polynomials

A function p is said to be a Polynomial function of degree n,

$p(x) = a_0 + a_1x + \ldots\ldots + a_nx^n$ where a_i's are real numbers such that $a_n \neq 0$ for some natural number n.

Then $\lim_{x \to x_0} p(x) = \lim_{x \to x_0} [a_0 + a_1x + \ldots\ldots\ldots + a_nx^n]$

$= a_0 + a_1x_0 + \ldots\ldots\ldots + a_nx_0^n$

$= p(x_0)$

Example: 1

Calculate: $\lim_{x \to 3} (x^3 - 4x + 1)$

Solution:

$P(x) = x^3 - 4x + 1$ is a polynomial.

Hence $\lim_{x \to 3} p(x) = p(3) = 3^3 - 4 \text{ x } 3 + 1 = 16$

Example: 2

Calculate: $\lim_{x \to x_0} (-1)$ for any real number x_0.

Solution:

$f(x) = -1$ is a polynomial of degree 0.

Hence $\lim_{x \to x_0} (-1) = f(x_0) = -1$

MATH FACT

The limit of a constant function is that constant.

Algebra of Limits:

Let f and g be any two functions such that both

$\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ exist then.

(i) $\lim_{x \to x_0} [f(x) \pm g(x)] = \lim_{x \to x_0} f(x) \pm \lim_{x \to x_0} g(x)$

(ii) $\lim_{x \to x_0} [f(x) \, . \, g(x)] = \lim_{x \to x_0} f(x) \, . \, \lim_{x \to x_0} g(x)$

(iii) $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$, provided $\lim_{x \to x_0} g(x) \neq 0$

(iv) $\lim_{x \to x_0} C f(x) = C \lim_{x \to x_0} f(x)$, C is a constant.

Indeterminate Form and Evaluation of Limits

Let f(x) and g(x) are any two functions in which the limits exist.

If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$

Then $\frac{f(x_0)}{g(x_0)}$ takes $\frac{0}{0}$ form which is meaningless

But does not imply that $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ is meaningless.

The indeterminate forms are $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \text{ x } \infty$, ∞^0, 0^0 and 1^∞.

Among all these, $\frac{0}{0}$ is the fundamental one.

Methods of Evaluation of Algebraic Limits

(i) Direct substitution

(ii) Factorization

(iii) Rationalisation

(iv) By using standard limits

Example: 1

Compute (i) $\lim_{x \to 4} (3x)$ (ii) $\lim_{x \to -2} \left(\frac{-5}{2} x\right)$

Solution:

(i) $\lim_{x \to 4} (3x) = 3 \lim_{x \to 4} x = 3\,(4) = 12$

(ii) $\lim_{x \to -2} \left(\frac{-5}{2} x\right) = \frac{-5}{2} \lim_{x \to -2} x = \frac{-5}{2} (-2) = 5$

Example: 2

Compute: $\lim_{x \to 0} \left[\frac{4x^3+x}{x} + 2\right]$

Solution:

$$\lim_{x \to 0} \left[\frac{4x^3+x}{x} + 2\right] = \lim_{x \to 0} \left(\frac{4x^3+x}{x}\right) + \lim_{x \to 0} (2)$$

$$= \lim_{x \to 0} (4x^2 + 1) + 2$$

$$= (0 + 1) + 2 = 3$$

Example: 3

Calculate : $\lim_{x \to -2} (x^4 - 3x + 2)(-x^3 + 1)$

Solution:

$$\lim_{x \to -2} (x^4 - 3x + 2)(-x^3 + 1) = \lim_{x \to -2} (x^4 - 3x + 2) \lim_{x \to -2} (-x^3 + 1)$$

$$= [(-2)^4 - 3(-2) + 2]\,[-(-2)^3 + 1]$$

$$= (16 + 6 + 2)\,(8 + 1) = 216.$$

Note:

If $\lim_{x \to x_0} f(x)$ exists then $\lim_{x \to x_0} [f(x)]^n = \left[\lim_{x \to x_0} f(x)\right]^n$

Example: 4

Calculate: $\lim_{x \to -2} (x^2 - 1)^{12}$

Solution:

$$\lim_{x \to -2} (x^2 - 1)^{12} = \left[\lim_{x \to -2} (x^2 - 1)\right]^{12} = [(-2)^2 - 1]^{12}$$

$$= 3^{12} = 531441$$

Example: 5

Compute $\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1}$

Solution:

We can't apply the quotient theorem immediately, Rationalise,

$$\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x \to 1} \frac{(\sqrt{x}-1)}{(\sqrt{x}-1)\ (\sqrt{x}+1)} = \lim_{x \to 1} \frac{1}{\sqrt{x}+1}$$

$$= \frac{\lim_{x \to 1} (1)}{\lim_{x \to 1} (\sqrt{x}+1)} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

Example: 6

Find $\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2}$

Solution:

Here we can't apply the quotient theorem immediately,

$$\lim_{x \to 0} \frac{\sqrt{x^2+9}-3}{x^2} = \lim_{x \to 0} \frac{(\sqrt{x^2+9}-3)\ (\sqrt{x^2+9}+3)}{x^2\ \sqrt{x^2+9}+3}$$

$$= \lim_{x \to 0} \frac{x^2+9-9}{x^2\ \sqrt{x^2+9}+3}$$

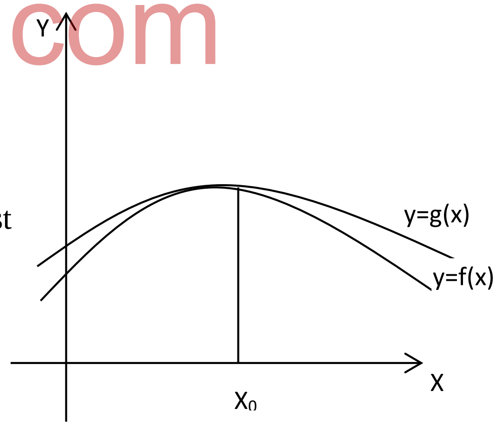
$$= \lim_{x \to 0} \frac{1}{\sqrt{x^2+9}+3} = \frac{1}{\sqrt{0+9}+3} = \frac{1}{3+3} = \frac{1}{6}$$

Note:

If f and g are any two real valued function with the same domain such that $f(x) \le g(x)$ for all x.

For some x_0, if both $\lim_{x \to x_0} f(x)$ and $\lim_{x \to x_0} g(x)$ exist

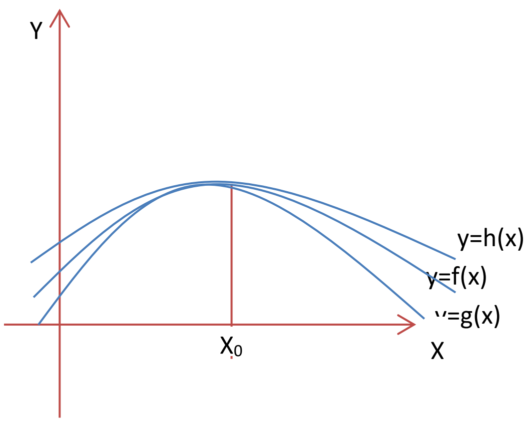
then $\lim_{x \to x_0} f(x) \le \lim_{x \to x_0} g(x)$



Theorem: (Sandwich theorem)

If $f, g, h : I \subseteq R \to R$ such that $g(x) \le f(x) \le h(x)$ for all x in a deleted neighbourhood of x_0 contained in I and if $\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = l$, then $\lim_{x \to x_0} f(x) = l$.

Note : Sandwich theorem is also known as squeeze theorem.



Example: 1

Evaluate $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$

Solution:

We know that $-1 \le \operatorname{Sin}\frac{1}{x} \le 1 \quad \Rightarrow -x^2 \le x^2 \operatorname{Sin}\frac{1}{x} \le x^2$

Take g (x) = - x^2, f (x) = $x^2 \operatorname{Sin}\left(\frac{1}{x}\right)$, h (x) = x^2

Then $\lim\limits_{x \to 0} g(x) = \lim\limits_{x \to 0} (-x^2) = 0$ and

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} (x^2) = 0$$

By Sandwich theorem, $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x^2 \operatorname{Sin}\frac{1}{x} = 0$

Example: 2

Prove that $\lim\limits_{x \to 0} \operatorname{Sin} x = 0$

Solution:

Since $-x \le \operatorname{Sin} x \le x$ for all $x \ge 0$

Take g (x) = - x f (x) = sin x and h(x) = x

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} -x = 0$$

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} x = 0$$

∴ By Sandwich theorem, $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \sin x = 0$

Example: 3

Show that $\lim\limits_{x \to 0^+} x\left[\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{2}{x} \right\rfloor + \cdots + \left\lfloor \frac{10}{x} \right\rfloor\right] = 55$

Solution:

We know that, $\frac{1}{x} - 1 \le \left\lfloor \frac{1}{x} \right\rfloor \le \frac{1}{x} + 1$

$$\frac{2}{x} - 1 \le \left\lfloor \frac{2}{x} \right\rfloor \le \frac{2}{x} + 1$$

$$\frac{10}{x} - 1 \le \left\lfloor \frac{10}{x} \right\rfloor \le \frac{10}{x} + 1$$

Summing we get, $\frac{55}{x} - 10 \le \left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{2}{x} \right\rfloor + \ldots + \left\lfloor \frac{10}{x} \right\rfloor \le \frac{55}{x} + 10$

$$\Rightarrow 55 - 10x \le x\left[\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{2}{x} \right\rfloor + \cdots + \left\lfloor \frac{10}{x} \right\rfloor\right] \le 55 + 10x$$

$$\Rightarrow \lim_{x \to 0^+} (55 - 10x) \le \lim_{x \to 0^+} x\left[\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{2}{x} \right\rfloor + \cdots + \left\lfloor \frac{10}{x} \right\rfloor\right] \le \lim_{x \to 0^+} 55 + 10x$$

$$\Rightarrow 55 \le \lim_{x \to 0^+} x\left[\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{2}{x} \right\rfloor + \cdots + \left\lfloor \frac{10}{x} \right\rfloor\right] \le 55$$

$$\Rightarrow \lim_{x \to 0^+} x\left[\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{2}{x} \right\rfloor + \cdots + \left\lfloor \frac{10}{x} \right\rfloor\right] = 55$$

Limits at Infinity

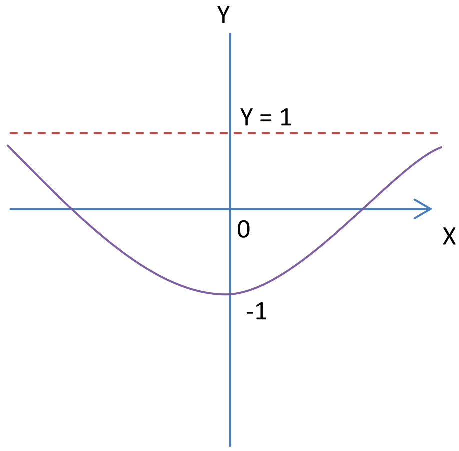
In the previous section, x approaches a number and the result was that the values of y became arbitrarily large (very large positive or negative). Here, we let x become arbitrarily large (positive or negative) and see what happens to y.

Consider the function f : R → R defined by $f(x) = \frac{x^2-1}{x^2+1}$ as x becomes large.

X	-4	-3	-2	-1	0	1	2	3	4
f (x)	0.882	0.8	0.6	0	-1	0	0.6	0.8	0.882

As x grows larger and larger you can see that the value of f (x) gets closer and closer to 1.

∴ we can write $\lim_{x \to \pm\infty} \frac{x^2-1}{x^2+1} = 1.$



MATH FACT

The line y = *l* is called a horizontal asymptote of the curve y = f (x) if either

$\lim_{x \to -\infty} f(x) = l$ or $\lim_{x \to +\infty} f(x) = l$

Method to find $\lim_{x \to \pm\infty}$ for Rational expressions

(i) Divide the numerator and denominator by the highest power of x that appears in the denominator.

(ii) Calculate the Limit as $x \to \infty$ (or $x \to \infty$) of both numerator and denominator.

Example: 1

Calculate $\lim_{x \to \infty} \frac{3x^2-4x+1}{x^2+2x-3}$

Solution:

$$\lim_{x \to \infty} \frac{3x^2-4x+1}{x^2+2x-3} = \lim_{x \to \infty} \left(\frac{3-\frac{4}{x}+\frac{1}{x^2}}{1+\frac{2}{x}-\frac{3}{x^2}}\right) \text{ [Divide the Nr. \& Dr. by } x^2]$$

$$= \frac{3-0+0}{1+0-0} = 3$$

(since $\frac{1}{x} \to 0$ as $x \to \infty$, $\frac{1}{x^2} \to 0$ as $x \to \infty$)

Example: 2

Calculate $\lim_{x \to \infty} \frac{x^3-x+2}{3x^2-5}$

Solution:

$$\lim_{x \to \infty} \frac{x^3-x+2}{3x^2-5} = \lim_{x \to \infty} \left(\frac{x-\frac{1}{x}+\frac{2}{x^2}}{3-\frac{5}{x^2}}\right) \text{ [∵ Divide the Nr. \& Dr. by } x^2]$$

$$\lim_{x \to \infty} \left(\frac{x-\frac{1}{x}+\frac{2}{x^2}}{3-\frac{5}{x^2}}\right) \to \infty \text{ as } x \to \infty.$$

[∵ The degree of Nr. is higher than that of Dr.]

The limit does not exist.

Example: 3

Calculate $\lim\limits_{x \to \infty} \frac{2 + x^3}{3x-2}$

Solution:

$$\lim\limits_{x \to \infty} \frac{2 + x^3}{3x-2} = \lim\limits_{x \to \infty} \left(\frac{\frac{2}{x} + x^2}{3 - \frac{2}{x}}\right)$$

$$\therefore \left(\frac{\frac{2}{x} + x^2}{3 - \frac{2}{x}}\right) \to \infty \text{ as } x \to \infty$$

$\therefore$ The Limit does not exist.

Note:

$f(x) \to \infty$ as $x \to a$ $\quad$ $f(x) \to \infty$ as $x \to \infty$

$f(x) \to -\infty$ as $x \to a$ $\quad$ and $\quad$ $f(x) \to -\infty$ as $x \to \infty$

means that the limits do not exist.

MATH FACT

The symbol ∞ does not represent a number and should not be treated as a number.

Limits of Rational Functions

If $R(x) = \frac{p(x)}{q(x)}$ and the degree of the polynomial p (x) is greater than the degree of q (x), then $\frac{p(x)}{q(x)} \to +\infty$ or $-\infty$ as $x \to \infty$.

If the degree of q (x) is greater than the degree of p (x) then $\lim\limits_{x \to \infty} \frac{p(x)}{q(x)} = 0$.

Finally, if the degree of p (x) is equal to the degree of q (x)

$$\text{then } \lim\limits_{x \to \infty} \frac{p(x)}{q(x)} = \frac{\text{co-efft. of highest power of x in p (x)}}{\text{co-efft. of highest power of x in q (x)}}$$

Example: 1

Calculate: $\lim\limits_{x \to \infty} \frac{5x^2 + 3x - 4}{-2x^3 - 7x + 1}$

Solution:

$$\lim\limits_{x \to \infty} \frac{5x^2 + 3x - 4}{-2x^3 - 7x + 1} = \lim\limits_{x \to \infty} \frac{x^2\,(5 + 3/x - 4/x^2)}{x^3\,(-2 - \frac{7}{x^2} + 1/x^3)}$$

$$= \lim\limits_{x \to \infty} \frac{1}{x} \cdot \lim\limits_{x \to \infty} \frac{(5 + 3/x - 4/x^2)}{(-2 - \frac{7}{x^2} + \frac{1}{x^3})}$$

$$= 0 \cdot \left(\frac{5 + 0 - 0}{-2 - 0 + 0}\right) = 0\,(5/-2) = 0.$$

Trigonometrical Limits:

(1) $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ $\qquad$ (2) $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

(3) $\lim\limits_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$ $\qquad$ (4) $\lim\limits_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$

Example: 1

Compute $\lim_{x \to 1} \frac{x^4 - 1}{x-1}$

Solution:

$$\lim_{x \to 1} \frac{x^4 - 1}{x-1} = \lim_{x \to 1} \frac{x^4 - 1^4}{x-1} = 4\,(1)^{4-1} = 4.$$

Example: 2

Compute $\lim_{t \to 1} \frac{\sqrt{t} - 1}{t - 1}$

Solution:

$$\lim_{t \to 1} \frac{\sqrt{t} - 1}{t - 1} = \lim_{t \to 1} \frac{t^{1/2} - 1^{1/2}}{t-1} = \frac{1}{2}\,(1)^{-½} = ½$$

Example: 3

Find $\lim_{x \to 0} \frac{(2+x)^6 - 64}{x}$

Solution:

Put $2 + x = y$ so that as $y \to 2$ as $x \to 0$

$$\lim_{x \to 0} \frac{(2+x)^6 - 64}{x} = \lim_{y \to 2} \frac{y^6 - 2^6}{y-2} = 6(2)^{6-1} = 192$$

Example: 4

Find the positive integer n so that $\lim_{x \to 3} \frac{x^n - 3^n}{x-3} = 27$

Solution:

Given, $\lim_{x \to 3} \frac{x^n - 3^n}{x-3} = 27$

$\Rightarrow\ n(3)^{n-1} = 27$

$\Rightarrow\ n\,(3)^{n-1} = 3 \times 3^2$

Comparing on both sides, we get

$n = 3$

Example: 5

Evaluate : $\lim_{\theta \to 0} \frac{\sin 6\theta}{\theta}$

Solution:

$$\lim_{\theta \to 0} \frac{\sin 6\theta}{\theta} = \lim_{\theta \to 0} 6 \cdot \left(\frac{\sin 6\theta}{6\theta}\right)$$

$$= 6 \cdot \lim_{\theta \to 0} \frac{\sin 6\theta}{6\theta} = 6 \times 1 = 6.$$

Example: 6

Evaluate : $\lim_{x \to 0} \frac{2\tan x}{5x}$

Solution:

$$\lim_{x \to 0} \frac{2\tan x}{5x} = \frac{2}{5} \lim_{x \to 0} \frac{\tan x}{x} = \frac{2}{5} \text{ x } 1 = \frac{2}{5}$$

Example: 7

Evaluate : $\lim_{x \to 0} \frac{1-\cos^2 x}{x \operatorname{Sin} 2x}$

Solution:

$$\lim_{x \to 0} \frac{1-\cos^2 x}{x \operatorname{Sin} 2x} = \lim_{x \to 0} \frac{\operatorname{Sin}^2 x}{x \,.\, 2 \operatorname{Sin} x \operatorname{Cos} x} = \lim_{x \to 0} \frac{\operatorname{Sin} x}{2x \,.\operatorname{Cos} x}$$

$$= \lim_{x \to 0} \left(\frac{\tan x}{2\,x}\right) = \frac{1}{2} \lim_{x \to 0} \left(\frac{\tan x}{x}\right)$$

$$= \frac{1}{2} \text{ x } 1 = \frac{1}{2}$$

Some Important Standard Limits:

(1) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(2) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a,\ a > 0$

(3) $\lim_{x \to 0} \frac{\log\,(1+x)}{x} = 1$

(4) $\lim_{x \to 0} \frac{\operatorname{Sin}^{-1} x}{x} = 1$

(5) $\lim_{x \to 0} \frac{\tan^{-1} x}{x} = 1$

(6) $\lim_{x \to 0} (1+x)^{1/x} = e$

(7) $\lim_{x \to \infty} (1 + \frac{1}{x})^x = e$

(7) $\lim_{x \to \infty} (1 + \frac{k}{x})^x = e^k$

Example: 1

Evaluate : $\lim_{x \to 0} (1 + \tan x)^{2 \cot x}$

Solution:

Let $\tan x = \frac{1}{y}$ as $x \to 0$, $y \to \infty$

$$\lim_{x \to 0} (1 + \tan x)^{2 \cot x} = \lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^{2y}$$

$$= \left[\lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^{y}\right]^2 = e^2.$$

Example: 2

Evaluate : $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{7x}$

Solution:

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{7x} = \left[\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}.\right]^7$$

$$= e^7$$

Example: 3

Evaluate : $\lim_{x \to 0} (1 + x)^{2/3x}$

Solution:

$$\lim_{x \to 0} (1 + x)^{2/3x} = \left[\lim_{x \to 0} (1 + x)^{\frac{1}{x}}\right]^{2/3} = e^{2/3}$$

Example: 4

Evaluate : $\lim_{x \to \frac{\pi}{4}} \frac{4\sqrt{2}-(\text{Cos } x + \text{Sin } x)^{5}}{1-\text{Sin } 2x}$

Solution:

Now, $\frac{4\sqrt{2}-(\text{Cos } x + \text{Sin } x)^{5}}{1-\text{Sin } 2x} = \frac{2^{5/2} - [(\text{Cos } x + \text{Sin } x)^{2}]^{5/2}}{1-\text{Sin } 2x}$

$$= \frac{2^{5/2} - (1+\text{Sin } 2x)^{5/2}}{2-(1+\text{Sin } 2x)}$$

$$\therefore \lim_{x \to \frac{\pi}{4}} \frac{2^{5/2} (\text{Cos } x + \text{Sin } x)^{5}}{1-\text{Sin } 2x} = \lim_{x \to \frac{\pi}{4}} \frac{2^{5/2}-(1+\text{Sin } 2x)^{5/2}}{2-(1+\text{Sin } 2x)}$$

Take y = 1 + Sin 2x as $x \to \frac{\pi}{4}$, $y \to 2$

$$= \lim_{y \to 2} \frac{2^{5/2}-y^{5/2}}{2-y} = \lim_{y \to 2} \frac{y^{5/2}-2^{5/2}}{y-2}$$

$$= \frac{5}{2}(2)^{5/2-1} = 5\sqrt{2}$$

Applications of Limits

Example: 1

The velocity in ft/sec of a falling object is modeled by $r(t) = -\sqrt{\frac{32}{k}} \cdot \frac{1-e^{-2t\sqrt{32k}}}{1+e^{-2t\sqrt{32k}}}$, where K is a constant that depends upon the size and shape of the object and the density of the air. Find the limiting velocity of the object, that is, find $\lim_{t \to \infty} r(t)$

Solution:

$$\lim_{t \to \infty} r(t) = \lim_{t \to \infty} -\sqrt{\frac{32}{k}} \cdot \frac{1-e^{-2t\sqrt{32k}}}{1+e^{-2t\sqrt{32k}}}$$

$$= -\sqrt{\frac{32}{k}} \lim_{t \to \infty} \frac{1-e^{-2t\sqrt{32k}}}{1+e^{-2t\sqrt{32k}}}$$

$$= -\sqrt{\frac{32}{k}} \frac{(1-0)}{(1+0)} = -\sqrt{\frac{32}{k}} \text{ ft / sec} \quad (\because t \to \infty \text{ then } e^{-2t} \to 0)$$

Example: 2

Suppose that the diameter of an animal's pupils is given by $f(x) = \frac{80x^{-0.2}+45}{2x^{-0.2}+9}$. Where x is the intensity of light and f (x) is in mm. Find the diameter of the pupils with (a) minimum light (b) maximum light.

Solution:

(a) For minimum light it is enough to find the limit of the function when $x \to 0^+$.

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{80x^{-0.2}+45}{2x^{-0.2}+9}$$

$$= \lim_{x \to 0^+} \frac{\frac{80}{x^{0.2}} + 45}{\frac{2}{x^{0.2}} + 9} = \lim_{x \to 0^+} \frac{80 + 45\,x^{0.2}}{2 + 9\,x^{0.2}}$$

$$= \frac{80}{2} = 40 \text{ mm}$$

(b) For maximum light, it is enough to find the limit of the function when $x \to \infty$.

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{80x^{-0.2} + 45}{2x^{-0.2} + 9}$$

$$= \lim_{x \to \infty} \frac{\frac{80}{x^{0.2}} + 45}{\frac{2}{x^{0.2}} + 9} = \frac{45}{9} = 5 \text{ mm}$$

That is, the pupils have a limiting size of 5 mm, as the intensity of light is very large.

Example: 3

A tank contains 4000 litres of pure water. Brine that contains 20 grams of salt per litre of water is pumped into the tank at a rate of 15 litres per minute. The concentration of salt water after t minutes in grams per litre is $C(t) = \frac{20\,t}{150+t}$. What happens to the concentration as $t \to \infty$.

Solution:

$$\lim_{t \to \infty} C(t) = \lim_{t \to \infty} \frac{20\,t}{150+t} = \lim_{t \to \infty} \left(\frac{20}{\frac{150}{t} + 1}\right) = 20.$$

Exercise

1) Evaluate : $\lim_{x \to 1} \frac{x^2 + 2x + 5}{x^2 + 1}$

2) Evaluate : $\lim_{x \to 0} \frac{6x^2 + 7x + 3}{4x^2 + 3x - 7}$

3) Evaluate : $\lim_{x \to 0} \frac{\log(1+x)}{x}$

4) Evaluate : $\lim_{x \to 1} \frac{x^m - 1}{x - 1}$

5) Evaluate : $\lim_{x \to 0} \frac{\sin^{-1}(-1)}{x}$

6) Evaluate : $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

7) Evaluate : $\lim_{x \to \infty} \left(1 + \frac{2}{x}\right)^x$

8) Evaluate : $\lim_{\theta \to \infty} \frac{1 - \cos\theta}{\theta}$

9) Evaluate : $\lim_{x \to 0} \frac{e^x - 1}{x}$

10) Evaluate : $\lim_{x \to 0} \frac{2^x - 1}{x}$

11) Evaluate : $\lim_{x \to 0} \frac{4x^2 - 3x}{5x^2 + 2x}$

12) Evaluate : $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{3x}$

13) Evaluate : $\lim_{x \to 0^+} \frac{\sin x}{\sqrt{x}}$

14) Evaluate : $\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - x - 2}$

15) Evaluate : $\lim_{x \to 0} \frac{\sin 7x}{9x}$

16) Evaluate : $\lim_{x \to 0} \frac{\sin 2x}{\sin 5x}$

17) Evaluate : $\lim_{x \to 2} \frac{x^3 - 2^3}{x - 2}$

18) Evaluate : $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$

19) Evaluate : $\lim_{x \to 0} \frac{x^{\tan x} - 1}{\tan x}$

20) Evaluate : $\lim_{x \to 1} \frac{\log x}{x - 1}$

21) Evaluate : $\lim_{\theta \to 0} \frac{5 \sin 6\theta}{3 \sin 2\theta}$

22) Evaluate : $\lim_{x \to 3} \frac{x^4 - 81}{x^2 - 9}$

23) Evaluate : $\lim_{x \to 4} \frac{x^3 - 64}{x^2 - 16}$

24) Evaluate : $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x}$

25) Evaluate : $\lim_{\theta \to 0} \frac{1 - \text{Cos}\,\theta}{\theta^2}$

26) Evaluate : $\lim_{x \to 3} \frac{e^x - e^3}{x - 3}$

27) Evaluate : $\lim_{x \to 0} \frac{e^x - \text{Sin}\,x - 1}{x}$

28) Evaluate : $\lim_{x \to 0} \frac{\log(1+n)}{x - 1}$

29) Evaluate : $\lim_{x \to 0} \frac{5^x - 6^x}{x}$

30) Evaluate :
$\lim_{x \to 0} \frac{3^x + 1 - \text{Cos}\,x - e^x}{x - 3}$

31) Evaluate : $\lim_{x \to 1/2} (1 + \text{Cos}\,x)^{3\,\text{sec}\,x}$

32) Evaluate : $\lim_{x \to 0^+} \frac{2^x - 1}{\sqrt{1+x} - 1}$

Chapter 4.2: DIFFERENTIATION

Consider a function y = f(x) of a variable x. The value of y depends upon the value of x. So x is called the independent variable and y is called the dependent variable.

Suppose x changes from an initial value x to a final value x_1. Then the increment in x is defined to be the amount of change in x and it is devoted by Δx (read as delta x).

$$\therefore\ \Delta x = x_1 - x$$

Thus, $x_1 = x + \Delta x$

As x changes from x to $x_1 = x + \Delta x$.

Y changes from f(x) to $f(x + \Delta x)$

We can put f(x) = y and $f(x + \Delta x) = y + \Delta y$. The increment in y namely Δy depends as the values of x and Δx.

If the increment Δy is divided by Δx, the quotient $\frac{\Delta y}{\Delta x}$ is called the average rate of change of y with respect of x, as x changes from x to $x + \Delta x$. The quotient is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

This fraction is called a difference quotient. The process of finding the derivative of a function is called Differentiation.

Derivative of a Function

Let f be defined on an open internal $I \subseteq R$ containing the point x and suppose that $\lim\limits_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$ exists. Then f is said to be differentiable at x and the derivative of f at x denoted by f ' (x), is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

For all x, for which this limit exists.

> The other notations to denote the derivative of y = f(x) are
>
> $f'(x), \frac{dy}{dx}, y', \frac{d}{dx}[f(x)], y_1$ and Dy.
>
> Here $\frac{d}{dx}$ or D is the differential operator.

Note:

f ' (x) is read as 'f dash of x'.

$\frac{dy}{dx}$ is read as "derivative of y with respect to x".

> The symbol $\frac{dy}{dx}$ is known as Leibnitz symbol.

MATH FACT

- A function is differentiable at x if its derivative exists at x.
- A function is differentiable on an open interval (a, b) if it is differentiable at every point in (a, b).

Derivatives of Basic Elementary Functions:

1. $\frac{d}{dx}(k) = 0$ (where k is constant)
2. $\frac{d}{dx}(x^n) = nx^{n-1}$
3. $\frac{d}{dx}(\sin x) = \cos x$
4. $\frac{d}{dx}(\cos x) = -\sin x$
5. $\frac{d}{dx}(\tan x) = \sec^2 x$
6. $\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$
7. $\frac{d}{dx}(\sec x) = \sec x \tan x$
8. $\frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x$
9. $\frac{d}{dx}(\log x) = \frac{1}{x}$
10. $\frac{d}{dx}(e^x) = e^x$
11. $\frac{d}{dx}(a^x) = a^x \log a$
12. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
13. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
15. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
16. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
17. $\frac{d}{dx}(\text{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

MATH FACT

$\frac{dy}{dx}$ does not mean $dy \div dx$

Differentiation Rules:

Using Differentiation Rules, we can find the derivative of any function without carrying out limit process.

Addition and Subtraction Rule

- The derivative of the sum or difference of two (or more) differentiable functions is equal to the sum or difference of their derivatives.
- If u and v are two differentiable function and $y = u \pm v$ then

$$\frac{dy}{dx} = \frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

1. If $y = 7x^2 + 9x + 3$. Find $\frac{dy}{dx}$

Solution:

Given, $y = 7x^2 + 9x + 3$

Differentiating with respect to x,

$$\frac{dy}{dx} = 14x + 9$$

2. If $y = x^4 - 2\sin x + \frac{5}{x^2}$ find $\frac{dy}{dx}$

Solution:

Given, $y = x^4 - 2\sin x + \frac{5}{x^2}$

$y = x^4 - 2\sin x + 5x^{-2}$

Differentiating with respect to x

$$\frac{dy}{dx} = 4x^3 - 2\cos x - 10x^{-3}$$

3. If $y = \sqrt{x} - \frac{1}{\sqrt{x}}$ find $\frac{dy}{dx}$

Solution:

Given, $y = \sqrt{x} - \frac{1}{\sqrt{x}}$

$$y = \sqrt{x} - x^{-1/2}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{2}\, x^{-½\ -1}$$

$$= \frac{1}{2\sqrt{x}} + \frac{1}{2}\, x^{-3/2}$$

Product Rule

If u and ν are differentiable functions and y = uν then

$$\frac{dy}{dx} = \frac{d}{dx}(u\nu) = u\frac{d\nu}{dx} + \nu\frac{du}{dx}$$

If u, ν and w are differentiable functions and y = uνw then

$$\frac{dy}{dx} = \frac{d}{dx}(u\nu w)$$

$$= \nu w\frac{du}{dx} + uw\frac{d\nu}{dx} + u\nu\frac{dw}{dx}$$

1. Find $\frac{dy}{dx}$ if $y = x^2 \cos x$.

Solution:

Given, $y = x^2 \cos x$

$$\frac{dy}{dx} = x^2\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(x^2)$$

$$= x^2(-\sin x) + \cos x\,(2x)$$

$$\frac{dy}{dx} = -x^2 \sin x + 2x \cos x$$

2. Find $\frac{dy}{dx}$ if $y = 8 \operatorname{cosec} x \log x$

Solution:

Given, $y = 8 \operatorname{cosec} x \log x$

$$\frac{dy}{dx} = 8 \operatorname{cosec} x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(8 \operatorname{cosec} x)$$

$$= 8 \operatorname{cosec} x\frac{1}{x} + \log x\,(-8 \operatorname{cosec} x \cot x)$$

$$\frac{dy}{dx} = \frac{8 \operatorname{cosec} x}{x} - 8 \log x \operatorname{cosec} x \cot x$$

3. Find $\frac{dy}{dx}$ if $y = e^x \tan x$

Solution:

Given, $y = e^x \tan x$

$$\frac{dy}{dx} = e^x\frac{d}{dx}(\tan x) + \tan x\frac{d}{dx}(e^x)$$

$$\frac{dy}{dx} = e^x \sec^2 x + \tan x\, e^x$$

4. If $y = x^2 e^x \sin x$. Find $\frac{dy}{dx}$

Solution:

Given, $y = x^2 e^x \sin x$

$$\frac{dy}{dx} = e^x \sin x \frac{d}{dx}(x^2) + x^2 \sin x \frac{d}{dx}(e^x) + x^2 e^x \frac{d}{dx}(\sin x)$$

$$= e^x \sin x\,(2x) + x^2 \sin x\,(e^x) + x^2 e^x \cos x$$

$$\frac{dy}{dx} = 2x e^x \sin x + x^2 e^x \sin x + x^2 e^x \cos x$$

5. If $y = \sqrt{x}\ \log x\, e^x$ find $\frac{dy}{dx}$

Solution:

Given, $y = \sqrt{x}\ \log x\, e^x$

$$\frac{dy}{dx} = \log x\, e^x \frac{d}{dx}(\sqrt{x}) + \sqrt{x}\, e^x \frac{d}{dx}(\log x) + \sqrt{x} \log x \frac{d}{dx}(e^x)$$

$$= \log x\, e^x \left(\frac{1}{2\sqrt{x}}\right) + \sqrt{x}\, e^x \left(\frac{1}{x}\right) + \sqrt{x} \log x\, e^x$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \log x\, e^x + \frac{\sqrt{x}}{x} e^x + \sqrt{x}\, e^x \log x$$

6. If $y = (x+1)(x+2)(x-3)$ find $\frac{dy}{dx}$

Solution:

Given, $y = (x+1)(x+2)(x-3)$

$$\frac{dy}{dx} = (x+2)(x-3)\frac{d}{dx}(x+1) + (x+1)(x-3)\frac{d}{dx}(x+2) + (x+1)(x+2)\frac{d}{dx}(x-3)$$

$$= (x+2)(x-3)(1) + (x+1)(x-3)(1) + (x+1)(x+2)(1)$$

$$\frac{dy}{dx} = (x+2)(x-3) + (x+1)(x-3) + (x+1)(x+2)$$

Quotient Rule

If u and v are two differentiable functions and $y = \frac{u}{v}$ with $v(x) \neq 0$ then

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

1. If $y = \frac{1+\cos x}{1-\sin x}$ find $\frac{dy}{dx}$

Solution:

Given $y = \frac{1+\cos x}{1-\sin x}$

$$\frac{dy}{dx} = \frac{(1-\sin x)\frac{d}{dx}(1+\cos x) - (1+\cos x)\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}$$

$$= \frac{(1-\sin x)(-\sin x) - (1+\cos x)(-\cos x)}{(1-\sin x)^2}$$

$$\frac{dy}{dx} = \frac{-\sin x + \sin^2 x + \cos x + \cos^2 x}{(1-\sin x)^2}$$

$$\frac{dy}{dx} = \frac{1 - \sin x + \cos x}{(1-\sin x)^2}$$

2. If $y = \frac{x^2 + \tan x}{x - \sin x}$ find $\frac{dy}{dx}$

Solution:

Given $y = \frac{x^2 + \tan x}{x - \sin x}$

$$\frac{dy}{dx} = \frac{(x - \sin x)\frac{d}{dx}(x^2 + \tan x) - (x^2 + \tan x)\frac{d}{dx}(x - \sin x)}{(x - \sin x)^2}$$

$$\frac{dy}{dx} = \frac{(x - \sin x)\ (2x + \sec^2 x) - (x^2 + \tan x)\ (1 - \cos x)}{(x - \sin x)^2}$$

3. If $y = \frac{1 + x + x^2}{1 - x + x^2}$ find $\frac{dy}{dx}$

Solution:

Given $y = \frac{1 + x + x^2}{1 - x + x^2}$

$$\frac{dy}{dx} = \frac{(1 - x + x^2)\frac{d}{dx}(1 + x + x^2) - (1 + x + x^2)\frac{d}{dx}(1 - x + x^2)}{(1 - x + x^2)^2}$$

$$= \frac{(1 - x + x^2)\ (1 + 2x) - (1 + x + x^2)\ (-1 + 2x)}{(1 - x + x^2)^2}$$

Chain Rule

Let y = f(u) be a function of u and in turn let u = g(x) be a function of x so that y = f [g (x)].

Then $\frac{dy}{dx} = \frac{d}{dx} f[g(x)]$

$= f'[g(x)]\, g'(x)$

To differentiate a function of function y = f [g (x)]

Take the derivative of the outer function f regarding the argument g(x) = u and multiply the derivative of the inner function g(x) with respect to the independent variable x.

The variable u is known as intermediate argument.

1. If $y = \sin(e^x)$ find $\frac{dy}{dx}$

Solution:

Given $y = \sin(e^x)$

Put $u = e^x$ $\quad y = \sin u$

Differentiating with respect to x,

$\frac{du}{dx} = e^x \qquad \frac{dy}{du} = \cos u$

Now, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \cos u\,(e^x) = e^x \cos(e^x)$

2. If y = log (sec x + tan x) find $\frac{dy}{dx}$

Solution:

Given, y = log (sec x + tan x)

Put u = sec x + tan x y = log u

$\frac{du}{dx}$ = sec x tan x + sec²x, $\frac{dy}{du} = \frac{1}{u}$

= sec x (tan x + sec x)

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{1}{u}$ sec x (tan x + sec x)

$= \frac{1}{(\sec x + \tan x)}$. sec x (tan x + sec x)

$\frac{dy}{dx}$ = sec x

3. If y = $\cos^4 x$, find $\frac{dy}{dx}$

Solution:

Given, y = $\cos^4 x$

Put u = cos x y = u^4

$\frac{du}{dx}$ = - sin x $\frac{dy}{du} = 4u^3$

Now, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 4u^3 \cdot (-\sin x)$

$\frac{dy}{dx} = -4\cos^3 x \sin x$

Simple Problems:

1. Find $\frac{dy}{dx}$ if $y = \frac{1}{x^4} + 7 \sin x$

Solution:

Given $y = \frac{1}{x^4} + 7 \sin x = x^{-4} + 7 \sin x$

Differentiating with respect to x

$\frac{dy}{dx} = -4x^{-5} + 7 \cos x$

2. Find $\frac{dy}{dx}$ if $y = \frac{1}{x^2} + \frac{1}{3x} - \frac{1}{\sin x} + \frac{1}{2}$

Solution:

Given $y = \frac{1}{x^2} + \frac{1}{3x} - \frac{1}{\sin x} + \frac{1}{2}$

$y = x^{-2} + \frac{x^{-1}}{3} - \operatorname{cosec} x + \frac{1}{2}$

Differentiating with respect to x

$$\frac{dy}{dx} = -2x^{-3} - \frac{x^{-2}}{3} + \text{cosec } x \cot x + 0$$

$$\frac{dy}{dx} = \frac{-2}{x^3} - \frac{1}{3x^2} + \text{cosec } x \cot x$$

3. Find $\frac{dy}{dx}$ if $y = \frac{1}{5x^3} + \frac{1}{\sec x} - \log x$

Solution:

Given, $y = \frac{1}{5}x^{-3} + \cos x - \log x$

Differentiating with respect to x

$$\frac{dy}{dx} = \frac{-3}{5}x^{-4} - \sin x - \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{-3}{5x^4} - \sin x - \frac{1}{x}$$

4. Find $\frac{dy}{dx}$ if $y = (1 + \sin x)(x - \cos x)$

Solution:

Given, $y = (1 + \sin x)(x - \cos x)$

$$\frac{dy}{dx} = (1 + \sin x)\frac{d}{dx}(x - \cos x) + (x - \cos x)\frac{d}{dx}(1 + \sin x)$$

$$\frac{dy}{dx} = (1 + \sin x)(1 + \sin x) + (x - \cos x)(\cos x)$$

5. Find $\frac{dy}{dx}$ if $y = e^x \sin x$

Solution:

Given, $y = e^x \sin x$

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(e^x)$$

$$\frac{dy}{dx} = e^x \cos x + \sin x \, e^x$$

6. Find $\frac{dy}{dx}$ if $y = x^2(1 + \log x)$

Solution:

Given, $y = x^2(1 + \log x)$

$$\frac{dy}{dx} = x^2\frac{d}{dx}(1 + \log x) + (1 + \log x)\frac{d}{dx}(x^2)$$

$$= x^2\frac{1}{x} + (1 + \log x)(2x)$$

$$\frac{dy}{dx} = x + 2x(1 + \log x)$$

7. Find $\frac{dy}{dx}$ if $y = x \sin x \log x$

Solution:

Given, $y = x \sin x \log x$

$$\frac{dy}{dx} = \sin x \log x \frac{d}{dx}(x) + x \log x \frac{d}{dx}(\sin x) + x \sin x \frac{d}{dx}(\log x)$$

$= \sin x \log x\,(1) + x \log x\,(\cos x) + x \sin x \left(\frac{1}{x}\right)$

$\frac{dy}{dx} = \sin x \log x + x \log x \cos x + \sin x$

8. Find $\frac{dy}{dx}$ if $y = (x^2 - 5) \cos x \log x$

Solution:

Given, $y = (x^2 - 5) \cos x \log x$

$\frac{dy}{dx} = \cos x \log x \frac{d}{dx}(x^2 - 5) + (x^2 - 5) \log x \frac{d}{dx}(\cos x) + (x^2 - 5) \cos x \frac{d}{dx} \log x$

$= \cos x \log x\,(2x) + (x^2 - 5) \log x\,(-\sin x) + (x^2 - 5) \cos x \frac{1}{x}$

$\frac{dy}{dx} = 2x \cos x \log x - (x^2 - 5) \log x \sin x + (x^2 - 5) \cos x \frac{1}{x}$

9. Find $\frac{dy}{dx}$ if $y = e^x \sqrt{x} \cos x$

Solution:

Given, $y = e^x \sqrt{x} \cos x$

$\frac{dy}{dx} = \sqrt{x} \cos x \frac{d}{dx}(e^x) + e^x \cos x \frac{d}{dx}(\sqrt{x}) + e^x \sqrt{x} \frac{d}{dx}(\cos x)$

$= \sqrt{x} \cos x\, e^x + e^x \cos x \frac{1}{2\sqrt{x}} + e^x \sqrt{x}\,(-\sin x)$

$\frac{dy}{dx} = \sqrt{x} \cos x\, e^x + \frac{e^x}{2\sqrt{x}} \cos x - e^x \sqrt{x} \sin x$

10. Find $\frac{dy}{dx}$ if $y = \frac{\sin x}{1-\cos x}$

Solution:

Given, $y = \frac{\sin x}{1-\cos x}$

$\frac{dy}{dx} = \frac{(1-\cos x)\frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(1-\cos x)}{(1-\cos x)^2}$

$= \frac{(1-\cos x) \cos x - \sin x\,(0 + \sin x)}{(1-\cos x)^2}$

$\frac{dy}{dx} = \frac{(1-\cos x) \cos x - \sin^2 x}{(1-\cos x)^2} = \frac{\cos x - 1}{(1-\cos x)^2} = \frac{-1}{1-\cos x}$

11. Find $\frac{dy}{dx}$ if $y = \frac{3x + 2}{5x-7}$

Solution:

Given, $y = \frac{3x + 2}{5x-7}$

$\frac{dy}{dx} = \frac{(5x - 7)\frac{d}{dx}(3x + 2) - (3x + 2)\frac{d}{dx}(5x - 7)}{(5x - 7)^2}$

$= \frac{(5x - 7)\,3 - (3x + 2)\,5}{(5x - 7)^2}$

$\frac{dy}{dx} = \frac{15x-21 - 15x - 10}{(5x - 7)^2} = \frac{-31}{(5x - 7)^2}$

12. Find $\frac{dy}{dx}$ if $y = \frac{e^x + 1}{e^x - 1}$

Solution:

Given, $y = \frac{e^x + 1}{e^x - 1}$

$$\frac{dy}{dx} = \frac{(e^x - 1)\frac{d}{dx}(e^x + 1) - (e^x + 1)\frac{d}{dx}(e^x - 1)}{(e^x - 1)^2}$$

$$\frac{dy}{dx} = \frac{(e^x - 1)e^x - (e^x + 1)e^x}{(e^x - 1)^2}$$

$$= \frac{(e^x)^2 - e^x - (e^x)^2 - e^x}{(e^x - 1)^2}$$

$$\frac{dy}{dx} = \frac{-2e^x}{(e^x - 1)^2}$$

13. Find $\frac{dy}{dx}$ if $y = \log(1 + \sin x)$

Solution:

Given, $y = \log(1 + \sin x)$

Put $u = 1 + \sin x$ $\qquad$ $y = \log u$

$\frac{du}{dx} = \cos x$ $\qquad$ $\frac{dy}{du} = \frac{1}{u}$

Now, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{1}{u} \cos x$

$$\frac{dy}{dx} = \frac{\cos x}{1 + \sin x}$$

14. Find $\frac{dy}{dx}$ if $y = \cos(x \log x)$

Solution:

Given, $y = \cos(x \log x)$

Put $u = x \log x$ $\qquad$ $y = \cos u$

$\frac{du}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$ $\qquad$ $\frac{dy}{du} = -\sin u$

$= 1 + \log x$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= -\sin u (1 + \log x)$

$$\frac{dy}{dx} = -\sin(x \log x)(1 + \log x)$$

15. Find $\frac{dy}{dx}$ if $y = (x^2 + x + 1)^4$

Solution:

Given, $y = (x^2 + x + 1)^4$

Put $u = x^2 + x + 1$, $\qquad$ $y = u^4$

$\frac{du}{dx} = 2x + 1$, $\qquad$ $\frac{dy}{du} = 4u^3$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 4u^3 (2x + 1)$$

$$\frac{dy}{dx} = 4(x^2 + x + 1)^3 (2x + 1)$$

EXERCISE

Differentiate the following with respect to x

1. $x^2 - 5x + 7$
2. $2 \sec x + 3 \cot x$
3. $\frac{1}{x^2} + \frac{2}{x} + \frac{3}{2}$
4. $x + \frac{1}{x}$
5. $e^x \log x$
6. $x^3 \log x$
7. $(1 + x^2)e^x$
8. $\frac{2x+3}{3x-2}$
9. $\frac{1+\cos x}{1-\cos x}$
10. $\frac{x+\cos x}{9+\sin x}$
11. $\frac{1+\tan x}{x-\sin x}$
12. e^{x^2}
13. $(3x + 6)^5$
14. $\text{Sin}^2 x$
15. $\text{Cos}^3 x$
16. $\log(\sec x)$
17. $\text{Sec}(\log x)$
18. $(2x^2 - 3x + 1)^2$
19. $\frac{5}{x^3} + \frac{2}{x^2} + \frac{7}{x} + \frac{3}{\cos x} + \frac{1}{3}$
20. $7e^x + 15\sqrt{x} + \frac{3}{x} - \frac{5}{x^2} + 11$
21. $\frac{x^6 - 4x^3 + 2x - 7}{x}$
22. $\frac{2x^3 + 3x^2 - 5x + 6}{x^2}$
23. $x^{-2}(1 + e^x)$
24. $x^{10} \tan x$
25. $(x - \cos x)(e^x + \tan x)$
26. $(x^2 + 2)(3 + \cot x)$
27. $xe^x \log x$
28. $e^x \log x \cos x$
29. $x^{-4} \sec x \log x$
30. $(3x^2 - 7x + 5) e^x \cot x$
31. $x^2 e^x \operatorname{cosec} x$
32. $\frac{x+\tan x}{5+\cot x}$
33. $\frac{2\sin x + 5\cos x}{3\sin x - 4\cos x}$
34. $\frac{x\log x}{e^x + \cot x}$
35. $\frac{x^2+3x}{x\cos x}$
36. $\frac{3\operatorname{cosec} x}{4+5\sec x}$
37. $\frac{x^2+\sin x}{\cos x - \sin x}$
38. $\log(\operatorname{cosec} x - \cot x)$
39. $\sqrt{\frac{1-\cos x}{1+\cos x}}$
40. $\text{Log}\left(\frac{1+\sin x}{1-\sin x}\right)$

Chapter 4.3: DIFFERENTIATION METHODS

Inverse Trigonometric functions

The inverse Trigonometric functions are the inverse functions of the trigonometric functions. If $x = \sin y$, then $y = \sin^{-1} x$ is the inverse function of $\sin x$. Similarly we can define $\cos^{-1} x$, $\tan^{-1} x$, $\text{cosec}^{-1} x$ and $\cot^{-1} x$, are the inverse trigonometric functions of cos x, tan x, cosec x and cot x.

Problems:

1. If $y = \sin^{-1}(\sqrt{x})$, find $\frac{dy}{dx}$

Solution:

Given, $y = \sin^{-1}(\sqrt{x})$

Put $u = \sqrt{x}$ $\quad y = \sin^{-1}(u)$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \qquad \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \quad \frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x}}\left(\frac{1}{2\sqrt{x}}\right)$$

2. Find $\frac{dy}{dx}$ if $y = \tan^{-1}(x^2)$

Solution:

Given, $y = \tan^{-1}(x^2)$

Put $u = x^2$ $\quad y = \tan^{-1}(u)$

$$\frac{du}{dx} = 2x \qquad \frac{dy}{du} = \frac{1}{1+u^2}$$

Now, $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{1+u^2} \quad 2x$$

$$\frac{dy}{dx} = \frac{1}{1+(x^2)^2} \quad 2x$$

$$\frac{dy}{dx} = \frac{2x}{1+x^4}$$

3. If $y = \tan^{-1}(\sec x)$, find $\frac{dy}{dx}$

Solution:

Given, $y = \tan^{-1}(\sec x)$

Put $u = \sec x$ $\quad y = \tan^{-1}(u)$

$$\frac{du}{dx} = \sec x \tan x \qquad \frac{dy}{du} = \frac{1}{1+u^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{1+u^2} \sec x \tan x$$

$$\frac{dy}{dx} = \frac{\sec x \tan x}{1+\sec^2 x}$$

4. If $y = \cos^{-1}(\sin x)$, find $\frac{dy}{dx}$

Solution:

Given, $y = \cos^{-1}(\sin x)$

Put $u = \sin x$ $\qquad$ $y = \cos^{-1}(u)$

$\frac{du}{dx} = \cos x$ $\qquad$ $\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}}$

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= -\frac{1}{\sqrt{1-u^2}} \cos x$$

$$\frac{dy}{dx} = \frac{-\cos x}{\sqrt{1-\sin^2 x}}$$

$$\frac{dy}{dx} = \frac{-\cos x}{\cos x} = -1$$

Substitution Method

Differentiation of certain functions seem to be very difficult, but by suitably substituting the independent variable with some trigonometric function or other functions, they can be differentiated easily. This method is called the Differentiation by substitution.

Consider the function $f(x) = \sqrt{a^2 + x^2}$

For this function $f'(x)$ can be found out by using function of function rule or Chain rule. But it is laborious. Instead we can use the substitution method.

5. Differentiate the following with respect to x.

(i) $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ $\qquad$ (ii) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ $\qquad$ (iii) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Solution:

Given, $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta$

$\Rightarrow \theta = \tan^{-1} x$

$\therefore y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$y = \sin^{-1}(\sin 2\theta)$ $\qquad$ $[\because \sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}]$

$y = 2\theta$

$y = 2\tan^{-1} x$

Differentiating with respect to x, we get

$$\frac{dy}{dx} = \frac{2}{1+x^2}$$

(ii) Given, $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta$

$\Rightarrow \theta = \tan^{-1} x$

$\therefore y = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$y = \cos^{-1}(\cos 2\theta)$ $\quad [\because \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}]$

$y = 2\theta$

$y = 2\tan^{-1} x$

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{2}{1+x^2}$

(iii) Given, $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Put $x = \tan\theta$

$\Rightarrow \theta = \tan^{-1} x$

$\therefore y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

$y = \tan^{-1}(\tan 2\theta)$ $\quad [\because \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}]$

$y = 2\theta$

$y = 2\tan^{-1} x$

Differentiating with respect to x, we get

$\frac{dy}{dx} = \frac{2}{1+x^2}$

6. If $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$, find $\frac{dy}{dx}$

Solution:

Given, $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

Put $x = \tan\theta$

$\Rightarrow \theta = \tan^{-1} x$

$\therefore y = \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$

$y = \tan^{-1}(\tan 3\theta)$ $\quad [\because \tan 3\theta = \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}]$

$y = 3\theta$

$y = 3\tan^{-1} x$

Differentiating w.r.to x, we get

$\frac{dy}{dx} = \frac{3}{1+x^2}$

Differentiation of Implicit functions

If the relation between x and y is given by an equation of the form f (x, y) = 0 and this equation is not easily solvable for y, then y is said to be an implicit function of x.

Explicit function of x

In case y is given in terms of x, then y is said to be an explicit function of x.

Implicit Differentiation process

This process consists of differentiating both sides of an equation with respect to x, using the rules of differentiation and then solving for $\frac{dy}{dx}$.

Problems:

1. Find $\frac{dy}{dx}$ for $xy = c^2$

Solution:

Given, $xy = c^2$

Differentiating both sides with respect to x

$$x\frac{dy}{dx} + y(1) = 0$$

$$x\frac{dy}{dx} + y = 0$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = \frac{-y}{x}$$

2. Find $\frac{dy}{dx}$ if $x^2 \sin y = c$

Solution:

Given, $x^2 \sin y = c$

Differentiating both sides with respect to x

$$x^2 \cos y \frac{dy}{dx} + \sin y\,(2x) = 0$$

$$x^2 \cos y \frac{dy}{dx} = -2x \sin y$$

$$\frac{dy}{dx} = \frac{-2x\sin y}{x^2 \cos y}$$

$$\frac{dy}{dx} = \frac{-2}{x}\tan y$$

3. Find $\frac{dy}{dx}$ if $x^2 + y^2 = a^2$

Solution:

Given, $x^2+y^2 = a^2$

Differentiating both sides with respect to x

$2x + 2y\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} = -2x$

$\frac{dy}{dx} = \frac{-2x}{2y}$

$\frac{dy}{dx} = \frac{-x}{y}$

4. Find $\frac{dy}{dx}$ if $x^3 + y^3 = 3\,axy$

Solution:

Given, $x^3+y^3 = 3\,axy$

Differentiating both sides with respect to x

$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y.1\right]$

$3x^2 + 3y^2\frac{dy}{dx} = 3ax\frac{dy}{dx} + 3\,ay$

$3y^2\frac{dy}{dx} - 3ax\frac{dy}{dx} = 3ay - 3x^2$

$(3y^2 - 3ax)\frac{dy}{dx} = 3ay - 3x^2$

$\frac{dy}{dx} = \frac{3\,(ay - x^2)}{3\,(y^2 - ax)}$

$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$

5. Find $\frac{dy}{dx}$ if $x^2 + y^2 + 2gx + 2fy + c = 0$

Solution:

Given, $x^2+y^2 + 2gx + 2fy + c = 0$

Differentiating both sides with respect to x

$2x + 2y\frac{dy}{dx} + 2g\,(1) + 2f\frac{dy}{dx} + 0 = 0$

$2x + 2y\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$

$2y\frac{dy}{dx} + 2f\frac{dy}{dx} = -2x - 2g$

$(2y + 2f)\frac{dy}{dx} = -2x - 2g$

$\frac{dy}{dx} = \frac{-2\,(x+g)}{2\,(y+f)}$

$\frac{dy}{dx} = \frac{-(x+g)}{y+f}$

6. Find $\frac{dy}{dx}$ if $ax^2 + 2hxy + by^2 = 0$

Solution:

Given, $ax^2+2hxy + by^2 = 0$

Differentiating both sides with respect to x

$$a(2x) + 2h\left[x\frac{dy}{dx} + y.1\right] + b(2y)\frac{dy}{dx} = 0$$

$$2ax + 2hx\frac{dy}{dx} + 2hy + 2by\frac{dy}{dx} = 0$$

$$2hx\frac{dy}{dx} + 2by\frac{dy}{dx} = -2ax - 2hy$$

$$(2hx + 2by)\frac{dy}{dx} = -2ax - 2hy$$

$$\frac{dy}{dx} = \frac{-2(ax + hy)}{2(hx+by)}$$

$$\frac{dy}{dx} = \frac{-(ax + hy)}{hx + by}$$

7. Find $\frac{dy}{dx}$ if $x \sin y + y \sin x = 0$

Solution:

Given, $x \sin y + y \sin x = 0$

Differentiating both sides with respect to x

$$x\cos y\frac{dy}{dx} + \sin y\,(1) + y\cos x + \sin x\frac{dy}{dx} = 0$$

$$(x\cos y + \sin x)\frac{dy}{dx} = -\sin y - y\cos x$$

$$\frac{dy}{dx} = -\frac{(\sin y + y\cos x)}{x\cos y + \sin x}$$

Differentiation of Logarithmic Functions

Consider the function $y = x^x$.

In order to find the derivative of the power – exponential function, take logarithm on both sides, we get

$$\log y = \log x^x$$

$$\log y = x \log x, \qquad x > 0$$

Differentiating both sides with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \log x\,(1) + x\left(\frac{1}{x}\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \log x + 1$$

Hence $\frac{dy}{dx} = y(\log x + 1)$

$$\frac{dy}{dx} = x^x(\log x + 1)$$

The operation consists of first taking the logarithm of the function f (x) (to base e) then differentiating is called logarithmic differentiation.

$$\therefore \frac{d}{dx}[\log(f(x))] = \frac{f'(x)}{f(x)}$$

is called the logarithmic derivative of f (x).

The advantage in this method is that the calculation of derivatives of complicated functions involving products, quotients or powers can often be simplified by taking logarithms.

This method is useful for the function having following forms.

(i) $y = [f(x)]^{g(x)}$

(ii) $y = \frac{f_1(x)\, f_2(x) \ldots\ldots}{g_1(x)\, g_2(x) \ldots\ldots}$ where $g_i(x) \neq 0$ and $i = 1,2,3.....$ and $f_i(x)$ and $g_i(x)$ are both differentiable.

1. Differentiate : $y = x^{\sqrt{x}}$

Solution:

Given, $y = x^{\sqrt{x}}$

Taking logarithm on both sides

$$\log y = \sqrt{x} \log x$$

Differentiating implicitly

$$\frac{1}{y}\frac{dy}{dx} = \sqrt{x}\left(\frac{1}{x}\right) + \log x\left(\frac{1}{2\sqrt{x}}\right)$$

$$\frac{dy}{dx} = \left(\frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{x}}\log x\right) y$$

$$\frac{dy}{dx} = \left(\frac{2 + \log x}{2\sqrt{x}}\right) x^{\sqrt{x}}$$

2. Differentiate : $y = x^{1/x}$

Solution:

Given, $y = x^{1/x}$

Taking logarithm on both sides

$$\log y = \frac{1}{x}\log x$$

Differentiating implicitly

$$\frac{1}{y}\frac{dy}{dx} = \log x\left(\frac{-1}{x^2}\right) + \frac{1}{x}\left(\frac{1}{x}\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{-\log x + 1}{x^2}$$

$$\frac{dy}{dx} = \frac{y}{x^2}\,(1 - \log x)$$

$$\frac{dy}{dx} = \frac{x^{1/x}}{x^2}\,(1 - \log x)$$

$$\frac{dy}{dx} = x^{\frac{1}{x} - 2}\,(1 - \log x)$$

$$\frac{dy}{dx} = x^{\left(\frac{1-2x}{x}\right)}\,(1 - \log x)$$

3. Differentiate : $y = x^{\log x}$

Solution:

Given, $y = x^{\log x}$

Taking logarithm on both sides

$$\log y = \log x \cdot \log x$$

Differentiating implicitly

$$\frac{1}{y}\frac{dy}{dx} = \log x\left(\frac{1}{x}\right) + \log x\left(\frac{1}{x}\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2\log x}{x}$$

$$\frac{dy}{dx} = y\left(\frac{2\log x}{x}\right)$$

$$\frac{dy}{dx} = x^{\log x}\left(\frac{2\log x}{x}\right)$$

4. Differentiate : $y = (e^x)^{e^x}$

Solution:

Given, $y = (e^x)^{e^x}$

Taking logarithm on both sides

$$\log y = e^x \log e^x$$

$$\log y = e^x \cdot x$$

Differentiating implicitly

$$\frac{1}{y}\frac{dy}{dx} = e^x(1) + xe^x$$

$$\frac{1}{y}\frac{dy}{dx} = e^x(1 + x)$$

$$\frac{dy}{dx} = y\,[e^x(1 + x)]$$

$$= (e^x)^{e^x}\,[e^x(1 + x)]$$

$$\frac{dy}{dx} = (e^x)^{e^x+1}(1 + x)$$

5. Differentiate : $y = (x-1)^2 (x-3)^5$

Solution:

Given, $y = (x-1)^2 (x-3)^5$

Taking logarithm on both sides

$\log y = \log [\,(x - 1)^2 (x-3)^5\,]$ (since $\log (mn) = \log m + \log n$)

$\log y = \log (x-1)^2 + \log (x-3)^5$

$\log y = 2\log (x-1) + 5 \log (x-3)$

Differentiating implicitly

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x-1} + \frac{5}{x-3}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x-1} + \frac{5}{x-3}\right)$$

$$\frac{dy}{dx} = (x-1)^2 (x-3)^5\left(\frac{2}{x-1} + \frac{5}{x-3}\right)$$

6. Differentiate : $y = \frac{(x-2)^2}{(x+5)^3}$

Solution:

Given, $y = \frac{(x-2)^2}{(x+5)^3}$

Taking logarithm on both sides

$\log y = \log\left[\frac{(x-2)^2}{(x+5)^3}\right]$ (since $\log\left(\frac{m}{n}\right) = \log m - \log n$)

$\log y = \log (x-2)^2 - \log (x+5)^3$

$\log y = 2 \log (x-2) - 3 \log (x+5)$

Differentiating implicitly

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x-2} - \frac{3}{x+5}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x-2} - \frac{3}{x+5}\right)$$

$$\frac{dy}{dx} = \frac{(x-2)^2}{(x+5)^3}\left[\frac{2}{x-2} - \frac{3}{x+5}\right]$$

Differentiation of Parametric Functions

Consider the equations x = f (t), y = g (t).

If two variables x and y are defined separately as a function of an intermediating variable t, then the specification of a functional relationship between x and y is described as parametric and the intermediate variable is known as parameter.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \quad \therefore \frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

Problems:

1. Find $\frac{dy}{dx}$ if x = a (t – sin t), y = a (1 – cos t)

Solution:

Given, x = a (t – sin t), y = a (1 – cos t)

Differentiating with respect to t,

$$\frac{dx}{dt} = a(1-\cos t), \frac{dy}{dt} = a \sin t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{a \sin t}{a(1-\cos t)}$$

$$\frac{dy}{dx} = \frac{\sin t}{1-\cos t}$$

2. Find $\frac{dy}{dx}$ if $x = a\cos^3 t,\ y = a\sin^3 t$

Solution:

Given, $x = a\cos^3 t,\ y = a\sin^3 t$

Differentiating with respect to t,

$$\frac{dx}{dt} = 3a\cos^2 t(-\sin t) \qquad \frac{dy}{dt} = 3a\sin^2 t(\cos t)$$

$$\frac{dx}{dt} = -3a\cos^2 t\sin t \qquad \frac{dy}{dt} = 3a\sin^2 t\cos t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\frac{\sin t}{\cos t}$$

$$\frac{dy}{dx} = -\tan t$$

3. Find $\frac{dy}{dx}$ if $x = r\cos t,\ y = r\,\mathrm{sint}$

Solution:

Given, $x = r\,\mathrm{cost}, \quad y = r\sin t$

Differentiating with respect to t,

$$\frac{dx}{dt} = -r\sin t, \qquad \frac{dy}{dt} = r\cos t$$

Now, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$$= \frac{r\cos t}{-r\sin t}$$

$$\frac{dy}{dx} = -\cot t$$

4. Find $\frac{dy}{dx}$ if $x = \sec\theta, \quad y = \log(\sec\theta + \tan\theta)$

Solution:

Given, $x = \sec\theta,\ y = \log(\sec\theta + \tan\theta)$

Differentiating with respect to θ,

$$\frac{dx}{d\theta} = \sec\theta\tan\theta, \qquad \frac{dy}{d\theta} = \frac{1}{\sec\theta + \tan\theta}(\sec\theta\tan\theta + \sec^2\theta)$$

$$= \frac{\sec\theta(\tan\theta + \sec\theta)}{\sec\theta + \tan\theta}$$

$$\frac{dy}{d\theta} = \sec\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sec\theta}{\sec\theta\tan\theta}$$

$$\frac{dy}{dx} = \frac{1}{\tan\theta} = \cot\theta$$

5. Find $\frac{dy}{dx}$ if $x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$

Solution:

Given, $x = a(\theta - \sin\theta), \quad y = a(1 - \cos\theta)$

Differentiating with respect to θ,

$\frac{dx}{d\theta} = a(1 - \cos\theta)$ $\qquad$ $\frac{dy}{d\theta} = a(0 + \sin\theta) = a\sin\theta$

$\frac{dx}{d\theta} = a \, . \, 2\sin^2\theta/2$ $\qquad$ $\frac{dy}{d\theta} = a \, . \, 2\sin\theta/2 \cos\theta/2$

$\frac{dx}{d\theta} = 2a\sin^2\theta/2$ $\qquad$ $\frac{dy}{d\theta} = 2a\sin\theta/2 \; \cos\theta/2$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{2a\sin\theta/2 \; \cos\theta/2}{2a\sin^2\theta/2} = \frac{\cos\theta/2}{\sin\theta/2}$$

$$\frac{dy}{dx} = \cot\theta/2$$

Differentiation of one function with respect to another function

If $y = f(x)$ is differentiable, then the derivative of y with respect to x is

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

If u and v are differentiable functions of x and if $\frac{dv}{dx} = v'(x) \neq 0$, then

$$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{u'(x)}{v'(x)}$$

Problems:

1. Find the derivative of x^x with respect to x log x.

Solution:

Take $u = x^x$, $\qquad v = x\log x$

$u = x^x$

Taking log on both sides

$\log u = x\log x$

Differentiating with respect to x

$$\frac{1}{u}\frac{du}{dx} = x\left(\frac{1}{x}\right) + (1)\log x$$

$$\frac{1}{u}\frac{du}{dx} = 1 + \log x$$

$$\frac{du}{dx} = (1 + \log x)u$$

$$\frac{du}{dx} = (1 + \log x)x^x$$

$v = x\log x$

Differentiating with respect to x

$$\frac{dv}{dx} = x\left(\frac{1}{x}\right) + (1)\log x$$

$$\frac{dv}{dx} = 1 + \log x$$

$$\frac{du}{dv} = \frac{du/dx}{dv/dx}$$

$$= \frac{(1+\log x)x^x}{1+\log x}$$

$$\frac{du}{dv} = x^x$$

2. Differentiate x^6 with respect to x^2.

Solution:

Take $u = x^6$, $v = x^2$

$$\frac{du}{dx} = 6x^5, \qquad \frac{dv}{dx} = 2x$$

$$\frac{du}{dv} = \frac{du/dx}{dv/dx}$$

$$= \frac{6x^5}{2x}$$

$$\frac{du}{dv} = 3x^4$$

3. Find derivative of $\sin(ax^2 + bx + c)$ with respect to $\cos(lx^2 + mx + n)$

Solution:

Take $u = \sin(ax^2 + bx + c)$, $v = \cos(lx^2 + mx + n)$

$$\frac{du}{dx} = \cos(ax^2 + bx + c)(2ax + b), \qquad \frac{dv}{dx} = -\sin(lx^2 + mx + n)(2lx + m)$$

$$\frac{du}{dv} = \frac{du/dx}{dv/dx}$$

$$\frac{du}{dx} = \frac{(2ax + b)\cos(ax^2 + bx + c)}{-(2lx + m)\sin(lx^2 + mx + n)}$$

4. Differentiate $\sin x$ with respect to $\cos x$

Solution:

Take $u = \sin x$, $v = \cos x$

$$\frac{du}{dx} = \cos x \qquad \frac{dv}{dx} = -\sin x$$

$$\frac{du}{dv} = \frac{du/dx}{dv/dx}$$

$$= \frac{\cos x}{-\sin x}$$

$$\frac{du}{dv} = -\cot x$$

5. Find the derivative of $\tan^{-1}(1 + x^2)$ with respect to $x^2 + x + 1$

Solution:

Take $u = \tan^{-1}(1 + x^2)$, $v = x^2 + x + 1$

$$\frac{du}{dx} = \frac{1}{1 + (1 + x^2)^2}(0 + 2x) \qquad \frac{dv}{dx} = 2x + 1$$

$$= \frac{2x}{1 + 1 + 2x^2 + x^4}$$

$$\frac{du}{dx} = \frac{2x}{x^4 + 2x^2 + 2}$$

$$\frac{du}{dv} = \frac{du/dx}{dv/dx}$$

$$\frac{du}{dv} = \frac{2x}{(x^4 + 2x^2 + 2)\,(2x+1)}$$

EXERCISE

Find $\frac{dy}{dx}$ for the following

1. $y = \tan^{-1}(\sqrt{x})$
2. $y = \sin^{-1}(3x)$
3. $y = e^{\sin^{-1}x}$
4. $y = \cos^{-1}(2x - 1)$
5. $y = \tan^{-1}(2x)$
6. $y = \cos^{-1}(\log x)$
7. $y = \sin^{-1}(\sin x)$
8. $y = \cot^{-1}(\cos x)$
9. $y = x^2 \sin^{-1} x$
10. $y = \cos^{-1}(\sqrt{x})$
11. $y = \cot^{-1}(\sec x)$
12. $y = e^x \tan^{-1} x$
13. $y = \cos^{-1}(1 - 2x)$

Find $\frac{dy}{dx}$ for the following

14. $y^2 = 4ax$
15. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
16. $x^2 + y^2 - 4x + 6y - 21 = 0$
17. $y^2 = x \sin y$
18. $y = a + xe^y$
19. $x^2 + y^2 = xy$
20. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$
21. $x^3 + y^3 = 3xy$
22. $x^2 + 3xy + y^2 = 4$
23. $x^2 + 2xy + y^2 = 0$
24. $y = x \sin(a + y)$
25. $y = x^{\cos x}$
26. $x^y = y^x$
27. $\sqrt{xy} = e^{(x-y)}$
28. $(\cos x)^{\log x}$
29. $x = a\cos\theta,\ \ y = a \sin\theta$
30. $x = a\sec\theta,\ \ y = b\tan\theta$
31. $x = a\sec^3 t,\ y = b\tan^3 t$
32. $x = 5t, \quad y = \frac{5}{t}$
33. $x = a\,(t + \cos t),\ \ y = a(1 + \sin t)$
34. $x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}$
35. $x = a\,(\cos t + t\sin t)\ ;\ y = a\,(\sin t - t\cos t)$
36. If $x = a\,(\theta + \sin\theta)$ $y = a\,(1 - \cos\theta)$ then show that $\frac{dy}{dx} = \tan\frac{\theta}{2}$
37. If $x = 3\sin t - \sin^3 t\ ;\ \ y = 3\cos t - \cos^3 t$ then prove that $\frac{dy}{dx} = -\tan^3 t$
38. Find the derivative of $\sin(x^2)$ with respect to x^2
39. Find the derivative of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\tan^{-1}x$
40. Find the derivative of $\log(\sin x)$ with respect to $\sqrt{\cos x}$
41. Find the derivative of $\log(1 + x^2)$ with respect to $\tan^{-1}x$
42. Find the derivative of $e^x\cos x$ with respect to $e^x \sin x$
43. Find the derivative of $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$ with respect to $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

Limits

Introduction to Differentiation

Differentiation Methods

Related Problems (Differentiation Methods)

Related Problems (Chain Rule)

UNIT - V DIFFERENTIAL CALCULUS-II

Chapter 5.1: SUCCESSIVE DIFFERENTIATION

The process of differentiating the same function again and again is called Successive Differentiation.

First Order Derivative:

If y = f (x) is a differentiable function then its derivative is

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The derivative of y with respect to x is called the first order derivative.

Second Order Derivative:

If f ' (x) is a differentiable, then its derivative is

$$\frac{d^2y}{dx^2} = f''(x) = \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x}$$

If f ' (x) is differentiable, then differentiating $\frac{dy}{dx}$ again with respect to x is called as second order derivative.

i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = f''(x)$

Similarly higher order derivatives can also be defined in the same way like $\frac{d^3y}{dx^3}$ represents third order derivative, $\frac{d^4y}{dx^4}$ represents fourth order derivative and so on.

Note:

$\frac{d^ny}{dx^n} = f^n(x)$ is called the n^{th} order derivative of y = f (x).

$\therefore f^n(x) = \frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$

Common notation of higher order derivatives of y = f (x).

1^{st} Derivative : $f'(x)$ or y' or y_1 or $\frac{dy}{dx}$ or Dy.

2^{nd} Derivative : $f''(x)$ or y'' or y_2 or $\frac{d^2y}{dx^2}$ or D^2y.

"

"

n^{th} Derivative : $f^n(x)$ or $y^{(n)}$ or y_n or $\frac{d^ny}{dx^n}$ or D^ny.

Worked Examples

1) If $y = 3x^2 + 2x + 8$ then find $\frac{d^2y}{dx^2}$

Solution:

$$y = 3x^2 + 2x + 8$$

Diff. w. r. to x,

$$\frac{dy}{dx} = 6x + 2 + 0$$

Again differentiate with respect to x

$$\frac{d^2y}{dx^2} = 6\,(1) + 0$$

$$\frac{d^2y}{dx^2} = 6$$

2) If $y = \frac{2}{3}x^2 - \frac{1}{3}x + 7$ then find y_2

Solution:

$$y = \frac{2}{3}x^2 - \frac{1}{3}x + 7$$

Diff. w. r. to x,

$$y_1 = \frac{2}{3}(2x) - \frac{1}{3}(1) + 0$$

Again Diff with respect to x

$$y_2 = \frac{2}{3}(2) - 0$$

$$y_2 = \frac{4}{3}$$

3) If $y = x^3 + \tan x$ find $\frac{d^2y}{dx^2}$

Solution:

$$y = x^3 + \tan x$$

Differentiate with respect to x on both sides,

$$\frac{dy}{dx} = 3x^2 + \sec^2 x$$

Again Differentiate with respect to x on both sides,

$$\frac{d^2y}{dx^2} = 6x + 2\sec x \sec x \tan x$$

$$\frac{d^2y}{dx^2} = 6x + 2\sec^2 x \tan x$$

4) Find $\frac{d^2y}{dx^2}$ of $y = \sin 7x$

Solution:

$$y = \sin 7x$$

Differentiate with respect to x on both sides,

$$\frac{dy}{dx} = \cos 7x \,.\, (7) = 7\cos 7x$$

Again Differentiate with respect to x on both sides

$$\frac{d^2y}{dx^2} = 7\,(-\sin 7x\,.\,(7))$$

$$\frac{d^2y}{dx^2} = -49 \sin 7x$$

5) If y = A sin x + B cos x then prove that $\frac{d^2y}{dx^2} + y = 0$

Solution:

Given, $y = A\sin x + B\cos x$ ________ (1)

Then $\frac{dy}{dx} = A\cos x + B(-\sin x)$

$\frac{dy}{dx} = A\cos x - B\sin x$ ________ (2)

$\frac{d^2y}{dx^2} = -A\sin x - B\cos x$

$\frac{d^2y}{dx^2} = -[A\sin x + B\cos x]$ ________ (3)

$\frac{d^2y}{dx^2} = -y$ [$\because$ from $y = A\sin x + B\cos x$]

$\frac{d^2y}{dx^2} + y = 0$

Hence it is proved.

6) Find the second order derivative of the function $e^{6x}\cos 3x$.

Solution:

let $y = e^{6x}\cos 3x$ ________ (1)

$\frac{dy}{dx} = e^{6x}(-\sin 3x\,.\,3) + \cos 3x\, e^{6x}\,.\,6$

$= e^{6x}\sin 3x\,(-3) + 6\cos 3x\, e^{6x}$

$\frac{dy}{dx} = -3\,e^{6x}\sin 3x + 6\cos 3x\, e^{6x}$

$\frac{d^2y}{dx^2} = -3\,[\,e^{6x}\cos 3x\,(3) + \sin 3x\, e^{6x}\,(6)\,] +$

$6\,[\cos 3x\, e^{6x}\,(6) + e^{6x}\,(-\sin 3x)\,(3)\,]$

$= -9\,e^{6x}\cos 3x - 18\sin 3x\, e^{6x} +$

$36e^{6x}\cos 3x - 18\sin 3x\, e^{6x}$

$\frac{d^2y}{dx^2} = 27\,e^{6x}\cos 3x - 36\sin 3x\, e^{6x}$

7) If $y = \sin^{-1}x$ show that, $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$

Solution:

Given $y = \sin^{-1}x$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\left(\sqrt{1-x^2}\right)\frac{dy}{dx} = 1$

Differentiate with respect to x on both side,

$$\sqrt{1-x^2}\ \frac{d}{dx}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\ .\ \frac{d}{dx}\ \left(\sqrt{1-x^2}\right) = 0$$

$$\sqrt{1-x^2}\ \frac{d^2y}{dx^2} + \frac{dy}{dx}\ \frac{1}{2}\ \left(\sqrt{1-x^2}\right)^{-½}(-\ 2x) = 0$$

$$\sqrt{1-x^2}\ \frac{d^2y}{dx^2} + \frac{dy}{dx}\ .\left(\frac{-x}{\sqrt{1-x^2}}\right) = 0$$

Multiplying all the terms by $\sqrt{1-x^2}$

$$(1 - x^{2})\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

Hence proved.

8) If $y = e^x (x + 1)$ then find y_2.

Solution:

$y = e^x (x + 1)$

$y_1 = e^x (1) + (x + 1) e^x$

$y_2 = e^x + (x + 1) e^x + e^x (1)$

$y_2 = 2e^x + (x + 1) e^x$

$y_2 = e^x\ [\ 2 + x + 1\ ]$

$y_2 = e^x\ [\ x + 3\ ]$

Part – B

Exercise Problem

1) Find the second order derivative of the following function.

(a) $x^2 + 6x - 15$

(b) $e^{2x} \cos 3x$

(c) $\tan^{-1} x$

(d) $x \sin 2x$

2) If $y = 5 \cos x - 3 \sin x$ prove that $\frac{d^2y}{dx^2} + y = 0$

3) If $y = 500\ e^{7x} + 600\ e^{-7x}$ show that $\frac{d^2y}{dx^2} - (m + n)\frac{dy}{dx} + mny = 0$

4) If $y = (\tan^{-1} x)^2$. show that $(x^2 + 1)^2 y_2 + 2x (x^2 + 1) y_1 = 2$.

5) If $y = x \cos x$ prove that $x^2y_2 - 2xy_1 + (x^2 + 2) y = 0$

6) If $xy = ae^x + be^{-x}$ prove that $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy$

7) If $y = \frac{\log x}{x^2}$ prove that $x^2y_2 + 5xy_1 + 4xy = 0$

8) If $y = e^x \sin x$ prove that, $y_2 - 2y_1 + 2y = 0$

9) The speed of a boat is given by $V = k\ (1/t - at)$ where k is the constant t is time, a is the acceleration of water, then find the second derivative of the given equation.

10) If the position of the particle is given by in the form $y = 2x^3 + 3x^2 + 6$ then find the second order derivative of the equation.

Definition:

Differential Equation:

An equation involving independent variable, dependent variable, derivatives of dependent variable with respect to independent variable and constants is called a differential equation.

Consider the function y = f (x) then $\frac{dy}{dx} = f'(x)$

Here x is an independent variable

y is a dependent variable

f ' (x) is derivative of dependent variable with respect to independent variable.

A differential equation is an equation with a function, y = f (x) and one or more of its derivatives, $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ ……….

Differential equations are of two types.

1. Ordinary Differential Equation
2. Partial Differential Equation

Ordinary Differential Equation:

An equation involving derivatives of the dependent variable with respect to only one independent variable is called an Ordinary Differential Equation.

In differential equations, order and degree are the main parameters for classifying different types of differential equations.

Order of a Differential Equation:

The order of a differential equation is defined as the order of highest derivative in the equation.

Consider the equation, $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3 = 0$.

Here $\frac{d^2y}{dx^2}$ is the highest order derivative.

∴ The order of the differential equation is 2.

Degree of Differential Equations

The degree of a differential equation is the highest power (or degree) of the derivative of the highest order of differential equation, after the equation is cleared of radicals or fractional power in its derivatives.

Consider the equation $3\left(\frac{d^2y}{dx^2}\right) = \left[4 + \left(\frac{dy}{dx}\right)\right]^3$

Here the highest order derivative is $\frac{d^2y}{dx^2}$

Degree of the highest order of derivative is 1.

∴ Degree of Differential equation is 1.

MATH FACT

- Order and degree of a differential equation are always positive integers.
- Order of the differential equation, cannot be more than the number of arbitrary constants in the equation.

Part – A

Exercise

1) For each of the following differential equations. Find the order and degree.

a) $\frac{d^2y}{dx^2}$ - 5y = $\left(\frac{dy}{dx}\right)^3$

b) $\sqrt{\frac{dy}{dx}}$ - 4 $\frac{dy}{dx}$ – 7x = 0

c) $\frac{d^2y}{dx^2}$ = xy + cos $\left(\frac{dy}{dx}\right)$

d) $\frac{dy}{dx}$ + xy = cot x

e) x = e^{xy} (dy/dx)

f) y $\left(\frac{dy}{dx}\right)^3$ = $\sqrt{1 + \left(\frac{d^2y}{dx^2}\right)^3}$

g) $\left(\frac{d^2y}{dx^2}\right)^3$ + 5 $\left(\frac{dy}{dx}\right)$ + 3y = 0

h) y'' = y – y' + $2(y')^2$

i) y' + y^2 = x + $\frac{3}{2}$

j) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$ = $\frac{d^2y}{dx^2}$

Formation of Ordinary Differential Equation:

Consider the equation f (x, y, C_1, C_2 …. C_n) = 0

Where x and y are the variables

C_1, C_2 …. C_n are the arbitrary constants.

Steps to form the differential equation.

- Differentiate the equation ‘n’ times with respect to x, we will get ‘n’ more relations between x, y, C_1, C_2 …. C_n and derivatives of y with respect to x.
- Eliminate the constants C_1, C_2 …. C_n from there ‘n’ relations and given equation.

Suppose we have two arbitrary constants C_1, C_2 in the equation f (x, y, C_1, C_2) = 0 then follow the steps to get the Differential equation.

Step 1	Find the first two successive derivatives.
Step 2	Eliminate C_1 and C_2 from the given function and the successive derivatives.

Note:

The order of the Differential equation to be formed is equal to the number of arbitrary constants present in the equation of the family of curves.

Model: 1 (Newton's Law)

We know from the Newton's second law of motion.

i.e. F = ma

If an object is released from a height of h (t) above the ground level, then the differential equation of second law of motion is

$$m\frac{d^2y}{dt^2} = f\left(t, \left(h\ (t)\right), \frac{dh}{dt}\right)$$

Here, m = mass of an object

h = height above the ground level

t = time taken to hit the ground

F = force applied

a = acceleration

This is a second order differential equation of an unknown height as a function of time.

Worked Examples

1) Form a differential equation to the equation $x^2 + y^2 = a^2$

Solution:

$$x^2 + y^2 = a^2$$

Diff. w. r. to x

$$2x + 2y\frac{dy}{dx} = 0$$

$$x + y\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\frac{dy}{dx} + \frac{x}{y} = 0$$

This is the required differential equation.

2) Form the differential equation for $y^2 = 4ax$

Solution:

$$y^2 = 4ax \qquad \_\_\_\_\_\_ (1)$$

Differentiate with respect to x.

$$2y\frac{dy}{dx} = 4a\ (1)$$

$$2y\frac{dy}{dx} = 4a \qquad \_\_\_\_\_\_ (2)$$

Substitute (2) in Equation (1)

$$y^2 = 2y\frac{dy}{dx}\ x.$$

$$y = 2x\frac{dy}{dx}$$

$$\frac{y}{2x} = \frac{dy}{dx}$$

$$\frac{dy}{dx} - \frac{y}{2x} = 0$$

3) Form a differential equation for y = A cos x + B sin x

Solution:

$$y = A\cos x + B\sin x \qquad \text{______ (1)}$$

Diff. w. r. to x

$$\frac{dy}{dx} = -A\sin x + B\cos x \qquad \text{______ (2)}$$

Again Diff. w. r. to x

$$\frac{d^2y}{dx^2} \Rightarrow -[A\cos x + B\sin x]$$

$$\frac{d^2y}{dx^2} \Rightarrow -[A\cos x + B\sin x] \qquad \text{_____ (3)}$$

Apply (1) in (3)

$$\frac{d^2y}{dx^2} \Rightarrow -y$$

$$\frac{d^2y}{dx^2} + y = 0$$

This is the required differential equation.

4) Form the differential equation of the family of circle having centre origin and radius is ‘a’ units.

Solution:

Equation of the circle with centre origin

$$x^2 + y^2 = a^2$$

Differentiate with respect to x

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

This is the required differential equation.

5) Find the differential equation for the family of all straight lines passing through the origin.

Solution:

We know the equation of straight line passing through the origin.

$$y = mx \qquad \text{______ (1)}$$

Differentiate with respect to x

$$\frac{dy}{dx} = m \qquad \text{\_\_\_\_\_\_\_ (2)}$$

Sub (2) in (1)

$$\because y = \frac{dy}{dx} x$$

This is the required differential equation.

Part – C

Exercise Problems

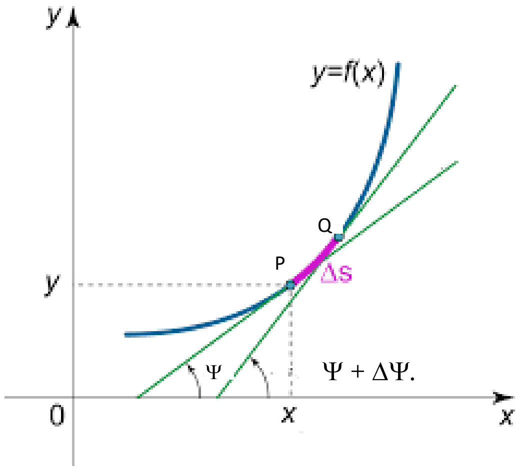
1) Form the differential equation of the curve represented by $xy = ae^{x} + be^{-x} + x^{2}$.
2) Form the differential equations by eliminating the arbitrary constants to the following problems.
   i. $y = ax + bx^2$
   ii. $y = a \cos (ax + b)$
   iii. $y = a \sin x + b \cos x$
   iv. $y = a \cos (\log x) + b \sin (\log x)$
   v. $y = ae^{mx}$
   vi. $xy = a^2$
   vii. $y = \sin (cx + d)$
   viii. $y = ax^2 + bx + c$

3) Find the differential equation of the equation of the straight line with the intercept form 5 and its passing through the origin.
4) Find the differential equation of the family of circle passing through the origin and their centres on the x – axis.

Chapter 5.2: GEOMETRICAL APPLICATIONS

Curvature of a Curve:

Curvature is a numerical measure of bending of a curve. Suppose that the tangent line is drawn to the curve y = f (x) at a point P (x, y), the tangent forms an angle Ψ with x axis.



At the displacement ΔS along the arc of the curve, the point P moves to the point Q. The position of the tangent line also changes. The angle of inclination of the tangent to the x axis at the point Q will be $\Psi + \Delta\Psi$.

Then the curvature is defined as the magnitude of rate of change of Ψ with respect to the arc length S.

$$\text{Curvature at P} = \lim_{\Delta S \to 0} \frac{\Delta\Psi}{\Delta S} = \frac{d\Psi}{dS}$$

MATH FACT

Smaller circle bends more sharply than larger circle. Thus, smaller circle has a larger curvature.

Note:

The curvature of a straight line is zero.

Radius of Curvature:

The reciprocal of the curvature of curve is called as the radius of curvature and usually denoted by ρ.

i.e. $\rho = \frac{ds}{d\Psi}$

The above relation between S and Ψ for any curve is called an intrinsic equation of the curve.

Radius of Curvature in Cartesian Form:

Let p (x, y) be any point on a given curve y = f (x) and Ψ be the angle made by the tangent at p with the axis, then the radius of curvature

$$\rho = \frac{ds}{d\Psi}$$

$$\therefore \tan\Psi = \frac{dy}{dx} \qquad \text{\_\_\_\_\_\_ (1)}$$

$$\cos\Psi = \frac{dx}{ds}$$

Differentiating (1) with respect to x

$$\text{Sec}^2\Psi \frac{d\Psi}{ds}\frac{ds}{dx} = \frac{d^2y}{dx^2}$$

i.e. $$\sec^2\Psi\left(\frac{1}{\rho}\right) \cdot \frac{1}{\cos\Psi} = \frac{d^2y}{dx^2}$$

$$\left(\frac{1}{\rho}\right)\sec^3\Psi = \frac{d^2y}{dx^2}$$

$$\rho\frac{d^2y}{dx^2} = \sec^3\Psi$$

$$\rho \cdot \frac{d^2y}{dx^2} = (1 + \tan^2\Psi)^{3/2} \qquad [\because \sec^2\Psi = 1 + \tan^2\Psi]$$

$$= \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

The radius of curvature is $\rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$

MATH FACT:

1. Curvature of the given curve is $= \frac{1}{\rho}$
2. Radius of curvature is always positive i.e. $\rho > 0$.

Worked Examples

1. Find the radius of the curvature at the point (s, Ψ) on the curve is $s = 4a \sin\Psi$.

Solution:

Given $s = 4a \sin\Psi$

$\rho = \frac{ds}{d\Psi} = 4a \cos\Psi$

2. Find ρ for the curve intrinsic equation is $s = c \log(\sec\Psi)$

Solution:

The given curve is

$S = c \log(\sec\Psi)$

$\rho = \frac{ds}{d\Psi} = \frac{c}{\sec\Psi} \text{ x } \sec\Psi \tan\Psi$

$\rho = c \tan\Psi$

3. What is the curvature of straight line?

Solution:

A straight line is a curve of infinite radius :

$\because$ curvature $= \frac{1}{\text{radius}}$

Hence, the curvature of straight line curvature $= \frac{1}{\infty} = 0$

4. Find the radius of curvature for $y = x^3 + 3x^2 + 2$ at the point (1, 2)

Solution:

$y = x^3 + 3x^2 + 2$

$y_1 = 3x^2 + 6x$

$y_2 = 6x + 6$

$(y_1)\ (1, 2) = 3\ (1)^2 + 6\ (1) = 9$

$(y_2)\ (1, 2) = 6\ (1) + 6 = 12$

$\because$ The radius of curvature is

$$\rho = \frac{(1+\ y_1^2)^{3/2}}{y_2}$$

$$= \frac{[1+\ (9)^2]^{3/2}}{12}$$

$$= \frac{(1+\ 81)^{3/2}}{12}$$

$$= \frac{(82)^{3/2}}{12}$$

$$= \frac{(82\ \sqrt{82})}{12} \qquad \because\ a^{3/2} = a\sqrt{a}$$

$$\rho = \frac{41\ \sqrt{82}}{6}$$

5. Find the radius of curvature for $y = x^2$ at $x = \frac{1}{2}$

Solution:

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$\frac{d^2y}{dx^2} \quad 2$$

Now, $\left(\frac{dy}{dx}\right)_{x=½} = 2\ (\frac{1}{2}) = 1$

$\therefore$ The radius of curvature is

$$\rho = \frac{(1+ {y_1}^2)^{3/2}}{y_2}$$

$$= \frac{(1+1)^{3/2}}{y_2}$$

$$= \frac{(2)^{3/2}}{2}$$

$$= \frac{2\ \sqrt{2}}{2}$$

$$\rho = \sqrt{2}$$

6. Find the radius of curvature for $y^2 = 4x$ at (1, 1)

Solution:

$$y^2 = 4x$$

$$2y\ \frac{dy}{dx} = 4$$

$$\frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

$$\frac{d^2y}{dx^2} = \frac{-2}{y^2}\ \frac{dy}{dx} = \frac{-2}{y^2} \cdot \frac{2}{y}$$

$$\frac{d^2y}{dx^2} = \frac{-4}{y^3}$$

Now at (1, 1)

$$\left(\frac{dy}{dx}\right)_{(1,1)} = \frac{2}{1} = 2$$

$$\left(\frac{d^2y}{dx^2}\right)_{(1,1)} = \frac{-4}{1^3} = -4$$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+2^2)^{3/2}}{-4}$$

$$= \frac{(5)^{3/2}}{-4}$$

$$\rho = \frac{5\sqrt{5}}{4}$$

$(\because \rho > 0)$

7. A beam is bent in the form of the curve $y = 2 \sin x - \sin 2x$. Find the radius of curvature at $x = \frac{\pi}{2}$.

Solution:

Given $y = 2 \sin x - \sin 2x$

$y_1 = 2 \cos x - 2 \cos 2x$

$y_2 = -2 \sin x - 2(-2 \sin 2x)$

$y_2 = -2 \sin x + 4 \sin 2x$

Now (y_1) at $x = \frac{\pi}{2} = 2 \cos\frac{\pi}{2} - 2 \cos 2\left(\frac{\pi}{2}\right) = 0 - 2(-1) = 2$

(y_2) at $x = \frac{\pi}{2} = -2 \sin\frac{\pi}{2} + 4 \sin 2\left(\frac{\pi}{2}\right) = -2(1) + 0 = -2$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+2^2)^{3/2}}{-2}$$

$$= \frac{(5)^{3/2}}{-2}$$

$$\rho = \frac{5\sqrt{5}}{2}$$

$\because \rho > 0$

8. A telegraph wire hangs in the form of curve $y = \log\left[\sec\left(\frac{x}{a}\right)\right]$. Prove that the curvature at any points is $\frac{1}{a} \cos\left(\frac{x}{a}\right)$.

Solution:

$$y = \log\left[\sec\left(\frac{x}{a}\right)\right]$$

$$y_1 = a\,\frac{1}{\sec\left(\frac{x}{a}\right)} \cdot \sec\left(\frac{x}{a}\right)\left(\tan\left(\frac{x}{a}\right)\right)\frac{1}{a}$$

$$y_1 = \tan\frac{x}{a}$$

$$y_2 = \sec^2 \frac{x}{a}\left(\frac{1}{a}\right)$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{[1+\tan^2\left(\frac{x}{a}\right)]^{3/2}}{\frac{1}{a}\cdot \sec^2\left(\frac{x}{a}\right)}$$

$$= \frac{a\,[\sec^2\left(\frac{x}{a}\right)]^{3/2}}{\sec^2\left(\frac{x}{a}\right)}$$

$$= \frac{a\sec^3\left(\frac{x}{a}\right)}{\sec^2\left(\frac{x}{a}\right)}$$

$$\rho = a\sec\left(\frac{x}{a}\right)$$

$\therefore$ curvature $\quad k = \frac{1}{\rho}$

$$k = \frac{1}{a\sec\left(\frac{x}{a}\right)}$$

$$k = \frac{1}{a}\cos\left(\frac{x}{a}\right)$$

9. Consider the curvature of the function. $f(x) = e^{ax}$ at $x = 0$. The graph is scaled up by a factor and the curvature is measured again at $x = 0$. What is the value of the curvature function at $x = 0$ if the scaling factor tends to infinity?

Solution:

We have to prove that curvature is zero. If the scaling factor is the function can be written in the form as

$$y = f(x) = e^{ax}$$

$$y = e^{ax}$$

$$y_1 = a\,e^{ax}$$

$$y_2 = a^2\,e^{ax}$$

Using radius of curvature formula

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$f = \frac{[1+(a\,e^{ax})^2]^{3/2}}{a^2\,e^{ax}}$$

$$= \frac{[1+a^2\,e^{2ax}]^{3/2}}{a^2\,e^{ax}}$$

Put $x = 0$, we get

$$\rho = \frac{(1+a^2)^{3/2}}{a^2}$$

To curvature:

$$k = \frac{1}{\rho} = \frac{a^2}{(1 + a^2)^{3/2}}$$

Now taking the limit as $a \to \infty$ we get,

$$k = \lim_{a \to \infty} \frac{a^2}{(1 + a^2)^{3/2}}$$

$$= \lim_{a \to \infty} \frac{a^2}{a^3(1 + \frac{1}{a^2})^{3/2}}$$

$$= \lim_{a \to \infty} \frac{1}{a\,(1 + \frac{1}{a^2})^{3/2}}$$

$$k = 0$$

$\therefore$ The curvature is zero.

Exercise

1. Find the radius of curvature at (s, Ψ) on the curve $S = c \tan \Psi$.
2. Find the radius of curvature at (s, Ψ) on the curve $S = 8a \sin^2 \frac{\Psi}{6}$
3. Find the radius of curvature for the following curves.
   - i) $y^2 = 4ax$ at $(9, 2a)$
   - ii) $y^2 = 4ax$ at $(0, 0)$
   - iii) $y^2 = 16x$ at $(4, 8)$
   - iv) $y = e^x$ at $(0, 1)$
   - v) $y = x^3$ at $(2, 1)$
   - vi) $x^2 + y^2 = 25$ at $(4, 3)$

Envelope

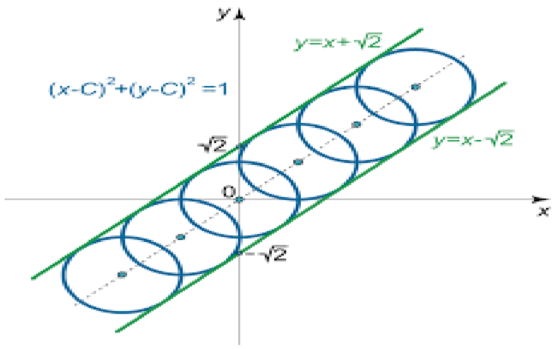
Definition for Envelope:

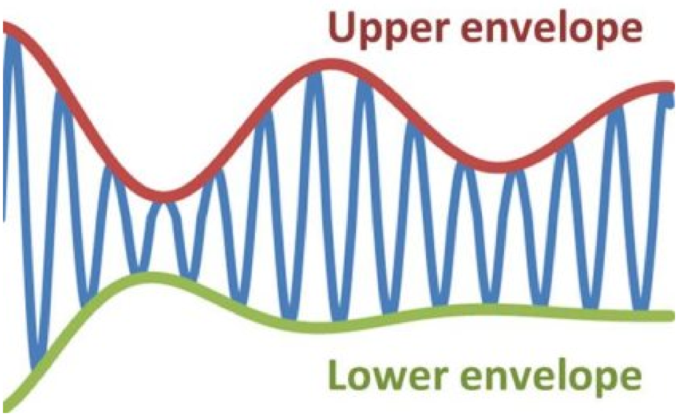
A Curve K which touches each member of the family of curves C and at each point on K is touched by some members of the family C is called the envelope.

An envelope for a family of curves $f(x, y, \alpha) = 0$

Where α is a parameter is $\frac{\partial}{\partial \alpha} f(x, y, \alpha) = 0$

A Curve which touches each member of a family of curves is called envelope of the that family





Procedure to find envelope for the family of curve:

Envelope of one parameter:

Let $y = f(x, y, \alpha)$ be the family of curves with α being one parameter.

Steps to find the envelope:

- Differentiate with respect to the parameter α to the given family of curves.
- Find the value of the parameter α.
- By substituting the value of parameter α in the given family of curves. Then we get the envelope of family of curves.

Worked examples:

1. Find the envelope of the family of lines x cos α + y sin α=P.

Solution:

Given $x \cos\alpha + y \sin\alpha = P$ ……………… (1)

Differentiate with respect to α, we get,

$-x \sin\alpha + y \cos\alpha = 0$ ……………… (2)

Squaring and adding (1) & (2) we get,

$(1) \Rightarrow x^2\cos^2\alpha + y^2\sin^2\alpha + 2xy \cos\alpha \sin\alpha = P^2$

$(2) \Rightarrow x^2\sin^2\alpha + y^2\cos^2\alpha - 2xy \cos\alpha \sin\alpha = 0$

$$x^2(\cos^2\alpha + \sin^2\alpha) + y^2(\cos^2\alpha + \sin^2\alpha) = P^2$$

$$x^2(1) + y^2(1) = P^2$$

$$x^2 + y^2 = P^2$$

which is required envelope.

2. Find the envelope of the family of circles $x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = c^2$

Solution:

Given $x^2 + y^2 - 2ax\cos\alpha - 2ay\sin\alpha = c^2$

Differentiate with respect to α , we get

$2ax\sin\alpha - 2ay\cos\alpha = 0$ -----------(2)

Equation (1) can be written as

$2ax\cos\alpha + 2ay\sin\alpha = X^2+Y^2- C^2$ --------(3)

Squaring and Adding (2) & (3), we get

(2) ⇒ $4a^2X^2\sin^2\alpha + 4a^2y^2\cos^2\alpha - \cancel{8a^2XY\sin\alpha\cos\alpha} = 0$

(3) ⇒ $4a^2X^2\cos^2\alpha + 4a^2y^2\sin^2\alpha + \cancel{8a^2XY\sin\alpha\cos\alpha} = (X^2+Y^2- C^2)^2$

$4a^2X^2(\sin^2\alpha + \cos^2\alpha) + 4a^2y^2(\cos^2\alpha + \sin^2\alpha) = (X^2+Y^2- C^2)^2$

$4a^2X^2+ 4a^2y^2 = (X^2+Y^2- C^2)^2$

$4a^2(X^2+ y^2) = (X^2+Y^2- C^2)^2$

Which is required envelope

3. Find the envelope of $x\cos^3\alpha + y\sin^3\alpha = a$ where α is the parameter

Solution:

Given, $x\cos^3\alpha + y\sin^3\alpha = a$ -------(1)

Differentiate with respect to α , we get

$-3x\cos^2\alpha\sin\alpha + 3y\sin^2\alpha\cos\alpha = 0$

$-3x\cos^2\alpha\sin\alpha = -3y\sin^2\alpha\cos\alpha$

$x\cos\alpha = y\sin\alpha$

$$\frac{x}{y} = \frac{\sin\alpha}{\cos\alpha}$$

$$\frac{x}{y} = \tan\alpha$$

$$\cos\alpha = \frac{y}{\sqrt{x^2+y^2}}, \quad \sin\alpha = \frac{x}{\sqrt{x^2+y^2}}$$

substitute sin α & cos α values in (1)

$$x\left(\frac{x}{\sqrt{X^2+Y^2}}\right)^3 + y\left(\frac{x}{\sqrt{X^2+Y^2}}\right)^3 = a$$

$$\frac{xy^3}{(X^2+Y^2)^{\frac{3}{2}}} + \frac{yx^3}{(X^2+Y^2)^{\frac{3}{2}}} = a$$

$$\frac{xy^3+yx^3}{(X^2+Y^2)^{\frac{3}{2}}} = a$$

$$\frac{xy(y^2+x^2)}{(x^2+y^2)\sqrt{x^2+y^2}} = a$$

$$\frac{xy}{\sqrt{x^2+y^2}} = a$$

$$xy = a\sqrt{x^2+y^2} \quad \text{(or)}$$

$$x^2y^2 = a^2(x^2+y^2)$$

Which is required envelope.

4. Find the Envelope of family of the curves $\frac{a^2\cos\theta}{x} - \frac{b^2\sin\theta}{y} = c$.

Solution:

Given $\frac{a^2\cos\theta}{x} - \frac{b^2\sin\theta}{y} = c$ …………….. (1)

Differentiate with respect to θ, we get

$-\frac{a^2}{x}\sin\theta - \frac{b^2}{y}\cos\theta = 0$ ……………… (2)

Squaring and adding (1) & (2) we get,

$$\left(\frac{a^2}{x}\right)^2 + \left(\frac{b^2}{y}\right)^2 = c^2$$

$$\frac{a^4}{x^2} + \frac{b^4}{y^2} = c^2$$

Which is required envelope.

Exercise problems:

1. Find the Envelope of family of the curves$(x-\alpha)^2 + y^2 = 4\alpha$ where α is a parameter.
2. Find the envelope of the family of straight lines x cos α + y sin α- C = 0.
3. Find the Envelope of family of the curves $\frac{x\cos\theta}{a} - \frac{y\sin\theta}{b} = 1$ where θ being a parameter

Chapter 5.3: PARTIAL DIFFERENTIATION

Introduction:

Partial derivatives of a function of several variables are its derivative with respect to one of those variables, keeping other variables as constant.

Let z = f (x, y) be a function of two independent variables x and y. The derivative of z with respect to x, keeping y as constant is called partial derivative of z with respect to x and denoted by $\frac{\partial z}{\partial x}$.

$$\therefore \frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,\ y) - f(x,\ y)}{\Delta x}$$

Provided the limit exists. Here Δx is small change in x.

Similarly the partial derivative of z with respect to y, keeping x as constant is denoted by $\frac{\partial z}{\partial y}$.

$$\therefore \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x,\ y+\Delta y) - f(x,\ y)}{\Delta y}$$

Provided the limit exists. Here Δy is small change in y.

Notation of first order partial derivatives.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y) \text{ or } \frac{\partial f}{\partial x} = f_x$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y) \text{ or } \frac{\partial f}{\partial y} = f_y$$

The process of finding a partial derivative is called Partial Differentiation.

MATH FACT

For a function of one variable its partial derivative is same as the ordinary derivative.

The symbol used to denote partial derivative is ∂.

$\frac{\partial z}{\partial x}$ read as “partial z by partial x”

Or

“dho z by dho x”

Second Order Partial Derivatives:

Function z = f (x, y) is said to be differentiable if $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ exist.

If the function is differentiable, its first order derivatives can be differentiated again and we can define the second order partial derivatives as follows:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\, \partial x} \text{ (or) } \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\, \partial y} \text{ (or) } \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ (or) } \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

Note: If z =f (x, y) and its partial derivate are continuous then the order of differential is immaterial. i.e. $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$

$\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$ are called the mixed partial derivatives.

Part – A

1. If $u = x^3 + y^3 + 3xy$, find (i) $\frac{\partial u}{\partial x}$ (ii) $\frac{\partial u}{\partial y}$

Solution :

$$u = x^3 + y^3 + 3xy \qquad \text{\_\_\_\_\_\_ (1)}$$

Differentiate (1) with respect to ‘x’,

$$\therefore \frac{\partial u}{\partial x} = 3x^2 + 3y$$

Differentiate (1) with respect to ‘y’

$$\frac{\partial u}{\partial y} = 3y^2 + 3x$$

2. Find $\frac{\partial u}{\partial x}$ if $u = x^3 \tan y$

Solution:

$$u = x^3 \tan y$$

$$\frac{\partial u}{\partial x} = 3x^2 \tan y$$

3. Find $\frac{\partial u}{\partial y}$ when $u = x^2 e^{5y}$

Solution:

$$u = x^2 e^{5y}$$

$$\frac{\partial u}{\partial y} = x^2 e^{5y} (5)$$

$$= 5x^2 e^{5y}$$

4. If $u = \log (x^2 + y^2)$ find $\frac{\partial u}{\partial x}$

Solution:

$$u = \log (x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2} (2x + 0)$$

$$= \frac{2x}{x^2 + y^2}$$

5. If $u = e^{x2+y2}$ Prove that $\frac{\partial u}{\partial x} = 2 x u$.

Solution:

$$u = e^{x2+y2}$$

Differentiate with respect to 'x'

$$\frac{\partial u}{\partial x} = e^{x^2+y^2}\ (2x)$$

$$= 2xu.$$

Part – B

1. If $u = (x-y)^2$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$

Solution:

$$u = (x-y)^2 \quad \_\_\_\_\_\_\_\_ (1)$$

Differentiate with respect to 'x'

$$\frac{\partial u}{\partial x} = 2(x-y)$$

Differentiate (1) with respect to 'y'

$$\frac{\partial u}{\partial y} = 2\,(x-y)\,(-1)$$

$$= -2\,(x-y)$$

$$\text{L.H.S. } \frac{\partial y}{\partial x} + \frac{\partial u}{\partial y} = 2\,(x-y) - 2\,(x-y)$$

$$= 0 = \text{RHS}$$

2. If $u = \tan^{-1}(y/x)$ find $\frac{\partial u}{\partial x}$.

Solution:

$$u = \tan^{-1}(y/x)$$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \left(\frac{-y}{x^2}\right)$$

$$= \frac{-y}{\frac{x^2+y^2}{x^2}} \ \frac{1}{x^2}$$

$$= \frac{-y}{x^2+y^2}$$

3. If $u = x^3 + y^3 + 3xy$ find $\frac{\partial^2 u}{\partial x^2}$

Solution:

$$u = x^3 + y^3 + 3xy \quad \_\_\_\_\_\_\_\_ (1)$$

Differentiate (1) with respect to 'x'

$$\frac{\partial u}{\partial x} = 3x^2 + 3y$$

Again Differentiate with respect to 'x'

$$\frac{\partial^2 u}{\partial x^2} = 6x$$

4. If $u = \log(x^3 + y^3)$, find $\frac{\partial^2 u}{\partial x^2}$

Solution:

$$u = \log(x^3 + y^3)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3} \; 3x^2$$

$$= \frac{3x^2}{x^3 + y^3}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{3x^2}{x^3 + y^3}\right)$$

$$= \frac{(x^3 + y^3)(6x) - (3x^2)(3x^2 + 0)}{(x^3 + y^3)^2}$$

$$= \frac{6x^4 + 6xy^3 - 9x^4}{(x^3 + y^3)^2}$$

$$= \frac{6xy^3 - 3x^4}{(x^3 + y^3)^2}$$

$$= \frac{3x(2x^3 - y^3)}{(x^3 + y^3)^2}$$

Part – C

1. If $u = \sin^{-1}(y/x)$ and $x \neq 0$, prove that $x\frac{\partial u}{\partial x} + y\left(\frac{\partial u}{\partial y}\right) = 0$

Solution:

$$u = \sin^{-1}(y/x)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \frac{y^2}{x^2}}} \; \frac{(-y)}{x^2}$$

$$= \frac{1}{\sqrt{\frac{x^2 - y^2}{x^2}}} \; \frac{(-y)}{x^2}$$

$$x\frac{\partial u}{\partial x} = \frac{1}{\sqrt{\frac{x^2 - y^2}{x}}} \; \frac{-yx}{x^2}$$

$$x\frac{\partial u}{\partial x} = \frac{-y}{\sqrt{x^2 - y^2}} \qquad \text{\_\_\_\_\_\_\_\_ (1)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \frac{y^2}{x^2}}} \; \frac{1}{x}$$

$$= \frac{x}{\sqrt{x^2 - y^2}} \; \frac{1}{x}$$

$$y\frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2 - y^2}} \qquad \text{\_\_\_\_\_\_\_\_ (2)}$$

(1) + (2) gives

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{-y}{\sqrt{x^2 - y^2}} + \frac{y}{\sqrt{x^2 - y^2}} = 0$$

2. If $u = \log(\tan x + \tan y + \tan z)$ prove that $\Sigma \; \sin 2x \frac{\partial u}{\partial x} = 2$.

Solution:

$$u = \log(\tan x + \tan y + \tan z)$$

Differentiate partially w. r. to x

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \sec^2 x$$

$$\text{Sin } 2x \frac{\partial u}{\partial x} = \frac{2 \sin x \cos x \frac{1}{\cos^2 x}}{\tan x + \tan y + \tan z}$$

$$= \frac{2 \frac{\sin x}{\cos x}}{\tan x + \tan y + \tan z}$$

$$\text{Sin } 2x \frac{\partial u}{\partial x} = \frac{2 \tan x}{\tan x + \tan y + \tan z} \qquad \text{---- (1)}$$

Similarly we can prove

$$\text{Sin } 2y \frac{\partial u}{\partial y} = \frac{2 \tan y}{\tan x + \tan y + \tan z} \qquad \text{---- (2)}$$

$$\text{Sin } 2z \frac{\partial u}{\partial z} = \frac{2 \tan z}{\tan x + \tan y + \tan z} \qquad \text{---- (3)}$$

(1) + (2) + (3) gives

$$\Sigma \text{ Sin } 2x \frac{\partial u}{\partial x} = 2 \frac{(\tan x + \tan y + \tan z)}{(\tan x + \tan y + \tan z)} = 2$$

3. If $u = 2x^3 + 3y^3$, find $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$

Solution:

Given $u = 2x^3 + 3y^3$ ---- (1)

Differentiate (1) with respect to x

$$\frac{\partial u}{\partial x} = 6x^2$$

$$\frac{\partial^2 u}{\partial x^2} = 12\,x$$

Differentiate (1) with respect to y

$$\frac{\partial u}{\partial y} = 9y^2$$

$$\frac{\partial^2 u}{\partial y^2} = 18\,y$$

4. If $u = \log (x^2 + y^2)$, find $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$

Solution:

$u = \log (x^2 + y^2)$ ---- (1)

Differentiate partially (1) with respect to x

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2} 2x$$

$$= \frac{2x}{x^2 + y^2}$$

Differentiate partially (1) with respect to y.

$$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2} 2y = \frac{2y}{x^2 + y^2}$$

To find $x \frac{\partial u}{\partial x} + y \left(\frac{\partial y}{\partial y}\right)$

$$= x \left(\frac{2x}{x^2 + y^2}\right) + y \left(\frac{2y}{x^2 + y^2}\right)$$

$$= \frac{2x^2}{x^2+y^2} + \frac{2y^2}{x^2+y^2}$$

$$= \frac{2(x^2+y^2)}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} + y\left(\frac{\partial y}{\partial y}\right) = 2$$

Exercises

Part – A

1. Find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ for the following functions.
   - (a) $u = x^3 + 3xy + y^2$
   - (b) $u = \sin 3x \cos 4y$
   - (c) $u = \frac{x}{y^2} - \frac{y}{x^2}$
   - (d) $u = \tan^{-1}(y/x)$
   - (e) $u = x^4 + y^3 + 3x^2y^2 + 3x^2y$
   - (f) $u = \tan^{-1}(x + y)$
   - (g) $u = \log(e^x + e^y)$
   - (h) $u = x^2y - x\sin(xy)$
   - (i) $u = x^3 + 5x^2y + y^3$
   - (j) $u = \log(x^2 + y^2)$
   - (k) $u = \log(x - y)$
   - (l) $u = x^3 + x^2y + xy^2 + y^3$
   - (m) $u = 2x^3 + 4y^3 + 2xy$
   - (n) $u = 2x^3 - 4y^3 + 3xy$
2. If $u = x^3 + y^3$ find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$
3. If $u = x^2 \sin y$, find $\frac{\partial u}{\partial x}$

Part – B

1. If $u = x^3 + 3xy + y^2$ find $\frac{\partial^2 u}{\partial x \partial y}$
2. If $u = x^2 + y^2$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$
3. If $u = e^{xy}$, find $\frac{\partial^2 u}{\partial x \partial y}$
4. Find $\frac{\partial^2 u}{\partial x^2}$ when $u = x^3 + y^3$
5. If $u = x^3 - x^2y - y^3$ find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

Part – C

1. If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$ find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$
2. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right)$. show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$
3. If $u = \log(x^2 + y^2) + \tan^{-1}\left(\frac{x}{y}\right)$. show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
4. If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$, prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$
5. If $u = x^3 - 2x^2y + 3xy^3$, find $\frac{\partial^2 u}{\partial x^2}$, and $\frac{\partial^2 u}{\partial y^2}$
6. If $u = x^2 + y^2 + 3xy$ find $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2}$
7. If $u = \tan^{-1}\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$
8. If $u = x^3 + y^3 + 3xy^2$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$
9. If $u = x^3 + y^3 + 3x^2y$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$

Jacobian and its Properties:

Jacobians:

Jacobians were introduced by the German Mathematician C.G. Jacobi in 1829. They are used in mechanise, probability, calculus etc., in evaluating multiple integral through transformation strategy.

Definition:

If u = u (x, y) and v = v (x, y) are two functions of independent variables x and y then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

is called the Jocobian of u, v with respect to x & y and it is usually denoted by $\frac{\partial\,(u,v)}{\partial\,(x,y)}$ or J $\left(\frac{u,v}{x,y}\right)$

In particular,

$$\frac{\partial\,(u,v,w)}{\partial\,(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Properties of Jacobians:

Property: 1

If u and v are the functions of x & y then

$$\frac{\partial\,(u,v)}{\partial\,(x,y)} \text{ x } \frac{\partial\,(x,y)}{\partial\,(u,v)} = 1$$

(Inverse property of Jacobians)

i.e. $JJ^1 = 1$

Property: 2

If u, v are functions of x, y and x, y are themselves functions of r, s

then $\frac{\partial\,(u,v)}{\partial\,(x,y)} \text{ x } \frac{\partial\,(x,y)}{\partial\,(r,s)} = \frac{\partial\,(u,v)}{\partial\,(r,s)}$

Property: 3

If u, v, w are functionally dependent functions of three independent variables x, y, x, then $\frac{\partial\,(u,v,w)}{\partial\,(x,y,z)} = 0$

Worked Examples

1. If x = u (1 + v) and y = v (1 + u) find $\frac{\partial\,(x,y)}{\partial\,(u,v)}$

Solution:

$$\frac{\partial\,(x,y)}{\partial\,(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$x = u(1+v) \qquad y = v(1+u)$

$\frac{\partial x}{\partial u} = 1 + v \qquad \frac{\partial y}{\partial u} = v$

$\frac{\partial x}{\partial v} = u \qquad \frac{\partial y}{\partial v} = 1 + u$

$$\therefore \frac{\partial (x,y)}{\partial (u,v)} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix}$$

$$= (1+v)(1+u) - uv$$

$$= 1 + u + v + uv - uv$$

$$\therefore \frac{\partial (x,y)}{\partial (u,v)} = 1 + u + v$$

2. Find the Jacobian of p, q, r with respect to x, y, z given

$p = x + y + z \qquad q = y + z \qquad r = z$

Solution:

$$J = \frac{\partial (p,q,r)}{\partial (x,y,z)} = \begin{vmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} & \frac{\partial q}{\partial z} \\ \frac{\partial r}{\partial z} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \end{vmatrix}$$

$P = x + y + z$	$q = y + z$	$r = z$
$\frac{\partial p}{\partial x} = 1$	$\frac{\partial q}{\partial x} = 0$	$\frac{\partial r}{\partial x} = 0$
$\frac{\partial p}{\partial y} = 1$	$\frac{\partial q}{\partial y} = 1$	$\frac{\partial r}{\partial y} = 0$
$\frac{\partial p}{\partial z} = 1$	$\frac{\partial q}{\partial z} = 1$	$\frac{\partial r}{\partial z} = 1$

$$\therefore J = \frac{\partial (p,q,r)}{\partial (x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1(1-0)$$

$$\therefore J = \frac{\partial (p,q,r)}{\partial (x,y,z)} = 1$$

3. Which one of the following is the definition of Jacobian of u, v with respect to x and y ?

(a) $J\left(\frac{x,y}{u,v}\right)$ (b) $J\left(\frac{u,v}{x,y}\right)$

(c) $\frac{\partial (x,y)}{\partial (u,v)}$ (d) $\frac{\partial (u,x)}{\partial (v,w)}$

Solution:

Option : (b)

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial (u,v)}{\partial (x,y)}$$

4. If x = r cos θ, y = r sin θ, Evaluate $\frac{\partial (x,y)}{\partial (r,\theta)}$

Solution:

$$\frac{\partial (x,y)}{\partial (r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$x = r\cos\theta$	$x = r\sin\theta$
$\frac{\partial x}{\partial r} = \cos\theta$	$\frac{\partial y}{\partial r} = \sin\theta$
$\frac{\partial x}{\partial \theta} = -r\sin\theta$	$\frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial (x,y)}{\partial (r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r \quad [\because \cos^2\theta + \sin^2\theta = 1]$$

5. If $u + v = e^x \cos y$ and $u - v = e^x \sin y$

Evaluate $J\left(\frac{u,v}{x,y}\right)$

Solution:

$u + v = e^x \cos y$ ______ (1)

$u - v = e^x \sin y$ ______ (2)

Solving (1) & (2)

We get $u = \frac{e^x}{2}(\cos y + \sin y)$

$v = \frac{e^x}{2}(\cos y - \sin y)$

$u = \frac{e^x}{2}(\cos y + \sin y)$

$\frac{\partial u}{\partial x} = \frac{e^x}{2}(\cos y + \sin y)$; $\frac{\partial u}{\partial y} = \frac{e^x}{2}(-\sin y + \cos y)$

$v = \frac{e^x}{2}(\cos y - \sin y)$

$\frac{\partial v}{\partial x} = \frac{e^x}{2}(\cos y - \sin y)$; $\frac{\partial v}{\partial y} = \frac{e^x}{2}(-\sin y - \cos y)$

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{e^x}{2}(\cos y + \sin y) & \frac{e^x}{2}(-\sin y + \cos y) \\ \frac{e^x}{2}(\cos y - \sin y) & \frac{e^x}{2}(-\sin y - \cos y) \end{vmatrix}$$

$$= \left[\frac{-e^x}{2}\frac{e^x}{2}(\cos y + \sin y)(\cos y + \sin y)\right]$$

$$- \left[\frac{+e^x}{2}\frac{e^x}{2}(\cos y - \sin y)(\cos y - \sin y)\right]$$

$$= \frac{-e^{2x}}{4}\ [(\cos y + \sin y)^2 + (\cos y - \sin y)^2]$$

$$= \frac{-e^{2x}}{4}\ [\cos^2 y + \sin^2 y) + 2\cancel{\cos y}\sin y + \cos^2 y + \sin^2 y - 2\cancel{\cos y}\sin y]$$

$$= \frac{-e^{2x}}{4}\ [2\cos^2 y + 2\sin^2 y]$$

$$= \frac{-e^{2x}}{2}\ [\cos^2 y + \sin^2 y]$$

$$= \frac{-e^{2x}}{2}\ (1)$$

$$\therefore J\left(\frac{u,v}{x,y}\right) = \frac{-e^{2x}}{2}$$

6. If x = u (1 – v), y = uv then compute J & J' and prove that JJ ' = 1

Solution:

We know that, $J = \frac{\partial\,(x,y)}{\partial\,(u,v)}$, $J' = \frac{\partial\,(u,v)}{\partial\,(x,y)}$,

Given x = u (1 – v) | y = uv

$\frac{\partial x}{\partial u} = 1 - v$ | $\frac{\partial y}{\partial u} = v$

$\frac{\partial x}{\partial v} = -u$ | $\frac{\partial y}{\partial v} = u$

$$J = \frac{\partial\,(x,y)}{\partial\,(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial x}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$$= u(1-v) + uv$$

$$= u - uv + uv$$

$$J = u \quad \text{-----------} \ (1)$$

Given x = u (1 – v) , y = uv

x = u – uv

x = u – y

x + y = u

y = uv

y = (x + y) v

$\therefore\ v = \frac{y}{x+y}$

u = x + y | $v = \frac{y}{x+y}$

$\frac{\partial u}{\partial x} = 1$ | $\frac{\partial v}{\partial x} = \frac{-y}{(x+y)^2}$

$\frac{\partial u}{\partial y} = 1$ | $\frac{\partial v}{\partial y} = \frac{(x+y)(1) - (y)(1)}{(x+y)^2}$

$= \frac{x}{(x+y)^2}$

$\frac{\partial v}{\partial y} = \frac{x}{(x+y)^2}$

$$J' = \frac{\partial (u,v)}{\partial (x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ \frac{-y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2}$$

$$= \frac{x+y}{(x+y)^2} = \frac{1}{x+y} = \frac{1}{u}$$

$$\therefore J' = \frac{1}{u} \qquad \text{---- (2)}$$

From (1) & (2) we get $JJ' = u \times \frac{1}{u} = 1$

7. If $u = x - y$, $v = y - z$, $w = z - x$ prove that $\frac{\partial (u,v,w)}{\partial (x,y,z)} = 0$

Solution:

$u = x - y$	$v = y - z$	$w = z - x$
$\frac{\partial u}{\partial x} = 1$	$\frac{\partial v}{\partial x} = 0$	$\frac{\partial w}{\partial x} = -1$
$\frac{\partial u}{\partial y} = -1$	$\frac{\partial v}{\partial y} = 1$	$\frac{\partial w}{\partial y} = 0$
$\frac{\partial u}{\partial z} = 0$	$\frac{\partial v}{\partial z} = -1$	$\frac{\partial w}{\partial z} = 1$

$$= \frac{\partial (u,v,w)}{\partial (x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{vmatrix}$$

$= 1\,(1 - 0) + 1\,(0 - 1) + 0$

$= 1 - 1$

$= 0 \qquad \therefore \frac{\partial (u,v,w)}{\partial (x,y,z)} = 0$

8. Show that the functions $u = x^2 + y^2 + z^2$, $v = x + y + z$ and $w = xy + yz + zx$ are dependent.

Solution:

We have to prove that $J\,(u, v, w) = \frac{\partial (u,v,w)}{\partial (x,y,z)} = 0$

$u = x^2 + y^2 + z^2$	$v = x + y + z$	$w = xy + yz + zx$
$\frac{\partial u}{\partial x} = 2x$	$\frac{\partial v}{\partial x} = 1$	$\frac{\partial w}{\partial x} = y + z$
$\frac{\partial u}{\partial y} = 2y$	$\frac{\partial v}{\partial y} = 1$	$\frac{\partial w}{\partial y} = x + z$
$\frac{\partial u}{\partial z} = 2z$	$\frac{\partial v}{\partial z} = 1$	$\frac{\partial w}{\partial z} = y + x$

$$= \frac{\partial (u,v,w)}{\partial (x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y & 2z \\ 1 & 1 & 1 \\ y+z & x+z & y+x \end{vmatrix}$$

$= 2x\,(y + x - x - z) - 2y\,(y + x - y - z) + 2z\,(x + z - y - z)$

$= 2x\,(y - z) - 2y\,(x - z) + 2z\,(x - y)$

$= 2xy - 2xz - 2xy + 2yz + 2xz - 2yz$

$\frac{\partial (u,v,w)}{\partial (x,y,z)} = 0$

$\therefore$ J (u, v, w) = 0

$\therefore$ u, v and w are functionally dependent

Exercises

1. If $x = uv$, $y = \frac{u}{v}$ prove that $\frac{\partial (x,y)}{\partial (u,v)} \times \frac{\partial (u,v)}{\partial (x,y)} = 1$
2. If $u = \frac{y^2}{x}$, $v = \frac{x^2}{y}$ find $\frac{\partial (u,v)}{\partial (x,y)}$
3. If $u = \frac{x+y}{x-y}$ and $v = \tan^{-1} x + \tan^{-1} y$ find the Jacobian $\frac{\partial (u,v)}{\partial (x,y)}$
4. Given $u = x + y$, $y = uv$ find $\frac{\partial (x,y)}{\partial (u,v)}$
5. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$; $w = \frac{xy}{z}$ then show that J (u, v, w) = 4
6. If $x = u^2 - v^2$, $y = 2uv$ find $\frac{\partial (x,y)}{\partial (u,v)}$
7. If $u = x + y$, $v = x - y$ find $\frac{\partial (x,y)}{\partial (u,v)}$
8. Find the Jacobian of the transformation

   i) $x = u - 2v$ $\quad y = 2u - v$

   ii) $x = 2u$, $\quad y = 3v^2$, $\quad z = 4w^3$

Homogenous Function:

In the previous section we have seen the partial differentiation. The mathematician EULER has found relations between homogenous functions and its partial derivatives. The study of Euler's theorem will help the learner to solve some of the problems without finding the partial derivatives for homogenous functions.

A polynomial equation with two variables x and y is said to be a homogenous if polynomial consisting of terms all of same degree.

Consider:

$p(x, y) = x^4 + 4x^3y + 6x^2y^2 - 4xy^3 + y^4$.

Here the degree of each term of the polynomial p (x, y) is 4. Hence p (x, y) is a homogenous function of degree 4.

MATH FACT

A polynomial of degree 0 is always homogeneous.

Definition : Homogenous Function:

A function f (x, y) is said to be homogenous of degree n if $f(kx, ky) = k^n f(x, y)$

Example

$$f(x, y) = \frac{x^4 + y^4}{x-y}$$

$$f(kx, ky) = \frac{(kx)^4 + (ky)^4}{kx-(ky)}$$

$$= \frac{k^4 (x^4 + y^4)}{k(x-y)}$$

$$= \frac{k^3 (x^4 + y^4)}{x-y}$$

$$= k^3 f(x, y)$$

Hence f (x, y) is a homogenous function of degree 3.

Euler's Theorem:

If f (x, y) is homogenous function of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n f(x, y)$$

Part – A

1. Verify whether the function $3x^2 + 2xy + y^2$ is homogenous.

Solution:

Since given function is polynomial in x and y and power of each term is 2.

$\therefore$ the given function is homogenous of degree 2.

Another Method:

$$\text{Let } f(x, y) = 3x^2 + 2xy + y^2$$

$$\therefore f(kx, ky) = 3(kx)^2 + 2(kx)(ky) + (ky)$$

$$= 3k^2x^2 + 2k^2xy + k^2y^2$$

$$= k^2(3x^2 + 2xy + y^2)$$

$$= k^2 f(x, y)$$

2 Show that $\frac{x^4 + y^4}{x^3 - y^3}$ is a homogenous functions

Solution:

$$\text{Let } f(x, y) = \frac{(kx)^4 - (ky)^4}{(kx)^3 - (ky)^3} = \frac{k^4x^4 + k^4y^4}{k^3x^3 - k^3y^3}$$

$$= \frac{k^4 (x^4 + y^4)}{k^3 (x^3 - y^3)}$$

$$= \frac{k (x^4 + y^4)}{x^3 - y^3} = kf (x, y)$$

Hence f (x, y) is a homogenous of degree 1.

3. Find the degree of the homogenous function = $\frac{x^3 + x^2y + y^3}{x+y}$

Solution:

$$\text{Let } f (x,y) = \frac{x^3 + x^2y + y^3}{x+y}$$

$$f (kx, ky) = \frac{(kx)^3 + (kx)^2 (ky) + (ky)^3}{kx + ky}$$

$$= \frac{k^3x^3 + k^3x^2y + k^3y^3}{kx + ky}$$

$$= \frac{k^3 (x^3 + x^2y + y^3}{k (x + y)}$$

$$= k^2 \frac{x^3 + x^2y + y^3}{x + y} = k^2 f (x, y)$$

The degree of the homogenous function is 2.

4. What is the degree of the homogenous functions = $\frac{1}{\sqrt{x^3 + y^3}}$

Solution:

$$\text{Let } f (x, y) = \frac{1}{\sqrt{x^3 + y^3}}$$

$$f (kx, ky) = \frac{1}{\sqrt{(kx)^3 + (ky)^3}}$$

$$= \frac{1}{\sqrt{k^3x^3 + k^3y^3}}$$

$$= \frac{1}{k^{3/2}\sqrt{x^3 + y^3}}$$

$$= k^{-3/2} f (x, y)$$

$\therefore$ Degree of homogenous function is $\frac{-3}{2}$

Note : In Euler theorem $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$. Here n indicates? Order of Z

Necessary condition of Euler's theorem is, z should be <u>homogeneous and of order n</u>.

Part - B

1) If $u = x^2 + 3xy + y^2$, verify Euler's theorems

Solution:

$$u = x^2 + 3xy + y^2$$

$$\frac{\partial u}{\partial x} = 2x + 3y$$

$$x\frac{\partial u}{\partial x} = 2x^2 + 3xy \qquad \text{------ (1)}$$

$$\frac{\partial u}{\partial y} = 3x + 2y$$

$$y\frac{\partial u}{\partial y} = 3xy + 2y^2 \qquad \text{------ (2)}$$

(1) + (2) $\Rightarrow x\frac{\partial u}{\partial y} + y\frac{\partial u}{\partial y}$

$$= 2x^2 + 3xy + 3xy + 2y^2$$

$$= 2x^2 + 6xy + 2y^2$$

$$= 2(x^2 + 3xy + y^2)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2u$$

2) Using Euler's theorem if $u = \frac{1}{x^2 + xy + y^2}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -2u$.

Solution:

Let $f(x, y) = u = \frac{1}{x^2 + xy + y^2}$

$f(kx, ky) = u = \frac{1}{(kx)^2 + (kx)(ky) + (ky)^2}$

$$= \frac{1}{k^2x^2 + k^2xy + k^2y^2}$$

$$= \frac{1}{k^2(x^2 + xy + y^2)}$$

$$= k^{-2} f(x, y)$$

$\therefore$ $u = f(x, y)$ is homogenous function degree = -2.

$\therefore$ By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -2u$.

3) Using Euler's theorem if $u = \tan^{-1}\left(\frac{x^3 + y^3}{x-y}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

Solution:

$$u = \tan^{-1}\left(\frac{x^3 + y^3}{x-y}\right)$$

$$\therefore \tan u = \frac{x^3 + y^3}{x-y}$$

Let $\tan u = z$

$$\therefore z(x, y) = \frac{x^3 + y^3}{x-y}$$

$$Z(kx, ky) = \frac{k^3x^3 + k^3y^3}{kx - ky}$$

$$= \frac{k^3(x^3 + y^3)}{k(x-y)}$$

$$= \frac{k^2 x^3 + y^3}{x-y}$$

$$= k^2 z(x, y)$$

$\therefore$ $z(x, y)$ is a homogenous function of degree 2.

$\therefore$ By Euler's theorem, $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2z.$

Put z = tan u in the above equation

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u.$$

$$x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u.$$

$$\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\tan u.$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u}$$

$$= \frac{2\sin u}{\cos u} \times \frac{\cos^2 u}{1}$$

$$= 2\sin u\cos u$$

$$= \sin 2u$$

4) Using Euler's theorem if $u = \sin^{-1}\left(\dfrac{x-y}{x+y}\right)$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$

Solution:

Let $f(x, y) = u = \sin^{-1}\left(\dfrac{x-y}{x+y}\right)$

$$f(x, y) = \sin u = \frac{x-y}{x+y}$$

$$f(kx, ky) = \frac{kx-ky}{kx+ky}$$

$$= k^o f(x, y)$$

$\therefore$ u = f (x, y) is a homogenous function of degree = 0.

$\therefore$ By Euler's theorem, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0\,(u)$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

5) If $u = \log\left(\dfrac{x^4+y^4}{\sqrt{x}+\sqrt{y}}\right)$, then prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{7}{2}$

Solution:

$$u = \log_e\left(\frac{x^4+y^4}{\sqrt{x}+\sqrt{y}}\right)$$

$$\therefore e^u = \frac{x^4+y^4}{\sqrt{x}+\sqrt{y}}$$

Let $e^u = z$

$$\therefore z(x,y) = \frac{x^4+y^4}{\sqrt{x}+\sqrt{y}}$$

$$z(kx, ky) = \frac{(kx)^4+(ky)^4}{\sqrt{kx}+\sqrt{ky}}$$

$$= \frac{k^4x^4+k^4y^4}{\sqrt{k}\sqrt{x}+\sqrt{k}\sqrt{y}}$$

$$= \frac{k^4\,[x^4 + y^4]}{\sqrt{k}\,[\sqrt{x} + \sqrt{y}]}$$

$$= k^{7/2}\,\frac{x^4 + y^4}{\sqrt{x} + \sqrt{y}}$$

$$= k^{7/2}\, z\,(x, y)$$

$\therefore$ z (x, y) is a homogenous function of degree 7/2.

By Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{7}{2} z$$

put $z = e^u$ in the above function.

$$x\frac{\partial}{\partial x}(e^u) + y\frac{\partial}{\partial y}(e^u) = \frac{7}{2}e^u$$

$$x\left(e^u\frac{\partial u}{\partial x}\right) + y\left(e^u\frac{\partial u}{\partial y}\right) = \frac{7}{2}e^u$$

$$e^u\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right] = \frac{7}{2}e^u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{7}{2}$$

Exercise

Part – A

1. Show that $u = x^3 - x^2y + xy^3$ is a homogenous function.
2. Find the degree of the homogenous function $f(x, y) = x^4 - 5x^3y + 4x^2y^2 - 3xy^3 + y^4$.
3. Verify whether the function $x^3 - y^3 - xy^2$ is homogenous function.
4. Verify whether the function $ax^2 + 2hxy + by^2$ is homogenous function.
5. Find the degree of the homogeneous function $\frac{1}{\sqrt{x^2 + y^2}}$
6. Find the degree of the homogenous functions $\frac{\sqrt{x} + \sqrt{y}}{x^2 + y^2}$
7. Verify whether the function $\frac{x^2 - xy + y^2}{x + y}$ is homogenous function.
8. Find the degree of the homogenous functions $\frac{1}{x^3 + y^3}$.

Part – B

1. Verify Euler's theorem for the function $u = x^3 + 5x^2y + 3xy^2 + 2y^3$
2. Verify Euler's theorem for $u = \sin\left(\frac{x}{y}\right)$
3. Verify Euler's theorem for u = tan (y/x)
4. Using Euler's theorem

   (a) $u = \tan^{-1}\left(\frac{x+y}{x-y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

   (b) $u = \cos^{-1}\left(\frac{x-y}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

(c) $u = xy^2 \sin\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$

(d) $z = \frac{x^2}{y^3} + \frac{y^2}{x^3}$ show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$

(e) $z = \frac{1}{\sqrt{x+y}}$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -\frac{1}{2}$

(f) $u = \sin^{-1}\left(\frac{x^2+y^2}{x-y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u$

(g) $u = \log\left(\frac{x^2+y^2}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3/2$

Partial Differentiation

Notes: