ENGINEERING MATHEMATICS - II

DIPLOMA COURSE IN ENGINEERING & TECHNOLOGY

SECOND SEMESTER

A Publication under
Government of Tamilnadu
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Untouchability is a sin
Untouchability is a crime
Untouchability is an inhuman

DIRECTORATE OF TECHNICAL EDUCATION
GOVERNMENT OF TAMILNADU

First Edition – 2020

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This book has been prepared by the Directorate of Technical Education

This book has been printed on 60 G.S.M Paper

Through the Tamil Nadu Text book and Educational Services Corporation

Printed by Web Offset at:

THE NATIONAL ANTHEM

FULL VERSION

Jana-gana-mana-adhinayaka jaya he
Bharata-bhagya-vidhata
Punjaba-Sindhu-Gujarata-Maratha-
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchhala-jaladhi-taranga
Tava Subha name jage,TavaSubhaasisa mage,
Gahe tava jaya-gatha.
Jana-gana-mangala-dayaka jaya he
Bharata-bhagya-vidhata.
Jaya he, jaya he, jaya he,
Jaya jaya jaya jaya he.

- **Rabindranath Tagore**

SHORT VERSION

Jana-gana-mana-adhinayaka jaya he
Bharata-bhagya-vidhata.
Jaya he, jaya he, jaya he,
Jaya jaya jaya jaya he.

AUTHENTIC ENGLISH TRANSLATION OF THE NATIONAL ANTHEM

Thou art the ruler of the minds of all people,
Thou dispenser of India's destiny.
Thy name rouses the hearts of the Punjab, Sind,
Gujarat and Maratha, of Dravida, Orissa and Bengal
It echoes in the hills of the Vindhyas and Himalayas,
mingles in the music of the Yamuna and Ganges
and is chanted by the waves of the Indian Sea.
They pray for Thy blessings and sing Thy praise
The saving of all people waits in Thy hand,
Thou dispenser of India's destiny.
Victory, Victory, Victory to Thee

THE NATIONAL INTEGRATION PLEDGE

"I solemnly pledge to work with dedication to preserve and strengthen the freedom and integrity of the nation."

"I further affirm that I shall never resort to violence and that all differences and disputes relating to religion, language, region or other political or economic grievances should be settled by peaceful and constitutional means."

INVOCATION TO GODDESS TAMIL

Bharat is like the face beauteous of Earth clad in wavy seas;
Deccan is her brow crescent-like on which the fragrant 'Tilak' is the blessed Dravidian land.
Like the fragrance of that 'Tilak' plunging the world in joy supreme reigns Goddess Tamil with renown spread far and wide.
Praise unto You, Goddess Tamil, whose majestic youthfulness, inspires awe and ecstasy

PREFACE

We take great pleasure in presenting this book of mathematics to the students of polytechnic colleges. This book is prepared in accordance with the new syllabus under 'N' scheme framed by the Directorate of Technical Education, Chennai.

This book has been prepared keeping in mind the aptitude and attitude of the students and modern methods of education. The lucid manner in which the concepts are explained, make the teaching and learning process more easy and effective. Each chapter in this book is prepared with strenuous effort to present the principles of the subject in the most easy to understand and the most easy to workout manner.

Each chapter and section is presented with QR code, an introduction, learning objective, definitions, theorems , explanation, worked examples, summary and exercises with answer given are for better understanding of concepts and in the exercises, problems have been given in view of enough practice for mastering the concept.

We hope that the book serve the purpose keeping in mind the changing needs of the society to make it lively and vibrating. The language used is very clear and simple which is up to the level of comprehension of students.

We extend our deep sense of gratitude to Thiru. K.Vivekanandan I.A.S., the Chairperson for giving valuable inputs and suggestions in bringing out this text book for the benefit of the student community. We also thank the Co-ordinator Dr. M.S. Padmanabhan, Principal(i/c), Central Polytechnic College, Chennai and Conveners who took sincere efforts in preparing and reviewing this book.

Valuable suggestions and corrections for the improvement of this book is most welcome and will be acknowledge most gratefully. Mail your suggestions to dote.nscheme@gmail.com.

- AUTHORS

ANNEXURE- I

STATE BOARD OF TECHNICAL EDUCATION &TRAINING, TAMILNADU

DIPLOMA IN ENGINEERING / TECHNOLOGY SYLLABUS

N-SCHEME

(Implemented from the Academic year 2020 - 2021 onwards)

Course Name : All branches of Diploma in Engineering and Technology and Special Programmes except DMOP, HMCT and Film &TV.

Subject Code : 40022

Semester : II

Subject Title : ENGINEERING MATHEMATICS - II

TEACHING AND SCHEME OF EXAMINATION

No of weeks per semester: 16 weeks

Subject	Instructions		Examination			
	Hours / Week	Hours / Semester	Marks			Duration
			Internal Assessment	Board Examinations	Total	
ENGINEERING MATHEMATICS II	4	64	25	100*	100	3 Hrs.

*** Examinations will be conducted for 100 marks and will be reduced to 75 marks.**

TOPICS AND ALLOCATION OF HOURS:

UNIT	Topics	Duration (Hrs)
I	Analytical Geometry	12
II	Vector Algebra - I	11
III	Vector Algebra - II	11
IV	Integral Calculus - I	12
V	Integral Calculus - II	11
Test & Model Exam		7
TOTAL		64

40022 ENGINEERING MATHEMATICS – II

DETAILED SYLLABUS

Contents: Theory

UNIT	NAME OF TOPICS	HOURS
I	**ANALYTICAL GEOMETRY**	
	1.1 ANALYTICAL GEOMETRY I:	3
	Introduction – Locus – Straight lines – angle between two straight lines – pair of straight lines – simple problems.	
	1.2 ANALYTICAL GEOMETRY II:	4
	Circles – General equation of a circle – Equation of tangents to circle. Family of circles-Concentric circles – Orthogonal circles (condition only) – contact of circles - simple problems.	
	1.3 CONICS	5
	Definition, general equation and classification of conics. sections of a cone - parabola, ellipse and hyperbola - standard equations – Applications of Conics – Simple problems.	
II	**VECTOR ALGEBRA – I**	
	2.1 VECTOR – INTRODUCTION	4
	Definition of vector – types, addition, subtraction and scalar multiplication of vector, properties of addition and subtraction. Position vector. Resolution of vector in three dimensions, distance between two points, Direction cosines and direction ratios. Simple problems.	
	2.2 PRODUCT OF TWO VECTORS	4
	Scalar product – Vector product – condition for parallel and perpendicular vectors, properties, angle between two vectors, unit vector perpendicular to two vectors –simple problems.	
	2.3 APPLICATION OF SCALAR AND VECTOR PRODUCT	3
	Geometrical meaning of scalar product and vector product, workdone, moment –simple problems.	

UNIT	NAME OF TOPICS	HOURS
III	**VECTOR ALGEBRA - II**	
	3.1 PRODUCT OF THREE AND FOUR VECTORS Scalar and vector triple product and product of four vectors-simple problems.	3
	3.2 VECTOR DIFFERENTIATION Vector point function and vector field, scalar point function and scalar field, vector differential operator – Gradient – basic properties of the gradient- simple problems.	4
	3.3 APPLICATION OF VECTOR DIFFERENTIATION Divergence, basic properties, solenoidal, curl of a vector function, irrotational vector- simple problems.	4
IV	**INTEGRAL CALCULUS – I**	
	4.1 INTEGRATION - DECOMPOSITION METHOD Historical approach for integration - Anti derivative - Definition of the integral as an anti-derivative - Fundamental rules for integration - Integration using decomposition method - simple problems based on Engineering Applications.	4
	4.2 METHODS OF INTEGRATION - INTEGRATION BY SUBSTITUTION Integrals of the form $\int [f(x)]^n \, f'(x) dx$, where $n \neq -1$, $\int \frac{f'(x)}{f(x)} dx$ and $\int F[f(x)] \, f'(x) dx$ - simple problems.	4
	4.3 STANDARD INTEGRALS Integrals of the form $\int \frac{dx}{a^2 \pm x^2}$, $\int \frac{dx}{x^2 - a^2}$, $\int \frac{dx}{\sqrt{a^2 - x^2}}$, $\int \sqrt{a^2 - x^2} \, dx$, $\int \sqrt{x^2 \pm a^2} \, dx$- Simple problems.	4

UNIT	NAME OF TOPICS	HOURS
V	**INTEGRAL CALCULUS - II**	
	5.1 METHODS OF INTEGRATION - INTEGRATION BY PARTS	4
	Integrals of the form $\int x\sin nx\,dx$, $\int x\cos nx\,dx$, $\int xe^{nx}\,dx$, $\int x^n \log x\,dx$, and $\int \log x\,dx$ - Simple problems.	
	5.2 BERNOULLI'S FORMULA	4
	Evaluation for the integrals $\int x^m \sin nx\,dx$, $\int x^m \cos nx\,dx$ *and* $\int x^m e^{nx}\,dx$ Where $m \le 3$using Bernoulli's formula - Simple problems.	
	5.3 DEFINITE INTEGRALS	3
	Definition of definite integral – Properties of definite integrals - Simple problems.	

Reference Books

1. Higher Secondary +1 Mathematics volume I&II. Tamilnadu Text book corporation.
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3. Engineering Mathematics V. Sundaram, R. Balasubramanian
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8. Engineering Mathematics – I Guruprasad Samanta, New age international publishers, 2nd edition 2015.
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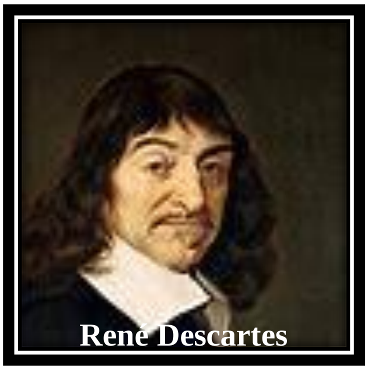
CONTENTS

ENGINEERING MATHEMATICS - II

UNIT - I ANALYTICAL GEOMETRY

Chapter 1.1 ANALYTICAL GEOMETRY - I

Introduction:



Rene Descartes

Pierre de Fermat

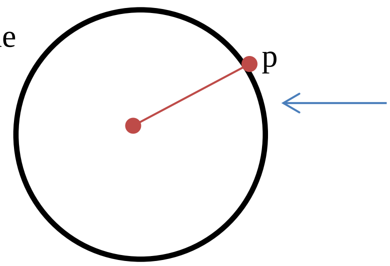
In classical Mathematics, analytical geometry also known as coordinate geometry (or) Cartesian geometry is the study of geometry using a coordinate system. This contrasts with synthetic geometry.

Analytical geometry used in physics and engineering and also in rocketry, space science and space flight. It is the foundation of most modern fields of geometry, including algebraic, differential, discrete and computational geometry. Usually the Cartesian coordinates system is applied to manipulate equations for planes, straight lines and squares often in two dimensions and sometimes three dimensions.

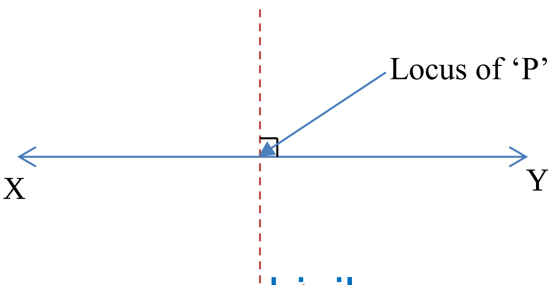
The importance of analytical geometry is that it establishes a correspondence between geometric curves and algebraic equations. This correspondence makes it possible to reformulate problems in geometry as equivalent problems in algebra and vice versa; The methods of either subject can then be used to solve problems in the other. For example, Computers create animations for display in games and films by manipulating algebraic equations. And it was the Analytical Geometry was invented and first used in 1637 by French Mathematician and Philosopher Rene Descartes (1596 – 1650). Later he showed how the methods of Algebra could be applied in to the study of Geometry. In lower classes analytical geometry can be explained more simply; it is concerned with the defining and representing geometrical shapes in a numerical way and extracting numerical informations from shapes numerical definitions and representations. This chapter is a continuation of the study of the concepts of Analytical Geometry to which the students has been introduced in earlier classes.

LOCUS

When a point moves in a plane according to some given conditions the path along which it moves is called a locus (plural of locus is loci). For example, a circle with center 'C' and radius 'r' formed a locus.



And also to construct a perpendicular bisector of the line xy of the point 'p'.



Examples:

1) Find the equation to the locus of a moving point which is always equidistant from the points (2, - 1) and (3, 2).

Solution:

Let A (2, - 1) and B (3, 2) be the given points and (x, y) be the coordinates of a point P on the required locus. Then $PA^2 = (x-2)^2 + (y+1)^2$ and $PB^2 = (x-3)^2 + (y-2)^2$

By given equidistant $PA^2 = PB^2$

(or) $(x-2)^2 + (y+1)^2 = (x-3)^2 + (y-2)^2$

(or) $x^2 - 4x + 4 + y^2 + 2y + 1 = x^2 - 6x + 9 + y^2 - 4y + 4$

(or) $2x + 6y = 8$

$\Rightarrow x + 3y = 4$

Which is the required equation to the locus of the moving point. And the locus of P is a straight line.

STRAIGHT LINES

A straight line is the set of all points between and extending beyond two points. Every straight line is associated with an equation. We recall the basic formulas of straight lines which was study in the lower class.

1) Equations of horizontal and vertical lines x = a and y = b.
2) Slope – intercept form equation of line $y = mx + c$.
3) Slope - one point form equation of line $y - y_1 = m(x - x_1)$
4) Two point form equation of line $\frac{y - y_1}{y_2 - y_1} = \frac{(x - x_1)}{(x_2 - x_1)}$
5) Intercept form $\frac{x}{a} + \frac{y}{b} = 1$.

Here we are going to discuss the other forms of straight line equations.

General Form of Straight Line:

The equation $ax + by + c = 0$ will always represent a straight line.

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be any 3 points on the locus represented by the equation $ax + by + c = 0$. Then

$$ax_1 + by_1 + c = 0 \quad \text{------ (1)}$$
$$ax_2 + by_2 + c = 0 \quad \text{------ (2)}$$
$$ax_3 + by_3 + c = 0 \quad \text{------ (3)}$$

equation (1) x $(y_2 - y_3)$ + equation (2) x $(y_3 - y_1)$ + equation (3) x $(y_1 - y_2)$

$\Rightarrow a[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 0$

Since $a \neq 0$, $x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] = 0$

(i.e.) (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear and hence they lie on a straight line. Thus the equation $ax + by + c = 0$ represents a straight line.

Perpendicular distance from a point to a straight line

The length of the perpendicular distance from the point P (x_1, y_1) to the line is $\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

Note:1 The length of the perpendicular from the origin to ax + by + c = 0 is $\left|\frac{c}{\sqrt{a^2 + b^2}}\right|$

Note: 2 The distance between two parallel straight lines is $d = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$.

Example:

1) Determine the equation of the straight line with slope 3 and y – intercept 4.

Solution:

The slope intercept form of straight line is y = mx + c.

Here slope m = 3; c = 4

$\therefore$ The required equation of the straight line is y = 3x + 4.

2) Determine the equation of the straight line passing through the point (-1, -2) and having slope $\frac{4}{7}$.

Solution:

Given point (-1, -2) and slope $\frac{4}{7}$

The point slope form is $y - y_1 = m(x - x_1)$

Here $(x_1, y_1) = (-1, -2)$

Slope $m = \frac{4}{7}$

$\therefore\ y + 2 = \frac{4}{7}(x + 1)$

(i.e.) 7y + 14 = 4x + 4

$\Rightarrow$ 4x – 7y – 10 = 0

3) Find the equation of the straight line joining the points (3, 6) and (2, -5).

Solution:

The equation of the straight line passing through two point is $\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$

Here $(x_1, y_1) = (3,6)$ and $(x_2, y_2) = (2, -5)$

Substituting the required values in the above straight line formula we get,

$$\frac{y-6}{6+5} = \frac{x-3}{3-2}$$

$$\frac{y-6}{11} = \frac{x-3}{1}$$

$1\,(y - 6) = 11\,(x - 3)$

$y - 6 = 11x - 33$

$11x - y - 27 = 0$ is the required equation of the straight line.

4) Find the length of the perpendicular from (3, 2) to the straight line $3x + 2y + 1 = 0$.

Solution:

Let $(x_1, y_1) = (3, 2)$

The perpendicular distance from (x_1, y_1) to the straight line $ax + by + c = 0$ is given by

$\left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right| = \left|\frac{3\,(3) + 2\,(2) + 1}{\sqrt{3^2 + 2^2}}\right| = \frac{14}{\sqrt{13}}$ units.

5) Find the equation of the straight line passing through the point (2, 1) and making intercepts on the co-ordinate axes which are in the ratio 2 : 3.

Solution:

The intercept form of straight line is $\frac{x}{a} + \frac{y}{b} = 1$ ------- (1)

The intercepts are in the ratio 2 : 3 $\therefore a = 2k;\ b = 3k$

Equation (1) becomes $\frac{x}{2k} + \frac{y}{3k} = 1$

$3kx + 2ky = 6k^2$

(i.e.) $3x + 2y = 6k$

Since (2, 1) lies on the above straight line is

$6 + 2 = 6k$

$8 = 6k$

$\boxed{k = \frac{4}{3}}$

Sub $k = 4/3$ in $3x + 2y = 6k$

Hence the required equation of the straight line is $3x + 2y = 8$.

Exercise : 1.1.1

1) Find the length of the perpendicular from (2, - 3) to the line $2x - y + 9 = 0$.

2) Find the equation of the straight line passing through the point (1, 2) and making intercepts on the co-ordinate axes which are in the ratio 2 : 3.

3) Determine the equation of the straight line passing through the points (1, 2) and (3, -4).

4) Determine the equation of the straight line passing through (-1, 2) and having slope $\frac{2}{7}$.

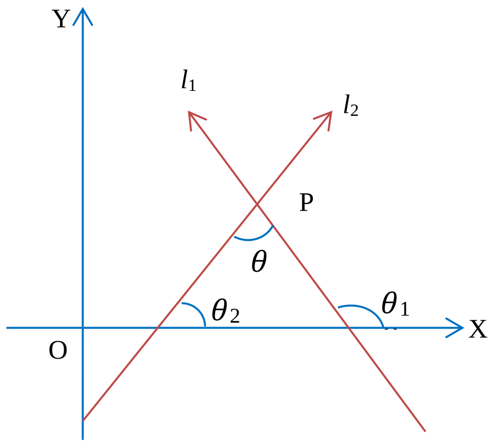
Exercise : 1.1.1 - Answers

(1) $\frac{16}{\sqrt{5}}$ units (2) $3x + 2y = 7$ (3) $3x + y = 5$ (4) $2x - 7y + 16 = 0$.

ANGLE BETWEEN TWO STRAIGHT LINES:

Whenever two straight lines intersect, they form two sets of angles. The intersection forms a pair of acute angles and another pair of obtuse angles. The absolute values of angles formed depend on the slopes of the intersecting lines.

Consider the diagram



Let l_1 be $y = m_1x + c_1$ and l_2 be $y = m_2x + c_2$, are two straight lines. They are intersecting at the point P which makes angle θ_1 and θ_2 with the positive direction of x – axis.

Then $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$ are slopes of the two straight lines.

Let 'θ' be the angle between the two straight lines.

From the diagram $\theta_1 = \theta + \theta_2$

$\therefore\ \theta = \theta_1 - \theta_2$

$\Rightarrow \quad \tan\theta = \tan(\theta_1 - \theta_2)$

$$= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \cdot \tan\theta_2} \left[\because \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\ \tan B}\right]$$

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 \cdot m_2}$$

Note that $\frac{m_1 - m_2}{1 + m_1 . m_2}$ is either positive (or) negative.

As convention we consider the acute angle as the angle between any two straight lines and hence we consider only the absolute value of tan θ.

Hence $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$

$\therefore\ \theta = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right|$

Corollary (1):

If the two straight lines are parallel, then their slopes are equal.

Proof:

Since the two straight lines are parallel, θ = 0 (there is no angle between them).

$\therefore \tan\theta = 0.$

$\Rightarrow \frac{m_1 - m_2}{1 + m_1 \cdot m_2} = 0$

$\Rightarrow\ m_1 - m_2 = 0. \ (\because 1 + m_1.m_2 \neq 0)$

(i.e.) $m_1 = m_2$

∴ If the straight lines are parallel, then the slopes are equal.

Corollary (2)

If the two straight lines are perpendicular then the product of their slopes is – 1.

Proof:

Since the two straight lines are perpendicular $\theta = 90^o$

$\therefore \tan\theta = \tan 90^o = \infty$

$\Rightarrow \tan\theta = \infty$

$\Rightarrow \frac{m_1 - m_2}{1 + m_1 \cdot m_2} = \infty$

This is possible only if the denominator is zero.

(i.e) $1 + m_1 \cdot m_2 = 0$

$\Rightarrow m_1 \cdot m_2 = -1$

$\therefore$ If the two straight lines are perpendicular then the product of their slopes is -1.

Note:1

If the straight lines are parallel, then the coefficients of x and y are proportional in their equations. In particular, the equations of two parallel straight lines differ only by constant term.

Note:2

The equation of the straight line perpendicular to the straight line $ax + by + c = 0$ is of the form $bx - ay + k = 0$ for some k.

Example:

1) Find the angle between the lines $2x - 3y + 7 = 0$ and $7x + 4y - 9 = 0$.

Solution:

Comparing the equations with the straight line equation $y = mx + c$.

Slope of $2x - 3y + 7 = 0$ is $m_1 = 2/3$ $\qquad$ $ax + by + c = 0$

Slope of $7x + 4y - 9 = 0$ is $m_2 = -7/4$ $\qquad$ $\because$ slope $m = \frac{-a}{b}$

Let θ be the angle between two lines, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 \cdot m_2}\right| = \left|\frac{\frac{2}{3} + \frac{7}{4}}{1 + \left(\frac{2}{3}\right)\left(-\frac{7}{4}\right)}\right| = \left|\frac{8 + 21}{12 - 14}\right|$$

$$\tan\theta = \left|\frac{29}{-2}\right|$$

$$\boxed{\theta = \tan^{-1}\left(\frac{29}{2}\right)}$$

2) Show that the straight lines $2x + y = 5$ and $x - 2y = 4$ are at right angles.

Solution:

Given $2x + y = 5$ ------ (1) and

$x - 2y = 4$ ------ (2)

Slope of equation (1) is $m_1 = \frac{-2}{1} = -2$

Slope of equation (2) is $m_2 = \frac{-1}{-2} = \frac{1}{2}$

Then $m_1 \cdot m_2 = (-2)\left(\frac{1}{2}\right) = -1$

$\therefore$ The two straight lines are at right angles.

3) Find the equation of the straight lines parallel to the line $3x + 2y - 7 = 0$ and passing through the point (1, -2).

Solution:

The straight line parallel to $3x + 2y - 7 = 0$ is of the form $3x + 2y + k = 0$ ------ (1)

The point (1, -2) satisfies the equation (1)

Hence $3(1) + 2(-2) + k = 0$

$3 - 4 + k = 0$

$-1 + k = 0$

$\boxed{k = 1}$

sub k = 1 in (1)

$\therefore$ The required equation of the straight line is $3x + 2y + 1 = 0$.

4) If the two straight lines $2x - 3y + 9 = 0$ and $6x + ky + 9 = 0$ are parallel, then find the value of 'k'. And also find the distance between them.

Solution:

Given $2x - 3y + 9 = 0$ ------ (1)

$6x + ky + 9 = 0$ ------ (2)

Slope of (1) is $m_1 = \frac{2}{3}$

Slope of (2) is $m_2 = \frac{-6}{k}$

Since lines are parallel $m_1 = m_2$

$\frac{2}{3} = \frac{-6}{k}$

$\Rightarrow \boxed{k = -9}$

sub k = - 9 in (2)

Hence the equations are $2x - 3y + 9 = 0$ and $6x - 9y + 9 = 0$ (i.e.) $2x - 3y + 3 = 0$.

The distance between the parallel line is $d = \left|\frac{C_1 - C_2}{\sqrt{a^2 + b^2}}\right|$

Here $a = 2$; $b = -3$; $c_1 = 9$; $c_2 = 3$.

$d = \left|\frac{9-3}{\sqrt{2^2 + (-3)^2}}\right| = \left|\frac{6}{\sqrt{13}}\right| \quad \therefore \boxed{d = \frac{6}{\sqrt{13}}}$ units

5) Find the equation of the straight line passing through the point (2, 1) and perpendicular to the straight line x + y = 9.

Solution:

The equation of any straight line perpendicular to x + y – 9 = 0 is of the form

x – y + k= 0.

The point (2, 1) lies on the straight line

$2 - 1 + k = 0.$

$\boxed{k = -1}$

$\therefore$ The required equation of the straight line is $x - y - 1 = 0$.

6) Show that the straight lines 3x + y + 4 = 0, 3x + 4y – 15 = 0 and 24x – 7y – 3 = 0 form an isosceles triangle.

Solution:

Given $3x + y + 4 = 0$ ------ (1)

$3x + 4y - 15 = 0$ ------ (2)

$24x - 7y - 3 = 0$ ------ (3)

Slope of (1) is $m_1 = \frac{-3}{1} = -3$

Slope of (2) is $m_2 = \frac{-3}{4}$

Slope of (3) is $m_3 = \frac{-24}{-7} = \frac{24}{7}$

Let θ_1 be the angle between (1) & (2)

$$\theta_1 = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 \, . \, m_2}\right|$$

$$= \tan^{-1}\left|\frac{-3 + \frac{3}{4}}{1 + (-3) + \left(\frac{-3}{4}\right)}\right|$$

$$\theta_1 = \tan^{-1}\left(\frac{9}{13}\right)$$

Let θ_2 be the angle between (2) & (3)

$$\theta_2 = \tan^{-1}\left|\frac{\left(\frac{-3}{4}\right) - \left(\frac{24}{7}\right)}{1 + \left(\frac{-3}{4}\right)\left(\frac{24}{4}\right)}\right| = \tan^{-1}\left(\frac{117}{44}\right)$$

Let θ_3 be the angle between (3) & (1)

$$\theta_3 = \tan^{-1}\left|\frac{\frac{24}{7} + 3}{1 + \left(\frac{24}{7}\right)(-3)}\right| = \tan^{-1}\left(\frac{45}{65}\right) = \tan^{-1}\left(\frac{9}{13}\right)$$

$\therefore \theta_1 = \theta_3$

Hence the triangle is isosceles triangle.

The conditions for the three straight lines to be concurrent:

Let the three straight lines be

$a_1x + b_1y + c_1 = 0$ ----- (1)

$a_2x + b_2y + c_2 = 0$ ----- (2)

$a_3x + b_3y + c_3 = 0$ ----- (3)

If the three straight lines are concurrent, then the point of intersection of any two straight lines lies on the third straight line.

Solving (1) & (2), the point of intersection is,

$$x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \text{ and } y = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$$

Substituting the values of x and y in equation (3).

We get, $a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2) = 0$.

(i.e) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$ is the condition for the three straight lines to be concurrent.

Example

1) Show that the straight lines $3x + 4y = 13$, $2x - 7y = -1$ and $5x - y = 14$ are concurrent.

Solution:

The condition of the concurrent is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 3 & 4 & -13 \\ 2 & -7 & 1 \\ 5 & -1 & -14 \end{vmatrix}$$

$= 3(99) - 4(-33) - 13(33)$

$= 297 + 132 - 429$

$= 429 - 429$

$= 0$

The given lines are concurrent.

2) Find 'a' so that the straight lines $x - 6y + a = 0$, $2x + 3y + 4 = 0$ and $x + 4y + 1 = 0$ are concurrent.

Solution:

Given straight lines are concurrent

$$\begin{vmatrix} 1 & -6 & a \\ 2 & 3 & 4 \\ 1 & 4 & 1 \end{vmatrix} = 0$$

$$1(-13) + 6(-2) + a(5) = 0$$
$$-25 + 5a = 0$$
$$5a = 25$$
$$\boxed{a = 5}$$

Exercise: 1.1.2

1) Find the angle between the straight lines

   a) $3x - 2y + 9 = 0$, $2x + y - 9 = 0$

   b) $2x + y = 4$, $x + 3y = 5$

2) Show that the triangle formed by straight lines $4x - 3y - 18 = 0$, $3x - 4y + 16 = 0$ and $x + y - 2 = 0$ is isosceles.

3) Show that the two straight lines whose equations $2x + 3y - 6 = 0$ and $2x + 3y + 7 = 0$ are parallel also find the distance between them.

4) Find the equation of the straight line perpendicular to straight line $3x + 4y + 28 = 0$ and passing through the point (-1, 4).

5) Show that the straight lines $3x + 4y = 13$, $2x - 7y + 1 = 0$ and $5x - y = 14$ are concurrent.

Exercise: 1.1.2 - Answers:

(1) (a) $\tan^{-1}(7/4)$ (b) $\pi/4$ (3) $d = \sqrt{13}$ units (4) $4x - 3y + 16 = 0$

PAIR OF STRAIGHT LINES

Introduction:

We know that every linear equation in x and y represents a straight line. That is $Ax + By + C = 0$, where A, B and C are constants, represents a straight line.

Consider two straight lines represented by the following equations:

$$l_1x + m_1y + n_1 = 0 \quad \text{........ (1)}$$

$$l_2x + m_2y + n_2 = 0 \quad \text{........ (2)}$$

Also consider the equation,

$$(l_1x + m_1y + n_1)(l_2x + m_2y + n_2) = 0 \quad \text{........ (3)}$$

If (x_1, y_1) is a point on the straight line given by (1), then $l_1x_1 + m_1y_1 + n_1 = 0$

This shows that (x_1, y_1) is also a point on the locus of (3). Therefore, every point on the line given by (1) is also a point on the locus of (3).

Similarly, every point on the line given by (2) is also a point on the locus of (3).

Therefore, (3) satisfies all points on the straight lines given by (1) and (2). Hence, we say (3) represents the combined equation of the straight lines given by (1) and (2).

It is possible to rewrite (3) as.

$$l_1l_2\,x^2 + (l_1m_2 + l_2m_1)\,xy + m_1m_2\,y^2 + (l_1n_2 + l_2n_1)\,x + (m_1n_2 + m_2n_1)\,y + n_1n_2 = 0 \quad \text{---- (4)}$$

(i.e) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ -------- (5)

Where, $l_1l_2 = a$, $m_1m_2 = b$, $n_1n_2 = c$

$l_1m_2 + l_2m_1 = 2h$, $l_1n_2 + l_2n_1 = 2g$, $m_1n_2 + m_2n_1 = 2f$

Homogenous Equation of Second Degree:

Every homogeneous equation of second degree in x and y represents a pair of straight lines passing through the origin.

Consider the equation, $ax^2 + 2hxy + by^2 = 0$, $a \neq 0$.

Dividing by x^2, we get $b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = 0$. This is a quadratic equation in $\frac{y}{x}$, and hence there are two values for $\frac{y}{x}$, say m_1 and m_2. Then,

$$b\left(\frac{y}{x}\right)^2 + 2h\left(\frac{y}{x}\right) + a = b\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right)$$

(i.e) $b\,(y - m_1x)\,(y - m_2x) = 0$

But $y - m_1x = 0$ and $y - m_2x = 0$ are straight lines passing through the origin.

Therefore, $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin.

Note: $ax^2 + 2hxy + by^2 = b\,(y - m_1x)\,(y - m_2x)$

Equating the coefficients of x^2 and xy, we get

$$m_1 + m_2 = \frac{-2h}{b}$$

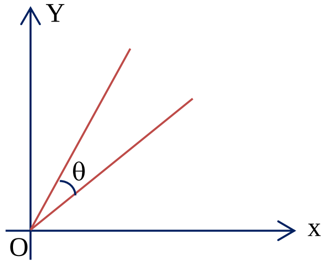
$$m_1 m_2 = \frac{a}{b}$$

Angle between the lines represented by $ax^2 + 2hxy + by^2 = 0$

Let $y - m_1x = 0$ and $y - m_2x = 0$ be the two lines represented by $ax^2 + 2hxy + by^2 = 0$.

Then, $m_1 + m_2 = \frac{-2h}{b}$

and $m_1 m_2 = \frac{a}{b}$



Let θ be the angle between the lines given by $ax^2 + 2hxy + by^2 = 0$. Then the angle between the lines is given by.

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \frac{\pm\sqrt{(m_1+m_2)^2 - 4m_1m_2}}{1 + m_1m_2} = \frac{\pm\sqrt{\left(\frac{-2h}{b}\right)^2 - 4\left(\frac{a}{b}\right)}}{1 + \frac{a}{b}}$$

$$\tan\theta = \frac{\pm 2\sqrt{h^2 - ab}}{a+b} \quad \text{------- (5)}$$

$$\therefore \theta = \tan^{-1}\left(\frac{\pm 2\sqrt{h^2 - ab}}{a+b}\right)$$

The positive sign gives the acute angle between the lines and negative sign gives the obtuse angle between them.

Note:

1. If the lines are parallel or coincident, then θ = 0. Then tan θ = 0. Therefore, from (5), we get $h^2 = ab$.
2. If the lines are perpendicular, then θ = π/2 and so we get from (6),

$$\tan\frac{\pi}{2} = \frac{\pm 2\sqrt{h^2 - ab}}{a+b}$$

This means a + b = 0.

Example 1:

The gradient of one of the lines $ax^2 + 2hxy + by^2 = 0$ is twice that of the other. Show that $8h^2 = 9ab$.

Solution:

The equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin.

Let the lines by $y - m_1x = 0$ and $y - m_2x = 0$.

Then $ax^2 + 2hxy + by^2 = b(y - m_1x)(y - m_2x)$.

Equating the coefficients of xy and x^2 on both sides, we get

$$m_1 + m_2 = \frac{-2h}{b} \qquad \text{----- (1)}$$

$$m_1m_2 = \frac{a}{b} \qquad \text{----- (2)}$$

Here, it has been given that $m_2 = 2m_1$

From (1) and (2), we get $3m_1 = \frac{-2h}{b}$

$$m_1 = \frac{-2h}{3b}$$

Also, $2m_1^2 = \frac{a}{b}$

$$2\left(\frac{-2h}{3b}\right)^2 = \frac{a}{b}$$

$\Rightarrow 8h^2 = 9ab$

Example: 2

Separate the equations $5x^2 + 6xy + y^2 = 0$.

Solution: We factorize this equation straight away as

$5x^2 + 6xy + y^2 = 0$

$5x^2 + 5xy + xy + y^2 = 0$

$5x(x + y) + y(x + y) = 0$

$(x + y)(5x + y) = 0$

So that the separate lines are $x + y = 0$ and $5x + y = 0$.

Alternate Method: Since the given equation is a homogenous equation, divide the given equation $5x^2 + 6xy + y^2 = 0$ by x^2

We get $5 + 6\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 = 0$

Substitute $\frac{y}{x} = m$ (slope of the lines for homogenous equation)

The above equation becomes $m^2 + 6m + 5 = 0$

Factorizing, we get $(m + 1)(m + 5) = 0$

$m = -1, \quad m = -5$

$\frac{y}{x} = -1, \quad \frac{y}{x} = -5$

$y = -x, \quad y = -5x$

That is, the lines are, $x + y = 0, 5x + y = 0$

Example: 3

If exists, find the straight lines by separating the equations $2x^2 + 2xy + y^2 = 0$.

Solution:

Since the given equation is a homogenous equation, divide the given equation $2x^2 + 2xy + y^2 = 0$ by x^2 and substituting $\frac{y}{x} = m$

We get $m^2 + 2m + 2 = 0$.

The values of m (slopes) are not real (complex number), therefore no line will exist with the join equation $2x^2 + 2xy + y^2 = 0$.

We sometimes say that the equation represents imaginary lines.

Note that in the entire plane, only (0, 0) satisfies this equation.

Example: 4

Find the combined equation of the two straight lines represented by $x + 2y = 0$ and $3x + y = 0$.

The given separate lines are $x + 2y = 0$ and $3x + y = 0$.

The combined equation of the two given straight lines is

$$(x + 2y)(3x + y) = 0$$

i.e., $3x^2 + xy + 6xy + 2y^2 = 0$

i.e., $3x^2 + 7xy + 2y^2 = 0$

Example: 5

Find the value of p if the lines represented by $px^2 - 5xy - 7y^2 = 0$ are perpendicular to each other.

Given: The pair of straight lines represented by $px^2 - 5xy - 7y^2 = 0$ are perpendicular.

$\therefore$ coefficient of x^2 + coefficient of $y^2 = 0$.

(i.e.) $p - 7 = 0$

$p = 7 \qquad (\because a + b = 0)$

Example: 6

Find the acute angle between the pair of lines represented by $2x^2 + 5xy + 3y^2 = 0$.

Solution: $2x^2 + 5xy + 3y^2 = 0$ is the given equation of pair of straight lines.

Let θ be the angle between the two straight lines.

$$\therefore \tan\theta = \frac{\pm 2\sqrt{h^2 - ab}}{a + b}, \quad a = 2, b = 3, h = 5/2$$

$$= \pm \frac{2\sqrt{\left(\frac{5}{2}\right)^2 - (2)(3)}}{2 + 3}$$

$$= \frac{\pm 2\sqrt{\frac{25}{4} - \frac{6}{1}}}{2 + 3} = \pm \frac{2\sqrt{\frac{1}{4}}}{5} = \pm \frac{2\left(\frac{1}{2}\right)}{5} = \frac{1}{5}$$

$$\therefore \tan\theta = \frac{1}{5}$$

$$\theta = \tan^{-1}\left(\frac{1}{5}\right)$$

Example: 7

Find the positive value of k such that the angle between the lines $2x^2 - 7xy + ky^2 = 0$ is 45°.

Solution: $2x^2 - 7xy + ky^2 = 0$ is the given equation of pair of straight lines.

Let θ be the angle between the lines.

$\therefore \theta = 45°$ (Given)

$\therefore \tan\theta = \tan 45 = 1$

(i.e) $\frac{\pm 2\sqrt{h^2 - ab}}{a+b} = 1$, here $a = 2$, $b = k$, $h = -\frac{7}{2}$

$\therefore \quad \frac{\pm 2\sqrt{\left(\frac{-7}{2}\right)^2 - 2k}}{2+k} = 1$

$\therefore \quad \pm\sqrt{\frac{49}{4} - 2k} = \frac{2+k}{2}$

Squaring both sides we get $\frac{49}{4} - 2k = \frac{(2+k)^2}{2^2}$

(i.e.) $\frac{49 - 8k}{4} = \frac{4 + k^2 + 4k}{4}$

(i.e.) $k^2 + 4 + 4k = 49 - 8k$

(i.e) $k^2 + 4k + 8k + 4 - 49 = 0$

(i.e.) $k^2 + 12k - 45 = 0$

(i.e.) $(k + 15)(k - 3) = 0$

(i.e) $k = 3$ or $k = -15$

$\boxed{k = 3}$ is the required value of for k. (∵ only positive value is asked)

Exercise: 1.1.3

1. The gradient of one of the lines $ax^2 + 2hxy + by^2 = 0$ is thrice that of the other. Show that $3h^2 = 4ab$.
2. Find the acute angle between the pair of line represented by the following equations.

   (i) $x^2 - 7xy + 12y^2 = 0$ [Ans: $\theta = \tan^{-1}(1/13)$]

   (ii) $y^2 - xy - 6x^2 = 0$ [Ans: $\theta = \pi/4$]

3) Find the combined equation representing the following pairs of straight lines.

   a) $2x + 3y = 0$ and $4x - 5y = 0$

   b) $4x + 5y = 0$ and $7x - 2y = 0$

   [Ans: a) $8x^2 + 2xy - 15y^2$ b) $28x^2 + 27xy - 10y^2$]

4) Find the separate equation of each of the straight lines represented by

   a) $2x^2 - 5xy - 3y^2 = 0$ [Ans: $2x + y = 0$, $x - 3y = 0$]

   b) $6x^2 - xy - y^2 = 0$ [Ans: $2x - y = 0$, $3x + y = 0$]

5) If the two straight lines represented by the equation $px^2 + 6xy - y^2 = 0$ are perpendicular to each other find the value of p. [Ans : 1]

6) Show that the equation $4x^2 - 12xy + 9y^2 = 0$ represents a pair of parallel straight lines.

7) Find the values of p if the two straight lines represented by $20x^2 + pxy + 5y^2 = 0$ are parallel to each other. [Ans: $p = \pm 20$]

8) Find the separate equation of the pair of straight lines $3x^2 + 8xy + 4y^2 = 0$. Also find the angle between these two lines. [Ans: $3x + 2y = 0$ & $x+2y = 0$; $\theta = \tan^{-1}(4/7)$]

Condition for general second degree equation to represent a pair of straight lines:

Consider the general equation of the second degree,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{----- (1)}$$

Let $lx + my + n = 0$ and $l_1x + m_1y + n_1 = 0$ be the equations of two lines represented by (1). Then,

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n)(l_1x + m_1y + n_1)$ comparing the co-efficients, we get

$$\left.\begin{array}{ll} ll_1 = a & lm_1 + l_1m = 2h \\ mm_1 = b & ln_1 + l_1n = 2g \\ nn_1 = c & mn_1 + m_1n = 2f \end{array}\right\} \qquad \text{----- (2)}$$

We know that,

$$\begin{vmatrix} l & l_1 & 0 \\ m & m_1 & 0 \\ n & n_1 & 0 \end{vmatrix} \times \begin{vmatrix} l_1 & l & 0 \\ m_1 & m & 0 \\ n_1 & n & 0 \end{vmatrix} = 0$$

By multiplying the two determinants, we get

$$\begin{vmatrix} 2ll_1 & lm_1 + l_1m & ln_1 + l_1n \\ l_1m + lm_1 & 2mm_1 & mn_1 + nm_1 \\ l_1n + ln_1 & m_1n + nm_1 & 2nn_1 \end{vmatrix} = 0 \qquad \text{------ (3)}$$

Substituting the values (2) in (3), we get

$$\begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0$$

(i.e.) $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Expanding the determinant, we get

$$a(bc - f^2) - h(hc - gf) + g(hf - bg) = 0$$

$$\Rightarrow \quad abc - af^2 - ch^2 + ghf + ghf - bg^2 = 0$$

$$\Rightarrow \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

This is the required condition.

Note:

The point of intersection of the lines represented by is $\left[\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right]$

Example 1:

Find λ, so that the equation $x^2 + 5xy + 4y^2 + 3x + 2y + \lambda = 0$ represents a pair of lines. Find also their point of intersection and the angle between them.

Solution:

Consider the second degree terms, $x^2 + 5xy + 4y^2$.

$$x^2 + 5xy + 4y^2 = (x + y)(x + 4y)$$

Let the two straight lines be $x + y + l = 0$ and $x + 4y + m = 0$. Then,

$$x^2 + 5xy + 4y^2 + 3x + 2y + \lambda = (x + y + l)\ (x + 4y + m)$$

Equating the coefficients of x, y and constant terms, we get

$$m + l = 3 \qquad \text{-----}\ (1)$$

$$m + 4l = 2 \qquad \text{-----}\ (2)$$

$$lm = \lambda \qquad \text{-----}\ (3)$$

Solving (1) and (2), we get $3l = -1$

$$\boxed{l = \frac{-1}{3}}$$

Sub $l = \frac{-1}{3}$ in (1)

$$m = 3 + 1/3 = 10/3$$

From (3), $\boxed{\lambda = -10/9}$

Then the two lines are $x + y \frac{-1}{3} = 0$, $x + 4y \frac{10}{3} = 0$ (or)

$$3x + 3y - 1 = 0,\ 3x + 12y + 10 = 0$$

Solving these two equations, we get the point of intersection as $\left(\frac{14}{9}, \frac{-11}{9}\right)$.

The angle between the lines is given by,

$$\tan\theta = \frac{\pm 2\sqrt{h^2 - ab}}{a+b} = \frac{\pm 2\sqrt{\left(\frac{5}{2}\right)^2 - (1)(4)}}{1+4}$$

$$= \frac{\pm 2\sqrt{\frac{25}{4} - 4}}{5} = \frac{\pm 2\sqrt{\frac{25-16}{4}}}{5}$$

$$= \frac{\pm 2\left(\frac{3}{2}\right)}{5}$$

$$\tan\theta = 3/5$$

$$\theta = \tan^{-1}\left(\frac{3}{5}\right)$$

Example 2:

Find the value of λ so that the equation $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines. Find also their point of intersection.

Solution:

$$\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$$

Comparing with the equation, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

We get	$a = \lambda,$	$2h = -10,$	$b = 12,\ 2g = 5,$	$2f = -16,$	$c = -3$
$\Rightarrow$	$a = \lambda,$	$h = \frac{-10}{2},$	$b = 12,\ g = 5/2,$	$f = \frac{-16}{2},$	$c = -3$
		$h = -5$		$f = -8$	

The condition for the given equation to represent a pair of straight lines is,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$-36\lambda + 200 - 64\lambda - 75 + 75 = 0$$

$$\boxed{\lambda = 2}$$

Then,

$$2x^2 - 10xy + 12y^2 + 5x - 16y - 3 = (2x - 4y + l)\ (x - 3y + m)$$

Equating the coefficients, of x, y and constant terms,

$2m + l = 5$ ----- (1)

$4m + 3l = 16$ ----- (2)

$lm = -3$ ----- (3)

Solving (1) and (2), we get $l = 6$, $m = -1/2$

Therefore, the two lines are $2x - 4y + 6 = 0$ and $x - 3y - ½ = 0$. Solving these two equations, we get the point of intersection as (-10, -7/2).

Example 3:

Find the value of λ so that the equation $x^2 - \lambda xy + 2y^2 + 3x - 5y + 2 = 0$ represents a pair of straight lines.

Solution:

$a = 1,\ b = 2,\ c = 2,\ f = -5/2,\ g = 3/2,\ h = -\lambda/2$

Condition for pair of straight lines., $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

$$\Rightarrow\ 4 + 2\left(\frac{-5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{-\lambda}{2}\right) - 1 \times \left(\frac{25}{4}\right) - 2\left(\frac{1}{4}\right) - 2\left(\frac{\lambda^2}{4}\right) = 0.$$

$$\Rightarrow\ 2\lambda^2 - 15\lambda + 27 = 0$$

$$\Rightarrow\ (2\lambda - 9)(\lambda - 3) = 0$$

$$\lambda = 3,\ 9/2$$

Exercise: 1.1.4

1. Show that the equation $6x^2 + 17xy + 12y^2 + 22x + 31y + 20 = 0$ represents a pair of straight lines and find their separate equations.

   [Ans: $2x + 3y + 4 = 0$, $3x + 4y + 5 = 0$]

2. Prove that the equation $3x^2 + 8xy - 7y^2 + 21x - 3y + 18 = 0$, represents two lines. Find their point of intersection and the angle between them.

   [Ans: $\left(\frac{-3}{2}, \frac{-5}{2}\right)$, $\pi/2$]

3. The equation $ax^2 - 2xy - 2y^2 - 5x + 5y + c = 0$ represents two straight lines perpendicular to each other. Find a and c.

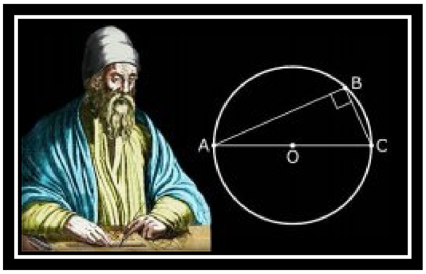
   [Ans: $a = 2$, $c = -3$]

4. Prove that the equations $8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0$ represents two parallel straight lines and find the distance between them.

   [Ans: $\frac{7\sqrt{5}}{2}$]

Chapter 1.2 ANALYTICAL GEOMETRY - II

CIRCLES:

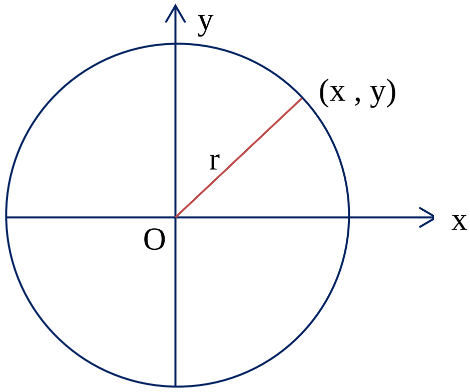


A circle is a round shaped figure that has no corner or edges.

In geometry, a circle can be defined as a closed, two – dimensional curved shape.

Center of a Circle:

The center of a circle is the center point in a circle from which all the distances to the points on the circle are equal. This distance is called the radius of the circle. Here O is the centre of the circle and r is the radius.



Equation of a Circle:

A circle is a set of points which are equidistant from a fixed point called the center.

The distance from the centre to any point on the circle is called the radius.

On the right is circle with centre (0,0) radius (r) and (x,y) any point on the circle.

Distance between (0, 0) and (x, y) are equals the radius r,

$\therefore \sqrt{(x-0)^2 + (y-0)^2} = r$ [Distance formula]

$\sqrt{x^2 + y^2} = r$

$x^2 + y^2 = r^2$ [Squares both sides]

Here, $x^2 + y^2 = r^2$ is said to be the equation of the circle.

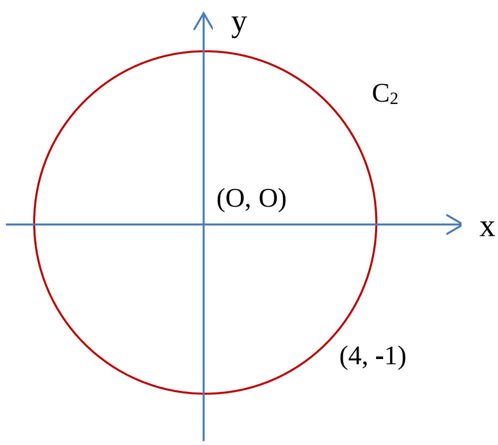
Equation of a circle with centre (0,0) and radius r, is $x^2 + y^2 = r^2$

Example: 1

Find the equation of the circles, each of centre (0,0):

(i) C_1, which has radius $\sqrt{13}$

Solution:



Centre is (0,0), Radius r = $\sqrt{13}$

Circle equation is, $x^2 + y^2 = r^2$

$x^2 + y^2 = (\sqrt{13})^2$

$x^2 + y^2 = 13$

The equation of the circle C_1 is, $x^2 + y^2 = 13$

(ii) C_2, which contains the point (4, -1)

Solution:

Here, the centre is (0,0).

$\therefore$ C_2 is the form of $x^2 + y^2 = r^2$ contains the point (4, -1)

$$x^2 + y^2 = r^2$$

$$(4)^2 + (-1)^2 = r^2$$

$$17 = r^2 \Rightarrow r = \sqrt{17}$$

Thus, the equation of the circle is $x^2 + y^2 = 17$.

Example: 2

Find the centre and radius for the following circles.

i) $x^2 + y^2 = 18$

In the form $x^2 + y^2 = r^2$

Centre is (0, 0)

$$r^2 = 18 \Rightarrow r = \sqrt{18}$$

$$r = 3\sqrt{2}$$

ii) $9x^2 + 9y^2 = 36$

Given, $9x^2 + 9y^2 = 36$

$x^2 + y^2 = \frac{36}{9}$ [Divide each side by 9]

In the form $x^2 + y^2 = r^2$

$\therefore$ The centre is (0,0)

$$r^2 = \frac{36}{9} = 4$$

$r = 2 \Rightarrow$ Radius = 2

Exercise : 1.2.1

1. Find the equation of the circles of centre (0,0) and

   i) radius $2\sqrt{3}$

   Ans: $x^2 + y^2 = 12$

   ii) radius 5

   Ans: $x^2 + y^2 = 25$

   iii) Containing the point (0, -3)

   Ans: $x^2 + y^2 = 9$

   iv) Containing the point (2, -5)

   Ans: $x^2 + y^2 = 29$

2. Write down the radius length of the each of the following circles:

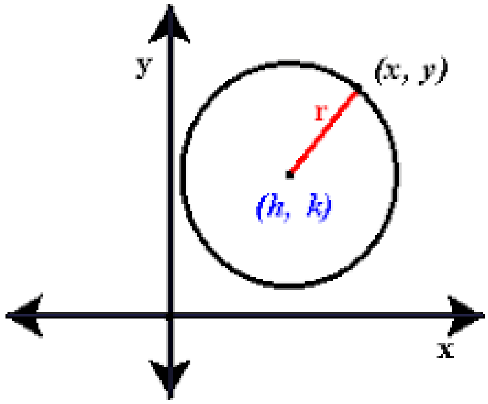
   i) $x^2 + y^2 = 29$

   Ans: $r = \sqrt{29}$

   ii) $16x^2 + 16y^2 = 1$

   Ans: $r = \frac{1}{4}$

EQUATION OF A CIRCLE, CENTRE (h, k) AND RADIUS (r):



On the right is a circle with centre (h, k) and radius r and (x, y) is any point on the circle.

Distance between (h, k) and (x, y) equals the radius r.

$\therefore \sqrt{(x-h)^2 + (y-k)^2} = r$ (Distance formula)

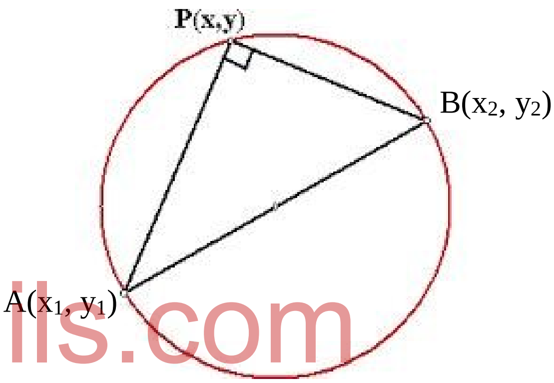
$(x - h)^2 + (y - k)^2 = r^2$ (Square both sides)

Hence,

$(x - h)^2 + (y - k)^2 = r^2$ is said to be the equation of the circle.

> The equation of a circle with centre (h, k) and radius r, is $(x - h)^2 + (y - k)^2 = r^2$

DIAMETER FORM OF A CIRCLE:



Let A (x_1, y_1), B(x_2, y_2) be the end points of the diameter of the circle. Let P(x, y) be any point on the circle.

The gradient (or) slope is

$m_{AP} = \frac{y - y_1}{x - x_1}$, $m_{BP} = \frac{y - y_2}{x - x_2}$

Since D_{APB} is a right angle (D is semi-circle)

$\therefore m_1 m_2 = -1$

$m_{AP}.m_{BP} = \left[\frac{y - y_1}{x - x_1}\right] \times \left[\frac{y - y_2}{x - x_2}\right] = -1$

We get the equation of the circle:

$$\boxed{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0}$$

Example : 3

i) Find the equation of the circle with centre (-3, 2) and radius $\sqrt{10}$.

Solution:

Centre (h, k) = (-3, 2) , Radius r = $\sqrt{10}$.

Here h = -3, k = 2, r = $\sqrt{10}$.

The equation is,

$(x - h)^2 + (y - k)^2 = r^2$

$(x + 3)^2 + (y - 2)^2 = (\sqrt{10})^2$

$(x + 3)^2 + (y - 2)^2 = 10$

$x^2 + 6x + 9 + y^2 - 4y + 4 = 10$

$x^2 + y^2 + 6x - 4y + 13 - 10 = 0$ (or)

$\therefore x^2 + y^2 + 6x - 4y + 3 = 0$

ii) Find the centre and radius of the circle $(x - 2)^2 + (y + 5)^2 = 9$.

Solution:

Given, $(x - 2)^2 + (y + 5)^2 = 9$

Circle equation, $(x - h)^2 + (y - k)^2 = r^2$.

Compare exactly to : $h = 2$, $k = -5$, $r^2 = 9 \Rightarrow r = 3$.

$\therefore$ Thus, centre (2, -5) and radius r = 3.

iii) Find the equation of the circle with centre (-1, -3) and containing the point (3, 0).

Solution:

Given, Centre = (-1, -3)

Containing point = (3, 0)

Equation : $(x - h)^2 + (y - k)^2 = r^2$

$(x + 1)^2 + (y + 3)^2 = r^2$ ------- (1)

$\Rightarrow (3, 0)$ $(3 + 1)^2 + (0 + 3)^2 = r^2$

$16 + 9 = r^2 \Rightarrow r^2 = 25$

$\boxed{r = 5} \Rightarrow$ In (1)

Circle equation is, $(x + 1)^2 + (y + 3)^2 = 25$.

Example: 4

If A(6,1) and B(-6, -1) are two points. Find the equation of the circle with AB as diameter.

Solution:

The equation of the circle, given two points $(x_1\ y_1)$ and (x_2, y_2) is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Here, the point $A(6,1) = (x_1, y_1)$ and

$B(-6, -1) = (x_2, y_2)$

$$(x - 6)(x + 6) + (y - 1)(y + 1) = 0$$

$$x^2 - 36 + y^2 - 1 = 0$$

$$x^2 + y^2 - 37 = 0$$

$$x^2 + y^2 = 37$$

Exercise : 1.2.2

1) Find the equation of the circle with centre (0, 2) and radius 3/2.

Ans: $x^2 + (y - 2)^2 = \left(\frac{3}{2}\right)^2$

2) Find the equation of the circle with centre (4,-2) and containing the point (3, 0).

Ans: $(x - 4)^2 + (y + 2)^2 = 5$.

3) Find the coordinates of the centre and the length of the radius of each of the circles:

(i) $(x - 5)^2 + (y - 7)^2 = 1$ (ii) $x^2 + (y - 5)^2 = 10$

Ans: Centre = (5, 7), r = 1 Ans : centre = (0, 5), $r = \sqrt{10}$

4) If A (2, 5) and B (3, -2) are two points. Find the equation of the circle with AB as diameter.

Ans: $\left(x - \frac{5}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{50}{4}$

GENERAL EQUATION OF A CIRCLE:

The general equation of a circle is written as :

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

When the equation of a circle is given in this form, we use the following method to find its centre and radius.

1. Make sure every time is on the left-hand side the co-efficient of $x^2 = y^2 = 1$.
2. Centre = (-g, -f)
3. Radius = $\sqrt{g^2 + f^2 - c}$ (Provide $g^2 + f^2 - c > 0$)

Note:

(1) If the point P (x_1, y_1) is on the circle then $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

(2) If the point P (x_1, y_1) is outside the circle then $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$

(3) If the point P (x_1, y_1) is inside the circle then $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$

Example : 5

The equation of a circle with radius 5 is $x^2 + y^2 - 6x + 4ky + 20 = 0$. $k \in z$.

i) Find the centre of the circle and the radius length interms of k.

ii) Find the values of k.

Solution:

i) $x^2 + y^2 - 6x + 4ky + 20 = 0$

$2g = -6, 2f = 4k, c = 20$

Centre (-g, -f) = (3, - 2k)

Radius r $= \sqrt{(3)^2 + (-2k)^2 - 20}$

(ii) $(5)^2 = 9 + 4k^2 - 20$

$25 = -11 + 4k^2 \Rightarrow 4k^2 = 36$

$k^2 = 9$

$k = \pm 3$

Example : 6

Find the centre and radius of the circle $x^2 + y^2 - 8y + 3 = 0$

Solution:

Given, $x^2 + y^2 + 0x - 8y + 3 = 0$

Circle equation $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre = $(-g,-f) = (0, 4)$

Radius = $\sqrt{(0)^2 + (4)^2 - 3} = \sqrt{13}.$

Example : 7

Determine whether the points (-3, -2) (5, -1) and (-2, 1) are inside, on or outside the circle $x^2 + y^2 - 2x + 8y - 8 = 0$

Solution:

$x^2 + y^2 - 2x + 8y - 8 = 0$ ------- (1)

sub (-3, -2) in (1) $\quad (-3)^2 + (-2)^2 - 2(-3) + 8(-2) - 8$

$= 9 + 4 + 6 - 16 - 8$

$= -5 < 0$

$\therefore$ (-3, -2) is inside the circle.

sub (5, -1) in (1) $\quad (5)^2 + (-1)^2 - 2(5) + 8(-1) - 8$

$= 25 + 1 - 10 - 8 - 8$

$= 0$

$\therefore$ (5, -1) is on the circle.

sub (-2, 1) in (1) $\quad (-2)^2 + (1)^2 - 2(-2) + 8(1) - 8$

$= 4 + 1 + 4 + 8 - 8$

$= 9 > 0$

$\therefore$ (-2, 1) is outside the circle.

Exercise : 1.2.3

1. Find the centre and radius length of each of the following circles.

   (i) $2x^2 + 2y^2 - 2x - 6y - 13 = 0.$ Ans: C = (1/2, 3/2) $\quad r = \sqrt{\frac{15}{2}}$

   (ii) $(x-3)(x + 3) + (y + 2)(y + 6) = 0$ Ans: C = (0, -4), $\quad r = \sqrt{13}$

2. A circle with centre (-3, -2) passes through the point (1, 1). Find the equation of the circle.
   Ans: $x^2 + y^2 + 6x + 4y - 12 = 0.$

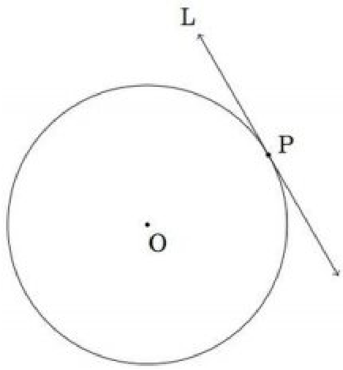
3. The equation of a circle with radius length 6 is $x^2 + y^2 - 2kx + 4y - 7 = 0.$

   (i) Find the centre of the circle and radius of in terms of k. Ans: Centre C = (1k, -2)

   (ii) Find the value of k. Ans: $k = \pm 5.$

4. The circle S has the equation $(x - 4)^2 + (y - 2)^2 = 13.$ The point (p, 0) lies on S. Find the two real values of p. Ans: p = 1, 7

Equation of the tangent to a circle:



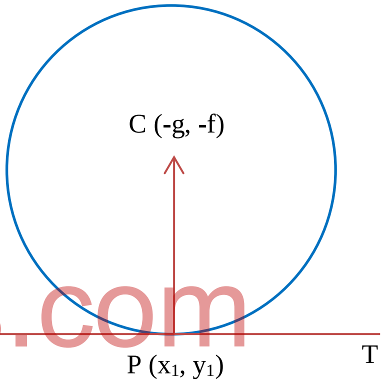
Tangent to a circle is a line that touches the circle at only one point, which is known as tangency. At the point of tangency, tangent to the circle is always perpendicular to its radius.

Let us consider a circle with centre at C and a straight line AB. This straight line can be related to the circle in three different positions.

Properties:

1. The tangent line never crosses the circle.
2. At the point of tangency, it is perpendicular to the radius.
3. A chord and tangent form an angle and this angle is same as that of tangents inscribed on the opposite side of the chord.

Equation of the tangent to a circle at a point (x_1, y_1):



Let the equation of the circle be
$x^2 + y^2 + 2gx + 2fy + c = 0.$

Let p (x_1, y_1) be a given point on it.

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

Let PT be the tangent at P.

The centre of the circle is (-g, -f)

Slope of the CP = $\frac{y_1 + f}{x_1 + g}$

Since CP is perpendicular to PT, slope of PT $\left(\frac{x_1 + g}{y_1 + f}\right)$

Equation of the tangent PT is $y - y_1 = m(x - x_1)$

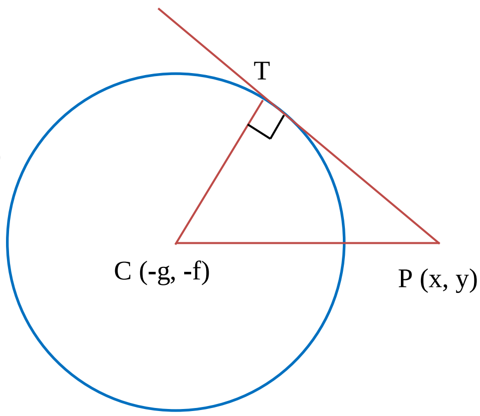
$y - y_1 = \left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$

on simplification it reduced to

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ which is the required equation of the tangent at (x_1, y_1).

Note: Equation of tangent to the circle $x^2 + y^2 = r^2$ at origin is $xx_1 + yy_1 = r^2$.

Length of the tangent to the circle from a point $P(x_1, y_1)$:



Let the equation of the circle be
$x^2 + y^2 + 2gx + 2fy + c = 0.$

Let PT be the tangent to the circle from $P(x_1, y_1)$ outside it. We know that the co-ordinate of the centre C is (-g, -f) and

Radius r = CT = $\sqrt{g^2 + f^2 - C}$

From the right angled triangle PCT,

$PT^2 = PC^2 - CT^2$

$= (x_1 + g)^2 + (y_1 + f)^2 - (g^2 + f^2 - c)$

$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$\therefore PT \quad = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$ which is the length of the tangent from the point (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Note:

(1) If the point P is on the circle then $PT^2 = 0$ (PT is zero).

(2) If the point P is outside the circle then $PT^2 > 0$ (PT is real).

(3) If the point P is inside the circle then $PT^2 < 0$ (PT is imaginary).

Example:8

Find the equation of the tangent to the circle $x^2 + y^2 - 4x + 8y - 5 = 0$ at (2, 1).

Solution:

The equation of the circle is $x^2 + y^2 - 4x + 8y - 5 = 0$.

$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ is the required equation of the tangent at (x_1, y_1).

$\therefore$ The equation of the tangent at (2, 1) is

$x(2) + y(1) + g(x + 2) + f(y+1) - 5 = 0$

$2x + y - 2(x + 2) + 4(y + 1) - 5 = 0$

$2x + y - 2x - 4 + 4y + 4 - 5 = 0$

$5y - 5 = 0$

$y - 1 = 0$

Example: 9

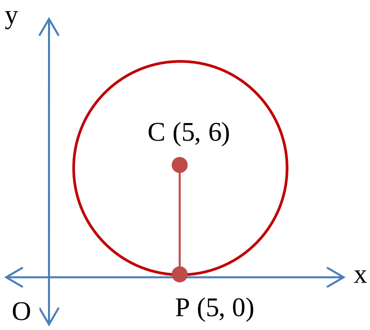
Find the equation of the circle which has its centre (5, 6) and touches x – axis.

Solution:

Let p be a point on x – axis where it touches the circle.

Given that the centre C is (5, 6) and p is (5, 0).

$r = CP = \sqrt{(5-5)^2 + (6-0)^2} = 6.$



The equation of the circle is,

$(x - h)^2 + (y - k)^2 = r^2$

$(x - 5)^2 + (y - 6)^2 = 6^2$

$x^2 - 10x + 25 + y^2 - 12y + 36 - 36 = 0$

$x^2 + y^2 - 10x - 12y + 25 = 0.$

Example:10

Find the length of the tangent from (2, 3) to the circle $x^2 + y^2 - 4x - 3y + 12 = 0$.

Solution:

The length of the tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ from the point (x_1, y_1) is $\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$

$\therefore$ Length of the tangent to the given circle is $\sqrt{x_1^2 + y_1^2 - 4x_1 - 3y_1 + 12}$

$$= \sqrt{2^2 + 3^2 - 4(2) - 3\,(3) + 12}$$

$$= \sqrt{8}$$

$$= 2\sqrt{2} \text{ units}$$

Exercise: 1.2.4

1. Find the length of the tangent from (1, 2) to the circle $x^2 + y^2 - 2x + 4y + 9 = 0$.

   Ans: $2\sqrt{5}$ units.

2. Find the value of 'p' so that the line $3x + 4y - p = 0$ is a tangent to $x^2 + y^2 - 64 = 0$.

   Ans: $p = \pm 40$

3. Find the coordinates of the point of intersection of the line $x + y = 2$ with the circle $x^2 + y^2 = 4$.

   Ans: (0, 2) , (2, 0)

4. Find the equation of the circle which has its centre at (2, 3) and touches x – axis.

   Ans: $x^2 + y^2 - 4x - 6y + 4 = 0$

5. Find the equation of the tangent to the circle $x^2 + y^2 - 4x + 2y - 21 = 0$ at (1,4).

   Ans: $x - 5y + 19 = 0$

FAMILY OF CIRCLES

Introduction:

A circle is a closed shape simple figure in each all the points lying on the surface of the circle joining its centre are equal and known as radius of the circle. A huge collection of circle is called a family of circles. There are various types of circles available around us.

For instance, if we are given two circles and we need to resolve the third circle touching the rest both the circle. However, the equation contains three unknown figures (i.e) g, f and c so we require at least three conditions to get a unique circle.

(1) When family of circles have a fixed centre:

The equation is $(x - h)^2 + (y - k)^2 = r^2$

Where, (h, k) is fixed and the only parameter that is varying is radius (r). The fixation of the radius will give a particular circle.

(2) Equation of family of circle passing through intersection of two circles $S_1 = 0$ and $S_2 = 0$.

The general equation of family of circles is passing through intersection of S_1 and S_2 which is given by $S_1 + KS_2 = 0$, where $k \neq -1$.

Again we are left with one parameter equation of the family of circles.

(3) Equation of the circle circumscribing a triangle whose slides are given by $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$.

This equation is given by $L_1L_2 + \lambda L_2L_3 + \mu L_3L_1 = 0$.

This provide the co-efficient of xy = 0 and coefficient of x^2 = cofficient of y^2.

Moreover, the particular value of the parameter λ and μ gives you a unique circle.

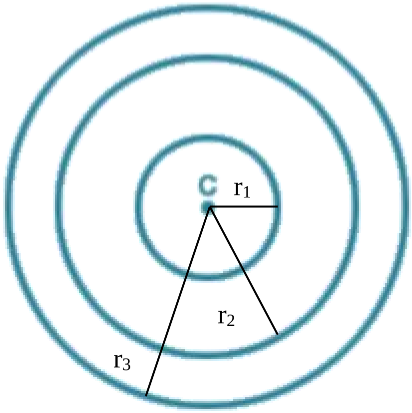
(4) Family of circle touching the circles S = 0 and the line L = 0 at their point of contact:

The family is given by equation $S + \lambda L = 0$, where λ is required family.

TYPES OF CIRCLES

1. CONCENTRIC CIRCLES:

When two or more circles have the common centre, then these circles are called concentric circles.



In the figure, there are three circles inside one another. All these circles are of different size and are having different radius. Consequently, if all the circles have the radius then the circles will not be concentric. As a result they will be lie on each, other and will not be able to see and treat each other as one single circle.

- Equation differ only by the constant term.

  (i.e) Equation of the concentric circle with the given circle

  $x^2 + y^2 + 2gx + 2fy + c = 0$ is,

  $x^2 + y^2 + 2gx + 2fy + k = 0$

Example : 11

Find the equation of the circle concentric with circle $x^2 + y^2 - 25 = 0$ and passing through (3, 0).

Solution:

Equation of the concentric circle with $x^2 + y^2 - 25 = 0$ is $x^2 + y^2 + k = 0$ which passes through (3, 0).

(i.e.) $(3)^2 + (0)^2 + k = 0$

$$\boxed{k = -9}$$

$\therefore$ Required equation of the circle is, $x^2 + y^2 - 9 = 0$

Exercise : 1.2.5

1) Find the equation of the circle, which is concentric with the circle $x^2 + y^2 - 4x - 6y - 9 = 0$ and passing through the point (-4, -5).

Ans: $x^2 + y^2 - 4x - 6y - 87 = 0$

2) Find the equation of the circle concentric with the circle $x^2 + y^2 - 2x - 6y + 4 = 0$ and having radius 7.

Ans: $x^2 + y^2 - 2x - 6y - 39 = 0$

CONTACT OF CIRCLES

When outer surface of two circles are touching, it is known as contact of circles.

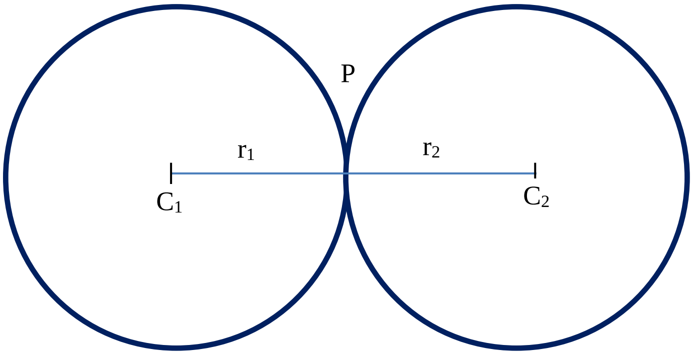
There may be two cases in contact of circles.

Case (i):

When two circles touching the other surface externally and where the distance between their centres is equal to the sum of their radii.

In this case, circles must satisfy the given equation.

Equation $C_1C_2 = r_1 + r_2$

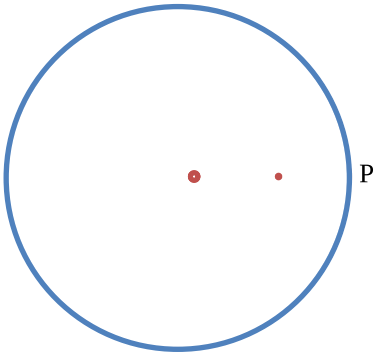


In the above figure,

There are two circles touching each other externally at point p. Both circles have different centres C_1 and C_2. The distance between both centres is the sum of their radii.

Case (ii):

When two circles touching each outer surface internally having their centres is equal to the difference of their radii.



Equation: $C_1C_2 = r_1 - r_2$ (or) $r_2 - r_1$.

In the figure, outer of the two circles are touching internally at P point. Similarly in case (i), both circles have different centres C_1 and C_2.

Example: 12

Show that the circles $x^2 + y^2 - 4x - 6y + 9 = 0$ and $x^2 + y^2 + 2x + 2y - 7 = 0$ touch each other.

Solution:

Given circles,

$x^2 + y^2 - 4x - 6y + 9 = 0$ and $x^2 + y^2 + 2x + 2y - 7$.

Centre $C_1 = (2, 3)$ and $C_2 = (-1, -1)$

Radius:

$r_1 = \sqrt{(2)^2 + (3)^2 - 9}$ $\qquad$ $r_2 = \sqrt{(-1)^2 + (-1)^2 + 7}$

$= \sqrt{4}$ $\qquad$ $r_2 = \sqrt{9}$

$r_1 = 2$ $\qquad$ $r_2 = 3$

Distance:

$$C_1C_2 = \sqrt{(2+1)^2+(3+1)^2}$$

$$= \sqrt{(3)^2+(4)^2} = \sqrt{25}$$

$C_1C_2 = 5$

$C_1C_2 = r_1 + r_2$

$5 = 2 + 3$

$\therefore$ The circles are touch each other externally.

Example 13

Show that the circles $x^2 + y^2 + 2x - 8 = 0$ and $x^2 + y^2 - 6x + 6y - 46 = 0$ touch each other.

Solution:

Given, $x^2 + y^2 + 2x - 8 = 0$ ----- (1)

$x^2 + y^2 - 6x + 6y - 46 = 0$ ----- (2)

Let C_1 and r_1 be the centre and radius of circle (1) and

C_2, r_2 be centre and radius of circle (2).

The general equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$.

To get the centre and radius

Centre $C_1(-g, -f) = (-1, 0)$

Radius $r_1 = \sqrt{g^2+f^2-c} = \sqrt{(1)^2+(0)^2-(-8)} = \sqrt{9} = 3$

Centre $C_2(-g, -f) = C_2\,(3, -3)$

$r_2 = \sqrt{g^2+f^2-c} = \sqrt{9+9+46} = \sqrt{64} = 8$

$|C_1C_2| = \sqrt{(3+1)^2+(-3)^2} = \sqrt{16+9} = \sqrt{25} = 5$

$r_2 - r_1 = 8 - 3 = 5$

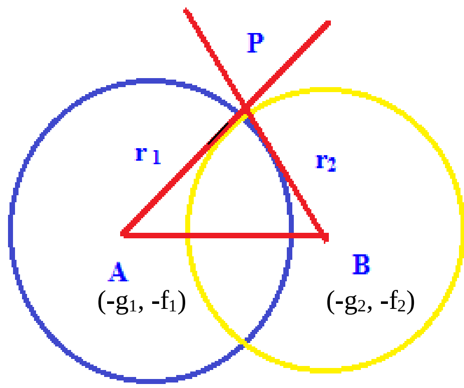
$C_1C_2 = r_2 - r_1$

$\therefore$ The circles are touch each other internally.

ORTHOGONAL CIRCLES

Two circles are said to be orthogonal if the tangents to the circle at the point of intersection of the circles are perpendicular to each other.

Condition for two circles to cut orthogonally



$x^2 + y^2 + 2gx + 2fy + c = 0$

$x^2 + y^2 + 2gx + 2fy + c = 0$

$A\,(-g_1, -f_1)$, $B\,(-g_2, -f_2)$

$AP = r_1 = \sqrt{{g_1}^2 + {f_1}^2 - C_1}$ and $BP = r_2$

$= \sqrt{{g_2}^2 + {f_2}^2 - C_2}$

$AB^2 = AP^2 + PB^2$

Expanding and simplifying, we get,

$2g_1g_2 + 2f_1f_2 = C_1 + C_2$ is the required condition for two circles to cut orthogonally.

Note: If centre of any one of the circles is at origin then the above condition becomes

$C_1 + C_2 = 0$

Example:14

Find the equation of the circle which passes through (1, 1) and cuts orthogonally each of the circles $x^2 + y^2 - 8x - 2y + 16 = 0$ and $x^2 + y^2 - 4x - 4y - 1 = 0$.

Solution:

Let the equation of circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ----- (1)

This passes through (1, 1)

(i.e) $1^2 + 1^2 + 2g(1) + 2f(1) + c = 0$

$2g + 2f + c = -2$ ------- (2)

Equation (1) orthogonal with the circle $x^2 + y^2 - 8x - 2y + 16 = 0$

$2g_1 = 2g$ $\quad$ $2f_1 = 2f$

$g_1 = g$ $\quad$ $f_1 = f$ $\quad$ $C_1 = C$

$2g_2 = -8$ $\quad$ $2f_2 = -2$

$g_2 = -4$ $\quad$ $f_2 = -1$ $\quad$ $C_2 = 16$

By Orthogonal condition

$2g_1g_2 + 2f_1f_2 = C_1 + C_2$

$2g(-4) + 2f(-1) = C + 16$

$8g + 2f + c = -16$ ------- (3)

Similarly, the circle $x^2 + y^2 - 4x - 4y - 1 = 0$ is orthogonal with the circle (1), we get

$\Rightarrow 4g + 4f + c = 1$ ------- (4)

Solve (2) and (3)

$g = \frac{-14}{6} = \frac{-7}{3}$

Solving (4) and (2)

$2g + 2f = 3$

$f = \frac{23}{6}$

$g = \frac{-7}{3}$ and $f = \frac{23}{6}$ in (2)

$2\left(\frac{-7}{3}\right) + 2\left(\frac{23}{6}\right) + c = -2$

Simplifying we get, $C = \frac{-15}{3}$

$\therefore$ Required equation of the circle is

$x^2 + y^2 - \frac{14}{3}x + \frac{23}{3}y + \left(\frac{-15}{3}\right) = 0$

$\Rightarrow 3x^2 + 3y^2 - 14x + 23y - 15 = 0$

Example:15

Prove that the circles $x^2 + y^2 - 8x + 6y - 23 = 0$ and $x^2 + y^2 - 2x - 5y + 16 = 0$ are orthogonal.

Solution:

The equations of the circle are

$x^2 + y^2 - 8x + 6y - 23 = 0$ ----- (1)

$x^2 + y^2 - 2x - 5y + 16 = 0$ ----- (2)

(1) $\Rightarrow$ $g_1 = -4$ $f_1 = 3$ $C_1 = -23$

(2) $\Rightarrow$ $g_2 = -1$ $f_2 = \frac{-5}{2}$ $C_2 = 16$

Condition for Orthogonality is

$2g_1g_2 + 2f_1f_2 = C_1 + C_2$

$2(-4)(-1) + 2(3)\left(\frac{-5}{2}\right) = -23 + 16$

$-7 = -7.$

$\therefore 2g_1g_2 + 2f_1f_2 = C_1 + C_2$

$\therefore$ The two circles cut orthogonally

Exercise: 1.2.6

1. Find the equation of the circle which passes through the point (1, 2) and cuts orthogonally each of the circles $x^2 + y^2 = 9$ and $x^2 + y^2 - 2x + 8y - 7 = 0$.

   Ans: $x^2 + y^2 - 10x - 2y + 9 = 0$

2. Find the circles which cuts orthogonally each of the following circles $x^2 + y^2 + 2x + 4y + 1 = 0$, $x^2 + y^2 - 4x + 3 = 0$ and $x^2 + y^2 + 6y + 5 = 0$.

   Ans: $x^2 + y^2 - 2x + 2y + 1 = 0$

3. Show that the circle $x^2 + y^2 - 8x - 6y + 21 = 0$ is orthogonal to the circle $x^2 + y^2 - 2y - 15 = 0$.

Chapter 1.3 CONICS

Introduction:

Analytical Geometry of two dimension is used to describe geometric objects such as **point, line, circle, parabola, ellipse, and hyperbola using Cartesian coordinate system.** Two thousand years ago (≈ 2- 1 BC (BCE)), the ancient Greeks studied **conic** curves, because studying them elicited ideas that were exciting, challenging, and interesting. They could not have imagined the applications of these curves in the later centuries.

Solving problems by the method of Analytical Geometry was systematically developed in the first half of the 17th century majorly, by Descartes and also by other great mathematicians like Fermat, Kepler, Newton, Euler, Leibniz, l'Hôpital, Clairaut, Cramer, and the Jacobis.

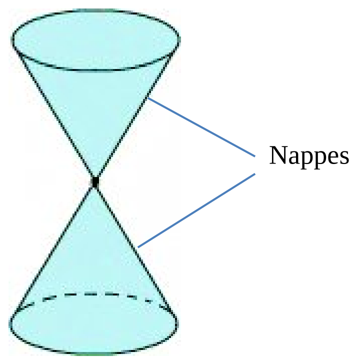
Analytical Geometry grew out of need for establishing **algebraic techniques** for solving **geometrical problems** and the development in this area has conquered industry, medicine, and scientific research.

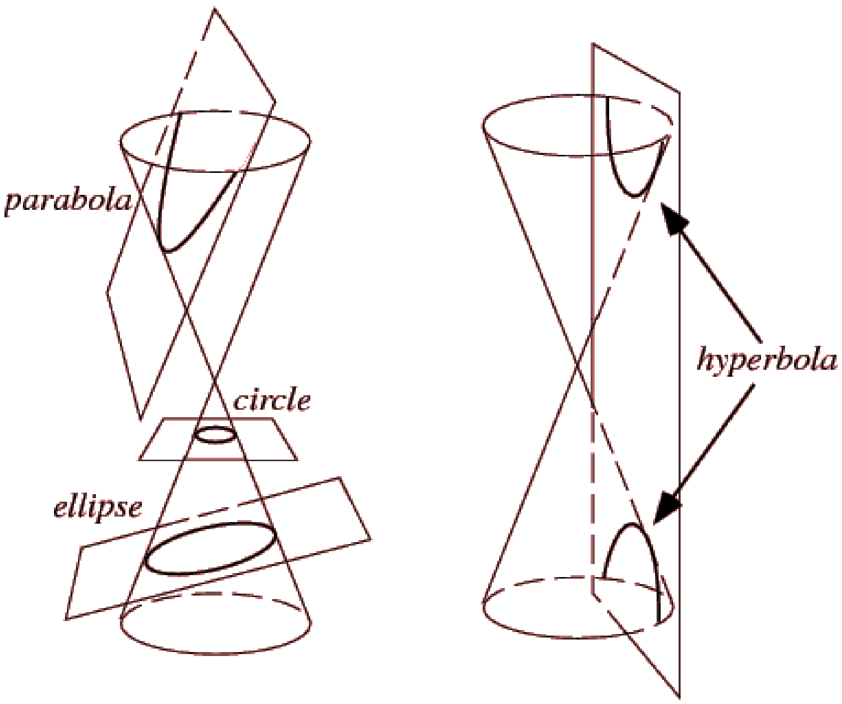
The theory of Planetary motions developed by Johannes Kepler, the German mathematician cum physicist stating that all the planets in the solar system including the earth are moving in elliptical orbits with Sun at one of a foci, governed by inverse square law paved way to established work in Euclidean geometry. Euler applied the co-ordinate method in a systematic study of space curves and surfaces, which was further developed by Albert Einstein in his theory of relativity.

Conic Section:

Conic section, also called conic, in geometry, any curve produced by the intersection of a plane and a right circular cone. Depending on the angle of the plane relative to the cone, the intersection is a circle, an ellipse, a parabola and the hyperbola.

Conic sections can be generated by intersecting a plane with a cone. A cone has two identically shaped parts called nappes. One nappe is what most people mean by "cone" and has the shape of a "party hat".



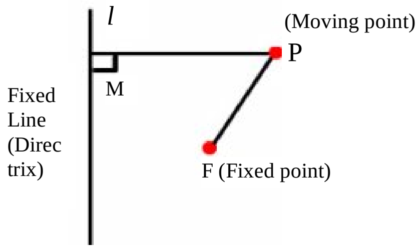


Conic sections are generated by the intersection of a plane with a cone. If the plane is parallel to the axis of revolution (the y – axis), then the conic section is a hyperbola.

If the plane is parallel to the generating line, the conic section is a parabola. If the plane is perpendicular to the axis of revolution, the conic section is a circle. If the plane intersects one nappe at an angle to the axis (other than 90^o), then the conic section is an ellipse.

The curves obtained by slicing the cone with a plane not passing through the vertex are called conic sections or simply conics.

Definition:



A conic is the locus of a point which moves in a plane, so that its distance from a fixed point bears a constant ratio to its distance from a fixed line not containing the fixed point.

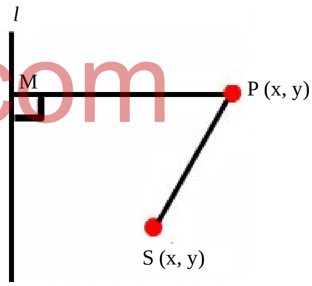
The fixed point is called **focus**, the fixed line is called **directrix** and the constant ratio is called **eccentricity**, which is denoted by **e**.

(i) If this constant $e = 1$ then the conic is called a parabola

(ii) If this constant $e < 1$ then the conic is called an ellipse

(iii) If this constant $e > 1$ then the conic is called a hyperbola

The general equation of a conic:



Let S(x, y) be the focus, l the directrix, and e be the eccentricity. Let P(x, y) be the moving point.

By definition of conic, we have

$$\frac{SP}{PM} = \text{constant} = e, \qquad \text{-----} (1)$$

Where $SP = \sqrt{(x_1 - x_2)^2 + (y - y_2)^2}$

and PM = perpendicular distance from P(x, y) to the line $lx + my + n = 0$. $= \left|\frac{lx+my+n}{\sqrt{l^2 + m^2}}\right|$

From (1) we get $SP^2 = PM^2$

$$(x - x_1)^2 + (y - y_1)^2 = e^2 \left[\frac{lx+my+n}{\sqrt{l^2 + m^2}}\right]$$

On simplification the above equation takes the form of general second degree equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

Classification with respect to the general equation of a conic:

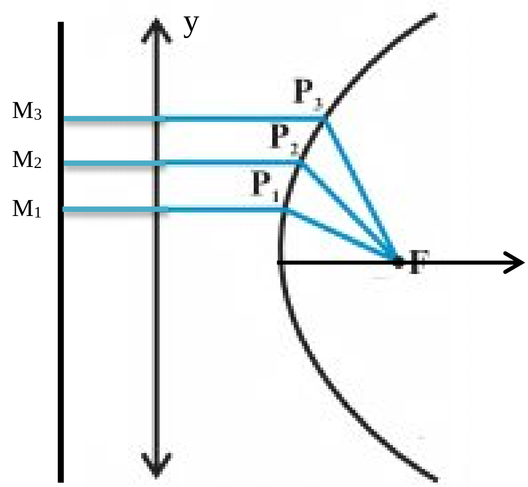
The equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ represents either a (non-degenerate) conic or a degenerate conic. If it is a conic, then it is

(i) a parabola if $B^2 – 4AC = 0$ $\Leftrightarrow$ $e = 1$

(ii) an ellipse if $B^2 – 4AC < 0$ $\Leftrightarrow$ $e < 1$

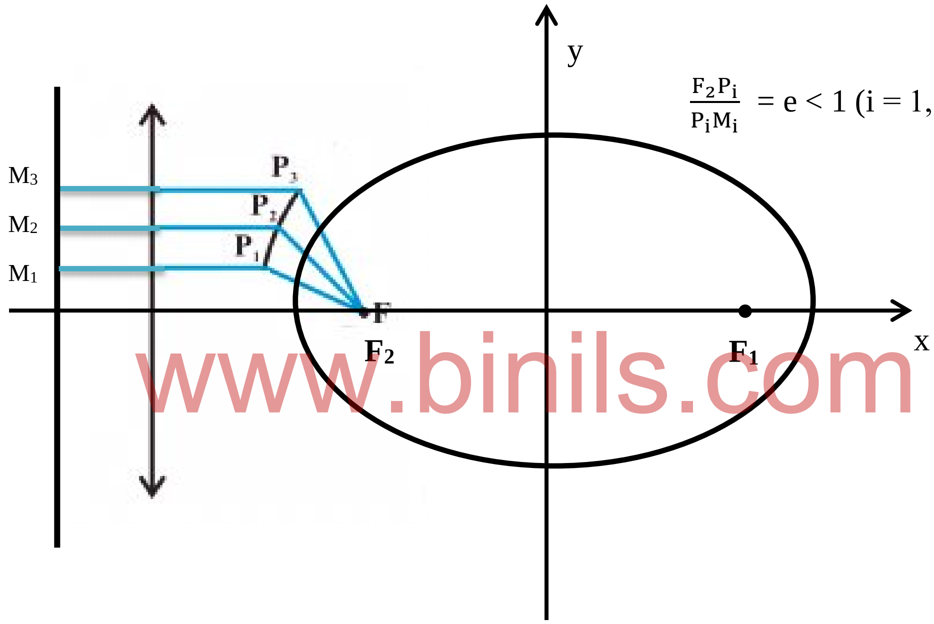
(iii) a hyperbola if $B^2 – 4AC > 0$ $\Leftrightarrow$ $e > 1$

(i)



$\frac{FP_i}{P_iM_i} = e = 1$ (i = 1, 2, 3, …..)

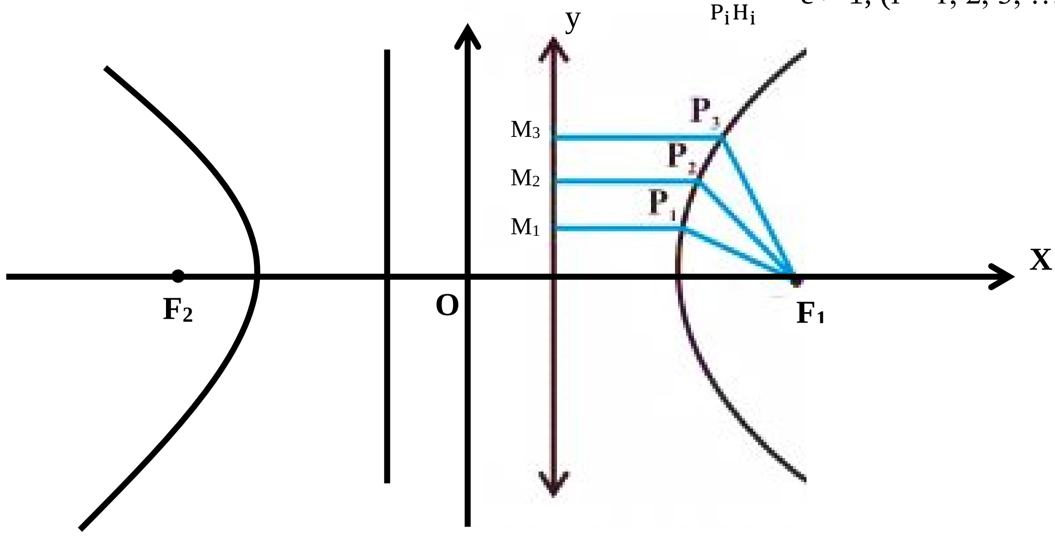
(ii)



$\frac{F_2P_i}{P_iM_i} = e < 1$ (i = 1, 2, 3, …..)

(iii)

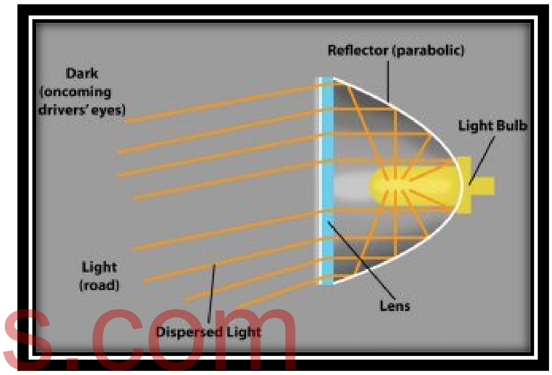
$\frac{F_1P_i}{P_iH_i} = e > 1$, (i = 1, 2, 3, …..)



Real Life Applications of Parabola:

Satellite Dish:

1. Parabolas are a set of points in one plane that form a U shaped curve, but the application of this curve is not restricted to the world of mathematics. It can also be seen in objects and things around us in our every day life.
2. The parabolic shape of a Satellite Dish helps receive and Transmit Signals.
3. A Satellite dish is a perfect example of the reflective properties of parabolas mentioned earlier. The signals that are received are directly sent to the focus, which are then correctly reflected to a receiver (signals are sent out parallel to the axis). These signals are then interpreted and are transmitted as channels on our TV. The same principle applies to radio frequencies too.



Headlight: This is the same principle like the one used in a torch. The inner surface is smooth and made of glass which makes it a powerful reflector. The principle used here is that the light source is at the focus, and the light rays will be reflected parallel to the axis. This is the reason one can see a thick focused beam of light emitting from a head light.

Suspension Bridge:

If one is to observe suspension bridges, the shape of the cables which suspend the bridge resemble a parabolic curve.

There has been sufficient confusion about whether the cables as suspended in a parabola or a catenary. Studies show that the shape is nearer to a parabola. The cables would have been hyperbolic, but when a uniform load (the horizontal deck) is present, they get deformed like a parabola.

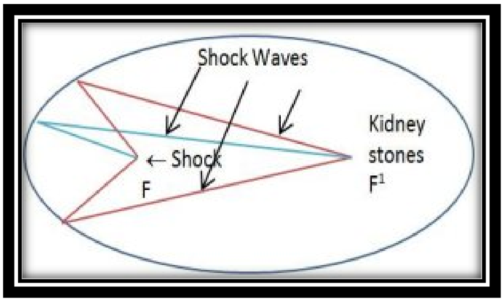
Fountains:

Fountains spray water in the air, the water jet propels upwards reaching a specific attitude, and then comes back. Again the path traced by the stream of water is similar to a parabola.

Real Life Applications of ellipse:

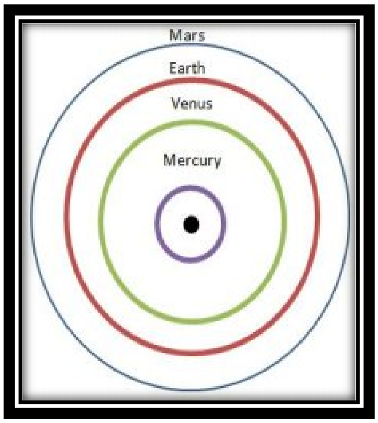
Many **real-world** situations can be represented by **ellipses**, including orbits of planets, satellites, moons and comets, and shapes of boat keels, rudders, and some airplane wings.

Kidney Stones: Ellipses have an important property that is used to reflect light and sound waves. Any light or sound wave that starts from one foci will be reflected to the other. This reflective property has been useful in medicine, to destroy kidney stones and gall stones called a lithotripter. It uses shockwaves to crush kidney stones into tiny pieces that are easier for the body to dispose of. This is a very useful and important application of the reflective property of are ellipse.



Satellite and Planet Orbits:

In Kepler's first law of planetary motion the path of each planet is an ellipse with the sun at one focus.



Real Life Examples of Hyperbola:

Cooling Towers of Nuclear Reactors:

The hyperboloid is the design standard for all nuclear cooling towers. It is structurally sound and can be built with straight steel beams.

Gear Transmission:

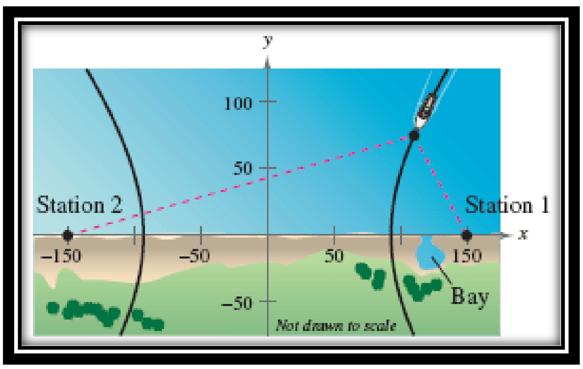
Two hyperboloids of revolution can provide gear transmission between two skew axes. The cogs of each gear are a set of generating straight lines.

Stones in a Lake:

When two stones are thrown simultaneously into a pool of still water, ripples move outward in concentric circles. These circles intersect in points which form a curve known as hyperbola.

Radio System:

Radio system's signals employ hyperbolic functions. One important radio system, LORAN, identified geographic positions using hyperbolas. Scientists and engineers established radio stations in positions according to the shape of a hyperbola in order to optimize the area covered by the signals from a station.



Parabola:

A parabola is the set of all points in a plane that are equidistant from a fixed line and a fixed point (not on the line) in the plane.

Since e = 1, (i.e) a parabola is a conic whose eccentricity is 1.

(i) Equation of a parabola in Standard form with Vertex at (0, 0)

Let S be the focus and l be the directrix.

Draw SZ perpendicular to the line l.

Let us assume SZ produced as x – axis and perpendicular bisector of SZ produced as y-axis.

The intersection of the perpendicular bisector with SZ be the origin O. Let SZ = 2a.

Then S is (a, 0) and the equation of the directrix is x + a = 0.

Let P(x, y) be the moving point in the locus that yield a parabola. Draw PM perpendicular to the directrix.

By definition, $e = \frac{SP}{PM} = 1 \Rightarrow SP^2 = PM^2$

Then, $(x - a)^2 + y^2 = (x + a)^2$

On simplifying, we get,

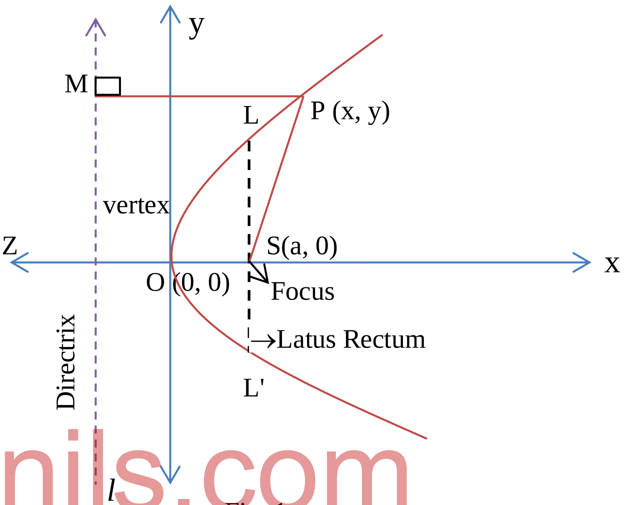
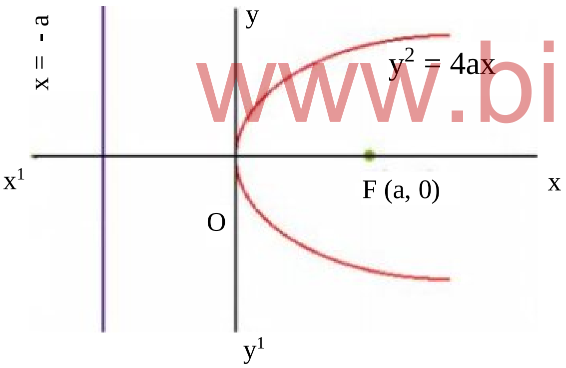


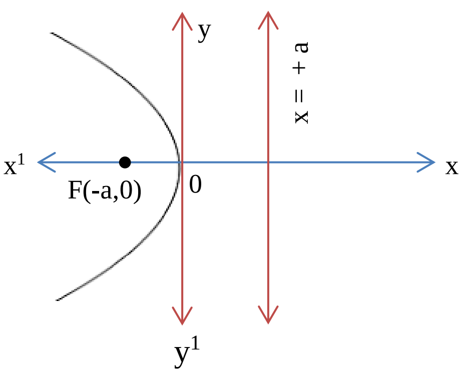
Fig. 1



$y^2 = 4ax$ which is the equation of the parabola in the standard form.

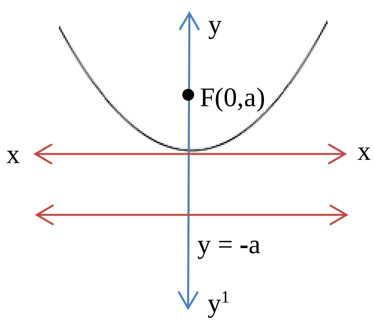
The other standard forms of a parabola are

$y^2 = -4ax$ [a > 0]



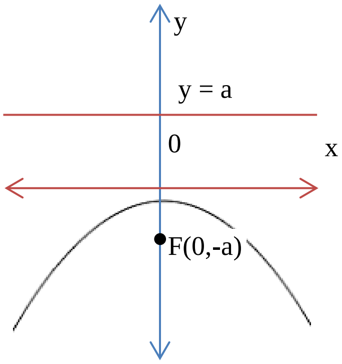
(i.e) the curve exist for $x \le 0$

$x^2 = 4ay$ [a > 0]



(i.e) the curve exist for $y \ge 0$

$x^2 = -4ay$ [a > 0]



(i.e) the curve exists for $y \leq 0$

Definitions:

- **Focus:** The fixed point used to draw the parabola is called the focus (F). Here, the focus is F (a, 0).
- **Directrix:** The fixed line used to draw a parabola is called the directrix of the parabola.

  $\therefore$ The equation of the directrix is x = -a.
- **Axis:** A line through the focus and perpendicular to the directrix is called the axis of the parabola.
- **Vertex:** The point of intersection of parabola with the axis is called the vertex of the parabola. Here, the vertex is V(0, 0).
- **Focal distance:** The food distance is the distance between a point on the parabola and its focus.
- **Focal Chord:** A chord which passes through the focus of the parabola is called the focal chord of the parabola.
- **Latus Rectum:** The length of the focal chord perpendicular to the axis is called the latus rectum of the parabola

  $\therefore$ Equation of the latus rectum is x = a.
- **Length of Latus rectum:**

  Equation of the parabola is $y^2 = 4ax$

  Latus rectum LL^1 passes through the focus (a, 0)

  Hence, the point L is (Fig. 1) (a, y_1)

  $\therefore y_1^2 = 4a^2$

  $\Rightarrow y_1 = \pm 2a$ $\therefore$ The end points of latus rectum are (a, 2a) & (a, -2a).

  $\therefore$ Length of the Latus rectum $LL^1 = 4a$.

Note: The standard form of the parabola $y^2 = 4ax$ has for its

- vertex (0, 0)
- axis as x – axis
- focus as (a, 0)
- lies completely on the non-negative of the x-axis.

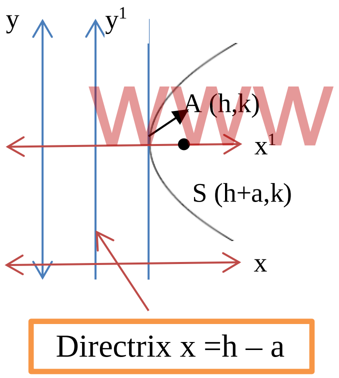
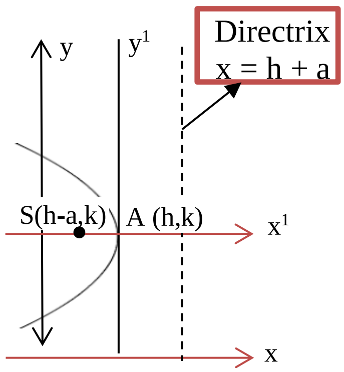
Replacing y by – y in $y^2 = 4ax$, the equation remains the same.

$\therefore$ $y^2 = 4ax$ is symmetric about x – axis.

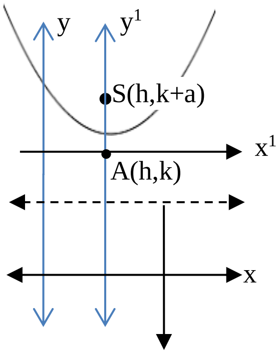
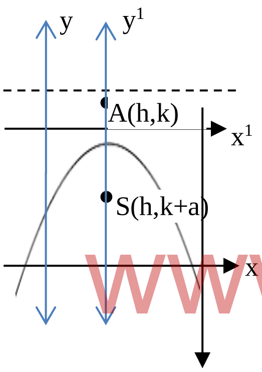
(i.e) x – axis divides the curve into two symmetrical parts.

(ii) Parabolas with vertex at (h, k) [The process of shifting the origin or translation of axes]

When the vertex is (h, k) and the axis of symmetry is parallel to x – axis, the equation of the parabola is either $(y-k)^2 = 4a(x-h)$ or $(y-k)^2 = -4a(x-h)$

Equation	Graph	Vertices	Focus	Axis of symmetry	Equation of directrix	Length of Latus rectum
$(y-k)^2 = 4a(x-h)$ Open right wards		(h, k)	(h+a, 0+k)	y = k	x=h-a	4a
$(y-k)^2 = -4a(x-h)$ Open left wards		(h, k)	(h-a, 0+k)	y = k	x=h+a	4a

When the vertex is (h, k) and the axis of symmetry is parallel to y – axis, the equation of the parabola is either $(x-h)^2 = 4a\,(y-k)$ or $(x-h)^2 = -4a\,(y-k)$

Equation	Graph	Verti-ces	Focus	Axis of symmetry	Equation of directrix	Length of Latus nectum
$(x-h)^2 = 4a\,(y-k)$ Open upwards	 Directrix y = k – a	(h, k)	(0+h, a+k)	x = h	y = k - a	4a
$(x-h)^2 = -4a\,(y-k)$ Open down-wards	 Directrix y = k + a	(h, k)	(0+h, -a+k)	x = h	y = k+a	4a

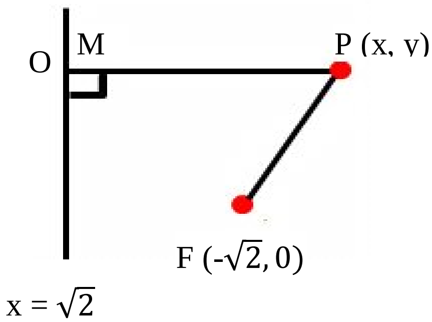
Note: To find the general form, replace x by x – h and y by y – k if the vertex is (h, k).

Remark: The above forms of equations do not have xy terms.

Example:

1) Find the equation of the parabola with focus $(-\sqrt{2}, 0)$ and directrix $x = \sqrt{2}$.

Solution: Let P(x, y) be any point on the parabola



Focus is perpendicular from P to the directrix.

$$\frac{FP}{PM} = e = 1 \quad \Rightarrow FP = PM$$

$$\Rightarrow FP^2 = PM^2$$

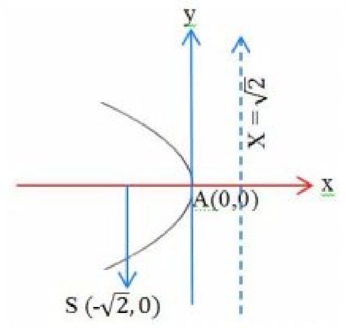
$$\Rightarrow (x+\sqrt{2})^2 + (y-0)^2 = \left[\pm\frac{x-\sqrt{2}}{\sqrt{1^2+0^2}}\right]^2$$

$$x^2 + 2\sqrt{2}x + 2 + y^2 = x^2 - 2\sqrt{2}x + 2$$

$$y^2 = -2\sqrt{2}x - 2\sqrt{2}x$$

$$\boxed{y^2 = -4\sqrt{2}x}$$

Alterative Method:



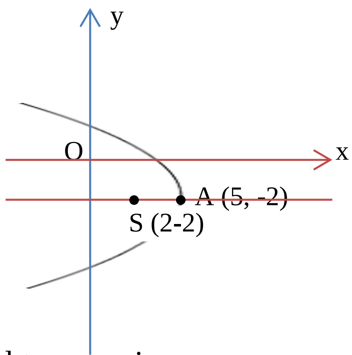
Parabola is open left and axis of symmetry as x – axis and vertex (0, 0).

Then the equation of the required parabola is of the form,

$(y - k)^2 = -4a\,(x - h) \quad \Rightarrow \quad (y - 0)^2 = -4\sqrt{2}\,(x - 0)$

$$\boxed{y^2 = -4\sqrt{2}x}$$

2) Find the equation of the parabola whose vertex is (5, - 2) & focus (2, -2)



Solution: Given vertex A(5, -2) and focus S(2, -2) and the focal distance

$$AS = a = 3$$

Parabola is open left and symmetric about the line parallel to x – axis.

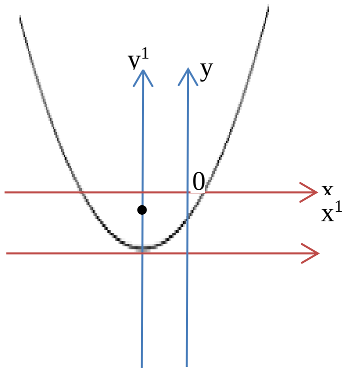
Then, the equation of the required parabola is,

$(y + 2)^2 \quad = -4\,(3)\,(x - 5)$

$y^2 + 4y + 4 = -12x + 60$

$\Rightarrow \boxed{y^2 + 4y + 12x - 56 = 0}$

3) Find the equation of the parabola with vertex (-1, -2), axis is parallel to y – axis and passing throgh (3, 6)



Solution: Since axis is parallel to y – axis the required equation of the parabola is

$(x + 1)^2 = 4a\,(y + 2)$

Since this passes through (3, 6), we get

$(3 + 1)^2 = 4a(6 + 2) \;\Rightarrow\; 16 = 32\,a$

$$\boxed{a = \tfrac{1}{2}}$$

Then the equation of parabola is $\Rightarrow \; (x + 1)^2 = 4\left(\tfrac{1}{2}\right)(y + 2)$

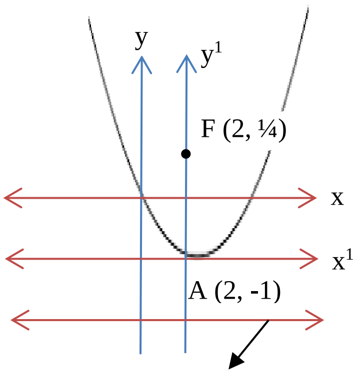
$\Rightarrow \; (x + 1)^2 = 2\,(y + 2)$

$x^2 + 2x + 1 = 2y + 4$

$$\boxed{x^2 + 2x - 2y - 3 = 0}$$

4) Find the vertex, focus, directrix and length of the latus rectum of the parabola $x^2 - 4x - 5y - 1 = 0$.

Solution: For the parabola



$x^2 - 4x - 5y - 1 = 0$

$x^2 - 4x = 5y + 1$

Add 4 on both sides,

$x^2 - 4x + 4 = 5y + 1 + 4$

$(x - 2)^2 = 5(y + 1)$

which is of the form $X^2 = 4aY$

Take $X = x - 2$ and $Y = y + 1$

$X^2 = 5Y$

$\therefore\ 4a = 5 \Rightarrow a = 5/4$

& the vertex is (2, -1) & the focus is $(2, \frac{1}{4})$

Equation of directrix is $y - k + a = 0$

$$y + 1 + \frac{5}{4} = 0 \Rightarrow 4y + 4 + 5 = 0$$

$$\boxed{4y + 9 = 0}$$

Length of Latus rectum = 4a = 5 units

5) Find the vertex, focus, axis, directrix and latus rectum of the parabola $2x^2 - 20y - 8x + 3 = 0$.

Solution: The given equation can be rewritten as

$2x^2 - 8x = 20y - 3$

Add 8 on both sides

$2x^2 - 8x + 8 = 20y - 3 + 8$

$2x^2 - 8x + 8 = 20y + 5$

$2[x^2 - 4x - 4] = 20y + 5$

$\div 2 \Rightarrow \quad x^2 - 4x - 4 = 10y + \frac{5}{2}$

$(x - 2)^2 = 10(y + \frac{1}{4})$

This is of the form $(x - h)^2 = 4a(y - k)$

$\Rightarrow \quad (h, k) = (2, -¼)$

It is a parabola with vertex as (2, - ¼)

Shift the origin to (2, - ¼)

$X = x - h = x - 2 \quad Y = y - k = y - (\frac{1}{4}) = y + \frac{1}{4}$

$\therefore$ The equation is $X^2 = 10Y$, This type is open upward

$\Rightarrow$ $4a = 10$ $\Rightarrow$ $a = 5/2$

	Referred to X, Y axes	**Referred to x, y axes where X = x – 2, Y = y + ¼**
Vertex	(0, 0)	$X = 0 \Rightarrow x - 2 = 0$ $x = 2$ $Y = 0 \Rightarrow y + \frac{1}{4} = 0$ $y = -\frac{1}{4}$ $\therefore$ Vertex is V (2, - ¼)
Focus	(0, a) (i.e.) $(0, \frac{5}{2})$	$X = 0 \Rightarrow x = 2$ $Y = \frac{5}{2} \Rightarrow \frac{5}{2} = y + \frac{1}{4}$ $\frac{5}{2} - \frac{1}{4} = y$ $y = \frac{9}{4}$ Focus is F(2, 9/4)
Axis	X = 0	$x - 2 = 0$ $\boxed{x = 2}$
Equation of Directrix	Y = - a	$Y = -5/2 \Rightarrow \frac{-5}{2} = y + \frac{1}{4}$ $\Rightarrow \boxed{y = -11/4}$
Equation of Latus Rectum	Y = a	$Y = +5/2 \Rightarrow \frac{5}{2} = y + \frac{1}{4}$ $\frac{5}{2} - \frac{1}{4} = y$ $\Rightarrow \boxed{y = 9/4}$
Length of Latus Rectum	$4a = 4 (\frac{5}{2})$ $= 10$	4a = 10

6) Find the axis, focus, vertex, equation of directrix, latus rectum, length of the latus rectum for the parabola $y^2 + 8x - 6y + 1 = 0$ and hence sketch their graphs.

Solution:

$y^2 + 8x - 6y + 1 = 0$

$y^2 - 6y = -8x - 1$

$y^2 - 6y + 9 = -8x - 1 + 9 = -8x + 8$ (adding 9 on both sides)

$(y - 3)^2 = -8(x - 1)$. This type is open leftward.

Comparing this equation with $y^2 = -4ax$ we get, $Y^2 = -4aX$,

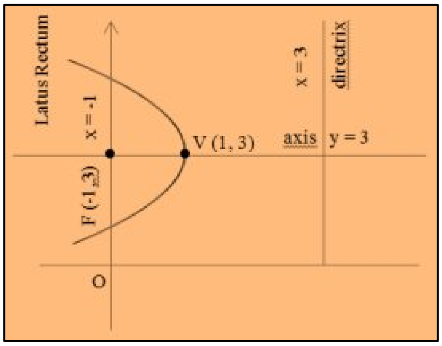
where $Y = y - 3$ and $X = (x - 1)$

$Y^2 = -8X$

$4a = 8$

$\boxed{a = 2}$

	Referred to X, Y axes	Referred to x, y axes where X = x – 1 & Y = y - 3
Axis	$Y = 0$	$Y = 0 \Rightarrow y - 3 = 0$ $\boxed{y = 3}$
Vertex	(0, 0)	$X = 0 \Rightarrow x - 1 = 0 \Rightarrow \boxed{x = 1}$ $Y = 0 \Rightarrow y - 3 = 0 \Rightarrow \boxed{y = 3}$ $\therefore$ Vertex is V(1, 3)
Focus	(-a, 0) (ie.) (-2, 0)	$X = -2 \Rightarrow x - 1 = -2$ $\Rightarrow X = -2 + 1 \Rightarrow \boxed{x = -1}$ $Y = 0 \Rightarrow y - 3 = 0 \Rightarrow \boxed{y = 3}$ $\therefore$ Focus is F(-1 , 3)
Equation of Directrix	$X = a$ (i.e.) $X = 2$	$X = 2 \Rightarrow x - 1 = 2 \Rightarrow x = 2 + 1 = 3$ $X = 2 \Rightarrow \boxed{x = 3}$
Equation of Latus Rectum	$X = -a$ $X = -2$	$X = -2 \Rightarrow x - 1 = -2 \Rightarrow x = -2 + 1 = -1$ $X = -2 \Rightarrow \boxed{x = -1}$
Length of the Latus Rectum	$4a = 4(2)$ $= 8$	8



Exercise: 1.3.1

1) Find the equation of the parabola in each of the cases given below:

   (i) focus (4, 0) and directrix x = - 4

   (ii) passes through (2, -3) and symmetric about y – axis.

   (iii) Vertex (1, -2) and focus (4, -2)

   (iv) End points of latus rectum (4, -8) and (4, 8)

   (v) Vertex (1, 2) and latus rectum : y = 5

2) Find the vertex, focus, equation of directrix and length of the latus rectum of the following parabolas

   (i) $y^2 = 16x$ (ii) $y^2 = -8x$

   (iii) $x^2 = 24y$ (iv) x2 – 2x + 8y + 17 = 0

   (v) $(x-4)^2 = 4(y+2)$ (vi) $y^2 - 4y - 8x + 12 = 0$

Exercise: 1.3.1 - Answers:

1) (i) $y^2 = 16x$ (ii) $3x^2 = -4y$ (iii) $(y+2)^2 = 12(x-1)$

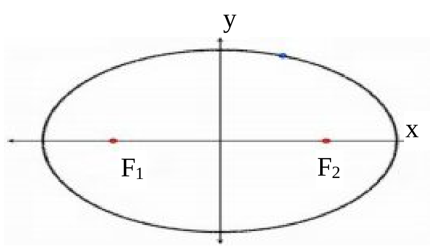
   (iv) $y^2 = 16x$ (v) $(x-1)^2 = 12(y-2)$

2)

	(i)	(ii)	(iii)	(iv)	(v)	(vi)
Vertex →	(0, 0)	(0, 0)	(0, 0)	(1, -2)	(4, -2)	(1, 2)
Focus →	(4, 0)	(-2 0)	(0, 6)	(1, -4)	(4, -1)	(3, 2)
Equation of directrix →	x = 4	y = -2	y = -6	y = 0	y = 0 – 3	x = -1
Length of latus rectum →	16	8	24	8	4	8

Ellipse:

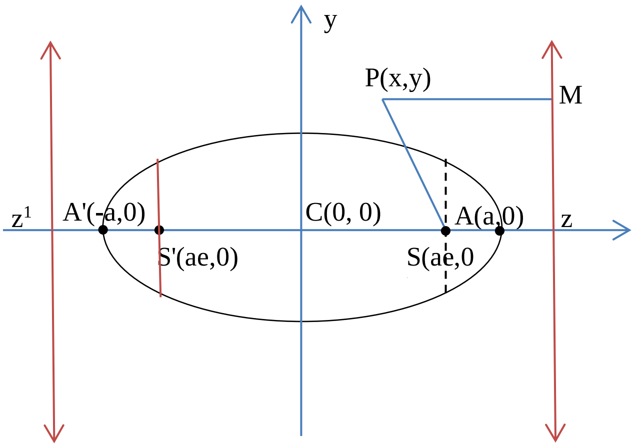
An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is constant.



The two fixed points are called the foci [plural of 'focus'] of the ellipse [0 < e < 1].

The locus of a point in a plane whose distance from a fixed point bears a constant ratio, less than one to its distance from a fixed line is called ellipse.

Equation of an Ellipse in Standard Form:



Let S be a focus, l be a directrix, **e** be the eccentricity ($0 < e < 1$) and P(x, y) be the moving point. Draw SZ and PM perpendicular to ‘l’.

Let A and A^1 be the points which divide SZ internally and externally in the ratio e:1 respectively.

Let AA'= 2a. Let the point of intersection of the perpendicular bisector with AA' be C.

$\therefore$ CA = a and CA' = a

Choose C as origin and CZ produced as x–axis and the perpendicualar bisector of AA' produced as y – axis.

By definition

$$\frac{SA}{AZ} = \frac{e}{1} \text{ and } \frac{SA'}{A'Z} = \frac{e}{1}$$

$$SA = eAZ$$

$$SA' = eA'Z$$

$$CA - CS = e(CZ - CA)$$

$$A'C + CS = e(A'C + CZ)$$

$$a - CS = e(CZ - a) \quad \text{------ (1)}$$

$$a + CS = e(a + CZ) \quad \text{------ (2)}$$

(2) + (1) gives $CZ = \frac{a}{e}$ and (2) – (1) gives $CS = ae$

$\therefore$ M is $\left(\frac{a}{e}, y\right)$ and S is (ae, 0)

By the definition of a conic, $\frac{SP}{PM} = e \Rightarrow SP^2 = e^2 PM^2$

$$(x - ae)^2 + (y - 0)^2 = e^2\left[\left(x - \frac{a}{e}\right)^2 + 0\right]$$

On simplification yields $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$

Since $1 - e^2$ is a positive quantity, write $b^2 = a^2(1 - e^2)$

Taking $ae = c$, $b^2 = a^2 - c^2$

Hence we obtain the locus of P as $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

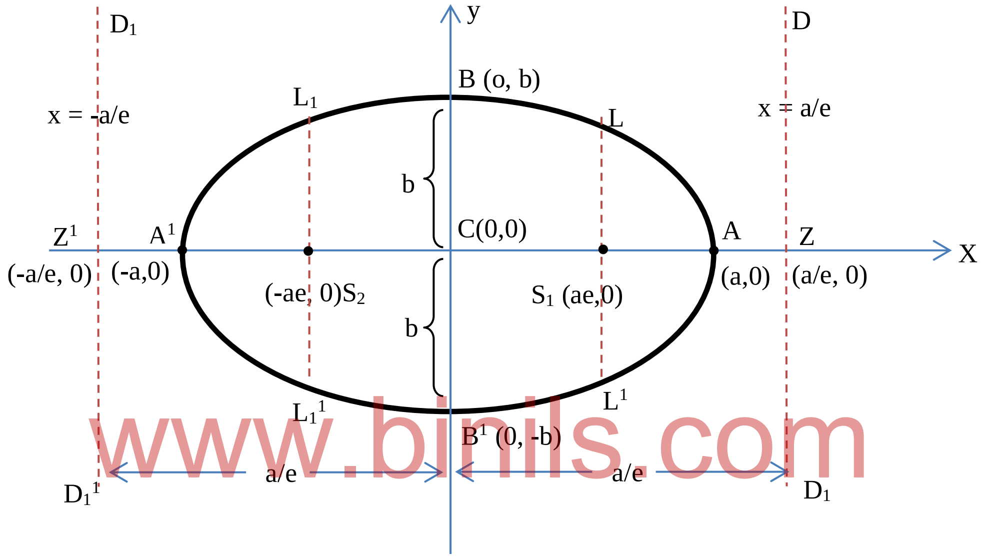
which is the equation of an ellipse in standard form and note that it is symmetrical about x and y axis.

Tracing of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ a > b$

- It does not pass through the origin.
- To find the points on x – axis, put y = 0, we get x = ± a. Therefore, the curve meets the x – axis at A (a, 0) and A' (-a, 0).
- To find the points on y – axis, put x = 0, we get y = ± b. Therefore, the curve meets the y – axis at B (0, b) and B' (0, -b).

Type : 1 Standard equation of an ellipse [Symmetrical about x & y axis]

If the major axis of the ellipse is along x – axis then the equation takes of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ provided a > b where $b^2 = a^2 (1 - e^2)$



Important Definitions Regarding Ellipse:

Faci : The fixed point which is used to draw ellipse.

∴ Co-ordinates of foci = (± ae, 0)

Directrices: The fixed straight line which is used to draw ellipse.

∴ Equation of directrices are X = ± a/e

Major Axis: The line segment AA' is called Major axis.

(i) Equation of Major axis (i.e.) x – axis ⇒ y = 0

(ii) Length of Major axis = 2a units.

Minor Axis: The line segment BB' is called minor axis.

(i) Equation of Minor axis is y – axis ⇒ x = 0

(ii) Length of Minor axis = 2b units.

Centre: The point of intersection of Major axis & Minor axis

∴ Co-ordinates of centre C = (0, 0).

Vertices: The point of intersection of the ellipse & the Major axis $(\pm a, 0)$

$\therefore$ Co-ordinates of vertices are A (a, 0) & A' (-a, 0)

Latus Rectum: It is a chord passing through the focus and perpendicular to Major axis.

(i) Equation of Latus rectum $x = \pm ae$

(ii) Length of Latus rectum = LLR = $\frac{2b^2}{a}$ units.

Eccentricity: $e = \sqrt{\frac{a^2 - b^2}{a^2}}$

Distance between the Foci = $S_1S_2 = 2ae$ [$\because F_1 + F_2 = ae + ae = 2ae$]

$CS_1 = CS_2 = ae$ [c = centre]

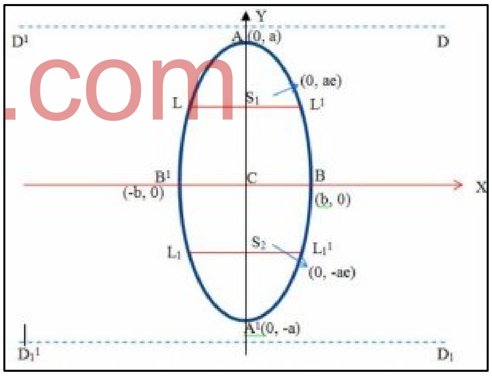
Length of the Major Axis = AA' = 2a

CA = CA' = a units

Distance between the directrices = $\frac{2a}{e}$

CZ = CZ' = a/e

Type: 2 Standard Equation of an Ellipse [Vertical Ellipse]



If the major axis of the ellipse is along y – axis then the equation of the ellipse takes the form

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b$

For this type of ellipse, we have the following as explained in the earlier ellipse.

Foci = (0, ae) (0, - ae)

Equation of Directrices $\Rightarrow y = \pm a/e$

Centre C = (0, 0)

Vertices $(0, \pm a)$

Equation of Latus Rectum $\Rightarrow y = \pm ae$

Length of Latus rectum $= \frac{2b^2}{a}$ units.

Eccentricity = $e = \sqrt{\frac{a^2 - b^2}{a^2}}$

Major axis : Equation of Major axis = y – axis $\Rightarrow$ x = 0

Length of Major axis = 2a units.

Minor axis : Equation of Minor axis = x – axis $\Rightarrow$ y = 0

Length of Minor axis = 2b units

Types of Ellipses with Centre at (h, k)

(a) Major axis Parallel to x – axis

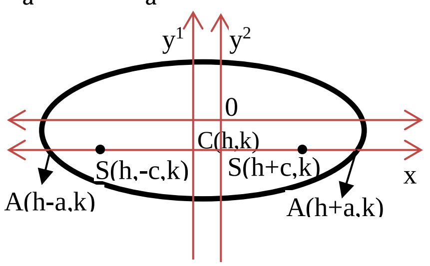
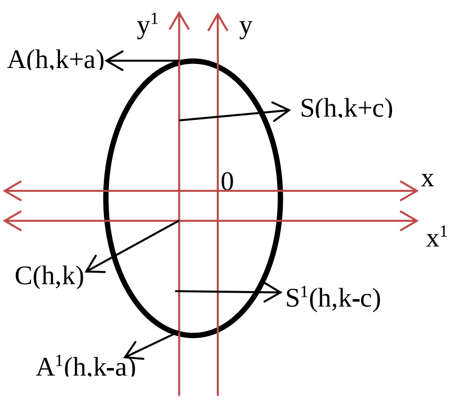
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,\ a > b.$$

The coordinates of the vertices are (h + a, k) and (h – a, k) and the coordinates of the foci are (h + c, k) and (h –c, k) where $c^2 = a^2 - b^2$.

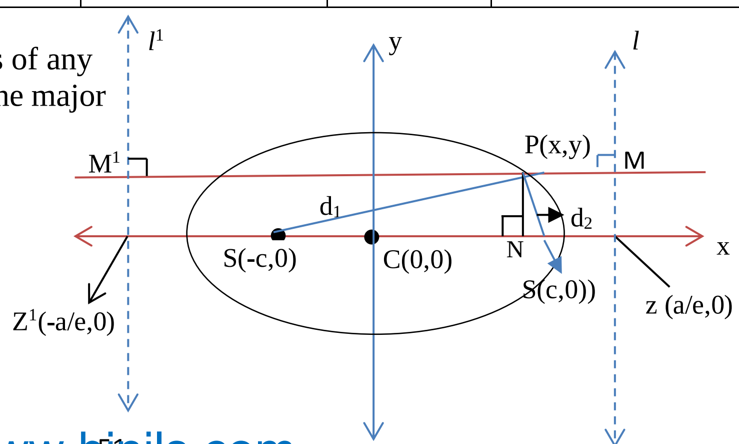
(b) Major axis Parallel to y – axis

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1,\ a > b$$

The coordinates of the vertices are (h, k+a) and (h, k-a) and the coordinates of the foci are (h, k + c) and (h, k –c) where $c^2 = a^2 - b^2$.

<table>
<tr><th>Equation</th><th>Centre</th><th>Major Axis</th><th>Vertices</th><th>Foci</th></tr>
<tr><td>$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,\ \boxed{a^2 > b^2}$

</td><td>(h, k)</td><td>Parallel to
x – axis</td><td>(h –a, k)
(h + a, k)</td><td>(h – c, k)
(h + c, k)
Foci are c units right and c units left of centre
where
$c^2 = a^2 - b^2$</td></tr>
<tr><td>$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1,\ \boxed{a^2 > b^2}$

</td><td>(h, k)</td><td>Parallel to
y – axis</td><td>(h, k – a)
(h, k + a)</td><td>(h, k – c)
(h, k + c)
Foci are c units right and c units left of centre
where
$c^2 = a^2 - b^2$</td></tr>
</table>

Theorem: The sum of the focal distances of any point on the ellipse is equal to length of the major axis.



Proof : Let P(x, y) be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Draw MM' through P, perpendicular to directrices l and l^1.

Draw PN ⊥r to x-axis.

By definition, $\quad SP = ePM$

$$= eNZ = e[CZ - CN] = e\left[\frac{a}{e} - x\right] = a - ex$$

$$SP' = ePM'$$

$$= e[CN + CZ'] = e\left[x + \frac{a}{e}\right] = ex + a$$

Hence, SP + SP' = a – ex + a + ex = 2a.

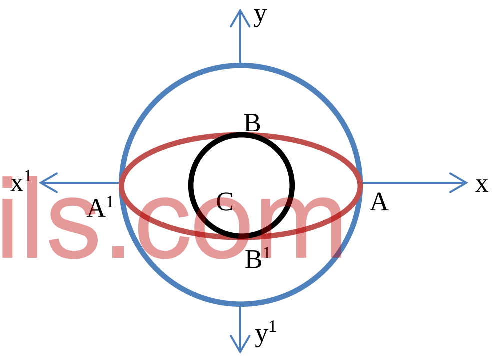
Remark:

(i) In the case of an ellipse $0 < e < 1$.

As $e \to 0$, $\frac{b}{a} \to 1$, $b \to a$ or the length of the minor and major axis are close in size, (i.e) the ellipse is close to being a circle.

(ii) As $e \to 1$, $\frac{b}{a} \to 0$ and the ellipse degenerates into a line segment (degenerate conic) (i.e) the ellipse is flat.

(iii) **Auxiliary** circle or circumcircle is the circle with length of major axis as diameter and **In circle** is the circle with length of minor axis as diameter. They are given by $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ respectively.



Example: 1

Find the length of Latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution:

The Latus rectum LL' (fig) of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through F (ae, 0).

Hence, L is (ae, y_1)

$$\therefore \frac{a^2e^2}{a^2} + \frac{{y_1}^2}{b^2} = 1$$

$$\frac{{y_1}^2}{b^2} = 1 - e^2 \left(\because e^2 = 1 - \frac{b^2}{a^2}\right)$$

$$\Rightarrow {y_1}^2 = b^2(1 - e^2) = b^2\left(\frac{b^2}{a^2}\right)$$

$$\Rightarrow {y_1}^2 = \frac{b^4}{a^2}$$

$$\Rightarrow y_1 = \pm \frac{b^2}{a}$$

i.e. the end points of Latus rectum L and L' are $\left(ae, \frac{b^2}{a}\right)$ & $\left(\pm ae, \frac{-b^2}{a}\right)$

Hence the length of Latum rectum LL' = $\frac{2b^2}{a}$.

Example: 2

Find the equation of the ellipse whose vertices are (± 3, 0) and foci (± 2, 0)

Solution:

W.K.T., $F_1F_2 = 2ae$ & $2ae = 4$

$\boxed{ae = 2}$

and $AA^1 = 2a \Rightarrow 2a = 6$

$\boxed{a = 3}$

$b^2 = a^2 - c^2 = 9 - 4 = 5$ $\because c = ae = 2$

Major axis is along x - axis, since a > b

[$\because$ centre is mid point of $AA^1 = \left(\frac{-3+3}{2}, \frac{0+0}{2}\right)$, C = (0, 0)

Centre is (0, 0) & Foci are (± 2, 0)

$\therefore$ The equation of the ellipse is $\frac{x^2}{9} + \frac{y^2}{5} = 1$.

Example: 3

Find the coordinates of the foci, the vertices, the length of major and minor axis and the eccentricity of the ellipse $9x^2 + 4y^2 = 36$.

Solution:

The given equation of the ellipse can be written in standard form as

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Since the denominator of $\frac{y^2}{9}$ is larger than the denominator of $\frac{x^2}{4}$, the major axis is along the y – axis, comparing the given equation with the standard equation

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, we have b = 2 & a = 3

Also $c = \sqrt{a^2 - b^2} = \sqrt{9 - 4} = \sqrt{5}$ (or) $e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{9-4}{9}} = \sqrt{\frac{5}{3}}$

$e = \frac{c}{a} = \frac{\sqrt{5}}{3}$

Hence the foci are $(0, \pm a)$ $(0, \sqrt{5})$ & $(0, -\sqrt{5})$

vertices are $(0, \pm a)$ $(0, 3)$ & $(0, -3)$

length of the Major axis is 2a = 6 units

& length of the Minor axis is 2b = 4 units.

& the eccentricity of the ellipse is $e = \frac{\sqrt{5}}{3}$

Example: 4

Find the equation of the ellipse, whose length of the major axis is 20 and foci are (0, ± 5)

Solution:

Since the foci are on y - axis, the major axis is along the y-axis, so, equation of the ellipse is of the form $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.

Given that, a = Semi – Major axis = $\frac{20}{2}$ = 10.

$$\Rightarrow \boxed{a = 10}$$

& the relation, $c^2 = a^2 - b^2$

$$\Rightarrow 5^2 = 10^2 - b^2 \quad \Rightarrow b^2 = 75$$

$\therefore$ Equation of ellipse is $\frac{x^2}{75} + \frac{y^2}{100} = 1$.

Example: 5

Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, one of the foci is (2, 3) and a directrix is x = 7. Also find the length of the Major and Minor axes of the ellipse.

Solution:

By the definition of a conic,

$$\frac{SP}{PM} = e \quad \text{(or)} \quad SP^2 = e^2\, PM^2$$

Then, $(x-2)^2 + (y-3)^2 = \frac{1}{4}(x-7)^2$

$$\Rightarrow \quad 3x^2 + 4y^2 - 2x - 24y + 3 = 0$$

$$\Rightarrow \quad 3\left(x - \tfrac{1}{3}\right)^2 + 4\,(y-3)^2 = 3\left(\tfrac{1}{9}\right) + 4(9) - 3 = \frac{100}{3}$$

$$\Rightarrow \quad \frac{\left(x-\frac{1}{3}\right)^2}{100/9} + \frac{(y-3)^2}{100/12} = 1$$ which is in the standard form.

Therefore, the length of Major axis $= 2a$

$$= 2\sqrt{\frac{100}{9}} = \frac{20}{3}$$

The length of Minor axis $= 2b$

$$= 2\sqrt{\frac{100}{12}} = \frac{10}{\sqrt{3}}$$

Example: 6

Find the equation of the ellipse if the major axis is parallel to y – axis, semi - major axis is 12, length of the latus rectum is 6 and the centre is (1, 12).

Solution:

Since the major axis is parallel to y - axis the equation of the ellipse is of the form

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$

The centre C(h, k) is (1, 12)

Semi major axis a = 12 $\Rightarrow a^2 = 144$

Length of the latus rectum $\frac{2b^2}{a} = 6 \Rightarrow \frac{2b^2}{12} = 6$

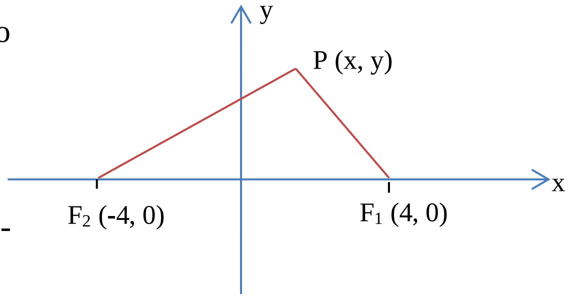
$$\therefore b^2 = 36$$

$\therefore$ The required equation is $\frac{(x-1)^2}{36} + \frac{(y-12)^2}{144} = 1$.

Example : 7

Find the equation of a point which moves so that the sum of its distances from

(-4, 0) and (4, 0) is 10.



Solution:

Let F_1 and F_2 be the fixed points (4, 0) and (-4, 0) respectively and P (x_1, y_1) be the moving point.

It is given that, $F_1P + F_2P = 10$

(i.e.) $\sqrt{(x_1 - 4)^2 + (y_1 - 0)^2} + \sqrt{(x_1 + 4)^2 + (y_1 - 0)^2} = 10$

We get, $9x_1^2 + 25y_1^2 = 225$

$\therefore$ The locus of (x_1, y_1) is $\frac{x^2}{25} + \frac{y^2}{9} = 1$

Example : 8

Find the equations of directrices, latus rectum and length of latus rectums of the ellipse $25x^2 + 9y^2 = 225$.

Solution:

$25x^2 + 9y^2 = 225$

$\frac{x^2}{9} + \frac{y^2}{25} = 1$

Here $a^2 = 25, b^2 = 9 \Rightarrow e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$

The equations of the directrices are $y = \pm \frac{a}{e} = \pm \frac{5}{4/5} = \pm \frac{25}{4}$

Equations of the latus rectum are $y = \pm ae = \pm 5 \times \frac{4}{5} = \pm 4$

Length of latus rectum is $\frac{2b^2}{a} = \frac{18}{5}$

Example: 9

Find the eccentricity, centre, foci, vertices of the ellipse $x^2 + 4y^2 - 8x - 16y - 68 = 0$ and draw the diagram.

Solution:

$(x^2 - 8x) + (4y^2 - 16y) = 68$

$(x^2 - 8x) + 4(y^2 - 4y) = 68$

$(x^2 - 8x + 16 - 16) + 4[y^2 - 4y + 4 - 4] = 68$

Method of completion of square

$(x^2 - 8x + 16) - 16 + 4[(y^2 - 4y + 4) - 4] = 68$

$(x - 4)^2 - 16 + 4[(y - 2)^2 - 4] = 68$

$(x - 4)^2 + 4(y - 2)^2 - 16 - 16 = 68$

$(x - 4)^2 + 4(y - 2)^2 = 68 + 32 = 100$

$(x - 4)^2 + 4(y - 2)^2 = 100$

$$\frac{(x-4)^2}{100} + \frac{4\,(y-2)^2}{100} = \frac{100}{100} = 1 \text{ (Dividing by 100]}$$

$$\frac{(x-4)^2}{100} + \frac{(y-2)^2}{25} = 1 \quad \text{----- (1)}$$

Equation (1) is not in standard form.

In order to bring in standard form, let us take

$x - 4 = X$

$y - 2 = Y$

$\therefore \frac{X^2}{100} + \frac{Y^2}{25} = 1$ ----- (2) [since $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$]

$\therefore$ The major axis is along x-axis and the minor axis is along y – axis.

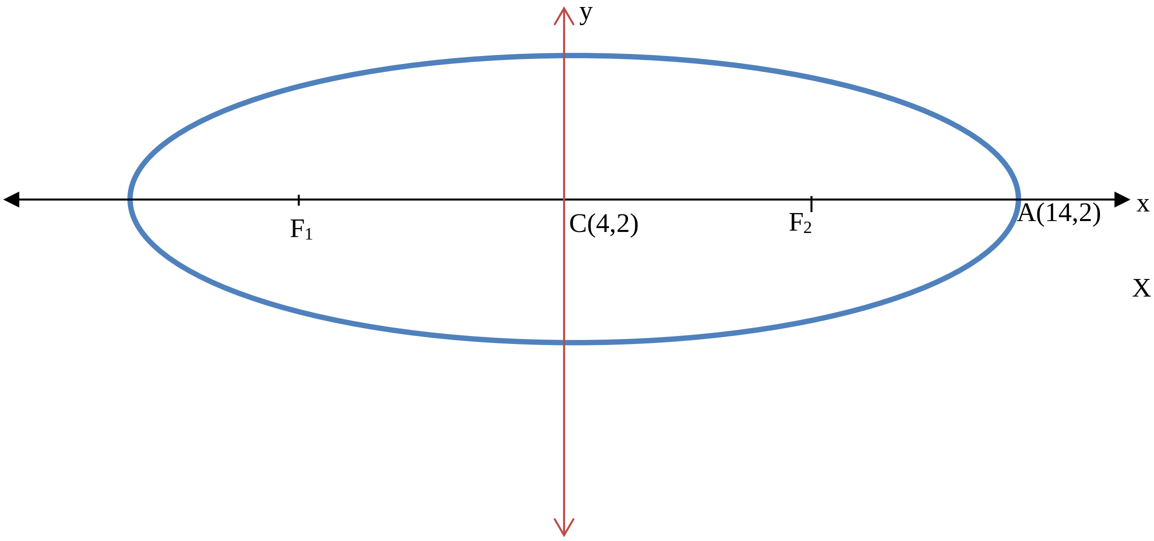
$a^2 = 100 \Rightarrow a = 10$

$b^2 = 25 \Rightarrow b = 5$

$$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{100-25}{100}} = \sqrt{\frac{75}{100}} = \sqrt{3/4} = \frac{\sqrt{3}}{2}$$

$$\boxed{\therefore\ e = \frac{\sqrt{3}}{2}}$$

Components	Referred to X, Y axes	Referred to x, y axes when X= x – 4 & Y = y – 2
Centre	X, Y (0, 0)	$X = 0 \Rightarrow x = 0 + 4 = 4$ $Y = 0 \Rightarrow y = 0 + 2 = 2$ $\therefore (x, y) = (4, 2)$
Vertice	$(\pm a, 0)$ $(\pm 10, 0)$ $(-10, 0)\ (+10, 0)$	$x = -10 + 4,\ y = 0 + 2 = 2$ $(x, y) = (-6, 2)$ $x = 10 + 4,\ y = 0 + 2 = 2$ $(x, y) = (14, 2)$
Foci	$(\pm ae, 0)$ $(\pm 5\sqrt{3}, 0)$ $(-5\sqrt{3}, 0)\ (5\sqrt{3}, 0)$	$x = -5\sqrt{3} + 4,\ y = 0 + 2 = 2$ $(x, y) = (-5\sqrt{3} + 4, 2)$ $x = 5\sqrt{3} + 4,\ y = 0 + 2 = 2$ $(x, y) = [+5\sqrt{3} + 4, 2]$
Eccentricity	$\sqrt{3}/2$	$\sqrt{3}/2$



Equation of directrices are = $\pm a/e = \pm \frac{10}{\sqrt{3/2}} = \pm \frac{20}{\sqrt{3}}$

Equation of latus rectum are x = $\pm ae = \pm 10 \times \frac{\sqrt{3}}{2} = \pm 5\sqrt{3}$

Length of latus rectum = $\frac{2b^2}{a} = \frac{2(25)}{10} = \frac{50}{10} = 5.$

Example: 10

The centre of the ellipse is (2, 3). One of the foci is (3, 3). Find the other focus.

Solution:

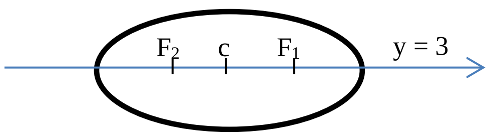
From the given data the major axis is parallel to the x - axis. Let F_1 be (3, 3).

Let F_2 be the point (x, y). Since C(2, 3) is the mid point of F_1 and F_2 on the major axis y = 3.

$$\frac{x+3}{2} = 2 \text{ and } \frac{y+3}{2} = 3$$

This gives x = 1 and y = 3

$\therefore$ Thus the other focus is (1, 3)



Exercise: 1.3.2

1) Find the equation of the ellipse in each of the cases given below:

   (i) foci ($\pm$ 3, 0), e = $\frac{1}{2}$

   (ii) foci (0, $\pm$ 4) and end points of major axis are (0, $\pm$ 5)

   (iii) length of latus rectum 8, eccentricity = $\frac{3}{5}$, centre (0, 0) and major axis on x – axis.

   (iv) Length of latus rectum 4, distance between foci $4\sqrt{2}$, centre (0, 0) and major axis as y – axis.

2) For the ellipse $4x^2 + y^2 + 24x - 2y + 21 = 0$, find the centre, vertices and the foci. Also prove that the length of latus rectum is 2.

3) Find the equations of axes and length of axes of the ellipse $6x^2 + 9y^2 + 12x - 36y - 12 = 0$

4) Find the eccentricity, centre, foci, vertices of the following ellipses.

(i) $\frac{(x+3)^2}{6} + \frac{(y-5)^2}{4} = 1$

(ii) $36x^2 + 4y^2 - 72x + 32y - 44 = 0$

Exercise: 1.3.2 - Answers:

(1) (i) $\frac{x^2}{36} + \frac{y^2}{27} = 1$ (ii) $\frac{x^2}{9} + \frac{y^2}{25} = 1$ (iii) $\frac{16x^2}{625} + \frac{y^2}{25} = 1$

(iv) $\frac{x^2}{8} + \frac{y^2}{16} = 1$

(2) Centre (-3, 1), vertices (3, ± 4 + 1), Foci (-3, ± $2\sqrt{3}$ + 1)

(3) Equation of major axis; y – 2 = 0, length of major axis = 6

Equation of minor axis x + 1 = 0 length of minor axis = $2\sqrt{6}$

(4) (i) $e = \frac{1}{\sqrt{3}}$, C(-3, 5), Foci : $F_1\ (-3 + \sqrt{2},\ 5)$, $F_2\ (-3 - \sqrt{2},\ 5)$ Vertices : $A(-3 + \sqrt{6},\ 5)$, $A^1(-3 - \sqrt{6},\ 5)$

(ii) $e = \frac{2\sqrt{2}}{3}$, C(1, -4), Foci : $F_1(1,\ 4\sqrt{2} - 4)$, $F_2(1, -4 - 4\sqrt{2})$ Vertices : $A\ (1, 2)$, $A^1(1, -10)$

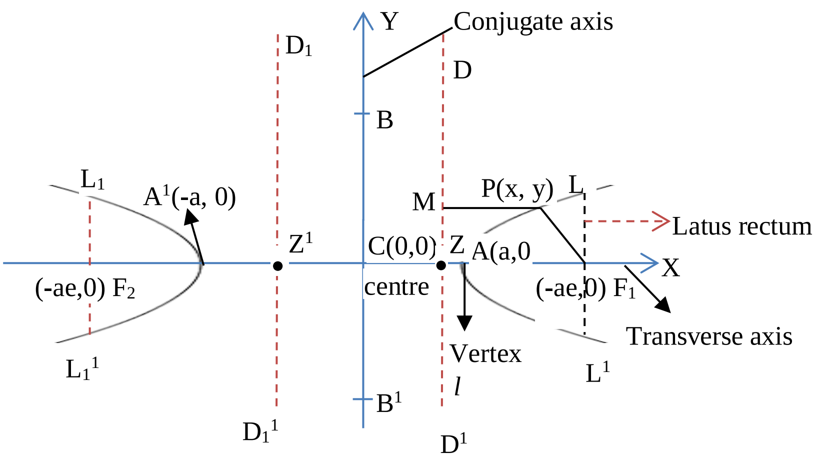
Hyperbola :

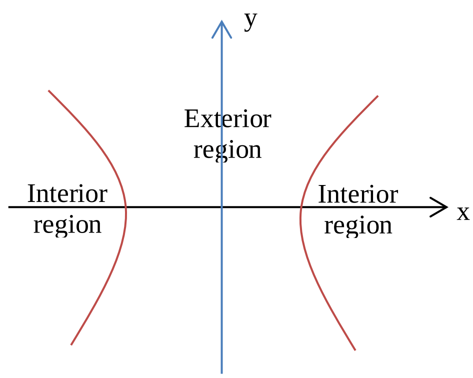
A hyperbola is the set of all points in a plane, the difference of whose distances from two fixed points in the plane is a constant. (e > 1)

(i) Equation of a Hyperbola in standard form with centre at (0, 0)

Let F be a focus, l be the directrix line, e be the eccentricity e > 1 and P(x, y) be the moving point. Draw FZ and PM perpendicular to l.

Let A and A^1 be the points which divide FZ internally and externally in the ratio e : 1 respectively.





Let AA' = 2a. Let the point of intersection of the perpendicular bisector with AA' be C. Then CA = CA' = a. choose C as origin and the line CZ produced as x-axis and the perpendicular bisector of AA' as y – axis.

By definition, $\frac{AF}{AZ} = e$ & $\frac{A'F}{A'Z} = e$

	AF = eAZ.	A'F = eA'Z
$\Rightarrow$	CF – CA = e (CA – CZ)	A'C + CF = e (A'C + CZ)
$\Rightarrow$	CF – a = e (a – CZ) ----(1)	a + CF = e (a + CZ) ---- (2)

(1) + (2) $\Rightarrow$ CS = ae

(2) – (1) $\Rightarrow$ CZ = $\frac{a}{e}$

Hence, the coordinates of F are (ae, 0). Since PM = x - $\frac{a}{e}$, Equation of directrix is x - $\frac{a}{e}$ = 0

Let P(x, y) be any point on the hyperbola.

By the definition of a conic, $\frac{FP}{PM} = e \Rightarrow FP^2 = e^2 PM^2$

$$(x - ae)^2 + (y - 0)^2 = e^2 \left(x - \frac{a}{e}\right)^2$$

$$(x - ae)^2 + y^2 = \frac{\cancel{e^2}(ex - a)^2}{\cancel{e^2}}$$

$$(e^2 - 1)\, x^2 - y^2 = a^2 (e^2 - 1)$$

$\Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$. Since e > 1, $a^2(e^2 - 1) > 0$

Setting $a^2(e^2 - 1) = b^2$

We obtain the locus of P as $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$which is the equation of a Hyperbola in standard form and note that it is symmetrical about x and y – axes.

Taking $\boxed{ae = c}$, we get $\boxed{b^2 = c^2 - a^2}$

Definition:

1) The line segment AA' is the transverse axis of length 2a
2) The line segment BB' is the conjugate axis of length 2b.
3) The line segment CA = the line segment CA' = semi transverse axis = a and the line segment CB = the line segment CB' = semi conjugate axis = b.
4) By symmetry, taking F' (-ae, 0), as focus and x = - $\frac{a}{e}$ as directrix l' gives the same hyperbola.

   Thus, we see that a hyperbola has two foci F (ae, 0) & F' (-ae, 0) two vertices A (a, 0) & A' (-a, 0) and two directrices x = $\frac{a}{e}$ & x = $-\frac{a}{e}$.
5) Latus recturm: i) Equation of L.R = x = ± ae

   ii) Length of Latus rectum = $\frac{2b^2}{a}$ units, which can be along lines as that of the ellipse.
6) Distance between the Foci = F_1F_2 = 2ae

7) Distance between the directrices = $\frac{2a}{e}$

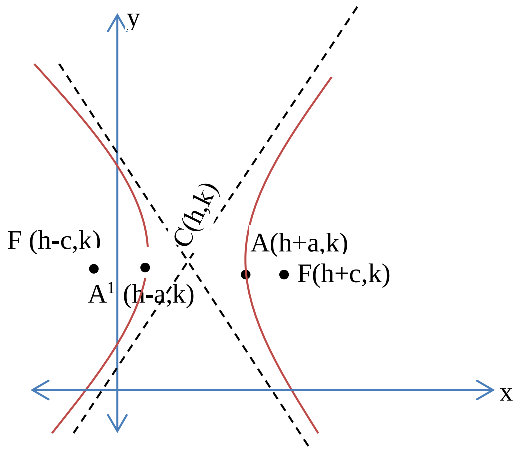
Asymptotes:

Let p(x, y) be a point on the curve defined by y = f(x), which moves further and further away from the origin such that the distance between P and some fixed lines tends to zero. The fixed line is called an asymptote.

Note that the hyperbolas admit asymptotes while parabola and ellipse do not.

ii) Types of Hyperbola with centre at (h, k):

(a) Transverse axis parallel to the x – axis

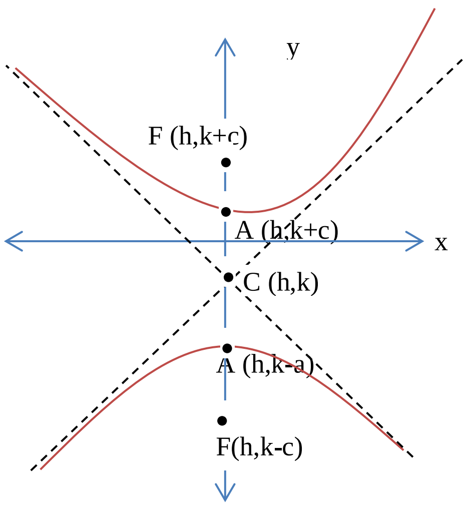


The equation of a hyperbola with centre C(h, k) and transverse axis parallel to the x - axis is given by $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

The coordinates of the vertices are A (h + a, k) & A^1 (h – a, k). The coordinates of the foci are F(h + c, k) and F'(h – c, k) where $c^2 = a^2 + b^2$.

The equations of directrices are $x = h \pm \frac{a}{e}$

(b) Transverse axis parallel to the y – axis:



The equation of a hyperbola with centre C(h, k) and transverse axis parallel to the y-axis is given by $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$.

The coordinates of the vertices are A(h, k + a) & A'(h, k – a).

The coordinates of the foci are F(h, k + c) & F'(h, k – c)

where $c^2 = a^2 + b^2$.

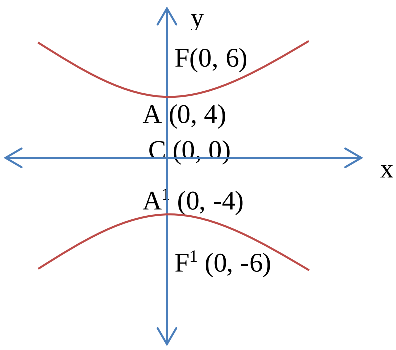
The equations of directrices are $y = k \pm \frac{a}{e}$

Examples:

1) Find the equation of the hyperbola with vertices (0, ± 4) and foci (0, ± 6)

Solution:

From the figure, the mid point of line joining foci is the centre C(0, 0),



Transverse axis is y – axis

AA' = 2a

8 = 2a

$8/2 = a \Rightarrow \boxed{a = 4}$

$b^2 = c^2 - a^2 = 36 - 16 = 20$

Hence the equation of the required hyperbola is $\frac{y^2}{16} - \frac{x^2}{20} = 1$

2) Find the vertices, foci for the hyperbola $9x^2 - 16y^2 = 144$

Solution:

Reducing $9x^2 - 16y^2 = 144$ to the standard form,

we have, $\frac{x^2}{16} - \frac{y^2}{9} = 1$

with the transverse axis is along x – axis,

Vertices are (-4, 0) and (4, 0) ; and $c^2 = a^2 + b^2 = 16 + 9 = 25$

$\Rightarrow \boxed{c = 5}$

Hence the foci are (-5, 0) & (5, 0)

3) Find the eccentricity, centre, foci and vertices of the hyperbola

$x^2 - 4y^2 + 6x + 16y - 11 = 0$

Solution:

$x^2 + 6x - 4y^2 + 16y = 11$

$x^2 + 6x - 4(y^2 - 4y) = 11$

$(x + 3)^2 - 9 - 4[(y - 2)^2 - 4] = 11$

$(x + 3) - 4(y - 2)^2 = 4$

$\div\ 4 \Rightarrow \frac{(x+3)^2}{4} - \frac{\cancel{4}(y-2)^2}{\cancel{4}} = 1$

$a^2 = 4$, $\quad b^2 = 1 \qquad$ where $X = x + 3$

$\Rightarrow a = 2 \qquad \Rightarrow b = 1 \qquad\qquad Y = y - 2$

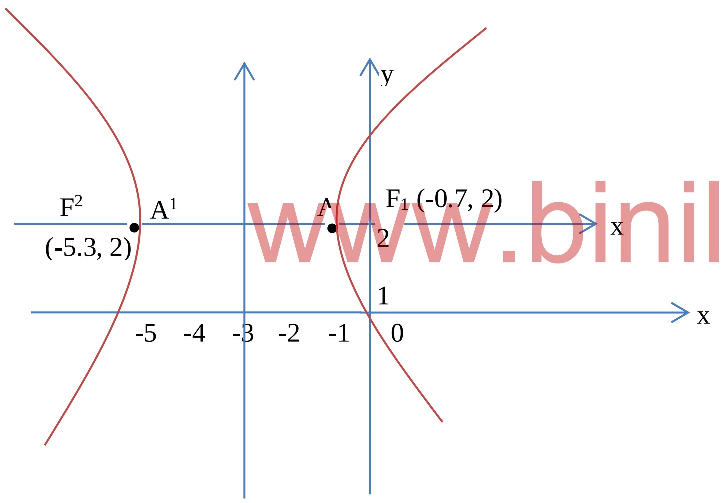
$c^2 = a^2 + b^2$

$c^2 = 5$

$c = \sqrt{5}$ & $ae = c = \sqrt{5}$

$e = \frac{c}{a} = \frac{\sqrt{5}}{2}$

	Referred to X, Y axis	Referred to x, y axis x= X – 3, y = Y + 2
Centre	(X, Y) = (0, 0)	(-3, 2) = (x, y)
Vertices	$(\pm a, 0) = (\pm 2, 0)$ = (-2, 0) (2, 0)	x = -2 – 3 = -5, y = 0 + 2 = 2 (-5, 2) x = 2 – 3 = -1, y = 0 + 2 = 2 (-1, 2)
Foci	$(\pm ae, 0) = (\pm\sqrt{5}, 0)$ $= (-\sqrt{5}, 0)\ (\sqrt{5}, 0)$	$x = -\sqrt{5} - 3,\ y = 0 + 2 = ((-\sqrt{5} - 3), 2)$ $x = -\sqrt{5} - 3,\ y = 0 + 2 = ((\sqrt{5} - 3), 2)$
Eccentricity	$\sqrt{5}/2$	$\sqrt{5}/2$



4) Find the equations, length of transverse and conjugate axes of the hyperbola $16y^2 - 9x^2 = 144$.

Solution:

$$\frac{y^2}{9} - \frac{x^2}{16} = 1$$

The centre is at the origin, the transverse axis is along y-axis and the conjugate axis is along x- axis.

$\therefore$ The transverse axis is y- axis, (i.e) x = 0

The conjugate axis is x-axis, (i.e.) y = 0

Here $a^2 = 9,\ b^2 = 16 \Rightarrow a = 3,\ b = 4$

$\therefore$ The length of transverse axis = 2a = 6

The length of conjugate axis = 2b = 8

5) Find the equation of the hyperbola whose centre is (1, 2). The distance between the directrices is $\frac{20}{3}$, the distance between the foci 30 and the transverse axis is parallel to y-axis.

Solution:

Since the transverse axis is parallel to y- axis, the equation is of the forms,

$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

Here centre C(h, k) is (1, 2)

The distance between the directrices is $\frac{2a}{e} = \frac{20}{3}$

$$\Rightarrow \boxed{\frac{a}{e} = \frac{10}{3}} \text{ ----- (1)}$$

The distance between the foci, 2ae = 30

From (1) and (2)

$\frac{a}{e}$ x ae = $\frac{10}{3}$ x 15

$$\boxed{ae = 15} \text{ ----- (2)}$$

$\Rightarrow \boxed{a^2 = 50}$

Also $\frac{ae}{a/e}$ $\Rightarrow \boxed{e^2 = 9/2}$

$b^2 = a^2 (e^2 - 1) \Rightarrow b^2 = 50\left(\frac{9}{2} - 1\right) = 175$

$\boxed{b^2 = 175}$

$\therefore$ The required equation is $\frac{(y-2)^2}{50} - \frac{(x-1)^2}{175} = 1$

Exercise: 1.3.3

1. Find the equation of the hyperbola in each of the cases given below:

(i) foci (± 2, 0), eccentricity $= \frac{3}{2}$

(ii) focus = (2, 3), directrix : x + 2y = 5, eccentricity = 2

(iii) centre (2, 1), one of the foci (8, 1) and corresponding directrix, x = 4

(iv) passing through (5, - 2) and length of the transverse axis along x axis and of length 8 units.

(v) Vertices (± 2, 0), foci (± 3, 0)

2. Find the coordinates of the foci, eccentricity, vertices and the length of the latus rectum of the following hyperbolas.

(i) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ (ii) $\frac{y^2}{9} - \frac{x^2}{27} = 0$ (iii) $9y^2 - 4x^2 = 36$

(iv) $16x^2 - 9y^2 = 576$

3. Find the eccentricity, centre, foci and vertices of the following hyperbolas and draw their diagrams.

(i) $25x^2 - 16y^2 = 400$ (ii) $\frac{y^2}{9} - \frac{x^2}{25} = 1$

(iii) $x^2 - 4y^2 + 6x + 16y - 11 = 0$ (iv) $x^2 - 3y^2 + 6x + 6y + 18 = 0$

Exercise: 1.3.3 - Answers:

1) (i) $\frac{9x^2}{16} - \frac{9y^2}{20} = 1$ (ii) $x^2 - 16xy - 11y^2 + 20x + 50y - 35 = 0$

(iii) $\frac{(x-2)^2}{12} - \frac{(y-1)^2}{24} = 1$ (iv) $\frac{x^2}{16} - \frac{9y^2}{64} = 1$ (v) $\frac{x^2}{4} - \frac{y^2}{5} = 1$

2) (i) Foci $(\pm 5, 0)$, $e = 5/4$, vertices $(\pm 4, 0)$, LR = 9/2

(ii) Foci $(0, \pm 6)$, $e = 2$, vertices $(0, \pm 3)$, LR = 18

(iii) Foci $(0, \pm\sqrt{13})$, $e = \sqrt{13}/2$, vertices $(0, \pm 2)$, LR = 9

(iv) Foci $(\pm 10, 0)$, $e = 5/3$, vertices $(\pm 6, 0)$, LR = 64/3

3) (i) $e = \sqrt{41}/4$, C(0, 0), Foci $(\pm\sqrt{41}, 0)$, V $(\pm 4, 0)$

(ii) $e = \sqrt{34}/3$, C(0, 0), F$(0, \pm\sqrt{34})$, V (0 ± 3)

(iii) $e = \sqrt{5}/2$, C(-3, 2), F$(\pm\sqrt{5} - 3, 2)$ A (-1, 2)

(iv) $e = 2$, C(-3, 2), F$(-3 \pm 2\sqrt{7} - 2)$ A $(-3, 2 \pm\sqrt{7})$

Applications of Conic Sections:

Conic sections are used in many fields of study, particularly to describe shapes. For example, they are used in astronomy to describe the shapes of the orbits or objects in space. Two massive objects in space that interact according to Newton's law of universal gravitation can move in orbits that are in the shape of conic sections. They could follow ellipses, parabolas, or hyperbolas, depending on their properties.

Parabola:

1) **Example:** A comet is moving in a parabolic orbit around the sun which is at the focus of a parabola. When the comet is 80 million kms from the sun, the line segment from the sun to the comet makes an angle of $\pi/3$ radians with the axis of the orbit, find (i) the equation of the comet's orbit (ii) how close does the comet come nearer to the sun? (Take the orbit as open rightward).

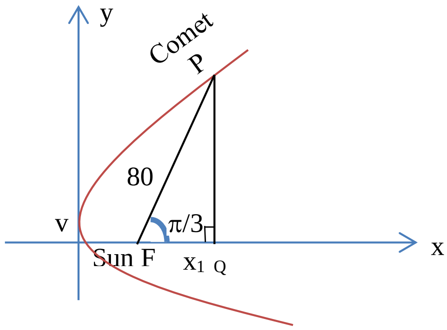
Solution:

Take the parabolic orbit as open rightward and the vertex at the origin.

Let P be the position of the comet in which FP = 80 million kms.

Draw a perpendicular PQ from P to the axis of the parabola.

Let $FQ = x_1$



From the triangle FQP,

$PQ = FP \times \sin\frac{\pi}{3}$

$= 80 \times \frac{\sqrt{3}}{2} = 40\sqrt{3}$

Thus, $FQ = x_1 = FP \cos\frac{\pi}{3} = 80 \text{ x } \frac{1}{2} = 40$

$\therefore$ VQ= a + 40 if VF = a.

P is (VQ, PQ) = $(a + 40, 40\sqrt{3})$

Since P lies on the parabola $y^2 = 4ax$

$(40\sqrt{3})^2 = 4a\,(a + 40)$

a = -60 or 20

a = -60 is not acceptable.

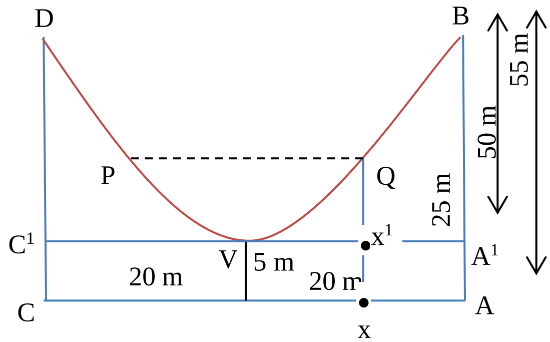
$\therefore$ The equation of the orbit is

$y^2 = 4 \times 20 \times x = 80\,x$

$\therefore$ The shortest distance between the sun and the comet is VF (i.e) a.

$\therefore$ The shortest distance is 20 million kms.

2) A cable of a suspension bridge is in the form of a parabola whose span is 40 mts. The road way in 5 mts below the lowest point of the cable. If an extra support is provided across the cable 30 mts above the ground level, find the length of the support if the height of the pillars are 55 mts.



Solution:

Let the vertex of the parabola is the lowest point on the cable.

Let AB, CD be the pillars. Given span of the parabola = 40 mts (i.e) distance between AB & CD = 40 mts.

$\therefore C^1V = VA = 20$ m

Height of each pillar = 55 m

$\Rightarrow$ AB = 55m

$\therefore \quad A^1B = 55 - 5 = 50$

$\therefore$ The point B on the parabola is B (20, 50)

Equation of the parabola is $x^2 = 4ay$

$(20)^2 = 4a\,(50)$

$$400 = 200\,a$$

$$a = 2\text{m}$$

$\therefore$ The equation of the suspension bridge is $x^2 = 4\,(2)\,y$

$$\Rightarrow \quad x^2 = 8y$$

Let PQ be the length of the extra support.

Then XQ = 30,

Given $xx^1 = 5$ m

$\Rightarrow \quad x^1Q = 30 - 5 = 25$ m

$\therefore$ Q (x, 25) is a point of the parabola

$\therefore \quad x^2 = 8 \times 25 = 200$

$\Rightarrow x = \sqrt{200} = 10\sqrt{2}$

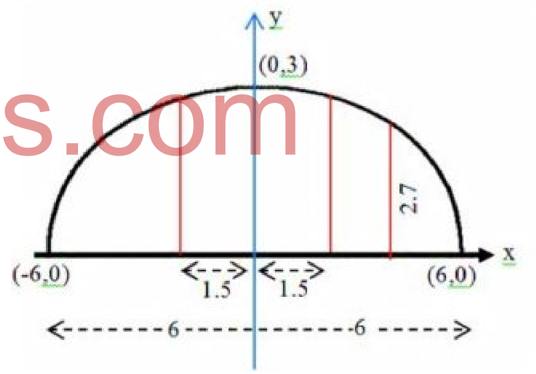
$\therefore$ PQ $= 2x = 2 \times 10\sqrt{2} = 20\sqrt{2}$.

Ellipse:

3) A semi elliptical arch way over a one – way road has a height of 3m and a width of 12m. The truck has a width of 3m and a height of 2.7 m. Will the truck clear the opening of the archway?

Solution:

Since the truck's width is 3m, to determine the clearance, we must find the height of the archway 1.5 m from the centre. If this height is 2.7 m or less the truck will not clear the arch way.



From the diagram a = 6 and b = 3 gives the equation of ellipse

as $\frac{x^2}{6^2} + \frac{y^2}{3^2} = 1$

The edge of the 3m wide truck corresponds to x = 1.5 m from centre. We will find the height of the arch way 1.5m from the centre by substituting x = 1.5 and solving for y.

$$\frac{\left(\frac{3}{2}\right)^2}{36} + \frac{y^2}{9} = 1$$

$$y^2 = 9\left(1 - \frac{9}{144}\right) = \frac{9\,(135)}{144} = \frac{135}{16}$$

$$y = \frac{\sqrt{135}}{4} = \frac{11.62}{4} = 2.90$$

$$\boxed{y = 2.90}$$

Thus the height of arch way 1.5 m from the centre is approximately 2.90 m. Since the truck's height is 2.7 m, the truck will clear the arch way.

4) A satellite is travelling around the earth in an elliptical orbit having the earth at a focus and of eccentricity ½. The shortest distance that the satellite gets to the earth is 400 kms. Find the longest distance that the satellite gets from the earth.

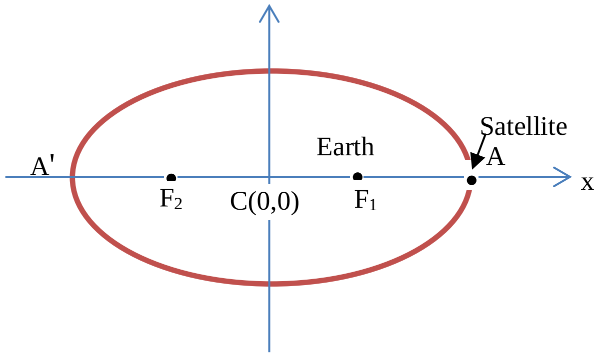
Solution:

Given, $e = \frac{1}{2}$

Shortest distance that the satellite gets to the earth is $F_1A = 400$ km.

Let the longest distance of the Satellite from the earth be F_1A'.

We know that, $CA = CA' = a$ (Semi Major axis)



$$CA = CF_1 + F_1A$$

$$CF_1 = ae = a\left(\frac{1}{2}\right) = \frac{a}{2}$$

$$\therefore CA = \frac{a}{2} + 400$$

$$a - \frac{a}{2} = 400$$

$$\frac{a}{2} = 400$$

$$\boxed{a = 800}$$

$$\therefore F_1A' = CF_1 + CA' = \frac{800}{2} + 800 = 1200 \text{ m}$$

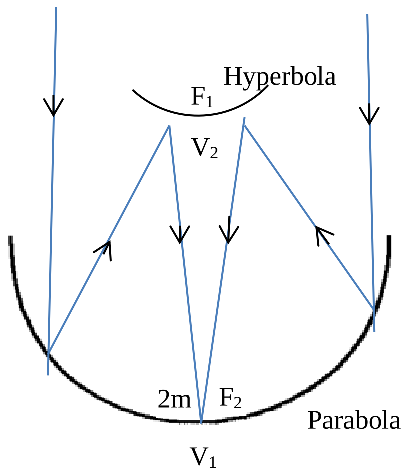
$\therefore$ The longest distance = 1200 km.

Hyperbola:

5) Certain telescopes contain both parabolic mirror and a hyperbolic mirror. In the telescope shown in figure the parabola and hyperbola share focus F_1 which is 14 m above the vertex of the parabola. The hyperbola's second focus F_2 is 2m above the parabola's vertex. The vertex of the hyperbolic mirror is 1m below F_1. Position a coordinate system with the origin at the centre of the hyperbola and with the foci on the y-axis. Then find the equation of the hyperbola.

Solution:

Let V_1 be the Vertex of the parabola and V_2 be the vertex of the hyperbola.



$\overline{F_1F_2} = 14 - 2 = 12$ m, $2c = 12 \Rightarrow c = 6$

The distance of centre to the vertex of the hyperbola is $a = 6 - 1 = 5$

$$b^2 = c^2 - a^2$$

$$= 36 - 25 = 11$$

$\therefore$ The equation of the hyperbola is $\frac{y^2}{25} - \frac{x^2}{11} = 1$

Exercise: 1.3.4

1) The girder of a railway bridge is in the parabolic form with span 100 ft, and the highest point on the arch is 10 ft. above the bridge. Find the height of the bridge at 10 ft to the left or right from the mid point of the bridge.
2) A reflecting telescope has a parabolic mirror for which the distance from the vertex to the focus is 9 mts. If the distance across (diameter) the top of the mirror is 160 cm, how deep is the mirror at the middle?
3) The orbit of the planet mercury around the sun is in elliptical shape with sun at a focus. The semi-major axis is of length 36 million miles and the eccentricity of the orbit is 0.206. Find (i) how close the mercury get to sun? (ii) the greatest possible distance between mercury and sun.
4) If the equation of the ellipse is $\frac{(x-11)^2}{484} + \frac{y^2}{64} = 1$ (x and y are measured in centimeters) where to the nearest centimetre, should the patient's kidney stone be placed so that the reflected sound hits the kidney stone?
5) Two coast guard stations are located 600 km apart at points A (0, 0) and B (0, 600). A distress signal from a ship at P is received at slightly different times by two stations. It is determined that the ship is 200 km farther from station A than it is from station B. Determine the equation of hyperbola that passes through the location of the ship.

Exercise: 1.3.4 - Answers

1) $9\frac{3}{5}$ ft.
2) $\frac{16}{9}$ cm

3) (i) 28.584 million miles

   (ii) 43.416 million miles
4) C = 20.5 cm
5) $\frac{(y-300)^2}{10000} - \frac{x^2}{80000} = 1$

Key points:

- Eccentricity is a parameter associated with every conic section, and can be thought of as a measure of how much the conic section deviates from being circular.
- The eccentricity of a conic section is defined to be the distance from any point on the conic section to its focus, divided by the perpendicular distance from that point to the nearest directrix.
- The value of e can be used to determine the type of conic section. If e = 1 it is a parabola, if e < 1 it is an ellipse, and if e > 1 it is a hyperbola.

Key Terms:

- **Eccentricity:** A parameter of a conic section that describes how much the conic section deviates from being circular.

Key points:

- Each conic section also has a degenerate form; these take the form of points and lines.

Key Terms:

- **degenerate:** A conic section which does not fit the standard form of equation.
- **asymptote:** A line which a curved function or shape approaches but never touches.
- **hyperbola:** The conic section formed by the plane being perpendicular to the base of the cone.
- **focus:** A point away from a curved line, around which the curve bends.
- **circle :**The conic section formed by the plane being parallel to the base of the cone.
- **ellipse:** The conic section formed by the plane being at an angle to the base of the cone.
- **parabola:** The conic section formed by the plane being parallel to the cone.
- **eccentricity:** A dimensionless parameter characterizing the shape of a conic section.
- **vertex:** The turning point of a curved shape.

Key points:

A conic section (or simply conic) is a curve obtained as the intersection of the surface of a cone with a plane, the three types are parabolas, ellipses and hyperbolas.

- A conic section can be graphed on a coordinate plane.
- Every conic section has certain features, including at least one focus and directrix. Parabolas have one focus and directrix, while ellipses and hyperbolas have two of each.
- A conic section is the set of points P whose distance to the focus is a constant multiple of the distance from P to the directrix of the conic.

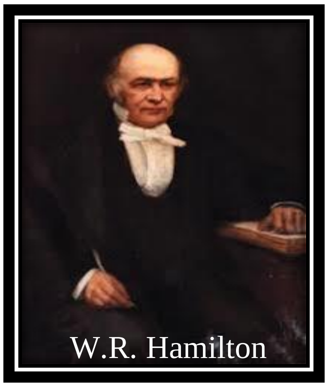
Key Terms:

- **vertex:** An extreme point on a conic section.
- **asymptote:** A straight line which a curve approaches arbitrarily closely as it goes to infinity.
- **locus:** The set of all points whose coordinates satisfy a given equation or condition.
- **focus:** A point used to construct and define a conic section, at which rays reflected from the curve converge (plural : foci)
- **nappe:** One half of a double cone.
- **conic section:** Any curve formed by the intersection of a plane with a cone of two nappes.
- **directrix:** A line used to construct and define a conic section; a parabola has one directrix; ellipses and hyperbolas have two (plural : directrices)

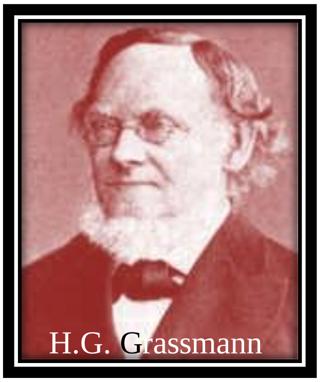
UNIT - II

VECTOR ALGEBRA - I

Chapter 2.1 VECTOR – INTRODUCTION


W.R. Hamilton

The development of the concept of vectors was influenced by the works of the German Mathematician H.G. Grassmann (1809 – 1877) and the Irish Mathematician W.R. Hamilton (1805 – 1865).


H.G. Grassmann

First we must know "What is a Vector?"

A vector has both magnitude and direction. We are using vector quantities daily in our lives.

Some applications of 'Vectors' in real life are mentioned below:

- To know the direction in which the force is attempting to move the body.
- To know, how the gravity exerts a force of attraction on a body to work.
- To calculate, the motion of a body which is confirmed to a plane.
- To describe the force acting on a body simultaneously in 3-D form.
- Vectors are used in Engineering where the force is much stronger than the structure will sustain, else it will be collapse.
- In various oscillators and wave propagations like sound propagation, vibration propagation, AC wave propagation.
- Vectors are used in 'Quantum Mechanics'.
- Velocity in a pipe (Like Fluid Mechanics) can be determined in terms of vector field.

It has a lot of applications along with calculus in physics, Engineering and Medicine, some of them are mentioned below.

- To calculate the volume of a parallelepiped, the scalar triple product is used.
- To find work done and torque in mechanics, the dot and cross products are respectively used.
- Curl and divergence of vectors are very much used in study of electromagnetism, hydrodynamics, blood flow, rocket launching and the path of a satellite.
- To calculate the distance between two aircrafts in the space and the angle between their paths, the dot and cross products are used.

- To install the solar panels by carefully considering the tilt of the roof and the direction of the Sun so that it generates more solar power, a simple application of scalar product of vectors is used.

Scalar and Vector

Definition:

A **scalar** is a quantity that is determined only by its magnitude.

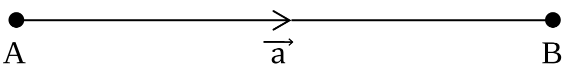
Eg.: distance, length, speed, temperature, voltage, mass, pressure etc.

A **Vector** is a quantity that is determined by both its magnitude and direction.

Eg.: force, displacement, velocity, acceleration, etc.

Representation of a Vector

Consider the diagram.



In line segment AB, arrow indicates the direction. The point A is called initial point or origin and the point B is called end point or terminal point. To denote a vector, we write the letter indicating its initial point and followed by letter indicating end point and put an arrow over two letters. As shown in the figure, the vector AB is simply denoted by

$$\overrightarrow{AB} = \vec{a}$$

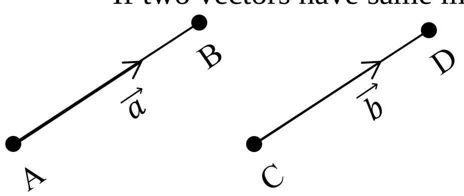
Modulus of a vector

The length or magnitude of the vector $\vec{a}$ is the length of the line segment AB. It is denoted by $|\vec{a}|$ or $|\overrightarrow{AB}|$

Different kinds of vectors

1. Equal vectors:

If two vectors have same magnitude and direction, then they are said to be equal vectors.



$$\overrightarrow{AB} = \overrightarrow{CD}$$

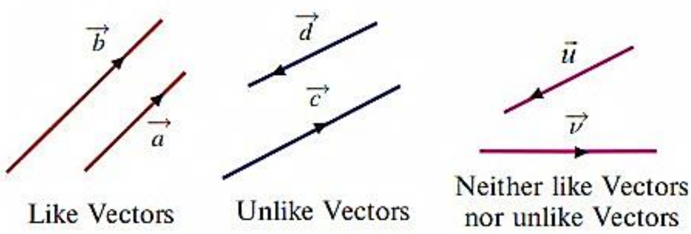
(or)

$$\vec{a} = \vec{b}$$

2. Like vectors and Unlike vectors:

Two vectors are said to be like vectors if they have the same direction.

Two vectors are said to be unlike vectors if they have opposite directions.



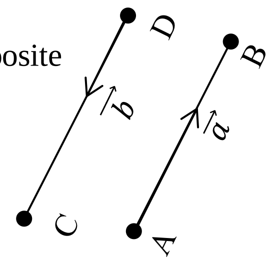
3. Collinear vectors:

Vectors which have the line of action parallel to one another or have the same line of action are called Collinear vectors.

Directions may be the same or opposite. [If $\vec{a}$ and $\vec{b}$ are collinear vectors then $\vec{b} = k\vec{a}$ where k is a scalar].

Eg. $\vec{a}$, 3 $\vec{a}$, - k $\vec{a}$ are all collinear vectors.

4. Negative vector:



If two vectors $\vec{a}$ and $\vec{b}$ have same magnitude but opposite direction, then each vector is negative vector of the other

(i.e.) $\vec{a} = -\vec{b}$ (or) $\vec{b} = -\vec{a}$

5. Unit vector:

A vector having unit magnitude is called unit vector. For any $\vec{a}$, unit vector in the direction of $\vec{a}$ is given by $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$|\vec{a}|$ is read as modulus of $\vec{a}$. It is nothing but magnitude of $\vec{a}$ and unit vector $\hat{a}$ is read as a cap'.

6. Zero vector (or) Void vector:

A vector of zero magnitude and arbitrary direction is called zero vector or null vector. It is denoted as $\vec{O}$. The initial point and terminal point of a null vector coincide. So, direction of null vector is indeterminate.

Properties of zero vector:

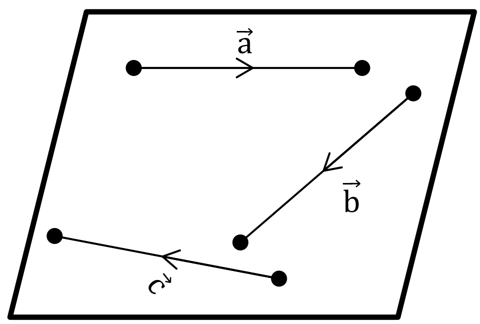
For any vector $\vec{a}$,

(i) $\vec{a} + \vec{O} = \vec{a}$ (ii) $\vec{a} - \vec{a} = \vec{O}$ (iii) $n\vec{O} = \vec{O}$

7. Proper vector:

Any vector of nonzero magnitude is called proper vector. (i.e.,) If $|\vec{a}| \neq 0$, then $\vec{a}$ is called as proper vector.

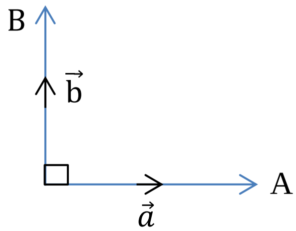
8. Co-planar vectors:



Vectors, lie in the same plane or parallel to the same, are called coplanar vectors. In figure $\vec{a}$, $\vec{b}$ & $\vec{c}$ are coplanar vectors.

If vectors lie in different planes they are called non-coplanar vectors.

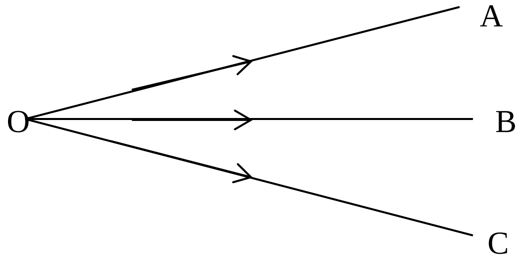
9. Orthogonal vectors:



If angle between two vectors is 90°, then the vectors are called orthogonal vectors.

Position vector:

The vector which specifies the position of a point with respect to some fixed point (like origin) is called Position Vector (P.V.).



In figure, 'O' is the fixed point and A, B & C are some points in space. Then the position vectors of A, B, C are $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

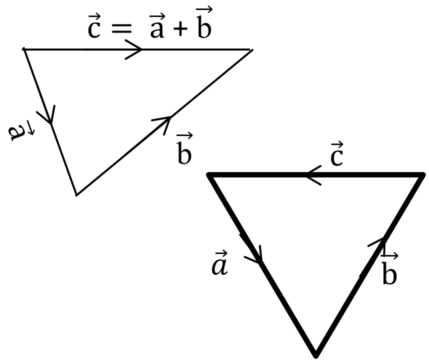
Addition of vectors:

Scalars can be added or subtracted by following the simple rules of algebra or arithmetic. But vectors do not follow the same rules, because while adding or subtracting vectors, their direction also has to be considered.

Let $\vec{a}$ and $\vec{b}$ be any two vectors. Draw $\vec{b}$ parallel to itself such that its initial point and the terminal point of $\vec{a}$ are same as shown in fig. Then the line segment draw from the initial point of $\vec{a}$ to terminal point of $\vec{b}$ represents addition of $\vec{a}$ and $\vec{b}$.

Triangle law of Addition

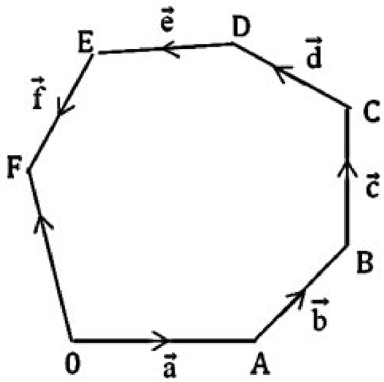
If two vectors are represented in magnitude and direction by the two sides of a triangle taken in order, then their sum is represented by the third side taken in the reverse order.



If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are the sides of a triangle taken in order then $\vec{a} + \vec{b} + \vec{c} = \vec{0}$.

Using the "Triangle Law of Addition", we can find the sum of any number of vectors.

When more vectors are represented both in magnitude and direction by the sides of a polygon taken in order, then the resultant is given by closing side of that polygon taken in the reverse order.



In fig, resultant $\overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF}$

Note: The above sum will be zero, if terminal point of the last vector coincides with the origin of first vector.

Scalar Multiplication of a Vector:

Let $\vec{a}$ be any vector and m be a scalar. Then $m\vec{a}$ is called the scalar multiplication of a vector $\vec{a}$ by the scalar m. If m is positive, then both $\vec{a}$ and $m\vec{a}$ have same direction. If m is negative, then $m\vec{a}$ and $\vec{a}$ have opposite direction.

Note: 1. Two vectors $\vec{a}$ & $\vec{b}$ are said to be parallel if $\vec{a} = m\,\vec{b}$ where m is a scalar.

2. If $m > 0$, they are in same direction. If $m < 0$, they are in opposite direction to each other.

Properties of vector:

For any two vectors $\vec{a}$ and $\vec{b}$.

1) Vector addition is commutative (i.e.) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

2) Vector addition is associative (i.e.) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

where m and n are scalars

3) $m(\vec{a} + \vec{b}) = m\,\vec{a} + m\,\vec{b}$

4) $m\,\vec{a} = \vec{a}\,m$

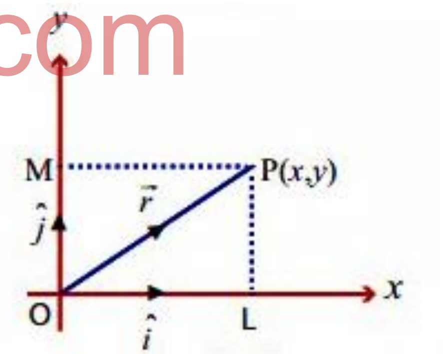
5) $m(n\vec{a}) = mn\vec{a} = n(m\vec{a})$

6) $(m + n)\,\vec{a} = m\,\vec{a} + n\,\vec{a}$

7) $m(\vec{a} + \vec{b}) = m\vec{a} + m\vec{b}$

Resolution of vectors in two dimensions:

It can be done for any finite dimension. But, we will discuss only in two and three dimensions.



Let P (x, y) be any point. Let L and M be the foots of the perpendiculars drawn from P to x and y axes. Then $\overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP} = \overrightarrow{OL} + \overrightarrow{OM}$

Since $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors, we have $\overrightarrow{OL} = x\hat{\imath}$ and $\overrightarrow{OM} = y\hat{\jmath}$.

Thus $\overrightarrow{OP} = \vec{r} = x\hat{\imath} + y\hat{\jmath}$.

To prove uniqueness, let $x_1\hat{\imath} + y_1\hat{\jmath}$ and $x_2\hat{\imath} + y_2\hat{\jmath}$ be two representations of same point P. Then $x_1\hat{\imath} + y_1\hat{\jmath} = x_2\hat{\imath} + y_2\hat{\jmath}$

$\Rightarrow (x_1 - x_2)\,\hat{\imath} - (y_2 - y_1)\,\hat{\jmath} = 0$

$\Rightarrow x_1 - x_2 = 0$ & $y_2 - y_1 = 0$

$\Rightarrow x_1 - x_2$ & $y_1 = y_2$

and hence uniqueness follows.

In ΔOLP, $OP^2 = OL^2 + LP^2$, hence $|\overrightarrow{OP}| = \sqrt{x^2 + y^2}$

(i.e.) $|\,\vec{r}\,| = r = \sqrt{x^2 + y^2}$

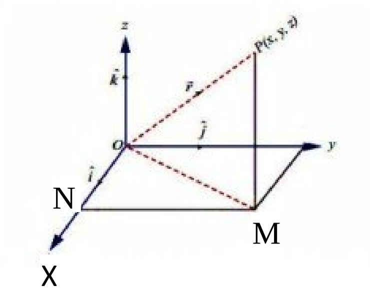
Resolution of vector in three dimensions:

OX, OY, OZ are three mutually perpendicular axes. Let $\vec{i}$, $\vec{j}$, $\vec{k}$ be the unit vectors in the direction of OX, OY, OZ respectively.

Let P (x, y, z) be any point in space. We can say $\overrightarrow{OP} = \vec{r}$

Draw PM perpendicular to the plane xoy and MN perpendicular to OX.

Then ON = x, NM = y and MP = z. By triangle law of addition,



$$\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP}$$

$$= \overrightarrow{ON} + \overrightarrow{NM} + \overrightarrow{MP} \quad (\therefore \overrightarrow{OM} = \overrightarrow{ON} + \overrightarrow{NM})$$

Thus $\overrightarrow{ON} = x\vec{i}$

$\overrightarrow{NM} = y\vec{j}$

$\overrightarrow{MP} = x\vec{k}$

$\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$

Now, $r^2 = OP^2 = OM^2 + MP^2$

$= ON^2 + NM^2 + MP^2$

$= x^2 + y^2 + z^2$

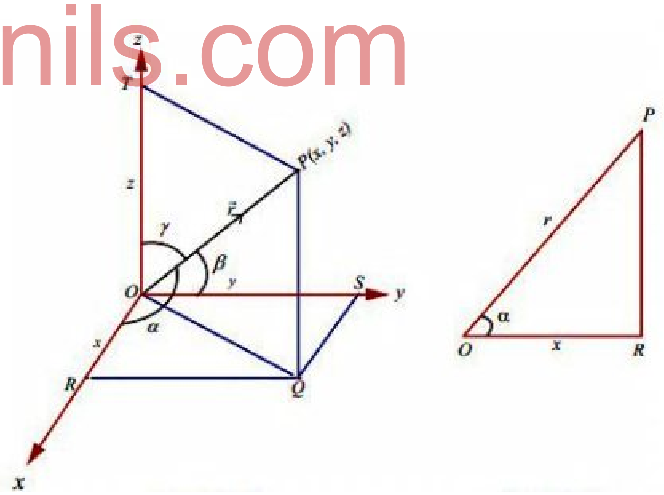
$$\therefore r = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore |\overrightarrow{OP}| = |\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

Direction Cosines and Direction Ratios:

Let P be any point in a space with co-ordinates (x, y, z) and of distance 'r' from the origin. Let R, S, T be foot of the perpendiculars drawn from P to x, y & z axes respectively.

Let α, β, γ be the angles made by the vector $\overrightarrow{OP}$ with positive x, y and z axes respectively.



That is

$\angle POR = \alpha$, $\angle POS = \beta$ & $\angle POT = \gamma$

In ΔOPR, $\angle PRO = 90^o$, $\angle POR = \alpha$, OR = x and OP = r.

$$\therefore \cos\alpha = \frac{OR}{OP} = \frac{x}{r}$$

In a similar way, we can find $\cos\beta = \frac{y}{r}$, $\cos\gamma = \frac{z}{r}$.

Here α, β, γ are the direction angles of $\overrightarrow{OP}$, and cos α, cos β, cos γ are called direction cosines of $\overrightarrow{OP}$.

Thus direction cosine of $\overrightarrow{OP} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$ where $r = \sqrt{x^2 + y^2 + z^2}$.

Any three numbers which are proportional to direction cosines of a vector are called Direction ratios.

From the result of direction cosines, we get

$$\frac{x}{\cos\alpha} = \frac{y}{\cos\beta} = \frac{z}{\cos\gamma} = r$$

$\Rightarrow$ x : y : z = cos α : cos β : cos γ.

$\therefore$ x : y : z is the direction ratios of the vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Result : For any vector, the sum of squares of direction cosines of $\vec{r}$ is 1.

i.e. $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

For example, l, m, n are the direction cosines of vector then $l^2 + m^2 + n^2 = 1$

Worked Examples

1. Find the modulus, direction cosines and direction ratios of $3\vec{i} + 2\vec{j} + 4\vec{k}$.

Solution:

Let $\vec{r} = 3\vec{i} + 2\vec{j} + 4\vec{k}$

Modulus of $\vec{r} = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$= \sqrt{3^2 + 2^2 + 4^2}$

$= \sqrt{29}$

Direction cosines are $\left(\frac{x}{|\vec{r}|}, \frac{y}{|\vec{r}|}, \frac{z}{|\vec{r}|}\right)$

$= \left(\frac{3}{\sqrt{29}}, \frac{2}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right)$

Direction ratios x : y : z = 3 : 2 : 4

2. Can a vector have direction angles 30°, 45°, 60°?

Solution:

The condition is $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Here $\alpha = 30^\circ, \beta = 45^\circ, \gamma = 60^\circ$

$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{3}{4} + \frac{1}{2} + \frac{1}{4} \neq 1$.

$\therefore$ They are not direction angles of any vector.

3. If the position vectors of the points A and B are $2\vec{i} + 3\vec{j} + 4\vec{k}$ and $3\vec{i} - 4\vec{j} + 5\vec{k}$. Find also modulus and direction cosines.

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (3\vec{i} - 4\vec{j} + 5\vec{k}) - (2\vec{i} + 3\vec{j} + 4\vec{k})$

$= \vec{i} - 7\vec{j} + \vec{k}$.

$|\overrightarrow{AB}| = \sqrt{1^2 + (-7)^2 + 1^2} = \sqrt{51}$

Direction cosines of $\overrightarrow{AB} = \left(\frac{1}{\sqrt{51}}, \frac{-7}{\sqrt{51}}, \frac{1}{\sqrt{51}}\right)$

4. Find the unit vector along with the vector $5\vec{i} - 3\vec{j} - \vec{k}$.

Solution:

Let $\vec{a} = 5\vec{i} - 3\vec{j} - \vec{k}$

Unit vector of $\vec{a} = \hat{a} = \frac{\vec{a}}{|\vec{a}|}$

$|\vec{a}| = \sqrt{5^2 + (-3)^2 + (-1)^2} = \sqrt{25 + 9 + 1} = \sqrt{35}$

$\therefore \hat{a} = \frac{5\vec{i} - 3\vec{j} - \vec{k}}{\sqrt{35}}$

5. Show that the points whose position vectors are $2\vec{i} - \vec{j} + 3\vec{k}$, $3\vec{i} - 5\vec{j} + \vec{k}$ and $-\vec{i} - 11\vec{j} + 9\vec{k}$ are collinear.

Solution:

Let $\overrightarrow{OA} = 2\vec{i} - \vec{j} + 3\vec{k}$

$\overrightarrow{OB} = 3\vec{i} - 5\vec{j} + \vec{k}$ and

$\overrightarrow{OC} = -\vec{i} + 11\vec{j} + 9\vec{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{i} - 4\vec{j} - 2\vec{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -4\vec{i} + 16\vec{j} + 8\vec{k}$

$= -4(\vec{i} - 4\vec{j} - 2\vec{k})$

$\overrightarrow{BC} = -4(\overrightarrow{AB})$ (or) $\overrightarrow{BC} = K\,\overrightarrow{AB}$, where $K = -4$.

$\therefore$ The points A, B and C are collinear.

6. If the vectors $\vec{i} + 2\vec{j} + \vec{k}$ and $-2\vec{i} + p\vec{j} - 2\vec{k}$ are collinear, find the value of 'p'.

Solution:

Given : $\vec{a} = \vec{i} + 2\vec{j} + \vec{k}$ and $\vec{b} = -2\vec{i} + p\vec{j} - 2\vec{k}$ and $\vec{a}$ and $\vec{b}$ are collinear.

Condition for collinear is

$\vec{a} = \lambda\,\vec{b}$ where λ is a scalar.

$\Rightarrow \vec{i} + 2\vec{j} + \vec{k} = \lambda(-2\vec{i} + p\vec{j} - 2\vec{k})$

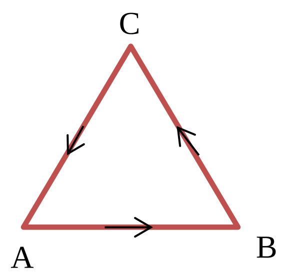
$= -2\lambda\vec{i} + p\lambda\vec{j} - 2\lambda\vec{k}$

$\Rightarrow 1 = -2\lambda$ and $2 = p\lambda$

$\Rightarrow \lambda = -\frac{1}{2}$ and $p = 2/\lambda$

$\Rightarrow p = \frac{2}{-1/2} = \frac{4}{-1} = -4$

7. Show that the points whose position vectors $2\vec{i} + 3\vec{j} + 4\vec{k}$, $3\vec{i} + 4\vec{j} + 2\vec{k}$ and $4\vec{i} + 2\vec{j} + 3\vec{k}$ form an equilateral triangle.



Solution:

Let $\overrightarrow{OA} = 2\vec{i} + 3\vec{j} + 4\vec{k}$

$\overrightarrow{OB} = 3\vec{i} + 4\vec{j} + 2\vec{k}$

$\overrightarrow{OC} = 4\vec{i} + 2\vec{j} + 3\vec{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{i} + \vec{j} - 2\vec{k}$

$AB = |\overrightarrow{AB}| = \sqrt{(1)^2 + (1)^2 + (-2)^2} = \sqrt{6}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \vec{i} - 2\vec{j} + \vec{k}$

$BC = |\overrightarrow{BC}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

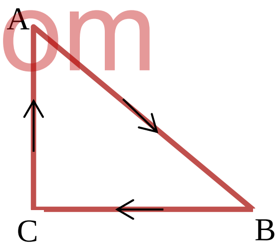
$= -2\vec{i} + \vec{j} + \vec{k}$

$AC = |\overrightarrow{CA}| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}$

Since AB = BC = AC = $\sqrt{6}$, the points A, B & C form an equilateral triangle.

8. Prove that the points whose position vectors $2\vec{i} + 4\vec{j} + 3\vec{k}$, $4\vec{i} + \vec{j} - 4\vec{k}$ and $6\vec{i} + 5\vec{j} - \vec{k}$ form right angled triangle.

Solution:



Let $\overrightarrow{OA} = 2\vec{i} + 4\vec{j} + 3\vec{k}$

$\overrightarrow{OB} = 4\vec{i} + \vec{j} - 4\vec{k}$

$\overrightarrow{OC} = 6\vec{i} + 5\vec{j} - \vec{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 2\vec{i} - 3\vec{j} - 7\vec{k}$

$AB = |\overrightarrow{AB}| = \sqrt{(2)^2 + (-3)^2 + (-7)^2} = \sqrt{62}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= 2\vec{i} + 4\vec{j} + 3\vec{k}$

$BC = |\overrightarrow{BC}| = \sqrt{2^2 + 4^2 + 3^2} = \sqrt{29}$

$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$= -4\vec{i} - \vec{j} + 4\vec{k}$

$CA = |\overrightarrow{CA}| = \sqrt{(-4)^2 + (-1)^2 + (4)^2} = \sqrt{33}$

By Pythagoras theorem,

$$hyp^2 = \text{op. side}^2 + \text{Adj. side}^2$$

$$\Rightarrow (\sqrt{62})^2 = (\sqrt{29})^2 + (\sqrt{33})^2$$

$$\Rightarrow 62 = 62$$

Hence, the point A, B & C form a right angled triangle.

Exercise: 2.1.1

1. State triangular law of addition of two vector.
2. Define: (i) unit vector (ii) null vector (iii) coplanar vector (iv) orthogonal vector.
3. Find the unit vector parallel to $2\vec{i} - \vec{j} + 4\vec{k}$.
4. If $5\vec{i} + 2\vec{j} + 7\vec{k}$ and $\vec{i} + \lambda\vec{j} + 3\vec{k}$ are in same direction, find 'λ'.
5. If $\vec{a} = \vec{i} - 2\vec{j} + \vec{k}$, $\vec{b} = 4\vec{i} - 3\vec{j} + \overrightarrow{5k}$ and $\vec{c} = -2\vec{i} + 3\vec{j} + 4\vec{k}$, find the magnitude and direction cosines of (i) $\vec{a} + \vec{b} + \vec{c}$ (ii) $|2\vec{a} - 5\vec{b} + 3\vec{c}|$
6. The position vectors of A, B, C and D, then find the value of

   (i) $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$ (ii) $\overrightarrow{OA} + \overrightarrow{AB} - \overrightarrow{CD} - \overrightarrow{DC}$
7. Is a vector have the following direction angles

   (i) $45^o, 90^o, 45^o$ (ii) $30^o, 60^o, 30^o$

8. Show that the following points are collinear.

   (i) $2\vec{i} + \vec{j} - \vec{k}$, $4\vec{i} + 3\vec{j} - 5\vec{k}$, $\vec{i} + \vec{k}$

   (ii) $\vec{i} + 2\vec{j} + 4\vec{k}$, $4\vec{i} + 8\vec{j} + \vec{k}$, $3\vec{i} + 6\vec{j} + 2\vec{k}$
9. Show that the points whose position vectors are $3\vec{i} - \vec{j} - 2\vec{k}$, $5\vec{i} + \vec{j} - 3\vec{k}$ and $6\vec{i} - \vec{j} - \vec{k}$ from an isosceles triangle.
10. Show that the following points form an equilateral triangle

    (i) $4\vec{i} + 5\vec{j} + 6\vec{k}$, $5\vec{i} + 6\vec{j} + 4\vec{k}$, $6\vec{i} + 4\vec{j} + 5\vec{k}$

    (ii) $2\vec{i} + 3\vec{j} + 4\vec{k}$, $3\vec{i} + 4\vec{j} + 2\vec{k}$, $4\vec{i} + 2\vec{j} + 3\vec{k}$
11. Prove that a right angled triangle is formed by the points whose position vectors are

    (i) $3\vec{i} + \vec{j} - 5\vec{k}$, $4\vec{i} + 3\vec{j} - 7\vec{k}$, $5\vec{i} + 2\vec{j} - 3\vec{k}$

    (ii) $3\vec{i} - \vec{j} + 6\vec{k}$, $5\vec{i} - 2\vec{j} + 7\vec{k}$, $6\vec{i} - 5\vec{j} + 2\vec{k}$
12. A carriage is pulled by four men. The components of the four forces F_1, F_2, F_3 and F_4 are

    $F_1 = (20\text{ N}, 25\text{ N})$

    $F_2 = (15\text{ N}, 5\text{ N})$

    $F_3 = (25\text{ N}, -5\text{ N})$

    $F_4 = (30\text{ N}, -15\text{ N})$. Calculate the Resultant force. Ans : (90 N, 10 N).

13. An aircraft is flying on a northerly course and its velocity relative to the air is V_1 = (okm/h, 300 km/h), calculate the velocity of the air craft relative to the ground for the following three different air velocities.

(a) V2 = (0, -50) km/h, head wind

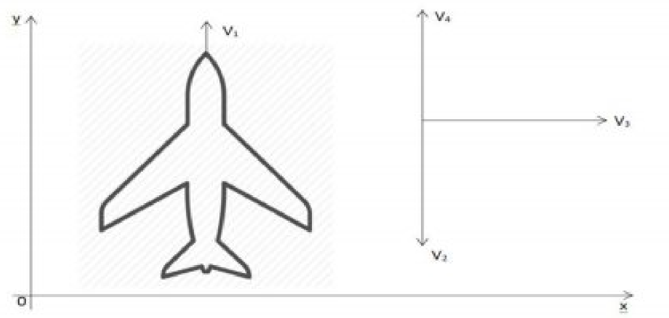
(b) V3 = (50, 0) km/h, crosswind

(c) V4 = (0, 50) km/h, tail wind

Also calculate the magnitude of the absolute velocity relative to the ground for the three cases.

(d) $| V_1 + V_2 |$ (e) $| V_1 + V_3 |$ (f) $| V_1 + V_4 |$

(Hint: Use resultant vector & magnitude)



Answer :

(a) (0, 250) km/h

(b) (50, 300) km/h

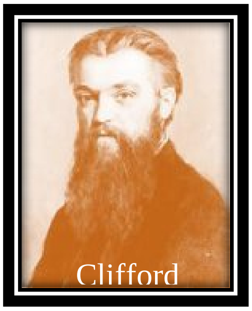
(c) (0, 350) km/h

(d) $| V_1 + V_2 |$ = 250 km/h

(e) $| V_1 + V_3 |$ = 304 km/h

(f) $| V_1 + V_4 |$ = 350 km/h

Chapter 2.2 PRODUCT OF TWO VECTORS



From the previous chapter, we are familiar with the concept of vectors. Great mathematicians Grassmann, Hamilton, Clifford and Gibbs were pioneers to introduce the dot and cross product of vectors.

Clifford (1845 – 1879), in his Elements of Dynamics (1878), broke down the product of two quaternions into two very different vector products, which he called the scalar product and the vector product. The term vectors was due to Hamilton and it was derived from the Latin word ‘to carry’. The theory of vector was also based on Grassman’s theory of extension.

In this chapter, we define two kinds of product of vectors.

1. Scalar product or Dot product
2. Vector product or Cross product.

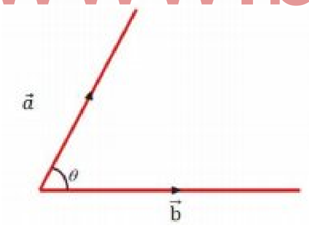
Scalar Product:

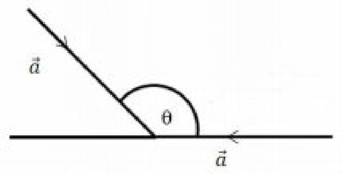
Dot product on two vectors is one of the most skills when developing computer game graphics. It is used to determine the work done on a moving object by a given force, used to find the distance between a point to an object. [Eg: distance between aircraft and a boat, earth and moon etc.] Also it is used to find projection of a point or line. To define such products, we need the angle between two vectors.

Angle between two vectors:

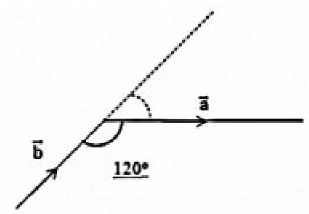
Let $\vec{a}$ and $\vec{b}$ be any two vectors.

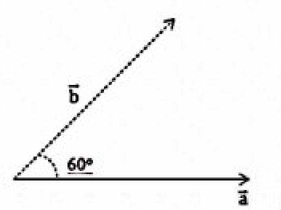
If directions of two vectors are either both converge or both diverge from their point of intersection.





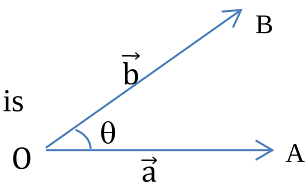
If vectors neither converge not diverge.





Here the angle between $\vec{a}$ & $\vec{b}$ is 60° but not 120°.

Definition:



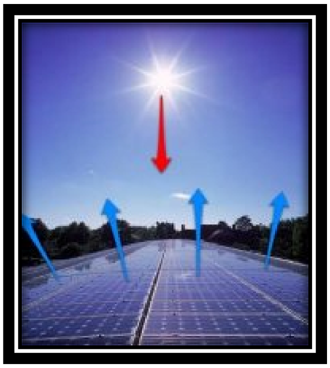
Let $\vec{a}$ & $\vec{b}$ be any two non-zero vectors and θ be the angle between the vectors, then scalar product (Dot product) is defined as

$$\vec{a}\,.\,\vec{b} = |\vec{a}|\;|\vec{b}|\cos\theta \text{ (or) } = a\,b\cos\theta$$

Note: Since the resultant of $\vec{a}\,.\,\vec{b}$ is a scalar, it is called scalar product. Further we use the symbol ‘dot’ (‘.’) and hence another name is dot product.

Example:

For the dot product: Energy Absorption



Consider the solar rays as one vector, the other where the solar panel is pointing (the normal vector).

1. Larger numbers mean stronger rays or a larger panel. How much energy is absorbed?

   Energy = Overlap in direction * Strength of rays * Size of panel

   $$\text{Energy} = \cos(\theta) \cdot |a| \cdot |b|$$

2. If you hold your panel sideways to the sun, no rays hit ($\cos(\theta) = 0$).

3. Solar rays are leaving the sun, and the panel is facing the sun, and the dot product is negative when vectors are opposed!.

Properties:

(i) Scalar product is commutative

$$\vec{a}.\vec{b} = \vec{b}.\vec{a}$$

(ii) Scalar product is positive or negative according to θ is acute or obtuse.

(iii) For any two non-zero vectors $\vec{a}$ & $\vec{b}$, $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a}$ is perpendicular to $\vec{b}$

(iv) $\vec{a}.\vec{a} = |\vec{a}|^2 = (\vec{a})^2 = \vec{a}^2 = a^2$

These representations are essential while solving problems.

(v) $\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1$ and

$$\vec{i}.\vec{j} = \vec{j}.\vec{k} = \vec{k}.\vec{i} = 0$$

(vi) For any two scalars λ and μ,

$$\lambda\,\vec{a}\,.\,\mu\,\vec{b} = \lambda\,\mu\,(\vec{a}.\vec{b}) = (\lambda\,\mu\,\vec{a}).\vec{b} = \vec{a}.(\lambda\,\mu\,\vec{b})$$

(vii) Scalar product is distributive over addition.

(i.e.) $\vec{a}.(\vec{b} + \vec{c}) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$ &

$$(\vec{a} + \vec{b}).\vec{c} = \vec{a}.\vec{c} + \vec{b}.\vec{c}$$

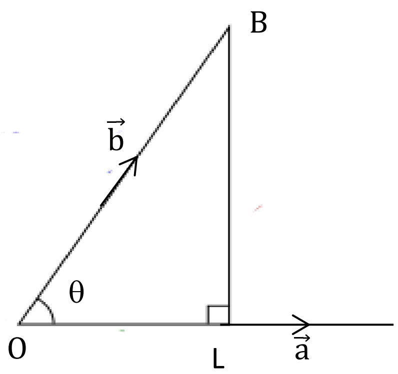
Similarly $\vec{a}.(\vec{b} - \vec{c}) = \vec{a}.\vec{b} - \vec{a}.\vec{c}$

(viii) If θ is the angle between the vectors $\vec{a}$ & $\vec{b}$ then

$$\text{Cos}\,\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} \quad \text{(or)}$$

$$\theta = \cos^{-1}\left[\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right]$$

(ix) **Geometrical meaning of scalar product:**



Let $\overline{OA} = \vec{a}$, $\overline{OB} = \vec{b}$ and θ be the angle between $\vec{a}$ & $\vec{b}$.

Draw BL $\perp$ OA.

From right angled triangle OLB, $\cos\theta = \frac{OL}{OB}$

$\Rightarrow$ $OL = OB\cos\theta = |\vec{b}|\cos\theta$

But OL is the projection of OB on OA ($\vec{b}$ on $\vec{a}$)

$\therefore \vec{a}\,.\,\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

$= |\vec{a}|\ OL$

$= |\vec{a}|$. projection of $\vec{b}$ on $\vec{a}$

$\Rightarrow$ projection of $\vec{b}$ and $\vec{a}$ $= \dfrac{\vec{a}\,.\,\vec{b}}{|\vec{a}|}$

(x) Working rule to find scalar product:

Let $\vec{a} = a, \vec{i} + a_2\vec{j} + a_3\vec{k}$ & $\vec{b}$ b $= \vec{i} + b_2\vec{j} + b_3\vec{k}$

$\vec{a}\,.\,\vec{b} = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k})\,.\,(b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$

$= a_1 b_1 \vec{i}\,.\,\vec{i} + a_1 b_2\vec{i}\,.\,\vec{j} + a_1b_3\vec{i}\,.\,\vec{k} +$

$a_2b_1\vec{j}\,.\,\vec{i} + a_2 b_2\vec{j}\,.\,\vec{j} + a_2b_3\vec{j}\,.\,\vec{k} +$

$a_3b_1\vec{k}\,.\,\vec{i} + a_3b_2\vec{k}\,.\,\vec{j} + a_3\,.\,b_3\vec{k}\,.\,\vec{k}$

$= a_1b_1 + a_2 b_2 + a_3 b_3$ (by property V)

Worked Examples

1) Find the scalar product of $\vec{i} + 2\vec{j} - 3\vec{k}$ and $-2\vec{j} + 4\vec{k}$

Solution:

Let $\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}$ & $\vec{b} = 0\vec{i} - 2\vec{j} + 4\vec{k}$

$\vec{a}\,.\,\vec{b} = (\vec{i} + 2\vec{j} - 3\vec{k})\,.\,(0\vec{i} - 2\vec{j} + 4\vec{k})$

$= 1(0) + 2(-2) - 3(4)$

$= 0 - 4 - 12 = -16.$

2) Prove that the vectors $3\vec{i} - \vec{j} + 5\vec{k}$ and $-6\vec{i} + 2\vec{j} + 4\vec{k}$ are perpendicular.

Solution:

Let $\vec{a} = 3\vec{i} - \vec{j} + 5\vec{k}$

$\vec{b} = -6\vec{i} + 2\vec{j} + 4\vec{k}$

$\vec{a}\,.\,\vec{b} = -18 - 2 + 20 = 0$

$\vec{a}\,.\,\vec{b} = 0$

$\therefore$ $\vec{a}$ & $\vec{b}$ are perpendicular.

3) Find 'p' if $2\vec{i} + p\vec{j} - \vec{k}$ and $3\vec{i} + 4\vec{j} + 2\vec{k}$ are perpendicular.

Solution:

Let $\vec{a} = 2\vec{i} + p\vec{j} - \vec{k}$

$\vec{b} = 3\vec{i} + 4\vec{j} + 2\vec{k}$

Given: $\vec{a}$ and $\vec{b}$ are perpendicular

$\therefore \vec{a} \cdot \vec{b} = 0$

Hence $\vec{a} \cdot \vec{b} = 6 + 4p - 2 = 0$

$\Rightarrow 4p = -4$

$p = -1$

4) Find the projection of $2\vec{i} + \vec{j}$ on $3\vec{i} - 4\vec{j} + \vec{k}$

Solution:

Let $\vec{a} = 2\vec{i} + \vec{j}$

$\vec{b} = 3\vec{i} - 4\vec{j} + \vec{k}$

$\vec{a} \cdot \vec{b} = 6 - 4 + 0 = 2$

$|\vec{b}| = \sqrt{9 + 16 + 1} = \sqrt{26}$

$\therefore$ Projection of $\vec{a}$ on $\vec{b} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \dfrac{2}{\sqrt{26}}$

5) Find the angle between $3\vec{i} + \vec{j} + 2\vec{k}$ and $2\vec{i} + 2\vec{j} + 4\vec{k}$

Solution:

Let $\vec{a} = 3\vec{i} + \vec{j} + 2\vec{k}$ &

$\vec{b} = 2\vec{i} + 2\vec{j} + 4\vec{k}$

$\vec{a} \cdot \vec{b} = 6 + 2 + 8 = 16$

$|\vec{a}| = \sqrt{9 + 1 + 4} = \sqrt{14}$

$|\vec{b}| = \sqrt{4 + 4 + 16} = \sqrt{24}$

$\text{Cos } \theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = \dfrac{16}{\sqrt{14} \cdot \sqrt{24}} = \dfrac{8}{\sqrt{84}}$

$\therefore \theta = \cos^{-1}\left(\dfrac{8}{\sqrt{84}}\right)$

6) Show that the vectors $2\vec{i} + 2\vec{j} + \vec{k}, \vec{i} + 2\vec{j} + 2\vec{k}$ and $2\vec{i} + \vec{j} - 2\vec{k}$ are mutually orthogonal.

Solution:

Let $\vec{a} = 2\vec{i} - 2\vec{j} + \vec{k}$

$\vec{b} = \vec{i} + 2\vec{j} + 2\vec{k}$ &

$\vec{c} = 2\vec{i} + \vec{j} + 2\vec{k}$

$\vec{a} \cdot \vec{b} = 2 - 4 + 2 = 0$

$\therefore$ $\vec{a}$ & $\vec{b}$ are perpendicular

$\vec{b} \cdot \vec{c} = 2 + 2 - 4 = 0$

$\therefore$ $\vec{b}$ & $\vec{c}$ are perpendicular

$\vec{a} \cdot \vec{c} = 4 - 2 - 2 = 0.$

$\therefore$ $\vec{a}$ & $\vec{c}$ are perpendicular

$\therefore$ $\vec{a}$, $\vec{b}$ & $\vec{c}$ are mutually orthogonal.

Exercise : 2.2.1

1) Define scalar product of two vectors.

2) What is the condition for two vector $\vec{a}$ and $\vec{b}$ to be perpendicular?

3) What is the projection of $\vec{b}$ on $\vec{a}$ and $\vec{a}$ on $\vec{b}$.

4) What is the value of

(i) $\vec{i} \cdot \vec{j}$ (ii) $\vec{i} \cdot \vec{k}$ (ii) $\vec{j} \cdot \vec{i}$

5) Find the scalar product of

(i) $3\vec{i} + 4\vec{j} + 5\vec{k}$ and $2\vec{i} - 3\vec{j} + 4\vec{k}$

(ii) $\vec{i} + \vec{j} + \vec{k}$ and $3\vec{i} - 3\vec{j} + 5\vec{k}$

(iii) $2\vec{i} - 3\vec{j}$ and $4\vec{j} + 7\vec{k}$

6) Prove the following vectors are perpendicular.

(i) $3\vec{i} - \vec{j} + 5\vec{k}$ and $-\vec{i} + 2\vec{j} + \vec{k}$

(ii) $3\vec{i} - \vec{j} + 5\vec{k}$ and $-6\vec{i} + 2\vec{j} + 4\vec{k}$

(iii) $8\vec{i} + 7\vec{j} - \vec{k}$ and $3\vec{i} - 3\vec{j} + 3\vec{k}$

7) If $|\vec{a}| = 2$, If $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 3$, find the angle between $\vec{a}$ and $\vec{b}$

8) If $\vec{a} = 3\vec{i} + 5\vec{j} - 9\vec{k}$, find $\vec{a} \cdot \vec{a}$ & $\vec{a} \cdot \vec{k}$

9) Find 'p' if the given set of vectors are perpendicular

(i) $p\vec{i} + 3\vec{j} + 2\vec{k}$ and $3\vec{i} + 2\vec{j} + 5\vec{k}$ (ii) $3\vec{i} + 4\vec{k}$ and $2\vec{i} - 4\vec{j} + 2p\vec{k}$

(iii) $2\vec{i} - 4\vec{j} + p\vec{k}$ and $4\vec{i} - 7\vec{j} + 6\vec{k}$

10) Find the projection of

(i) $\vec{i} + 3\vec{j} - 4\vec{k}$ on $2\vec{i} - 3\vec{j}$ (ii) $4\vec{j} + 6\vec{k}$ on $3\vec{i} + 6\vec{k}$

(iii) $\vec{i} + \vec{j} + \vec{k}$ on $5\vec{i} - 4\vec{j} + 3\vec{k}$

11) Find the angle between the following vectors

(i) $3\vec{i} - 2\vec{j} - 3\vec{k}$ and $-\vec{i} - \vec{j} - \vec{k}$ (ii) $4\vec{i} + 3\vec{j} - 5\vec{k}$ and $3\vec{i} + 2\vec{j} - \vec{k}$

12) Prove that the vectors $\vec{i} + 2\vec{j} + \vec{k}$, $\vec{i} + \vec{j} - 3\vec{k}$ and $7\vec{i} - 4\vec{j} + \vec{k}$ are perpendicular to each other.

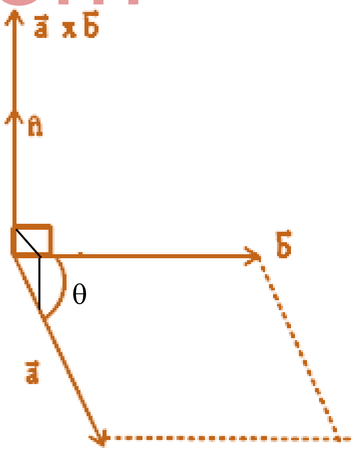
VECTOR PRODUCT (OR) CROSS PRODUCT

The force extended on a moving charge in a magnetic field determines the torque of an object. Some more uses are :

- to find out the curling up of the vector field.
- to find how much torque applied to a rotating system
- to find a vector which is perpendicular to the plane spanned by two vectors.

Also it has many applications when dealing with rotating bodies.

Definition:



Vector product of any two non-zero vectors $\vec{a}$ & $\vec{b}$ is written as $\vec{a} \times \vec{b}$ and is defined as $\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta\, \hat{n}$. Where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$, $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ & $\vec{b}$.

It is written as $\vec{a} \times \vec{b}$ (read as $\vec{a}$ cross $\vec{b}$). That is $\vec{a} \times \vec{b}$ is normal to the plane containing $\vec{a}$ and $\vec{b}$.

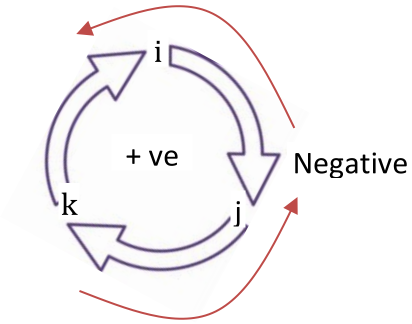
Note : 1. Here, the order of the vector is important to decide the direction of $\hat{n}$.

2. Since the resultant is a vector, this product is named as vector product. Also we use the symbol cross 'x' to define such a product, hence it has another name 'cross product'.

Properties:

(i) Vector product is non-commutative. (i.e.) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.

(ii) If two vectors are collinear or parallel iff $\vec{a} \times \vec{b} = 0$



(iii) $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$ &

$\vec{i} \times \vec{j} = \vec{k}, \quad \vec{j} \times \vec{k} = \vec{i}, \quad \vec{k} \times \vec{i} = \vec{j}$

$\vec{j} \times \vec{i} = -\vec{k}, \quad \vec{k} \times \vec{j} = -\vec{i}, \quad \vec{i} \times \vec{k} = -\vec{j}$

(iv) For any scalar λ & μ,

$$\lambda\vec{a} \times \mu\vec{b} = \lambda\mu\,(\vec{a} \times \vec{b}) = \vec{a} \times \lambda\mu\vec{b}$$

$$= \lambda\mu\,\vec{a} \times \vec{b} = \mu\vec{a} \times \lambda\vec{b}$$

(v) Vector product is distributive over addition and subtraction

(i.e.) $\vec{a} \times (\vec{b} \pm \vec{c}) = (\vec{a} \times \vec{b}) \pm (\vec{a} \times \vec{c})$ &

$(\vec{a} \pm \vec{b}) \times \vec{c} = (\vec{a} \times \vec{c}) \pm (\vec{b} \times \vec{c})$

$\vec{a} \times (\vec{b} + \vec{c} + \vec{d}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{a} \times \vec{d}$

(vi) Working rule:

$$\vec{a} \times \vec{b} = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \times (b_1\vec{i} + b_2\vec{j} + b_3\vec{k})$$

$$= (a_1b_1\vec{i} \times \vec{i} + a_1b_2\vec{i} \times \vec{j} + a_1b_3\vec{i} \times \vec{k}$$

$$+ a_2b_1\vec{j} \times \vec{i} + a_2b_2\vec{j} \times \vec{j} + a_2b_3\vec{j} \times \vec{k}$$

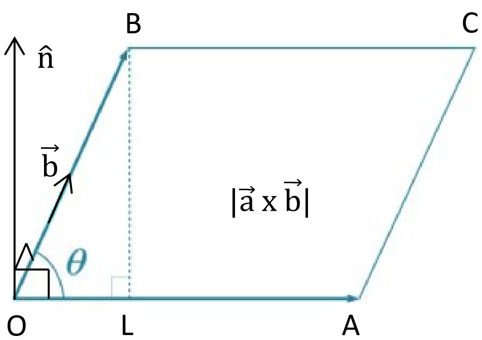
$$+ a_3b_1\vec{k} \times \vec{i} + a_3b_2\vec{k} \times \vec{j} + a_3b_3\vec{k} \times \vec{k}$$

$$= (a_2b_3 - a_3b_2)\vec{i} - (a_1b_3 - a_3b_1)\vec{j} + (a_1b_2 - a_2b_1)\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(vii) If θ is the angle between $\vec{a}$ and $\vec{b}$ then $\theta = \sin^{-1}\left[\frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}\right]$

(viii) The unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$



(ix) Geometrical meaning of cross product:

Let $\overline{OA} = \vec{a}$ and $\overline{OB} = \vec{b}$

Complete the parallelogram OABC, with adjacent sides $\vec{a}$ & $\vec{b}$

Let 'θ' be the angle between $\vec{a}$ & $\vec{b}$.

Draw BL $\perp$r OA

From right angled triangle OBL,

$$\text{Sin } \theta = \frac{BL}{OB}$$

$$\Rightarrow BL = OB \,.\, \text{Sin } \theta = |\vec{b}| \text{ Sin } \theta$$

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \text{ Sin } \theta \quad = OA \,.\, BL$$

= base & height

= Area of a parallelogram OACB

Note:

1. Vector area of triangle with adjacent sides $\vec{a}$ and $\vec{b} = \frac{1}{2}|\vec{a} \times \vec{b}|$
2. Condition for collinear of three given points A, B, C is $\overrightarrow{AB} \times \overrightarrow{BC} = 0$.
3. If $\overrightarrow{d_1}$ and $\overrightarrow{d_2}$ are the diagonals of the parallelogram then its area is $\frac{1}{2}|\overrightarrow{d_1} \times \overrightarrow{d_2}|$

Worked Examples

1) Find the vector product of $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$ and $\vec{b} = \vec{j} - 2\vec{k}$.

Solution:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 0 & 1 & -2 \end{vmatrix}$$

$$= \vec{i}\,(-6 + 1) - \vec{j}\,(-4 + 0) + \vec{k}\,(2 - 0)$$

$$= -5\vec{i} + 4\vec{j} + 2\vec{k}$$

2) Prove that $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = 2\,(\vec{b} \times \vec{a})$

Solution:

$$\text{L.H.S} = (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})$$

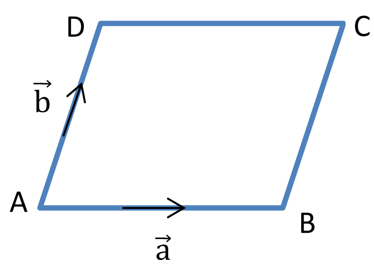
$$= \vec{a} \times \vec{a} - \vec{a} \times \vec{b} + \vec{b} \times \vec{a} - \vec{a} \times \vec{b}$$

$$= 0 + \vec{b} \times \vec{a} + \vec{b} \times \vec{a} - 0$$

$$= 2\,(\vec{b} \times \vec{a}) = \text{R.H.S.}$$

3) Find the area of a parallelogram whose adjacent sides are $\vec{i} + \vec{j} + \vec{k}$ and $3\vec{i} - \vec{k}$

Solution:



Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$\vec{b} = 3\vec{i} - \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 3 & 0 & -1 \end{vmatrix}$$

$$= \vec{i}\,(-1 - 0) - \vec{j}\,(-1 - 3) + \vec{k}\,(0 - 3)$$

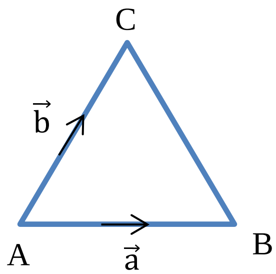
$\vec{a} \times \vec{b} = -\vec{i} + 4\vec{j} - 3\vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{1+16+9} = \sqrt{26}$

$\therefore$ Area of a parallelogram $= |\vec{a} \times \vec{b}| = \sqrt{26}$ sq. unit.

4) Find the area of the triangle whose adjacent sides are $\vec{i} + \vec{j} + \vec{k}$ and $\vec{i} + 2\vec{j} - 3\vec{k}$

Solution:



Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$\vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & 2 & -3 \end{vmatrix}$$

$= \vec{i}\ (-3 - 2) - \vec{j}\,(-3 - 1) + \vec{k}\ (2 - 1)$

$= 5\vec{i} + 4\vec{j} + \vec{k}$

$|\vec{a} \times \vec{b}| = \sqrt{25+16+1} = \sqrt{42}$

$\therefore$ Area of a triangle $= \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}\sqrt{42}$ sq. unit.

5) Prove that $(\vec{a} \times \vec{b})^2 + (\vec{a}\,.\,\vec{b})^2 = |\vec{a}|^2\ |\vec{b}|^2$

Solution:

L.H .S $= (\vec{a} + \vec{b})^2 \times (\vec{a}\,.\vec{b})^2$

$= [|\vec{a}|\,|\vec{b}|\sin\theta.\,\hat{n}]^2 + [|\vec{a}|\,|\vec{b}|\cos\theta]^2$

$= |\vec{a}|^2\ |\vec{b}|^2 \sin^2\theta. + |\vec{a}|^2\,|\vec{b}|^2\cos^2\theta\ [\because \hat{n}^2 = 1]$

$= |\vec{a}|^2\ |\vec{b}|^2\ [\sin^2\theta + \cos^2\theta]$ $\quad [\because \sin^2\theta + \cos^2\theta = 1]$

$= |\vec{a}|^2\ |\vec{b}|^2 =$ RHS.

6) What is the unit vector perpendicular to each of the vectors $2\vec{i} - \vec{j} + \vec{k}$ and $3\vec{i} + 4\vec{j} - \vec{k}$. Also calculate the sine of the angle between these two vectors.

Solution:

Let $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$ & $\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 3 & 4 & -1 \end{vmatrix}$$

$= \vec{i}\,(1 - 4) - \vec{j}\,(-2 - 3) + \vec{k}\ (8 + 3)$

$|\vec{a} \times \vec{b}| = \sqrt{9+25+121} = \sqrt{155}$

$|\vec{a}| = \sqrt{4+1+1} = \sqrt{6}$

$|\vec{b}| = \sqrt{9+16+1} = \sqrt{26}$

$$\text{Unit vector } \hat{n} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \frac{-3\vec{i} + 5\vec{j} + 11\vec{k}}{\sqrt{155}}$$

$$\sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|} = \frac{\sqrt{155}}{\sqrt{6}.\sqrt{26}} = \sqrt{\frac{155}{156}} \text{ (or)}$$

$$\theta = \sin^{-1}\sqrt{\frac{155}{156}}$$

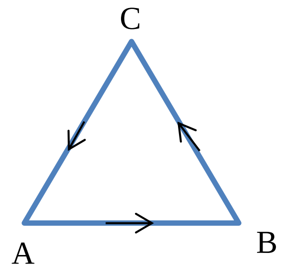
7) Find the area of the triangle formed by the points whose position vectors are $2\vec{i} + 3\vec{j} + 4\vec{k}$, $3\vec{i} + 4\vec{j} + 2\vec{k}$ and $4\vec{i} + 2\vec{j} + 3\vec{k}$.

Solution:

Let $\overrightarrow{OA} = 2\vec{i} + 3\vec{j} + 4\vec{k}$, $\overrightarrow{OB} = 3\vec{i} + 4\vec{j} + 2\vec{k}$ & $\overrightarrow{OC} = 4\vec{i} + 2\vec{j} + 3\vec{k}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{i} + \vec{j} - 2\vec{k}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \vec{i} - 2\vec{j} + \vec{k}$$



$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -2 \\ 1 & -2 & -1 \end{vmatrix}$$

$$= \vec{i}(1 - 4) - \vec{j}(1 + 2) + \vec{k}(-2 - 1)$$

$$= -3\vec{i} - 3\vec{j} - 3\vec{k}$$

$$|\overrightarrow{AB} \times \overrightarrow{BC}| = \sqrt{9 + 9 + 9} = \sqrt{27}$$

$$\therefore \text{Area of triangle} = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{BC}| = \frac{\sqrt{27}}{2} \text{ sq. unit.}$$

8) Prove that $\vec{i} - 2\vec{j} + 3\vec{k}$, $2\vec{i} + 3\vec{j} - 4\vec{k}$ and $-7\vec{j} + 10\vec{k}$ are collinear.

Solution:

$$\overrightarrow{OA} = \vec{i} - 2\vec{j} + 3\vec{k}$$

$$\overrightarrow{OB} = 2\vec{i} + 3\vec{j} - 4\vec{k} \text{ \& } \overrightarrow{OC} = -7\vec{j} + 10\vec{k}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{i} + 5\vec{j} - 7\vec{k}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -2\vec{i} - 10\vec{j} + 14\vec{k}$$

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 5 & -7 \\ -2 & -10 & 14 \end{vmatrix}$$

$$= \vec{i}(70 - 70) - \vec{j}(14 - 14) + \vec{k}(-10 + 10)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = 0$$

$$\overrightarrow{AB} \times \overrightarrow{BC} = 0$$

$\therefore$ A, B, C are collinear.

Exercise: 2.2.2

1. Define cross (or) vector product of two vectors.
2. What is the geometrical meaning of $\vec{a} \times \vec{b}$?
3. If $\vec{a}$ and $\vec{b}$ are adjacent sides of a parallelogram, what is the area of the parallelogram ?
4. What is the unit vector $\perp$ r to the plane of $\vec{a}$ and $\vec{b}$?
5. Find the vector product of

   i) $2\vec{i} + 3\vec{j} - 4\vec{k}$ and $\vec{i} + 2\vec{j} - \vec{k}$

   ii) $\vec{i} + \vec{j} + \vec{k}$ and $3\vec{i} + 5\vec{j} + 4\vec{k}$
6. Show that the vectors, $\vec{i} - 2\vec{j} + 4\vec{k}$ and $3\vec{i} - 6\vec{j} + 12\vec{k}$ are parallel.
7. If the vectors $\vec{a} = 2\vec{i} - 3\vec{j}$ and $\vec{b} = 6\vec{i} + m$ are collinear, find values of m.
8. If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $|\vec{a} \times \vec{b}| = 7$, find the angle between $\vec{a}$ and $\vec{b}$.
9. Find the area of parallelogram whose adjacent sides are

   i) $2\vec{i} - 3\vec{j}$ and $\vec{i} + 2\vec{j} - 3\vec{k}$

   ii) $\vec{i} + \vec{j} + \vec{k}$ and $3\vec{i} - \vec{k}$

   iii) $\vec{i} + \vec{j} - 2\vec{k}$ and $2\vec{i} - \vec{j} + \vec{k}$
10. Find the area as the parallelogram whose diagonals are represented by $3\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{i} - 3\vec{j} + 4\vec{k}$.
11. Find the area of the triangle two of whose sides are

    i) $-\vec{i} + 2\vec{j} + 4\vec{k}$ and $\vec{i} - \vec{j} - \vec{k}$

    ii) $2\vec{i} - \vec{j} + 2\vec{k}$ and $10\vec{i} - 2\vec{j} + \vec{k}$

    iii) $3\vec{i} + 4\vec{j} + \vec{k}$ and $3\vec{i} - \vec{j} - \vec{k}$
12. Find the unit vector perpendicular to the plane of the vectors

    i) $2\vec{i} + 3\vec{j} - \vec{k}$ and $3\vec{i} + 2\vec{j} + 3\vec{k}$

    ii) $\vec{i} - \vec{j} + \vec{k}$ and $\vec{i} + 2\vec{j} + 3\vec{k}$
13. Find the area of triangle whose vertices are

    i) $\vec{i} + 2\vec{j} - \vec{k}$, $2\vec{i} + 3\vec{k}$, $3\vec{i} + 2\vec{j} + \vec{k}$

    ii) $3\vec{i} + 2\vec{j} - \vec{k}$, $2\vec{i} - 3\vec{j} + \vec{k}$, $5\vec{i} + \vec{j} + 3\vec{k}$

    iii) (3, 1, 2), (1, -1, -3) and (4, -3, 1)

14. Find the unit vector perpendicular to the plane of the vectors also find the sine of the angle between the vectors.

i) $2\vec{i} - \vec{j} + 2\vec{k}$ and $10\vec{i} - 2\vec{j} + 7\vec{k}$

ii) $-\vec{i} + \vec{j} + \vec{k}$ and $-4\vec{i} + 3\vec{j} + 2\vec{k}$

iii) $\vec{i} + 2\vec{j} + \vec{k}$ and $3\vec{i} + 4\vec{j} + 5\vec{k}$

15. Show that the following points are collinear

i) $2\vec{i} + \vec{j} + \vec{k}$, $4\vec{i} + 3\vec{j} - 5\vec{k}$, $6\vec{i} + 5\vec{j} - 9\vec{k}$

ii) $\vec{i} + 2\vec{j} + 4\vec{k}$, $4\vec{i} + 8\vec{j} + \vec{k}$, $3\vec{i} + 6\vec{j} + 2\vec{k}$

iii) $2\vec{i} - \vec{j} + 3\vec{k}$, $3\vec{i} - 5\vec{j} + \vec{k}$, $-\vec{i} + 11\vec{j} + 9\vec{k}$

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Chapter 2.3 APPLICATION OF SCALAR AND VECTOR PRODUCT

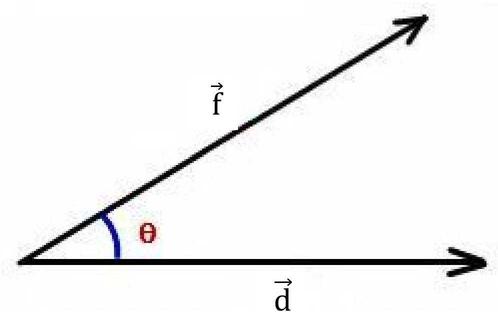
Work done:

The work done may be positive (or) zero (or) negative will be depending on the angle between the force and the displacement.

Positive work:

If a force has an acute angle, then the work done is positive.

For example, motion of ball, falling towards ground where displacement of ball is in the direction of force of gravity.



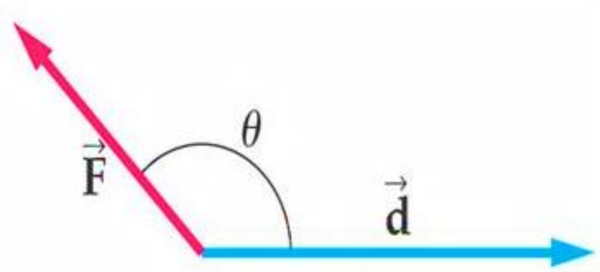
$\theta < 90^o$

$W > 0$

Negative work:

If the force has an obtuse angle, then the work done is said to be an negative.

For example, a ball is thrown in upwards direction, its displacement would be in upwards direction, but the force due to earth's gravity is in the downward direction.



$\theta > 90^o$

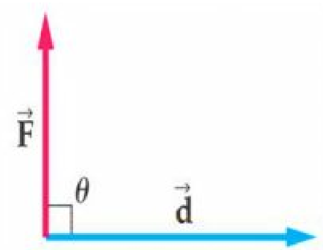
$W < 0$

Zero work:

If the directions of force and the displacement are perpendicular to each other, the work done by the force on the object is zero.

For example, when we push hard against a wall, the force we are exerting on the wall does no work because. In this case the displacement of the wall is d = 0.

However, in this process, our muscles are using our internal energy and as a result we get tired.

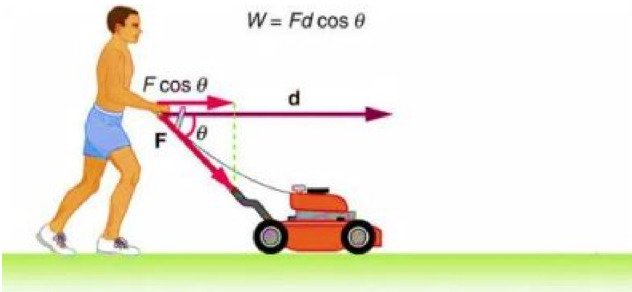


$\theta = 90^o$

$W = 0$

Definition: Work done

If $\vec{d}$ is the displacement vector of a particle moved from a point to another point after applying a constant force $\vec{f}$ on the particle, then the work done by th force on the particle is $w = \vec{f}\,.\,\vec{d}$



Note:

- While heating a gas, the work done (gas expand) is a positive work.
- Working of piston is an example of negative work

Work done in our daily life

1) Any work doing by a person like moving a table, push / pull the door, lifting a thing, throwing a ball are some examples of work done (i.e.) displacement of a particle moved from one point to another point. But trying to move is not a work done. For example: A boy try to lift a rock is not work done.
2) Work done is using in designing works for Civil, Electrical, Chemical Engineering fields likes bridges, dams, roads and railways, electrical circuits (current, voltage and resistance) oil refining, man-made fibres and products.

Worked Examples

Part - A

1. Find the work done by the force $2\vec{i} - 5\vec{j} + \vec{k}$ when the displacement is $3\vec{i} + 2\vec{j} - 3\vec{k}$.

Solution:

Let, force $= \vec{f} = 2\vec{i} - 5\vec{j} + \vec{k}$

displacement $= \vec{d} = 3\vec{i} + 2\vec{j} - 3\vec{k}$

$\therefore$ work done $= \vec{f}\,.\,\vec{d}$

$= (2\vec{i} - 5\vec{j} + \vec{k})\,.\,(3\vec{i} + 2\vec{j} - 3\vec{k})$

$= (2)(3) + (-5)(2) + (1)(-3)$

$= 6 - 10 - 3$

$= -7$

$= 7$ units

Part – B

2. Find the work done by the force $\vec{i} + 3\vec{j} - \vec{k}$ when it displaces a particle from the point (1, -2, 5) to (3, 4, 6).

Solution:

Let $\vec{f} = \vec{i} + 3\vec{j} - \vec{k}$

$\overrightarrow{OA} = \vec{i} - 2\vec{j} + 5\vec{k}$

$\overrightarrow{OB} = 3\vec{i} + 4\vec{j} + 6\vec{k}$

Displacement $\vec{d} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (3\vec{i} + 4\vec{j} + 6\vec{k}) - (\vec{i} - 2\vec{j} + 5\vec{k})$

$= 3\vec{i} + 4\vec{j} + 6\vec{k} - \vec{i} + 2\vec{j} - 5\vec{k})$

$\vec{d} = 2\vec{i} + 6\vec{j} + \vec{k}$

Work done $= \vec{f} . \vec{d}$

$= (\vec{i} + 3\vec{j} - \vec{k}) . (2\vec{i} + 6\vec{j} + \vec{k})$

$= (1)(2) + (3)(6) + (-1)(1)$

$= 19$ units

Part – C

1. A particle acted upon by constant forces $2\vec{i} + 5\vec{j} + 6\vec{k}$ and $-\vec{i} - 2\vec{j} - \vec{k}$ is displaced from the point (4, -3, -2) to the point (6, 1, -3). Find the total work done by the force.

Solution:

Given forces, $\vec{f}_1 = 2\vec{i} + 5\vec{j} + 6\vec{k}$

$\vec{f}_2 = -\vec{i} - 2\vec{j} - \vec{k}$

$\therefore \vec{f} = \vec{f}_1 + \vec{f}_2$

$= (2\vec{i} + 5\vec{j} + 6\vec{k}) + (-\vec{i} - 2\vec{j} - \vec{k})$

$= 1\vec{i} + 3\vec{j} + 5\vec{k}$

and $\overrightarrow{OA} = 4\vec{i} - 3\vec{j} - 2\vec{k}$

$\overrightarrow{OB} = 6\vec{i} + 1\vec{j} - 3\vec{k}$

$\therefore$ displacement $= \vec{d} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (6\vec{i} + 1\vec{j} - 3\vec{k}) - (4\vec{i} - 3\vec{j} - 2\vec{k})$

$= 2\vec{i} + 4\vec{j} - \vec{k}$

$\therefore$ work done $= \vec{f} . \vec{d}$

$= (\vec{i} + 3\vec{j} + 5\vec{k}) . (2\vec{i} + 4\vec{j} - 4\vec{k})$

$= (1)(2) + (3)(4) + (5)(-1)$

$= 9$ units.

2. A conveyor belt generates a force $F = 5\hat{i} - 3\hat{j} + \hat{k}$ that moves a suitcase from point (1, 1, 1) to point (9, 4, 7) along a straight line. Find the work done by the conveyor belt. The distance is measured in meters and the force is measured in newtons.

Solution

The displacement vector $\overrightarrow{PQ}$ has initial point (1,1,1) and terminal point (9, 4, 7):

$$\overrightarrow{PQ} = (9-1, 4-1, 7-1) = (8, 3, 6) = 8\hat{i} + 3\hat{j} + 6\hat{k}.$$

Work is the dot product of force and displacement:

$$W = \vec{F}.\overrightarrow{PQ}$$

$$W = \vec{F}\,.\,\overrightarrow{PQ} = (5\vec{i} - 3\vec{j} + \vec{k})\,.\,(8\vec{i} + 3\vec{j} + 6\vec{k}) = 5\,(8) + (-3)\,(3) + (1)\,(6)$$

$$= 37 \text{ J}$$

3. A constant force of 30 lb is applied at an angle of 60° to pull a handcart 10 ft across the ground. What is the work done by this force?

Solution:

Force : 30 lb, d = 10 ft, Q = 60°

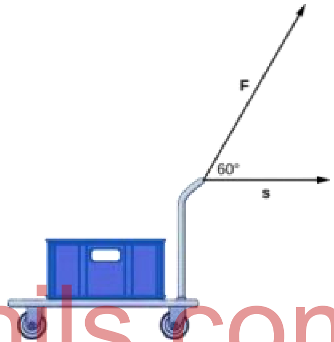
Work done = Force x displacement

= Fd cos θ

= 30 x 10 x cos 60

= $30 \times 10 \times \frac{1}{2}$

= 150 ft . lb



4. Suppose a child is pulling a wagon with a force having a magnitude of 8 lb on the handle at an angle of 55°. If the child pulls the wagon 50 ft, find the work done by the force

Solution:

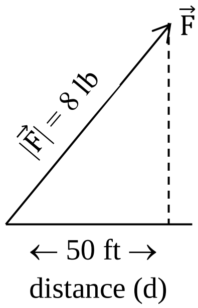
Work done $= \vec{F}\ .\ \vec{d}$

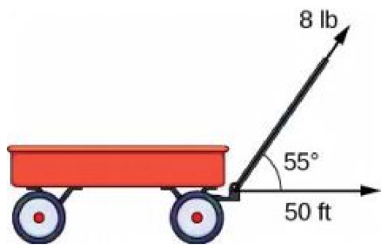
$= |\vec{F}|\,|\vec{d}|\cos\theta$

$= 8 \times 50 \times \cos 55°$

$= 229.43$

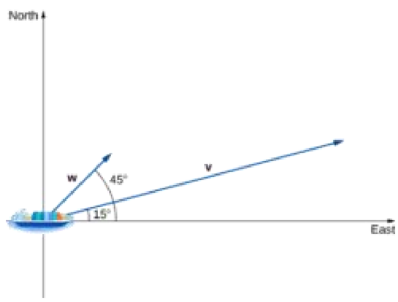
$\simeq$ 229 ft lb.





5. **Application Problems for Projection**

A container ship leaves port traveling 15° north of east. Its engine generates a speed of 20 knots along that path (see the following figure). In addition, the ocean current moves the ship northeast at a speed of 2 knots. Considering both the engine and the current, how fast is the ship moving in the direction 15° north of east? Round the answer to two decimal places.



Solution:

Let $\vec{v}$ be the velocity vector generated by the engine, and let w be the velocity vector of the current. We already know $|\vec{V}| = 20$ along the desired route.

$$\text{Projection of } \vec{w} \text{ on } \vec{v} = \frac{\vec{V} \cdot \vec{w}}{|\vec{V}|}$$

$$= \frac{|\vec{v}|\,|\vec{w}|\cos(30^\circ)}{|\vec{V}|} = |\vec{w}|\cos(30^\circ)$$

$$= 2\,\frac{\sqrt{3}}{2} = \sqrt{3} \approx 1.73 \text{ knots.}$$

The ship is moving at 21.73 knots in the direction 15° north of east.

Exercise: 2.3.1

1. State the formula to find work done by the force $\vec{f}$ in displacing the particle from the point A to B.
2. Find the work done by the force $\vec{i} - 7\vec{j} + 2\vec{k}$ when the displacement is $3\vec{i} - 5\vec{j} - 4\vec{k}$.
3. A particle acted on by constant forces $8\vec{i} + 2\vec{j} - 6\vec{k}$ and $6\vec{i} + 2\vec{j} - 2\vec{k}$ is displaced from the point (1, 2, 3) to the point (5, 4, 1). Find the total work done by the forces.
4. A particle is displaced from the point $5\vec{i} - 5\vec{j} - 7\vec{k}$ to the point $6\vec{i} + 2\vec{j} - 2\vec{k}$ under the action of constant forces $10\vec{i} - \vec{j} + 11\vec{k}$, $4\vec{i} + 5\vec{j} + 6\vec{k}$ and $-2\vec{i} + \vec{j} - 9\vec{k}$ calculate the total work done by the force.
5. Forces of magnitude $5\sqrt{2}$ and $10\sqrt{2}$ units acting in the directions $3\vec{i} + 4\vec{j} + 5\vec{k}$ and $10\vec{i} + 6\vec{j} - 8\vec{k}$ respectively, act on a particle which is displaced from the point with position vector $4\vec{i} - 3\vec{j} - 2\vec{k}$ to the point with position vector $6\vec{i} + \vec{j} - 3\vec{k}$. Find the work done by the forces.
6. A particle is acted upon by the forces $3\vec{i} - 2\vec{j} + 2\vec{k}$ and $2\vec{i} + \vec{j} - \vec{k}$ is displaced from the point (1, 3, -1) to the point (4, -1, 2m). If the work done by the forces is 14 units. Find the value of 'm'.

   Note:

   In fluid mechanics on the topic impact of jets, we are using the formula work done = (force) (distance) for finding the efficiency of moving vane velocity.
7. A jet of water 0.25 m diameter is discharging under a constant head of 53m. Find the force exerted by the jet on a plate which is moving with a velocity 12m/sec. in the direction of jet. Take $C_1 = 0.93$.
8. A jet of water 80 mm diameter move with a velocity of 15 m/sec and strikes a series of vanes moving with a velocity of 10 m/s. Find (i) force exerted by the jet (ii) work done by the jet/s (iii) efficiency of the jet.

Moment Vs Torque:

Moment is the perpendicular distance between the point of rotation and the forces line of action.

Moment is a static force and is used in non-rotational events.

Torque is a measure of the turning forces of an object.

Torque is a movement force and is used there is rotation and a pivot.

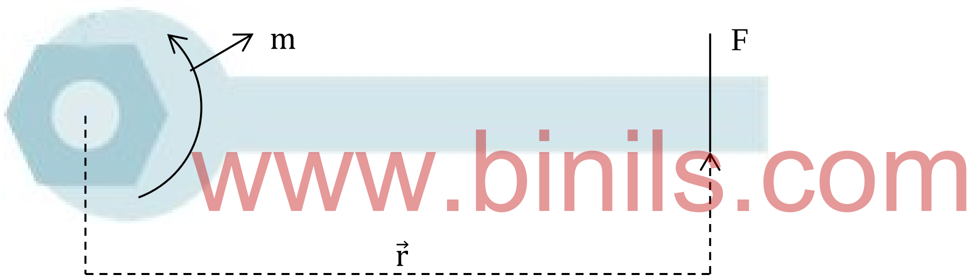
Definition:

The moment of a force is the turning effect about a pivot point. To develop a moment, the force must act upon the body to attempt to rotate it. A moment is can occur when forces are equal and opposite but not directly in line with each other.

The moment of a force acting about a point (or) axis is found by multiplying the force (F) by the perpendicular distance from the axis (r), called the lever arm.

$\therefore$ moment = perpendicular distance x force

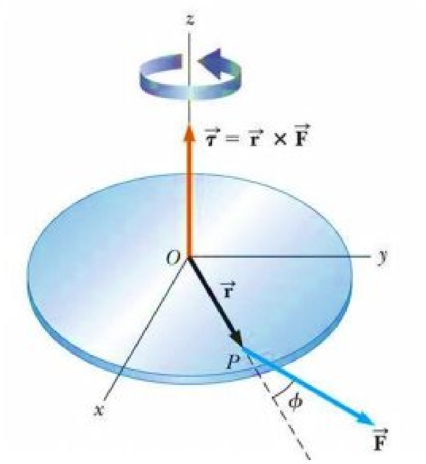
$= \vec{r} \times \vec{F}$



Torque in everyday life:

- The force in the piston applied on the crankshaft make the wheels to turn.
- Twisting force that tends to cause rotation induces torque.
- Push a key and rotating it, to open (or) close the door.
- A few objects which experience Torque are see – saw, wrenches, vacuum cleaner , computer printers, dish washers etc.
- Twist a bottle lid or tap, to open (or) close.
- When you spin a top by pulling on the thread in a swift motion.
- To find mass of a rigid body.

Definition of Moment or Torque:



Let O be any point and $\vec{r}$ be the position vector relative to the point O of any point P on the line of action of the force $\vec{F}$.

The moment of the force = $\vec{r} \times \vec{F}$.

The magnitude of the moment is $|\vec{r} \times \vec{F}|$.

The moment of the force is also called as Torque of the force.

Worked Examples

Example 1:

Find the magnitude of the torque about the point (2, 0, -1) of a force $2\vec{i} + \vec{j} - \vec{k}$, whose line of action passes through the origin.

Solution:

Let A be the point (2, 0, -1)

Position Vector of A = $\overrightarrow{OA} = 2\vec{i} - \vec{k}$,

Position Vector of P = $\overrightarrow{OP} = 0\vec{i} + 0\vec{j} + 0\vec{k}$

force $\vec{F} = 2\vec{i} + \vec{j} - \vec{k}$,

$\therefore \vec{r} = \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = 2\vec{i} + \vec{k}$

Torque of the force = $\vec{r} \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= -\vec{i} - 2\vec{k}$$

$\therefore$ Magnitude of the Torque $= |\vec{r} \times \vec{F}|$

$$= \sqrt{(-1)^2 + (-2)^2}$$

$$= \sqrt{5}$$

Examples 2:

Find the magnitude of the moment about the point $\vec{i} + 2\vec{j} - \vec{k}$ of a force represented by $3\vec{i} + \vec{k}$ acting through the point $2\vec{i} + \vec{j} - 3\vec{k}$.

Solution:

Given $\vec{F} = 3\vec{i} + \vec{k}$

P.V. of P = $\overrightarrow{OP} = 2\vec{i} - \vec{j} - 3\vec{k}$

P.V. of A $= \overrightarrow{OA} = \vec{i} + 2\vec{j} - \vec{k}$

$\therefore \vec{r} = \overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$= (2\vec{i} - \vec{j} - 3\vec{k}) - (\vec{i} + 2\vec{j} - \vec{k})$

$= \vec{i} - 3\vec{j} - 2\vec{k}$

$\therefore$ Moment of the force $= \vec{r} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & -2 \\ 3 & 0 & 1 \end{vmatrix}$

$= -3\vec{i} - 7\vec{j} + 9\vec{k}$

$\therefore$ Magnitude of a moment $= |\vec{r} \times \vec{F}|$

$= \sqrt{(-3)^2 + (-7)^2 + (9)^2}$

$= \sqrt{139}$ units.

Example 3:

If $\vec{F} = 2\vec{i} - 3\vec{j} + \vec{k}$ and $\vec{r} = \vec{i} + 2\vec{j} + 4\vec{k}$, find torque.

Solution:

Given $\vec{F} = 2\vec{i} - 3\vec{j} + \vec{k}$

$\vec{r} = \vec{i} + 2\vec{j} + 4\vec{k}$

Torque of the force $= \vec{r} \times \vec{F}$

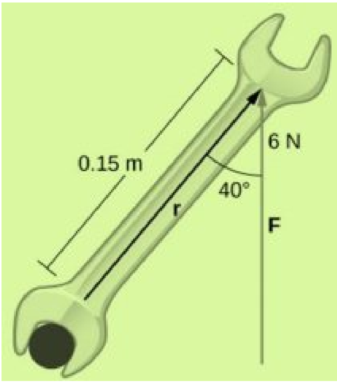
$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 4 \\ 2 & -3 & 1 \end{vmatrix}$

$= 14\vec{i} + 7\vec{j} - 7\vec{k}$

Example 4 :

Application of Torque:

A bolt is tightened by applying a force of 6 N to a 0.15-m wrench. The angle between the wrench and the force vector is 40°. Find the magnitude of the torque about the center of the bolt. Round the answer to two decimal places.



Torque describes the twisting action of the wrench

Solution:

Substitute the given information into the equation defining torque:

$\vec{\tau} = |\vec{r} \times \vec{F}|$

$= |\vec{r}|\,|\vec{F}| \sin\theta$

$= (0.15\text{ m})(6\text{N}) \sin 40°$

≈ 0.58 N·m.

Example 5:

Calculate the force required to produce 15 N·m torque at an angle of 30° from a 150-cm rod.

Solution:

Given: Torque of the force 15 N. m

$\theta = 30$

Calculation: Required force Torque $= |\vec{r} \times \vec{F}|$

$$15 \text{ N. m} = |\vec{r}|\ |\vec{F}|\ \sin\theta$$

$$15 \text{ N. m} = 1.5 \times F \times \frac{1}{2}$$

$$30 = 1.5\, F$$

$$F = \frac{30}{1.5} = 20$$

$$\boxed{F = 20 \text{ N}}$$

Note:

1) Moment (or) Torque helps in strength of material related calculations [(i.e.) Power, Diameter, Shear stress, etc.)]
2) In Civil Engineering, Torque is used to calculate the strength of the beam (or) bending moment.

Exercise: 2.3.2

1. Find the magnitude of the moment of a force represented by $3\vec{i} + 4\vec{j} - 5\vec{k}$ about the point with position vector $2\vec{i} - 3\vec{j} + 4\vec{k}$ acting through a point whose position vector is $4\vec{i} + 2\vec{j} - 3\vec{k}$.
2. Find torque of the resultant of the forces represented by $-3\vec{i} + 6\vec{j} - 3\vec{k}$, $4\vec{i} - 10\vec{j} + 12\vec{k}$ and $4\vec{i} + 7\vec{j}$ acting at the point with position vector $8\vec{i} - 6\vec{j} - 4\vec{k}$ about the point with position vector $18\vec{i} + 3\vec{j} - 9\vec{k}$.
3. Find the torque about the point (4, 3, 2) of the force represented by $\vec{i} + 2\vec{j} - \vec{k}$ acting through the point (0, 1, -1).
4. Find the magnitude and direction cosine of the moment about the point (1, -2, 3) of the force $2\vec{i} + 3\vec{j} + 6\vec{k}$ whose line of action passes through origin.
5. Find the Torque of the force $3\vec{i} + 4\vec{j} + \vec{k}$ acting through the point $\vec{i} - 2\vec{j} + 3\vec{k}$ about the point $4\vec{i} - 3\vec{j} + \vec{k}$.
6. Calculate the power transmitted by a shaft of 100 mm diameter running at 250 rpm, if exceed 75 N /mm^2.
7. Determine the diameter of a solid shaft which will transmit 60 KW at 150 rpm. The maximum torque likely to exceed the mean torque by 30% for the maximum shear stress limited to 60 N/mm^2. Take C = 0.8 x 10^5 N / mm^2.

8. A cantilever of span 4 metre with right end fixed carries an uniformly distributed load of 3 k N/m throughout its length. Determine the positions and magnitudes of the maximum shear force and maximum bending moment in the cantilever. Draw the S.F. and B.M. diagrams.

9. A horizontal beam of 12 m length, simply supported at its ends, is carrying vertical concentrated loads of magnitude 10 KN, 20 KN and 25 KN at distances 3 m, 7 m and 10 m respectively from the left support. Draw the shear force and bending moment diagrams indicating the values at salient points.

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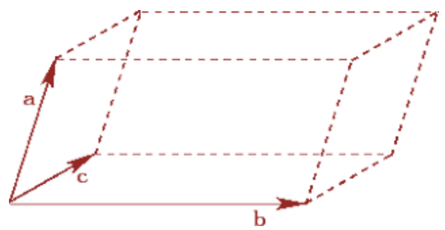
UNIT - III   VECTOR ALGEBRA - II

Chapter 3.1 PRODUCT OF THREE AND FOUR VECTORS

Definition: Scalar Triple Product

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be any three vectors. The scalar Triple product is defined by scalar product of $\vec{a} \times \vec{b}$ with $\vec{c}$.

It is written as $(\vec{a} \times \vec{b}) \cdot \vec{c}$



Note:

1. The scalar triple product also written as

   $\vec{a} \cdot (\vec{b} \times \vec{c})$ (or) $[\,\vec{a}\ \ \vec{b}\ \ \vec{c}\,]$

2. $[\,\vec{a}\ \ \vec{b}\ \ \vec{c}\,]$ is read as the box product of $\vec{a}$ $\vec{b}$ and $\vec{c}$

Scalar Triple Product can be expressed in the form of a determinant:

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

$\vec{c} = c_1\vec{i} + c_2\vec{j} + c_3\vec{k}$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (b_2c_3 - b_3c_2)\vec{i} - (b_1c_3 - b_3c_1)\vec{j} + (b_1c_2 - b_2c_1)\vec{k}$$

$$\therefore \vec{a} \cdot (\vec{b} \times \vec{c}) = (a_1\vec{i} + a_2\vec{j} + a_3\vec{k}) \cdot [(b_2c_3 - b_3c_2)\vec{i} - (b_1c_3 - b_3c_1)\vec{j} + (b_1c_2 - b_2c_1)\vec{k}]$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1)\vec{j} + a_3(b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\therefore \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Properties of Scalar Triple Product:

(1) The dot and cross are interchangeable in Scalar Triple Product.

i.e. $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$

(2) From the properties of determinants, $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$

i.e $\quad [\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

(3) The Scalar Triple Product of three vectors is zero, if any two of them are equal.

i.e. $\quad [\vec{a}\ \vec{b}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{a}] = [\vec{a}\ \vec{c}\ \vec{c}] = 0$

(4) If any two vectors are equal (or) parallel then the value of Scalar Triple Product is zero (i.e.) $\vec{a}$ and $\vec{b}$ are parallel.

Then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(5) If $\vec{i}, \vec{j}, \vec{k}$ are unit vectors then $[\vec{i}, \vec{j}, \vec{k}] = 1$

(6) The Scalar Triple Product changes in its sign for every change in the cyclic order

i.e. $\quad [\vec{a}, \vec{b}, \vec{c}] = -[\vec{a}, \vec{c}, \vec{b}]$

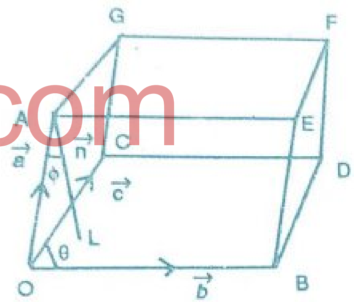
(7) If the three vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar, then the Scalar Triple Product is zero.

i.e. $\quad [\vec{a}\ \vec{b}\ \vec{c}] = 0$

Geometrical Interpretation of Scalar Triple Product:

In parallelepiped,

$\overrightarrow{OA} = \vec{a}, \ \overrightarrow{OB} = \vec{a}, \ \overrightarrow{OC} = \vec{c}$



Let $\hat{n}$ be the unit vector along LA and ϕ be the angle between $\hat{n}$ & $\vec{a}$, θ be the angle between $\vec{b}$ & $\vec{c}$.

Draw AL perpendicular to the plane OBDC.

Cross product of $\vec{b}$ and $\vec{c}$ $= bc \sin\theta\, \hat{n}$

$$\begin{aligned} \therefore \vec{a} \cdot (\vec{b} \times \vec{c}) &= \vec{a} \cdot (bc \sin\theta\, \hat{n}) \\ &= a\, bc \sin\theta \cos\phi \\ &= (bc \sin\theta)(a \cos\phi) \\ &= \text{Area of the base x height} \\ &= \text{Volume of the parallelepiped} \end{aligned}$$

Note:

1. The volume of the parallelepiped is $\vec{a} \cdot (\vec{b} \times \vec{c})$
2. If $\vec{a}, \vec{b}, \vec{c}$ are coplanar, then the volume of parallelepiped is zero.

Worked Examples:

Example 1:

Find the volume of the parallelepiped whose co-terminus edges are $2\vec{i} + 3\vec{j} + 4\vec{k}$, $\vec{i} - 2\vec{j} - \vec{k}$, $3\vec{i} + \vec{j} + 2\vec{k}$.

Solution:

Let $\vec{a} = 2\vec{i} + 3\vec{j} + 4\vec{k}$

$\vec{b} = \vec{i} - 2\vec{j} - \vec{k}$

$\vec{c} = 3\vec{i} + \vec{j} - 2\vec{k}$

$\therefore$ Volume of parallelepiped $= [\vec{a}, \vec{b}, \vec{c}]$

$$= \begin{vmatrix} 2 & 3 & 4 \\ 1 & -2 & -1 \\ 3 & 1 & -2 \end{vmatrix}$$

$= 35$ m. units

Example 2:

If $\vec{a} = 2\vec{i} - 3\vec{j}$, $\vec{b} = \vec{i} + \vec{j} - \vec{k}$ and $\vec{c} = 3\vec{i} - \vec{k}$. Find $\vec{a} \cdot (\vec{b} \times \vec{c})$.

Solution:

Given $\vec{a} = 2\vec{i} - 3\vec{j}$, $\vec{b} = \vec{i} + \vec{j} - \vec{k}$ and $\vec{c} = 3\vec{i} - \vec{k}$

$$\therefore \vec{b} \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix}$$

$= -\vec{i} - 2\vec{j} - 3\vec{k}$

$\therefore \vec{a} \cdot (\vec{b} \times \vec{c}) = (2\vec{i} - 3\vec{j}) \cdot (-\vec{i} - 2\vec{j} - 3\vec{k})$

$= (2)(-1) + (-3)(-2) + (0)(-3)$

$= 4$

Example 3:

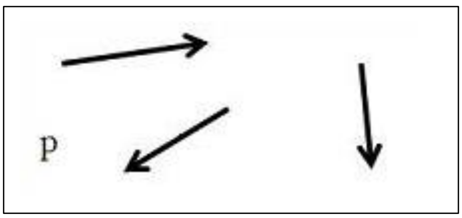
Show that the vectors $\vec{i} + 2\vec{j} - 3\vec{k}$, $2\vec{i} - \vec{j} + 2\vec{k}$ and $3\vec{i} + \vec{j} - \vec{k}$ are coplanar.

Solution:

Let $\vec{a} = \vec{i} + 2\vec{j} - 3\vec{k}$

$\vec{b} = 2\vec{i} - \vec{j} + 2\vec{k}$

$\vec{c} = 3\vec{i} + \vec{j} - \vec{k}$



We know that, the condition for coplanar, $[\vec{a}, \vec{b}, \vec{c}] = 0$.

$$\therefore [\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix}$$

$$= 1(1-2) - 2(-2-6) - 3(2+3)$$

$$= 0$$

$\therefore$ the vectors are coplanar.

Example 4:

Find the value of 'm' so that $2\vec{i} - \vec{j} + \vec{k}$, $\vec{i} + 2\vec{j} - 3\vec{k}$ and $3\vec{i} + m\vec{j} + 5\vec{k}$ are coplanar.

Solution:

Let $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}$, $\vec{c} = 3\vec{i} + m\vec{j} + \vec{k}$ given the vectors are coplanar.

$$\therefore [\vec{a}, \vec{b}, \vec{c}] = 0$$

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & m & 5 \end{vmatrix} = 0$$

$2(10 + 3m) + 1(5 + 9) + 1(m - 6) = 0$

$20 + 6m + 5 + 9 + m - 6 = 0$

$7m + 28 = 0$

$m = -4$

Example 5:

Find the value of $[\vec{i} + \vec{j},\ \vec{j} + \vec{k},\ \vec{k} + \vec{i}]$

Solution:

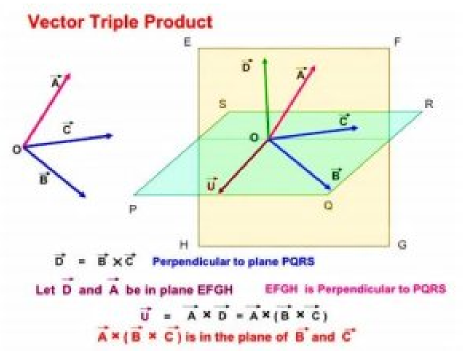
$$[\vec{i} + \vec{j},\ \vec{j} + \vec{k},\ \vec{k} + \vec{i}] = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 1(1-0) - 1(0-1) + 0(0-1)$$

$$= 2$$

Vector Triple Product

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be the any three vectors. Then the vector triple product is defined by vector product of $\vec{a}$ and $\vec{b} \times \vec{c}$. It is written as $\vec{a} \times (\vec{b} \times \vec{c})$.



Note:

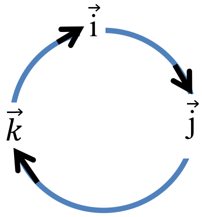
1. $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$.

Theorem (statement only)

If $\vec{a}$, $\vec{b}$, $\vec{c}$ are any three vectors, then $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{b} \cdot \vec{c})\,\vec{a}$

Worked Examples:

Example 1: Find $\vec{i} \times (\vec{j} \times \vec{k})$



Solution:

$$\vec{i} \times (\vec{j} \times \vec{k}) = \vec{i} \times \vec{i}$$
$$= \vec{0}$$

Example 2:

If $\vec{a} = -2\vec{i} + 3\vec{j} - 2\vec{k}$, $\vec{b} = 3\vec{i} + \vec{j} + 3\vec{k}$, $\vec{c} = 2\vec{i} - 5\vec{j} + \vec{k}$ find $(\vec{a} \times \vec{b}) \times \vec{c}$

Solution:

Given $\vec{a} = -2\vec{i} + 3\vec{j} - 2\vec{k}$, $\vec{b} = 3\vec{i} + \vec{j} + 3\vec{k}$, $\vec{c} = 2\vec{i} - 5\vec{j} + \vec{k}$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 3 & -2 \\ 3 & 1 & 3 \end{vmatrix}$$

$$= 11\,\vec{i} - 11\,\vec{k}$$

$$\text{and } (\vec{a} \times \vec{b}) \times \vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 11 & 0 & -11 \\ 2 & -5 & 1 \end{vmatrix}$$

$$= -55\,\vec{i} - 33\,\vec{j} - 55\,\vec{k}$$

Exercise: 3.1.1

1. Find the value of $[\vec{i}, \vec{j}, \vec{k}]$
2. Find the value of $[\vec{i} - \vec{j}, \vec{j} - \vec{k}, \vec{k} - \vec{i}]$
3. If $\vec{a} = \vec{i} - 2\vec{j} + 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}$, $\vec{c} = 3\vec{i} + 2\vec{j} + \vec{k}$ find $\vec{a} \cdot (\vec{b} \times \vec{c})$
4. Find the value of $\vec{k} \times (\vec{j} \times \vec{i})$, $\vec{j} \times (\vec{i} \times \vec{k})$
5. If $\vec{a} = \vec{i} - \vec{j}$, $\vec{b} = 2\vec{i} + \vec{j} + \vec{k}$, $\vec{c} = \vec{i} + 3\vec{j} + \vec{k}$, find $\vec{a} \times \left(\vec{b} \times \vec{c}\right)$
6. Find the volume of parallelepiped whose edges are $4\vec{i} - 8\vec{j} + \vec{k}$, $2\vec{i} - \vec{j} - 2\vec{k}$, $3\vec{i} - 4\vec{j} + 12\vec{k}$.
7. Find the value of 'λ', when the volume of the parallelepiped whose coterminous edges are $7\vec{i} + \lambda\vec{j} - 3\vec{k}$, $\vec{i} + 2\vec{j} - \vec{k}$, $-3\vec{i} + 7\vec{j} + 5\vec{k}$ is 90 cubic units.
8. Determine whether the three vectors $2\vec{i} + 3\vec{j} + \vec{k}$, $\vec{i} - 2\vec{j} + 2\vec{k}$ and $3\vec{i} + \vec{j} + 3\vec{k}$ are coplanar.

9. Show that $[\vec{a} - \vec{b},\ \vec{b} - \vec{c},\ \vec{c} - \vec{a}] = 0$

10. Prove that $\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$

11. Let the four points A (6, -7, 0), B (16, -19, -4), C (0, 3, -6) and D (2, -5, 10), then prove that $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar.

12. If $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{b} = 3\vec{i} + 5\vec{j} + 2\vec{k}$ and $\vec{c} = -\vec{i} - 2\vec{j} + 3\vec{k}$, verify that $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\,\vec{b} - (\vec{b} \cdot \vec{c})\,\vec{a}$

Product of More Vectors

Product of four vectors by Scalar Product

1. Let, $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be any four vectors, then the scalar

   Product of four vectors is $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

   This $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ can be expressed as in the form of determinant

   i.e. $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$

2. Product of four vectors by cross product

   Let, $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ be any four vectors, then the vector product of four vector is $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$.

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\,\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\,\vec{d}$$

Note:

1. If the four vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are coplanar then $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = 0$.

Worked Examples:

Example 1:

If $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{k}$, $\vec{c} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{d} = 3\vec{i} + 2\vec{j} + \vec{k}$, find $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$

Solution:

Let $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + \vec{k}$, $\vec{c} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{d} = 3\vec{i} + 2\vec{j} + \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= \vec{i} + \vec{j} - 2\vec{k}$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= -4\vec{i} + 8\vec{j} - 4\vec{k}$$

$$\therefore (\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{i} + \vec{j} - 2\vec{k}) \cdot (-4\vec{i} + 8\vec{j} - 4\vec{k})$$

$$= (1)(-4) + (1)(8) + (-2)(-4)$$

$$= 12$$

Example 2:

If $\vec{a} = \vec{i} - \vec{j}$, $\vec{b} = \vec{i} + \vec{j} - 4\vec{k}$, $\vec{c} = 3\vec{j} - \vec{k}$ and $\vec{d} = 2\vec{i} + 5\vec{j} + \vec{k}$, find the value of $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$

Solution:

Let $\vec{a} = \vec{i} - \vec{j}$

$\vec{b} = \vec{i} - \vec{j} - 4\vec{k}$

$\vec{c} = 3\vec{j} - \vec{k}$

$\vec{d} = 2\vec{i} + 5\vec{j} + \vec{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 1 & 1 & -4 \end{vmatrix} = 4\vec{i} + 4\vec{j} + 2\vec{k}$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 3 & -1 \\ 2 & 5 & 1 \end{vmatrix} = 8\vec{i} - 2\vec{j} - 6\vec{k}$$

$$\therefore (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 4 & 2 \\ 8 & -2 & -6 \end{vmatrix}$$

$$= -20\vec{i} + 40\vec{j} - 40\vec{k}$$

Example 3:

Prove that $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) + (\vec{b} \times \vec{c}) \cdot (\vec{a} \times \vec{d}) + (\vec{c} \times \vec{a}) \cdot (\vec{b} \times \vec{d}) = 0$

Solution:

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

$$= (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{d})$$

and

$$(\vec{b} \times \vec{c}) \cdot (\vec{a} \times \vec{d}) = \begin{vmatrix} \vec{b} \cdot \vec{a} & \vec{b} \cdot \vec{d} \\ \vec{c} \cdot \vec{a} & \vec{c} \cdot \vec{d} \end{vmatrix}$$

$$= (\vec{b} \cdot \vec{a})(\vec{c} \cdot \vec{d}) - (\vec{c} \cdot \vec{a})(\vec{b} \cdot \vec{d})$$

and

$$(\vec{c} \times \vec{a}).(\vec{b} \times \vec{d}) = \begin{vmatrix} \vec{c}\,.\,\vec{b} & \vec{c}\,.\,\vec{d} \\ \vec{a}\,.\,\vec{b} & \vec{a}\,.\,\vec{d} \end{vmatrix}$$

$$= (\vec{c}\,.\,\vec{b})(\vec{a}\,.\,\vec{d}) - (\vec{a}\,.\,\vec{b})(\vec{c} \times \vec{d})$$

$$\therefore \text{LHS} = (\vec{a}\,.\,\vec{c})(\vec{b}\,.\,\vec{d}) - (\vec{b}\,.\,\vec{c})(\vec{a}\,.\,\vec{d}) + (\vec{b}\,.\,\vec{a})(\vec{c}\,.\,\vec{d})$$
$$- (\vec{c}\,.\,\vec{a})(\vec{b}\,.\,\vec{d}) + (\vec{c}\,.\,\vec{b})(\vec{a}\,.\,\vec{d}) - (\vec{a}\,.\,\vec{b})(\vec{c}\,.\,\vec{d})$$

$$= 0$$

Example 4:

If $\vec{a} = \vec{i} + \vec{j} + \vec{k}$, $\vec{b} = \vec{i} - \vec{j} - \vec{k}$, $\vec{c} = -\vec{i} + \vec{j} + 2\vec{k}$, $\vec{d} = 2\vec{i} + \vec{j}$, find $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$ and also verify = $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\,\vec{c} - [\vec{a}, \vec{b}, \vec{c}]\,\vec{d}$

Solution:

given $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

$\vec{b} = \vec{i} - \vec{j} - \vec{k}$

$\vec{c} = -\vec{i} + \vec{j} + 2\vec{k}$

$\vec{d} = 2\vec{i} + \vec{j}$

$$\therefore \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -1 & -1 \end{vmatrix} = 2\vec{j} - 2\vec{k}$$

$$\vec{c} \times \vec{d} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 2 \\ 2 & 1 & 0 \end{vmatrix} = -2\vec{i} + 4\vec{j} - 3\vec{k}$$

$$\therefore (\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & -2 \\ -2 & 4 & -3 \end{vmatrix}$$

$$= 2\vec{i} + 4\vec{j} + 4\vec{k} \qquad \text{-------} (1)$$

$$\text{and } [\vec{a}, \vec{b}, \vec{d}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 1 & 0 \end{vmatrix}$$

$$= 1(0+1) - 1(0+2) + 1(1+2)$$

$$= 2$$

$$\text{and } [\vec{a}, \vec{b}, \vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & 2 \end{vmatrix}$$

$$= 1(-2+1) - 1(2-1) + 1(1-1) = -2$$

$\therefore [\vec{a}, \vec{b}, \vec{d}]\, \vec{c} - [\vec{a}, \vec{b}, \vec{c}]\, \vec{d} = 2(-\vec{i} + \vec{j} + 2\vec{k}) - (-2)(2\vec{i} + \vec{j})$

$= -2\vec{i} + 2\vec{j} + 4\vec{k} + 4\vec{i} + 2\vec{j}$

$= 2\vec{i} + 4\vec{j} + 4\vec{k}$ ------ (2)

From (1) and (2)

$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\, \vec{c} - [\vec{a}, \vec{b}, \vec{d}]\, \vec{d}$

Exercise: 3.1.2

1. If $\vec{a} = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{b} = \vec{j} + \vec{k}$, $\vec{c} = \vec{i} + \vec{k}$, $\vec{d} = \vec{i} + \vec{j} + \vec{k}$. Evaluate $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$.

2. If $\vec{a} = 2\vec{i} - \vec{j} + \vec{k}$, $\vec{b} = -\vec{i} - \vec{j} - \vec{k}$, $\vec{c} = 2\vec{i} + 3\vec{j} - \vec{k}$, $\vec{d} = \vec{i} + \vec{j} - \vec{k}$. Evaluate $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d})$.

3. If $\vec{a} = \vec{i} - \vec{j} + \vec{k}$, $\vec{b} = 2\vec{i} + 3\vec{j} - 5\vec{k}$, $\vec{c} = 2\vec{i} + \vec{j} - 2$, $\vec{d} = 3\vec{i} - \vec{j} + 4\vec{k}$. Show that $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$

4. If $\vec{a} = \vec{i} + \vec{j}$, $\vec{b} = \vec{j} + \vec{k}$, $\vec{c} = \vec{k} + \vec{i}$, $\vec{d} = \vec{i} + \vec{j} + \vec{k}$. Verify that $(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) = [\vec{a}, \vec{b}, \vec{d}]\, \vec{c} - [\vec{a}, \vec{b}, \vec{c}]\, \vec{d}$

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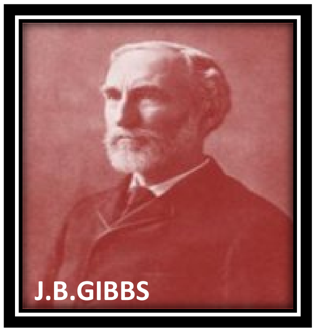
Chapter 3.2 VECTOR DIFFERENTIATION

Introduction



The best features of Quaternion Calculus and Cartesian Geometry were united, largely through the efforts of the American Mathematician J.B. Gibbs (1839 - 1903) and O.Heaviside (1850 - 1925) of England and new subject called Vector Algebra was created. The development of the algebra of vectors and vector analysis as we know it today was first revealed in sets of remarkable notes made by Gibbs.

In this chapter we have to study the basics of vector calculus comprising of vector point function and vector field, scalar point function and scalar field and gradient.

Scalar Point Function and Scalar Field

If each point (x, y, z) of a region D in space there corresponds a number or a scalar ϕ (x, y, z) then ϕ is called a scalar point function and the region D is called a scalar field.

Example of Scalar Point Function:

1. The temperature at any point within or on the surface of earth at a given time defines a scalar field.
2. $\phi(x, y, z) = x^2 + y^2 + z^2$ defines a scalar field. Density, potential are also examples of scalar point functions and for their scalar fields.
3. The pressure P in a field varies according to its depth and P (x, y, z) is a Scalar Point Function.

Vector Point Function and Vector Field

If to each point (x, y, z) of a Region D in space there corresponds a vector $\vec{F}(x, y, z)$ then $\vec{F}$ is called a vector point function and the Region D is called a vector filed.

Examples

1. The velocity at any point (x, y, z) within a moving field at time 't' is a vector point function and hence a vector field is defined in this region of field.
2. $\vec{F}(x, y, z) = 2x\vec{i} + xy\vec{j} + xyz\vec{k}$ defines a vector field. Force, electric intensity etc. are also examples of vector point functions.

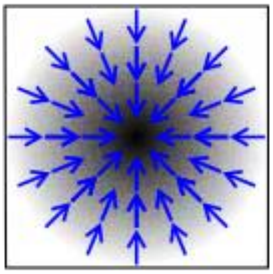
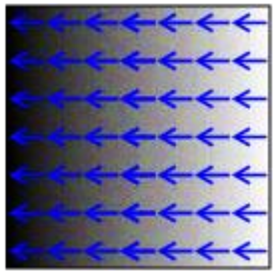
Vector differential operator

The vector differential operator 'DEL' or 'NABLA', denoted as '∇' is defined by

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Where $\vec{i}$, $\vec{j}$, $\vec{k}$ are unit vectors in x, y, z directions.

The Gradient

A gradient is a vector differential operator on a scalar field like temperature. Every point in space having a specific temperature. The gradient is a differential operator that gives a vector field, which in every point shows in what direction in 2 or 3 dimensional space the field of values is increasing the fastest. By moving in the opposite direction of the gradient are seeing the fastest decline of the scalar value (temperature). By moving perpendicularly to the gradient are staying at the same scalar value. So gradient operates on a scalar field to give a vector field.

Let φ (x, y, z) be a scalar point function differentiable at each point (x, y, z) in a certain region of space. Then the gradient of φ is defined as

$$\text{grad } \varphi \text{ (or) } \nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \Sigma\, \vec{i}\frac{\partial\varphi}{\partial x}$$

The gradient of φ is a vector field.

Note: grad φ = ∇φ is a vector whose three components are $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}$

Properties of Gradient

If ϕ and ψ are scalar point functions and c is constant then

1) $\nabla c = 0$, where C is a constant
2) $\nabla (c\phi) = c\,\nabla\phi$ where c is constant
3) $\nabla (\phi + \psi) = \nabla\phi + \nabla\psi$
4) $\nabla (\phi\psi) = \phi\,\nabla\psi + \psi\,\nabla\phi$
5) $\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}$

Physical significance of 'grad p'

If φ (x, y, z) = c (c being a constant) represents a surface, then 'grad φ' represents the normal vector to the surface at the point (x, y, z).

For, if $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, is the position vector of the point (x, y, z) on the surface, we have $\overrightarrow{dr} = (dx)\vec{i} + (dy)\vec{j} + (dz)\vec{k}$ which is in the tangent plane to the surface of (x, y, z).

Again, $\nabla\varphi \cdot \overrightarrow{dr} = \left[\frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k}\right] \cdot [\,dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}\,]$

$$= \left(\frac{\partial\varphi}{\partial x}\right) dx + \left(\frac{\partial\varphi}{\partial y}\right) dy + \left(\frac{\partial\varphi}{\partial z}\right) dz$$

$$= d\varphi = 0 \quad (\because \varphi = c)$$

$\therefore$ The vector $\nabla\varphi$ which is perpendicular to the tangent plane is the normal vector to $\varphi = c$ at (x, y, z).

Worked Examples:

1. If $\varphi = xyz$ find $\nabla\varphi$

Solution:

Given $\varphi = x\,y\,z$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,yz + \vec{j}\,xz + \vec{k}\,xy$$

$$= yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$$

2. If $\varphi = \log(x^2 + y^2 + z^2)$ find $\nabla\varphi$.

Solution:

Given $\varphi = \log(x^2 + y^2 + z^2)$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\left[\frac{2x}{x^2+y^2+z^2}\right] + \vec{j}\left[\frac{2y}{x^2+y^2+z^2}\right] + \vec{k}\left[\frac{2z}{x^2+y^2+z^2}\right]$$

$$= \frac{2}{x^2+y^2+z^2}\left[x\vec{i} + y\vec{j} + 2\vec{k}\right]$$

$$= \frac{2}{x^2+y^2+z^2}\left[\vec{r}\right]$$

3. Find $\nabla(\log r)$

Solution:

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, $|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$

$$r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla(\log r)$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\log r)$$

$$= \vec{i}\frac{\partial}{\partial x}(\log r) + \vec{j}\frac{\partial}{\partial y}(\log r) + \vec{k}\frac{\partial}{\partial z}(\log r)$$

$$= \vec{i}\,\frac{1}{r}\frac{\partial r}{\partial x} + \vec{j}\,\frac{1}{r}\frac{\partial r}{\partial y} + \vec{k}\,\frac{1}{r}\frac{\partial r}{\partial z}$$

$$= \frac{1}{r}\left[\vec{i}\,\frac{\partial r}{\partial x} + \vec{j}\,\frac{\partial r}{\partial y} + \vec{k}\,\frac{\partial r}{\partial z}\right]$$

$$= \frac{1}{r}\left[\vec{i}\,\frac{x}{r} + \vec{j}\,\frac{y}{r} + \vec{k}\,\frac{z}{r}\right]$$

$$= \frac{1}{r^2}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$= \frac{\vec{r}}{r^2}$$

4. Prove that $\nabla\left(e^{x^2+y^2+z^2}\right) = 2\,e^{r^2}\,\vec{r}$

Solution:

We know that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial x}{\partial r} = \frac{x}{r}, \quad \frac{\partial y}{\partial r} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla\left(e^{x^2+y^2+z^2}\right) = \nabla\left(e^{r^2}\right) = \Sigma\,\vec{i}\,\frac{\partial}{\partial x}\left(e^{r^2}\right)$$

$$= \Sigma\,\vec{i}\,e^{r^2}\,2r\,\frac{\partial r}{\partial x}$$

$$= \Sigma\,\vec{i}\,e^{r^2}.\,2\,r.\,\frac{x}{r}$$

$$= \Sigma\,\vec{i}\,e^{r^2}\,2x = 2e^{r^2}\,\Sigma\,x\vec{i}$$

$$= 2\,e^{r^2}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$= 2\,e^{r^2}\,\vec{r}$$

5. If $f = x^2\,y\,z$, find grad f at the point (1, -2, 1)

Solution:

Given $f = x^2\,y\,z$

$$\because \text{grad } f = \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

$$\text{grad } f = \frac{\partial}{\partial x}(x^2 y z)\,\vec{i} + \frac{\partial}{\partial y}(x^2 y z)\,\vec{j} + \frac{\partial}{\partial z}(x^2 y z)$$

$$= (2xyz)\,\vec{i} + (x^2 z)\,\vec{j} + (x^2 y)\,\vec{k}$$

$\therefore$ At the point (1, -2, 1).

$$\text{grad } f = -4\,\vec{i} + \vec{j} - 2\vec{k}$$

6. Find the unit normal to the surface xy + yz + zx = 3 at the point (1, 1, 1).

Solution:

If $\varphi = c$ is a surface, $\nabla\varphi$ is the normal to it.

Here f = xy + yz + zx

$\therefore$ normal to $\varphi = \nabla\varphi = \frac{\partial\varphi}{\partial x}\vec{i} + \frac{\partial\varphi}{\partial y}\vec{j} + \frac{\partial\varphi}{\partial z}\vec{k}$

$= \left[\frac{\partial}{\partial x}(xy + yz + zx)\right]\vec{i} + \left[\frac{\partial}{\partial y}(xy + yz + zx)\right]\vec{j} + \left[\frac{\partial}{\partial z}(xy + yz + zx)\right]\vec{k}$

$= (y + z)\vec{i} + (x + z)\vec{j} + (y + x)\vec{k}$

$\therefore$ normal at (1, 1, 1)

$\nabla\phi$ at (1, 1, 1) $= 2\vec{i} + 2\vec{j} + 2\vec{k}$

$\therefore$ unit normal $= \frac{2\vec{i} + 2\vec{j} + 2\vec{k}}{\sqrt{2^2 + 2^2 + 2^2}}$

$= \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$

7. Find the acute angle between the surface $x y^2 z = 2$ and $x^2 + y^2 + z^3 = 6$ at the point (2, 1, 1)

Solution:

Let $f = xy^2z = 2$ be the surface I normal vector to I at (2, 1, 1).

$\nabla f = |\, y^2z\,\vec{i} + 2xyz\,\vec{j} + xy^2\,\vec{k}\,|$

∇f at (2, 1, 1) $= \vec{i} + 4\vec{j} + 2\vec{k} = \vec{a}$ (say)

Let $g = x^2 + y^2 + z^3 = 6$ be the surface II normal vector to II at (2, 1, 1)

$\nabla g = |\, 2x\,\vec{i} + 2y\,\vec{j} + 3x^2\,\vec{k}\,|$

∇g at (2, 1, 1) $= 4\vec{i} + 2\vec{j} + 3\vec{k} = \vec{b}$ (say)

$\therefore$ Angle between the surfaces

= Angle between the normal to them

= Angle between $\vec{a}$ and $\vec{b}$

$= \cos^{-1}\frac{\vec{a}\,.\,\vec{b}}{|\vec{a}|\,.\,|\vec{b}|}$

$= \cos^{-1}\left|\frac{4 + 8 + 6}{\sqrt{1 + 16 + 4}\,\sqrt{16 + 4 + 9}}\right|$

$= \cos^{-1}\left|\frac{16}{\sqrt{21}\,\sqrt{29}}\right|$

8. Find the constants p and q such that the surfaces $px^2 - qyz = (p + 2)\,x$ and $4x^2y + z^3 = 4$ are orthogonal at the point (1, -1, 2).

Solution:

Let $f = px^2 - qyz - (p + 2)\,x = 0$ be the surface I and $g = 4x^2y + z^3 = 4$ be the surface II.

Normal to I at (1, -1, 2)

$$= \nabla f \text{ at } (1, -1, 2)$$

$$= \left(\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right) \text{ at } (1, -1, 2)$$

$$= [\,(2px - p - 2)\,\vec{i} - (qz)\,\vec{j} - (q\,y)\,\vec{k}\,] \text{ at } (1, -1, 2)$$

$$= (p - 2)\,\vec{i} - (2q)\,\vec{j} - (q)\,\vec{k} = \vec{a} \quad \text{(say)}$$

Normal to II at (1, -1, 2) $= \nabla g$ at (1, -1, 2)

$$= [\,8xy\,\vec{i} + 4x^2\,\vec{j} + 3z^2\,\vec{k}\,] \text{ at } (1, -1, 2)$$

$$= -8\vec{i} + 4\vec{j} + 12\vec{k} = \vec{b} \quad \text{(say)}$$

Since the surfaces I and II are orthogonal.

$$\therefore \vec{a} \cdot \vec{b} = 0$$

$$\therefore -8\,(p - 2) + 4\,(-2q) + 12q = 0$$

$$\Rightarrow -8p + 16 - 8q + 12q = 0$$

$$\Rightarrow -8p + 4q + 16 = 0$$

$$\Rightarrow 2p - q = 4 \qquad \text{----- (1)}$$

Since the point (1, -1, 2) lies on surface I, we get

$$P - q = (P + 2)\,(1)$$

$$P - q = P + 2$$

$$-q = 2$$

$$\boxed{q = -2}$$

Sub q = -2 in equation (1)

We get

$$2P + 2 = 4$$

$$2P = 4 - 2 = 2$$

$$\boxed{P = 1}$$

$\therefore$ P = 1 and q = -2

9. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $|\,\vec{r}\,|$ $\sqrt{x^2 + y^2 + z^2}$ show that grad $(r^3) = 3r\,\vec{r}$.

Solution:

$$\text{Let } \varphi = r^3 = (x^2 + y^2 + z^2)^{\frac{3}{2}}$$

$$\text{Then } \frac{\partial \varphi}{\partial x} = \frac{3}{2}\,(x^2 + y^2 + z^2)^{\frac{3}{2}-1} \cdot 2x$$

$$= 3xr$$

Similarly

$$\frac{\partial \varphi}{\partial y} = 3\,yr \text{ and } \frac{\partial \varphi}{\partial z} = 3zr$$

$$\text{grad } \varphi = \frac{\partial \varphi}{\partial x}\vec{i} + \frac{\partial \varphi}{\partial y}\vec{j} + \frac{\partial \varphi}{\partial z}\vec{k}$$

$$= 3xr\,\vec{i} + 3yr\,\vec{j} + 3zr\,\vec{k}$$

$$= 3r\,(x\,\vec{i} + y\,\vec{j} + z\,\vec{k})$$

` $$= 3r\,.\,\vec{r}$$

10. The temperature at a point (x, y, z) in space given by T (x, y, z) = $x^2 + y^2 - z$. A mosquito located at (4, 4, 2) to fly in such a direction that it gets cooled faster. Find the direction in which it should fly ?

Solution:

Given T (x, y, z) = $x^2 + y^2 - z$

$$\nabla T = \vec{i}\,\frac{\partial T}{\partial x} + \vec{j}\,\frac{\partial T}{\partial y} + \vec{k}\,\frac{\partial T}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla T \text{ at } (4, 4, 2) = 8\vec{i} + 8\vec{j} - \vec{k}$$

$\therefore$ Mosquito will fly in the direction of maximum rate of decrease is $8\vec{i} + 8\vec{j} - \vec{k}$.

Exercise: 3.2.1

1. Find $\nabla\varphi$ if $\varphi = x^2 + y^2 + z^2$
2. Find the grad φ where φ is $(x^2 + y^2 + z^2)^{5/2}$
3. Find $\nabla\varphi$ if $\varphi = \frac{1}{2}\log(x^2 + y^2 + z^2)$
4. If $\varphi = 2xz^3 - 3x^2yz$, find $\nabla\varphi$ at the point (2, 2, -1)
5. If $V = 2x\,\vec{i} - 3y^2\,\vec{j} + 2^3\,\vec{k}$ and $\varphi = 2xyz - 3z^2$, find V. $\nabla\varphi$ and V x $\nabla\varphi$ at the point (1, 2)
6. Find the unit normal vector to the surface $z = x^2 + y^2$ at the point(1, -2, 5).
7. Find the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point (1, -2, 1).
8. Find the angle between the surfaces $x^2 - y^2 - z^2 = 11$ and $xy + yz - zx - 18 = 0$ at the point (6, 4, 3).
9. Find the angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point (1, -2, 1).
10. Find the acute angle between the normal to the surface $xy = z^2$ at the points (1, 9, 3) and (3, 3, -3).
11. Find the values of 'a' and 'b' so that the surfaces $ax^3 - by^2z = (a + 3)\,x^2$ and $4x^2y - z^3 = 11$ may cut orthogonally at (2, -1, -3).

Chapter 3.3 APPLICATION OF VECTOR DIFFERENTIATION

Introduction

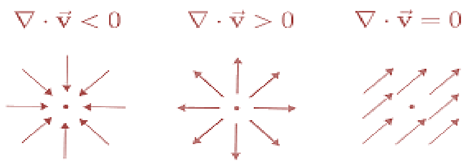
In the chapter we have to study basics of vector calculus comprising of divergence and curl, solenoidal and irrotational vectors.

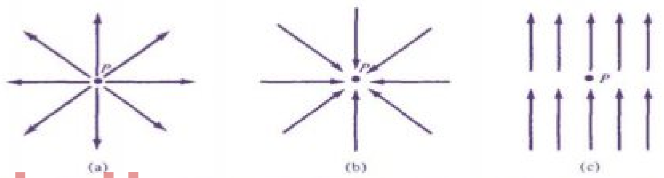
Four friends float down a river, each marking a corner of a square. If the square is getting bigger, the river has positive divergence. If it's shrinking, negative divergence

Next, four friends are rigidly connected the square can't change shape. If the square starts rotating like a Frisbee as it goes along, the river has curl. Positive curl is counter clockwise rotation Negative curl is clockwise

Divergence

Divergence is a differential operator that acts on a vector field to give a scalar field, so divergence is the opposite of gradient. If a vector field like a gravitational field or an electric field, by taking the divergence get the scalar field that is proportional to the mass or charge density "causing" that field. Everywhere in space where we do have gravitational or electric fields, but where no mass or charge is located, the divergence is zero.





(a) Positive divergence, (b) negative divergence, (c) zero divergence.

Let $\vec{F}$ be a vector point function. The divergence of $\vec{F}$ denoted by div $\vec{F}$ is defined as

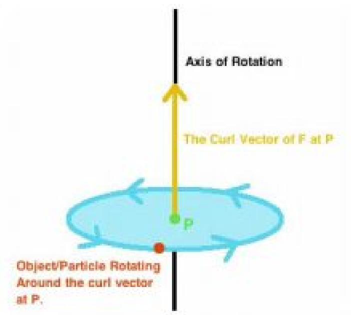
$$\text{div } \vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial T}{\partial z}\right) . \vec{F}$$

$$= \vec{i}\frac{\partial \vec{F}}{\partial x} + \vec{j}\frac{\partial \vec{F}}{\partial y} + \vec{k}\frac{\partial \vec{F}}{\partial z}$$

Note: div $\vec{F}$ = $\nabla.\vec{F}$ is defined only for vector functions and div $\vec{F}$ is a scalar function.

Curl

The curl is a differential operator acting on a vector field to give another vector field. Curl alludes to something round or rotation, which is why it is also called rotation. The curl measures the net boost that an element affected by the vector field that the curl operator acts upon would get when going in a small closed loop in a specific plane.



The curl of the vector function $\vec{F}$ is denoted by curl $\vec{F}$ and it is defined as

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{F}$$

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

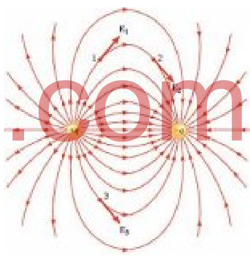
Note: Curl $\vec{F} = \nabla \times \vec{F}$ is a vector function and curl is defined only for vector functions.

Properties of Divergence and Curl

1. $\text{div}(\vec{F} \pm \vec{G}) = \text{div}\,\vec{F} \pm \text{div}\,\vec{G}$
2. $\text{curl}(\vec{F} \pm \vec{G}) = \text{curl}\,\vec{F} \pm \text{curl}\,\vec{G}$
3. $\nabla.(\varphi\vec{F}) = \nabla\varphi.\vec{F} + \varphi(\nabla.\vec{F})$ (or) $(\text{div}(\varphi\vec{F}) = (\text{grad}\,\varphi).\vec{F} + \varphi(\text{div}\,\vec{F})$ φ is a scalar function)
4. $\text{curl}(\varphi\vec{F}) = (\text{grad}\,\varphi) \times \vec{F} + \varphi(\text{curl}\,\vec{F})$ (or) $\nabla \times (\varphi\vec{F}) = \nabla\varphi.\vec{F} + \varphi(\nabla.\vec{F})$
5. $\text{div}(\vec{F} \times \vec{G}) = (\text{curl}\,\vec{F})\,\vec{G} - \vec{F}\,(\text{curl}\,\vec{G})$ (or) $\nabla.(\vec{F} \times \vec{G}) = (\nabla \times \vec{F}).\vec{G} - (\nabla \times \vec{G}).\vec{F}$

Solenoidal vector

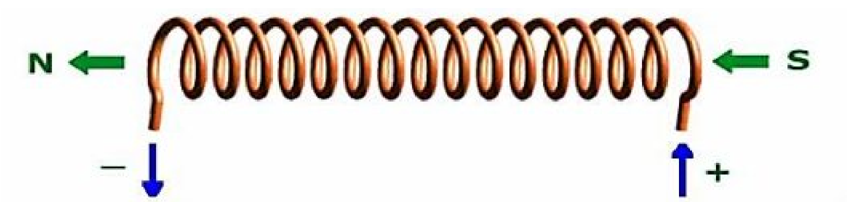
A vector function $\vec{F}$ is said to be a solenoidal vector if $\nabla.\vec{F} = \text{div}\,\vec{F} = 0$.



Example :

Solenoid creates magnetic field

Winding a metal wire into a helix and passing electrical current through the wire results in the creation directed magnetic field, such that the ends of the helix or solenoid have N (north-seeking) and S (south-seeking) magnetic poles.



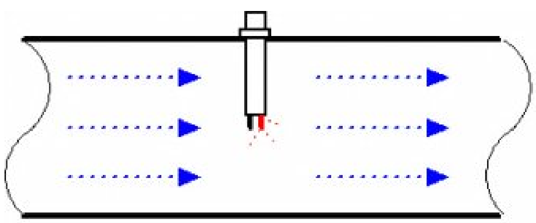
Current through solenoid creates magnetic field

Irrotational vector

A vector function $\vec{F}$ is said to be a irrotational vector if $\nabla \times \vec{F} = 0$ (curl $\vec{F} = 0$)

Example:

Flow of water in a pipe is an example of irrotational



Physical significance of the divergence

If A represents the velocity of fluid in a fluid flow, div A represents the rate of fluid flow through unit volume (or) Div A gives the rate at which fluid is originating at a point per unit volume.

Similarly if A represents the Electric flux or heat flux, Div A represents the amount of electric flux or heat flux that diverges per unit volume in unit time.

Physical significance of curl

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ be the position vector of a point p (x, y, z) of a rigid body rotating about a fixed axis about the origin 0 with an angular velocity $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$. Then the velocity V of the particle p is given by.

$$V = \vec{w} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= (w_2z - w_3y)\,\vec{i} + (w_3x - w_1z)\,\vec{j} + (w_1y - w_2z)\,\vec{k}$$

$$\text{Curl } \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_2z - w_3y & w_3x - w_1z & w_1y - w_2x \end{vmatrix}$$

$$= \vec{i}\,(w_1 + w_1) + \vec{j}\,(w_2 + w_2) + \vec{k}\,(w_3 + w_3)$$

$$= 2\vec{w}.$$

Thus the curl of velocity vector is twice the angular velocity of rotation.

Applications:

Divergence

The easiest way to think about this is water coming out of a hose, the water will have a magnitude and a direction. This is essentially what divergence is. It can tell how fast and in what direction a fluid is moving.

E.g: At the rim, we have positive divergence as the water is moving away from the rim.

Gradient

A gradient essentially tells how much a surface or some quantity changes from one point in space/time to another.

E.g.: Think of moving over a bed sheet, in different areas it will be more bumpy (many patches raised above the rest) than others, and how much it changes is gradient

Curl

Curl gives the measure of angular velocity of an object. If Curl is zero, it means object is not rotating. If Curl is not zero, its magnitude represents the speed of the object and its direction denotes the axis of rotation.

Every radio and TV broadcast, almost every electric motor or dynamo, almost every transformer operates according to Maxwell's equations, which are all based on gradient, divergence and curl.

Grad, div and curl are also fundamental for fluid flow, particularly compressible mediums (gasses). Fluid flow operates from pipelines to jet engines.

They are almost everywhere in the "real world".

Worked Examples:

1. If $\vec{F} = x^2z\,\vec{i} - 2y^3z\,\vec{j} + xy^2z\,\vec{k}$ then find div $\vec{F}$ and curl $\vec{F}$.

Solution:

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(x^2z\,\vec{i} - 2y^3z\,\vec{j} + xy^2z\,\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2y^3z) + \frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz - 6y^2z + xy^2$$

$$\text{Curl } \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2z & -2y^3z & xy^2z \end{vmatrix}$$

$$= \vec{i}\,[\,2xyz + 2y^3\,] - \vec{j}\,[\,y^2z - x^2\,] + \vec{k}\,[\,0\,]$$

$$= \vec{i}\,[\,2xyz + 2y^3\,] - \vec{j}\,[\,y^2z - x^2\,]$$

2. Find $\nabla \times (\nabla\varphi)$ where $\varphi = x^2 + y^2 + z^2$.

Solution:

$$\text{Given } \varphi = x^2 + y^2 + z^2$$

$$\nabla\varphi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)$$

$$\nabla\varphi = 2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}$$

$$\nabla \times (\nabla\varphi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & 2z \end{vmatrix}$$

$$= \vec{i}\,[\,0\,] - \vec{j}\,[\,0\,] + \vec{k}\,[\,0\,]$$

$$= 0$$

3. If $\vec{F} = (x^2 - y^2 + 2xz)\,\vec{i} + (xz - xy + yz)\,\vec{j} + (z^2 + x^2)\,\vec{k}$ then find $\nabla \times \vec{F}$ and $\nabla \times (\nabla \times \vec{F})$.

Solution:

$$\text{Given } \vec{F} = (x^2 - y^2 + 2xz)\,\vec{i} + (xz - yx + yz)\,\vec{j} + (z^2 + x^2)\,\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 + 2xz & xz - xy + yz & x^2 + z^2 \end{vmatrix}$$

$= \vec{i}\,[\,0 - (x + y)\,] - \vec{j}\,[\,2x - 2x\,] + \vec{k}\,[\,z - y + 2y\,]$

$= -(x + y)\,\vec{i} + 0\vec{j} + (z + y)\,\vec{k}$

$$\vec{\nabla} \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -(x+y) & 0 & y+z \end{vmatrix}$$

$= \vec{i}\,(1) - \vec{j}\,(0) + \vec{k}\,(1)$

$= \vec{i} + \vec{k}$

4. Find div $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = \nabla\,(x^3 + y^3 + z^3 - 3xyz)$

Solution:

Given $\vec{F} = \nabla\,(x^3 + y^3 + z^3 - 3xyz)$

$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2)$

$= (3x^2 - 3yz)\,\vec{i} + (3y^2 - 3xz)\,\vec{j} + (3z^2 - 3xy)\,\vec{k}$

div $\vec{F} = \nabla \cdot \vec{F}$

$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot$

$[\,(3x^2 - 3yz)\,\vec{i} + (3y^2 - 3xz)\,\vec{j} + (3z^2 - 3xy)\,\vec{k}$

$= 6x + 6y + 6z$

$= 6\,(x + y + z)$

$$\text{Curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix}$$

$= \vec{i}\,(-3x + 3x) - \vec{j}\,(-3y + 3y) + \vec{k}\,(-3z + 3z)$

$= 0$

5. If $\vec{r}$ is position vector of any point then prove that (i) div $\vec{r}$ = 3 (ii) curl $\vec{r}$ = 0

Solution:

Here $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\text{div}\,\vec{r} = \nabla \cdot \vec{r}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= 1 + 1 + 1 = 3$$

i.e. div $\vec{r}$ = 3

$$\text{Curl}\ \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= 0$$

6. If $\vec{V} = \nabla\varphi$ where $\varphi = x^3\ y^3\ z^3$, find div $\vec{V}$ and curl $\vec{V}$

Solution:

Given $\varphi = x^3\ y^3\ z^3$

$$\vec{V} = \nabla\varphi = \left(\vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}\right)$$

$$\vec{V} = 3x^2y^3z^3\ \vec{i} + 3x^3y^2z^3\ \vec{j} + 3x^3y^3z^2\ \vec{k}$$

$$\text{div}\,\vec{V} = \nabla \cdot \vec{V}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left((3x^2y^3z^3)\ \vec{i} + (3x^3y^2z^3)\ \vec{j} + (3x^3y^3z^2)\ \vec{k}\right)$$

$$= 6xy^3z^3 + 6x^3yz^3 + 6x^3y^3z$$

$$\text{Curl}\ \vec{V} = \nabla \times \vec{V}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y^3z^3 & 3x^3y^2z^3 & 3x^3y^3z^2 \end{vmatrix}$$

$$= \vec{i}\ (9x^3y^2z^2 - 9x^3y^2z^2) - \vec{j}\ (9x^2y^3z^2 - 9x^2y^3z^2) + \vec{k}\ (9x^2y^3z^3 - 9x^2y^2z^3)$$

$$= 0.$$

7. Show that $\vec{F} = (x + 3y)\ \vec{i} + (x - 3z)\ \vec{j} + (x - 2z)\ \vec{k}$ is solenoidal

Solution:

Given $\vec{F} = (x + 3y)\ \vec{i} + (x - 3z)\ \vec{j} + (x - 2z)\ \vec{k}$

To prove $\nabla \cdot \vec{F} = 0$

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot$$

$$[\,(x + 3y)\ \vec{i} + (x - 3z)\ \vec{j} + (x - 2z)\ \vec{k}$$

$$= 1 + 1 - 2$$

$$= 0.$$

$\therefore \vec{F}$ is solenoidal

8. If $\vec{F} = (x + 3y)\,\vec{i} + (x - 2z)\,\vec{j} + (x + \lambda z)\,\vec{k}$ is solenoidal then find the value of λ.

Solution:

Given $\vec{F}$ is solenoidal

i.e $\nabla . \vec{F} = 0$

$$= \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right). \quad [\,(x + 3y)\,\vec{i} + (x - 2z)\,\vec{j} + (x + \lambda z)\,\vec{k} = 0$$

$$\frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + \lambda z) = 0$$

$$1 + 1 + \lambda = 0$$

$$\lambda + 2 = 0 \quad \Rightarrow \lambda = -2$$

9. Show that $\vec{F} = (3x^2 + 2y + 1)\,\vec{i} + (4xy - 3y^2z - 3)\,\vec{j} + (2 - y^3)\,\vec{k}$ is irrotational.

Solution:

If $\nabla \times \vec{F} = 0$ then the field $\vec{F}$ is Irrotational

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2 + 2y + 1) & (4xy - 3y^2z - 3) & (2 - y^3) \end{vmatrix}$$

$$= \vec{i}\,[-3x^2 + 3y^2]\; - \vec{j}\;[0 - 0] + \vec{k}\;[4y - 4y]$$

$$\Rightarrow \nabla \times \vec{F} = 0$$

$\therefore \vec{F}$ is irrotational

10. Find the constants a, b, c so that

$\vec{F} = (x + 2y + az)\,\vec{i} + (bx - 3y - z)\,\vec{j} + (4x + cy + 2z)\,\vec{k}$ is irrotational.

Solution:

Given $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$$

$$\Rightarrow \vec{i}\,(c + 1) - \vec{j}\,(4 - a) + \vec{k}\,(b - 2) = 0\,\vec{i} - 0\,\vec{j} + 0\,\vec{k}$$

i.e. $c + 1 = 0$, $4 - a = 0$, $b - 2 = 0$

$c = -1$, $a = 4$, $b = 2$

$\therefore$ a = 4, b = 2 and c = -1.

Exercise: 3.3.1

1. If $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ then find div $\vec{F}$ and curl $\vec{F}$.
2. If $\vec{F} = xyz\,\vec{i} + 3x^2y\,\vec{j} + (xz^2 - zy^3)\,\vec{k}$ then find div $\vec{F}$ and curl $\vec{F}$.
3. Find $\nabla \times (\nabla \varphi)$ where $\varphi = 3x^2y + 4y^3z + 3x^3yz$.
4. If $\varphi = x^3 + y^3 + 3xyz$; find curl (grad φ).
5. If $\vec{V} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, find div (curl $\vec{V}$).
6. If $\vec{F} = (2x + 2y + 2z)\,\vec{i} + (xy + yz + zx)\,\vec{j} + (3xz)\,\vec{k}$ then find $\nabla \times \vec{F}$ and $\nabla \times (\nabla \times \vec{F})$.
7. If $\vec{F} = (x^2 - y^2 + 2xz)\,\vec{i} + (xz - xy + yz)\,\vec{j} + (z^2 + x^2)\,\vec{k}$ then find $\nabla \times \vec{F}$ and $\nabla \times (\nabla \times \vec{F})$.
8. Show that the following vectors are solenoidal

   i) $\vec{F} = (x + 3y)\,\vec{i} + (y - 3z)\,\vec{j} + (x - 2z)\,\vec{k}$

   ii) $\vec{F} = 5y^4z^3\,\vec{i} + 8xz^2\,\vec{j} + y^2x\,\vec{k}$

   iii) $\vec{F} = (2x - 5y)\,\vec{i} + (x - y)\,\vec{j} + (3x - z)\,\vec{k}$

9. Find 'a' such that, $(3x - 2y + z)\,\vec{i} + (4x + ay - z)\,\vec{j} + (x - y - 2z)\,\vec{k}$ is solenoidal.

   [Ans. a = - 5)

10. Show that the following vectors are irrotational.

   i) $\vec{F} = (2xy - z^2)\,\vec{i} + (x^2 + 2yz)\,\vec{j} + (y^2 - 2zx)\,\vec{k}$

   ii) $\vec{F} = (3x^2 + 2y^2 + 1)\,\vec{i} + (4xy - 3y^2z - 3)\,\vec{j} + (2 - y^3)\,\vec{k}$

   iii) $\vec{F} = (y^2 + 2xz^2)\,\vec{i} + (2xy - z)\,\vec{j} + (2x^2z - y + 2z)\,\vec{k}$

   iv) $\vec{F} = (\sin y + z)\,\vec{i} + (x\cos y - z)\,\vec{j} + (x - y)\,\vec{k}$

11. Show that $\vec{F} = (y^2 - z^2 + 3yz - 2x)\,\vec{i} + (3xz + 2xy)\,\vec{j} + (3xy - 2xz + 2z)\,\vec{k}$ is both solenoidal and irrotational.
12. Find the constants a, b, c so that the following vector is irrotational

   i) $\vec{F} = (x + 2y + az)\,\vec{i} + (bx - 3y - z)\,\vec{j} + (4x + cy + 2z)\,\vec{k}$

   [Ans. a = 4, b = 2, c = -1]

   ii) $\vec{F} = (axy + bz^3)\,\vec{i} + (3x^2 - z)\,\vec{j} + (3xz^2 - y)\,\vec{k}$ [Ans. a = 6, b = 1, c = -1]

   iii) $\vec{F} = (axy - z^3)\,\vec{i} + (a - 2)\,x^2\,\vec{j} + (1 - a)\,xz^2\,\vec{k}$ [Ans. a =4]

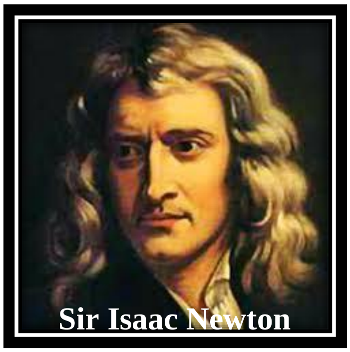
UNIT - IV INTEGRAL CALCULUS - I

Chapter 4.1 INTEGRATION - DECOMPOSITION METHOD

Introduction


Gottfried Wilhelm Leibniz

Integral calculus is an important part of calculus, as important as differential calculus. In differential calculus we study the relationship between two quantities, let's say between distance and time. For this relationship we usually use the rate of change between two variables.


Sir Isaac Newton

In Integral calculus, however, we take the inverse process of the relationship between two quantities. This is known as integration, anti-differentiation or anti-derivative. The most important application of integral calculus is to compute the area or volume. In ancient times, the informal concepts were developed by the Greek mathematicians Archimedes (287 BC - 212 BC) and Eudoxus (410 BC - 347 BC). They developed the approximate area of different geometric shapes and these basic methods were also developed by Chinese mathematician Liu Hui around the 3rd century to find the area of a circle. In the 17th Century John Kepler further developed some important concepts regarding astronomical investigations to find the area of a sector and the area of an ellipse. The concept of integral calculus was formally developed further by Isaac Newton (1643-1727) and Gottfried Leibniz (1646-1716) they developed basic concepts to find area and volume.

In Integral calculus, we encounter different concepts such as the area of various geometric shapes, the area under the curve by using the definite integral, the indefinite integral and various practical applications. We also encounter the most important theorem of calculus called the "Fundamental Theorem of Calculus."

One cannot imagine a world without differentiation and integration. In this century, we witnessed remarkable scientific advancement owing to the ingenious application of these two basic components of Mathematics. Calculus serve as unavoidable tool for finding solution to the variety of problems that arise in Physics, Astronomy, Engineering, Chemistry, Geology and Biology.

Calculus deals principally with two geometric problems.

i) The problem of finding **SLOPE** of the tangent line to the curve, is studied by the limiting process known as **differentiation** and

ii) The Problem of finding the **AREA** of a region under a curve is studied by another limiting process called **Integration**.

Introduction to Integration

The Integral calculus is concerned with the inverse problem namely given the derivative of a function to find the function. In symbol we require to find f(x) where

$$\frac{d}{dx} f(x) = g(x) \text{ and } g(x) \text{ is given.}$$

Then we write as $f(x) = \int g(x)\, dx$. Thus we define integration as follows.

The integral of the function g(x) with respect to x is the function whose derivative with respect to x is g(x) and is written as $\int g(x)\, dx$.

Illustrations

$\int 4x^3\, dx \quad = x^4$ since $\frac{d}{dx} x^4 = 4x^3$.

$\int \cos x\, dx \quad = \sin x$ since $\frac{d}{dx}(\sin x) = \cos x$.

$\int \frac{1}{x}\, dx \quad = \log x$ since $\frac{d}{dx}(\log x) = \frac{1}{x}$

Arbitrary Constant

Since the derivative of a constant is zero, there is no exact value for the integral. In particular, the above three results can be expressed in a more general way.

$\int 4x^3\, dx \quad = x^4$ since $\frac{d}{dx}(x^4 + c) = 4x^3$.

$\int \cos x\, dx \quad = \sin x + c$ since $\frac{d}{dx}(\sin x + c) = \cos x$.

$\int \frac{1}{x}\, dx \quad = \log x$ since $\frac{d}{dx}(\log x + c) = \frac{1}{x}$

Hence an arbitrary constant is always added to the result of an integration.

i.e. $\int g(x)\, dx = f(x) + c$

This is known as the indefinite integral of g(x). The 'dx' which appears in the integral indicates that the integration is with respect to x. In indefinite integrals an arbitrary constant c is always added.

General Rules:

(i) $\int k\, f(x)\, dx = k \int f(x)\, dx$

(ii) $\int [a\, f(x) + h\, g(x)]\, dx = a \int f(x)\, dx + b \int g(x)\, dx$.

FUNDAMENTAL RULES OF INTEGRATION

STANDARD RESULTS

DERIVATIVES	ANTIDERIVATIVES
$\frac{d}{dx}(c) = 0$, Where 'c' is a constant	$\int o\,dx = c$, Where 'c' is a constant
$\frac{d}{dx}(kx) = k$, Where 'k' is a constant	$\int k\,dx = kx + c$, Where 'c' is a constant
$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$	$\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$
$\frac{d}{dx}(\log x) = \frac{1}{x}$	$\int \frac{1}{x}\, dx = \log x + c$
$\frac{d}{dx}(-\cos x) = \sin x$	$\int \sin x\, dx = -\cos x + c$

DERIVATIVES	ANTIDERIVATIVES
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x \, dx = \sin x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x \, dx = \tan x + c$
$\frac{d}{dx}(-\cot x) = \mathrm{cosec}^2 x$	$\int \mathrm{cosec}^2 x \, dx = -\cot x + c$
$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$	$\int \sec x \cdot \tan x \, dx = \sec x + c$
$\frac{d}{dx}(-\mathrm{cosec}\, x) = \mathrm{cosec}\, x \cdot \cot x$	$\int \mathrm{cosec}\, x \cdot \cot x \, dx = -\mathrm{cosec}\, x + c$
$\frac{d}{dx}(e^x) = e^x$	$\int e^x \, dx = e^x + c$
$\frac{d}{dx}(x) = 1$	$\int dx = x + c$
$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{\sqrt{x}} \, dx = 2\sqrt{x} + c$
$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + c$
$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} \, dx = \tan^{-1} x + c$
$\frac{d}{dx}(\sin mx) = m \cos mx$	$\int \cos mx \, dx = \frac{\sin mx}{m} + c$
$\frac{d}{dx}(-\cos mx) = m \sin mx$	$\int \sin mx \, dx = \frac{-\cos mx}{m} + c$
$\frac{d}{dx}(e^{mx}) = m e^{mx}$	$\int e^{mx} \, dx = \frac{e^{mx}}{m} + c$

Note:

1) $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a\,(n+1)} + c$

2) $\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

3) $\int \frac{1}{ax+b} \, dx = \frac{1}{a} \log(ax+b) + c$

4) $\int \sin(ax+b) \, dx = \frac{-1}{a} \cos(ax+b) + c$

5) $\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + c$

INTEGRATION USING DECOMPOSITION METHOD:

In integration there is no rule for multiplication (or) division of algebraic (or) trigonometric function as we have in differentiation. Such functions are to be decomposed into addition and subtraction before integration.

For example,

$\frac{\cos^2 x}{1-\sin x}$ can be decomposed as follows.

$$\frac{\cos^2 x}{1-\sin x} = \frac{1-\sin^2 x}{1-\sin x} = \frac{1^2-\sin^2 x}{1-\sin x} = \frac{(1+\sin x)(1-\cancel{\sin} x)}{1-\cancel{\sin} x} = 1 + \sin x.$$

Note:

In doing problem, when there is a product of two or more polynomial functions, multiply them and then integrate. If a product of two trigonometric functions of the type sin2x cos3x, use product formula and then integrate.

Worked Examples:

1) **Evaluate:** $\int (x+3)(x+1)dx$

Solution:

Given, $\int (x+3)(x+1)\,dx$

$$= \int (x^2 + x + 3x + 3)\,dx$$
$$= \int (x^2 + 4x + 3)\,dx$$
$$= \int x^2\,dx + 4\int x\,dx + 3\int dx$$
$$= \frac{x^3}{3} + 4\,\frac{x^2}{2} + 3x + c$$
$$= \frac{x^3}{3} + 2x^2 + 3x + c$$

2) **Evaluate:** $\int (2 + x^3)^2\,dx$

Solution:

Given, $\int (2 + x^3)^2\,dx$

$$(a + b)^2 = a^2 + b^2 + 2ab$$
$$= \int (4 + x^6 + 4x^3)\,dx$$
$$= 4\int dx + \int x^6\,dx + 4\int x^3dx$$
$$= 4x + \frac{x^7}{7} + 4\,\frac{x^4}{4} + c$$
$$= 4x + \frac{x^7}{7} + x^4 + c$$

3) **Evaluate:** $\int \frac{3x^3 - x^2 + 5x + 2}{x^5}\, dx$

Solution:

Given, $\int \frac{3x^3 - x^2 + 5x + 2}{x^5}\, dx$

$$= \int \left(\frac{3x^3}{x^5} - \frac{x^2}{x^5} + \frac{5x}{x^5} + \frac{2}{x^5}\right) dx$$

$$= \int \left(\frac{3}{x^2} - \frac{1}{x^3} + \frac{5}{x^4} + \frac{2}{x^5}\right) dx$$

$$= 3\int \frac{1}{x^2} dx - \int \frac{1}{x^3} dx + 5\int \frac{1}{x^4} dx + 2\int \frac{1}{x^5}\, dx$$

$$= 3\int x^{-2}\, dx - \int x^{-3}\, dx + 5\int x^{-4}\, dx + 2\int x^{-5}\, dx$$

$$= 3 . \left(\frac{x^{-2+1}}{-2+1}\right) - \frac{x^{-3+1}}{-3+1} + 5\left(\frac{x^{-4+1}}{-4+1}\right) + 2\left(\frac{x^{-5+1}}{-5+1}\right) + c$$

$$= \frac{3x^{-1}}{-1} - \frac{x^{-2}}{-2} + \frac{5x^{-3}}{-3} + \frac{\cancel{2}x^{-4}}{\underset{-2}{\cancel{-4}}} + c$$

$$= \frac{-3}{x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{1}{-2x^4} + c$$

$$= \frac{-3}{x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{1}{-2x^4} + c$$

4) **Evaluate:** $\int \tan^2 x\, dx$

Solution:

Given, $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx$

$= \int \sec^2 x\, dx - 1\int dx$ $\quad [\because \tan^2\theta = \sec^2\theta - 1]$

$= \tan x - x + c$

5) **Evaluate:** $\int (\tan x + \cot x)^2\, dx$

Solution:

Given, $\Rightarrow \int (\tan x + \cot x)\, dx$

$$\Rightarrow \int (\tan^2 x + 2\tan x . \cot x + \cot^2 x)\, dx$$

$$= \int \left(\tan^2 x + 2\cancel{\tan x}\, \frac{1}{\cancel{\tan x}} + \cot^2 x\right) dx$$

$$= \int \tan^2 x\, dx + 2\int dx + \int \cot^2 x\, dx$$

$$= \int (\sec^2 x - 1)\, dx + 2\int dx + \int (\mathrm{cosec}^2 x - 1)\, dx$$

$$= \int \sec^2 x\, dx - \int dx + 2\int dx + \int \mathrm{cosec}^2 x - \int dx$$

$$= \tan x - x + 2x - \cot x - x + c$$

$$= \tan x - \cot x + c$$

6) **Evaluate:** $\int \frac{\sin^2 x}{1-\cos x}\, dx$

Solution:

Given, $\int \frac{\sin^2 x}{1-\cos x}\, dx$

$= \int \frac{1-\cos^2 x}{1-\cos x}\, dx$

$= \int \frac{(1+\cos x)(1-\cancel{\cos} x)}{(1-\cancel{\cos} x)}\, dx$

$= \int (1+\cos x)\, dx$

$= \int dx + \int \cos x\, dx$

$= x + \sin x + c$

7) **Evaluate:** $\int \sqrt{1+\sin 2x}\, dx$

Solution:

Given, $\int \sqrt{1+\sin 2x}\, dx$

$= \int \sqrt{\sin^2 x + \cos^2 x + 2\sin x \cos x}\, dx$

$= \int \sqrt{(\sin x + \cos x)^2}\, dx$ $\quad \because \sin 2x = 2 \sin x \cos x$

$= \int (\sin x + \cos x)\, dx$ $\quad \because \sin^2 x + \cos^2 x = 1$

$= \int \sin x\, dx + \int \cos x\, dx$

$= -\cos x + \sin x + c$

8) **Evaluate:** $\int \frac{1}{1+\sin x}\, dx$

Solution:

Given, $\int \frac{1}{1+\sin x}\, dx$

$= \int \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\, dx$

$= \int \frac{1-\sin x}{1-\sin^2 x}\, dx$

$= \int \frac{1-\sin x}{\cos^2 x}\, dx$

$= \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos x \cdot \cos x}\right) dx$

$= \int \left(\sec^2 x - \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}\right) dx$

$= \int (\sec^2 x - \sec x . \tan x)\, dx$

$= \int \sec^2 x\, dx - \int \sec x . \tan x\, dx$

$= \tan x - \sec x + c$

9) **Evaluate:** $\int \sin^3 x\, dx$

Solution:

Given, $= \int \sin^3 x\, dx$

$= \int \left(\frac{3}{4} \sin x - \frac{1}{4} \sin 3x\right) dx$

$= \frac{3}{4} \int \sin x\, dx - \frac{1}{4} \int \sin 3x\, dx$

$= \frac{3}{4}(-\cos x) - \frac{1}{4} \int \left(-\frac{\cos 3x}{3}\right) + c$

$= \frac{-3\cos x}{4} + \frac{\cos 3x}{12} + c$

$\sin 3x = 3\sin x - 4\sin^3 x$

$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$

$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

10) **Evaluate:** $\int \cos^3 x\, dx$

Solution:

Given, $= \int \cos^3 x\, dx$

$= \int \left(\frac{3}{4} \cos x + \frac{1}{4} \cos 3x\right) dx$

$= \frac{3}{4} \int \cos x\, dx + \frac{1}{4} \int \cos 3x\, dx$

$= \frac{3}{4} \sin x + \frac{1}{4} \left(\frac{\sin 3x}{3}\right) + c$

$= \frac{3}{4} \sin x + \frac{1}{12} \sin 3x + c$

$\cos 3x = 4\cos^3 x - 3\cos x$

$\cos^3 x = \frac{\cos 3x + 3\cos x}{4}$

11) **Evaluate:** $\int \sin^2 x\, dx$

Solution:

Given, $= \int \sin^2 x\, dx$

$\int \sin^2 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right) dx$

$= \frac{1}{2} [\int dx - \int \cos 2x\, dx]$

$= \frac{1}{2} \left[x - \frac{\sin 2x}{2}\right] + c$ $\qquad \because \sin^2 x = \frac{1-\cos 2x}{2}$

$= \frac{x}{2} - \frac{\sin 2x}{4} + c$

12) **Evaluate:** $\int \sin 3x \cos 2x\, dx$

Solution:

Given, $\int \sin 3x \cos 2x\, dx$

$2\sin A \cos B = \sin(A+B) + \sin(A-B)$

$\text{Sin } 3x \cos 2x = \frac{1}{2}[\sin(3x+2x) + \sin(3x-2x)]$

$= \frac{1}{2}[\sin 5x + \sin x]$

$\int \sin 3x \cos 2x\, dx = \frac{1}{2}[\int [\sin 5x + \sin x]\, dx]$

$= \frac{1}{2}[\int \sin 5x\, dx + \int \sin x\, dx]$

$$= \frac{1}{2}\left[\left(\frac{-\cos 5x}{5}\right) - \cos x\right] + c$$

$$= -\frac{-\cos 5x}{10} - \frac{\cos x}{2} + c$$

13) **Evaluate:** $\int \sin 7x \sin 4x\, dx$

Solution:

Given, $\int \sin 7x \sin 4x\, dx$

$$2 \sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$= \frac{1}{2}[\cos(7x - 4x) - \cos(7x + 4x)]$$

$$= \frac{1}{2}[\cos(3x) - \cos(11x)]$$

$$\int \sin 7x \sin 4x\, dx = \frac{1}{2}\left[\int (\cos 3x - \cos 11x)\, dx\right]$$

$$= \frac{1}{2}\left[\int \cos 3x\, dx - \int \cos 11x\, dx\right]$$

$$= \frac{1}{2}\left[\frac{\sin 3x}{3} - \frac{\sin 11x}{11}\right] + c$$

Simple Applications:

1) If $f'(x) = 3x^2 - 4x + 5$ and $f(1) = 3$, then find $f(x)$

Solution:

Given, $= f'(x) = \frac{d}{dx}[f(x)]$

$$f'(x) = 3x^2 - 4x + 5$$

Integrating on both sides, will respect to 'x', we get,

$$\int f'(x)\, dx = \int (3x^2 - 4x + 5)\, dx$$

$$f(x) = 3\int x^2\, dx - 4\int x\,dx + 5\int dx$$

$$= \cancel{3}\left(\frac{x^3}{\cancel{3}}\right) - \overset{2}{\cancel{4}}\left(\frac{x^2}{\cancel{2}}\right) + 5x + c$$

$$f(x) = x^3 - 2x^2 + 5x + c.$$

Since $f(1) = 3$.

$$f(1) = 3 \Rightarrow 3 = (1)^3 - 2(1)^2 + 5(1) + c$$

$$3 = 1 - 2 + 5 + c$$

$$3 = 4 + c$$

$$c = 3 - 4$$

$$\boxed{c = -1}$$

$$\therefore \boxed{f(x) = x^3 - 2x^2 + 5x - 1}$$

2) A train started from Madurai Junction towards Coimbatore at 3 pm (time t = 0) with velocity V (t) = 20 t + 50 km/Hour, where 't' is measured in hours. Find the distance covered by the train at 5 p.m.

Solution:

In Calculus terminology, Velocity V = ds/dt is rate of change of position with respect to time, where S is the distance.

The velocity of the train ns given by

$$V(t) = 20t + 50$$

$$\therefore \frac{ds}{dt} = 30t + 50$$

To find the distance function S one has to integrate the derivative function.

i.e $\quad S = \int (20t + 50)\, dt$

$$S = 20 \int t\,dt + 50 \int dt$$

$$= \cancel{20}^{\,10} \left(\frac{t^2}{\cancel{2}}\right) + 50t + c$$

$$S = 10t^2 + 50t + c$$

Since, the distance covered by the train is zero when time is zero.

Let us use this initial condition S = 0 at t = 0. To find the value 'c' of the constant of integration.

$$\Rightarrow S = 10t^2 + 50t + c \Rightarrow c = 0.$$

$$\therefore S = 10t^2 + 50t.$$

The distance covered by the train in 2 hours.

(5 pm – 3 pm) is given by, substituting t = 2 in the above equation, we get

$$S = 10(2)^2 + 50(2) = 140 \text{ km.}$$

Exercise: 4.1.1

1. Evaluate: $\int \frac{dx}{1 - \cos x}$
2. Evaluate: $\int \frac{\cos^2 x}{1 + \sin x}\, dx$
3. Evaluate: $\int (x - 2)(x + 3)^2\, dx$
4. Evaluate: $\int \frac{dx}{1 + \cos x}\, dx$
5. Evaluate: $\int \frac{\sin^2 x}{1 + \cos x}\, dx$
6. Evaluate: $\int (\sin x + \cos x)^2\, dx$
7. Evaluate: $\int x(x^2 - 1)\, dx$
8. Evaluate: $\int \sin 7x \,.\, \sin 3x\, dx$
9. Evaluate: $\int \sin 5x \,.\, \cos 3x\, dx$
10. Evaluate: $\int \frac{dx}{1 - \sin x}\, dx$
11. Evaluate: $\int (1 + x + x^2)(1 - x - x^2)\, dx$
12. Evaluate: $\int \sin 3x \,.\, \sin nx$
13. Evaluate: $\int \left(\frac{2x^2 - 3x^2 + 4x + 6}{x^3}\right) dx$

Simple application - problems

14. If $f'(x) = 4x - 5$ and $f(2) = 1$, find $f(x)$.

15. If $f'(x) = 9x^2 - 6x$ and $f(0) = -3$, find $f(x)$.

Exercise: 4.1.1 - Answers:

1) $-\cot x - \operatorname{cosec} x + c$

2) $x + \cos x + c$

3) $\frac{x^4}{4} + \frac{4x^3}{3} - \frac{3x^2}{2} - 18x + c$

4) $\tan x - \sec x + c$

5) $x - \sin x + c$

6) $x - \frac{\cos 2x}{2} + c$

7) $\frac{x^4}{4} - \frac{x^2}{2} + c$

8) $\frac{\sin 4x}{8} - \frac{\sin 10x}{20} + c$

9) $\frac{-\cos 8x}{16} - \frac{\cos 2x}{4} + c$

10) $\tan x + \sec x + c$

11) $x - \frac{x^3}{3} - \frac{x^4}{2} - \frac{x^5}{5} + c$

12) $\frac{\sin 2x}{4} - \frac{\sin 4x}{8} + c$

13) $2 \log x - 3x - 4/x - 3/x^2 + c$

14) $f(x) = 2x^2 - 5x + 3$

15) $f(x) = 3(x^3 - x^2 - 1)$

Chapter 4.2 METHODS OF INTEGRATION - INTEGRATION BY SUBSTITUTION

The method of substitution (change of variable)

This method is used to reduce a seemingly complex integrand to a known simple form.

Consider complex integral $\int F(h(x))\, h'(x)\, dx$,

Let $h(x) = z$ [Proceeding with the new variable we can successfully integrate the resulting function]

$h'(x)\, dx = dz$

$\Rightarrow \int F(h(x))\, h'(x)\, dx = \int F(z)\, dz$

Note:

(i) If the integrand is of the form $\frac{f'(x)}{f(x)}$ we put $f(x) = t$ and $f'(x)\ dx = dt$

$$\text{Thus, } \int \frac{f'(x)}{f(x)} dx = \int \frac{dt}{t}$$

$$= \log t + c$$

$$\boxed{\int \frac{f'(x)}{f(x)} dx = \log f(x) + c}$$

(ii) When the integrand is of the form $\frac{f'(x)}{\sqrt{f(x)}}$, we put $f(x) = t$ and $f'(x)\, dx = dt$

$$\text{Thus } \int \frac{f'(x)}{\sqrt{f(x)}} dx = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + c$$

$$\int \frac{f^1(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$$

(iii) When the integrand is of the form $[f(x)]^n\ f'(x)$, we put $f(x) = t$ and $f'(x)dx = dt$

$$\text{Thus } \int [f(x)]^n\ f'(x)\, dx = \int t^n\, dt + c$$

$$= \frac{t^{n+1}}{n+1} + c,\ n \neq -1$$

$$\boxed{\int [f(x)]^n\ f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1} + c}$$

Example 1:

Evaluate: $\int \frac{2x}{x^2+1}\ dx$

Solution:

$$\int \frac{2x}{x^2+1}\ dx$$

Put $t = x^2 + 1$

$\frac{dt}{dx} = 2x$

$dt = 2xdx$

$$\therefore \int \frac{2x}{x^2+1}\ dx = \int \frac{dt}{t}$$

$= \log t + c$

$= \log (x^2 + 1) + c$

Example 2:

Evaluate: $\int \frac{2x+9}{x^2+9x+30}\, dx$

Solution:

$\int \frac{2x+9}{x^2+9x+30}\, dx$

Put $t = x^2 + 9x + 30$

$\frac{dt}{dx} = 2x + 9 (1) + 0$

$dt = (2x + 9)\, dx$

$= \int \frac{dt}{t}$

$= \log t + c$

$= \log (x^2 + 9x + 30) + c$

Example 3:

Evaluate: $\int \frac{4x+2}{\sqrt{x^2+x+1}}\, dx$

Solution:

$\int \frac{4x+2}{\sqrt{x^2+x+1}}\, dx$

$= 2 \int \frac{(2x+1)}{\sqrt{x^2+x+1}}\, dx$

Put $t = x^2 + x + 1$

$\frac{dt}{dx} = 2x + 1 + 0$

$dt = (2x + 1)\, dx$

$= 2 \int \frac{dt}{\sqrt{t}}$

$= 2 (2\sqrt{t}) + c$

$= 4\sqrt{t} + c$

$= 4\sqrt{x^2 + x + 1} + c$

Example 4:

Evaluate: $\int \frac{x}{\sqrt{4-x^2}}\, dx$

Solution:

$\int \frac{x}{\sqrt{4-x^2}}\, dx$

$t = 4 - x^2$

$dt = -2x\, dx$

$$\frac{dt}{-2} = xdx$$

$$= \int \frac{\frac{-1}{2}\,dt}{\sqrt{t}}$$

$$= \frac{1}{-2}\, 2\sqrt{t} + c$$

$$= -\sqrt{4 - x^2} + c$$

Example 5:

Evaluate: $\int (4x+3)\sqrt{4x^2+6x+1}\;dx$

Solution:

$$\int (4x+3)\sqrt{4x^2+6x+1}\;dx$$

Put t $= 4x^2 + 6x + 1$

$$\frac{dt}{dx} = 8x + 6\,(1) + 0$$

$$\frac{dt}{dx} = (8x + 6)\,dx$$

$$\frac{dt}{dx} = 2\,(4x + 3)\,dx$$

$$\frac{dt}{2} = (4x + 3)\,dx$$

$$= \int \sqrt{t}\,.\,\frac{dt}{2}$$

$$= \int \frac{1}{2}\,t^{1/2}\;dt$$

$$= \frac{1}{2}\left[\frac{t^{3/2}}{3/2}\right] + c$$

$$= \frac{1}{3}\,t^{3/2} + c$$

$$= \frac{1}{3}\,(4x^2 + 6x + 1)^{3/2} + c$$

Example 6:

Evaluate: $\int x^3\sqrt{3+5x^4}\;dx$

Solution:

$$\int x^3\sqrt{3+5x^4}\;dx$$

Put t $= 3 + 5x^4$

$$\frac{dt}{dx} = 0 + 20x^3$$

$$dt = 20x^3\,dx$$

$$\frac{dt}{20} = x^3dx$$

$$= \int \sqrt{t}\,\frac{dt}{20}$$

$$= \frac{1}{20}\int t^{1/2}\;dt$$

$$= \frac{1}{20} \frac{t^{3/2}}{3/2} + c$$

$$= \frac{1}{30} (3 + 5x^4)^{3/2} + c$$

Example 7:

Evaluate: $\int \frac{1}{x \log x} dx$

Solution:

$$\int \frac{1}{x \log x} dx$$

$$= \int \frac{\frac{1}{x}}{\log x} dx$$

Put $t = \log x$

$$\frac{dt}{dx} = \frac{1}{x}$$

$$dt = \frac{1}{x} dx$$

$$= \int \frac{dt}{t}$$

$$= \log t + c$$

$$= \log (\log x) + c$$

Example 8:

Evaluate: $\int \frac{1}{1 + e^x} dx$

Solution:

$$\int \frac{1}{1 + e^x} dx$$

Multiply & divide by e^{-x}

$$= \int \frac{e^{-x} dx}{e^{-x} (1 + e^{-x})}$$

$$= \int \frac{e^{-x} dx}{e^{-x} + e^0}$$

$$= \int \frac{e^{-x} dx}{e^{-x} + 1}$$

Put $t = e^{-x} + 1$

$$\frac{dt}{dx} = -e^{-x} + 0$$

$$dt = -e^{-x} dx$$

$$-dt = e^{-x} dx$$

$$\therefore \int \frac{e^{-x} dx}{e^{-x} + 1} = \int \frac{-dt}{t}$$

$= -\log t + c$

$= -\log (e^{-x} + 1) + c$

Example 9:

Evaluate: $\int \frac{\sin x}{1-\cos x}\, dx$

Solution:

$\int \frac{\sin x}{1-\cos x}\, dx$

Put $t = 1 - \cos x$

$\frac{dt}{dx} = 0 - (-\sin x)$

$dt = \sin x\, dx$

$= \int \frac{dt}{t}$

$= \log t + c$

$= \log (1 - \cos x) + c$

Example 10:

Evaluate the following integral. (i) $\int \tan x\, dx$, (ii) $\int \cot x\, dx$, (iii) $\int \sec x\, dx$, (iv) $\int \operatorname{cosec} x\, dx$

Solution:

(i) $\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$

Put $t = \cos x$

$\frac{dt}{dx} = -\sin x$

$dt = -\sin x\, dx$

$-dt = \sin x\, dx$

$= \int \frac{-dt}{t}$

$= -\log t + c$

$= -\log (\cos x) + c$

$= \log (\cos x)^{-1} + c$

$= \log \left(\frac{1}{\cos x}\right) + c$

$$\boxed{\int \tan x\, dx = \log \sec x + c}$$

(ii) $\int \cot x\, dx = \int \frac{\cos x}{\sin x}\, dx$

Put $t = \sin x$

$\frac{dt}{dx} = \cos x$

$dt = \cos x\, dx$

$$= \int \frac{dt}{t}$$

$$= \log t + c$$

$$\boxed{\int \cot x \, dx = \log(\sin x) + c}$$

(iii) $\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$

Put $t = \sec x + \tan x$

$$\frac{dt}{dx} = \sec x \ \tan x + \sec^2 x$$

$$dt = \sec x(\tan x + \sec x) \, dx$$

$$= \int \frac{dt}{t}$$

$$= \log t + c$$

$$\boxed{\int \sec x \, dx = \log(\sec x + \tan x) + c}$$

(iv) $\int \operatorname{cosec} x \, dx = \int \frac{\operatorname{cosec} x(\operatorname{cosec} x + \cot x)}{\operatorname{cosec} x + \cot x} dx$

Put $t = \operatorname{cosec} x + \cot x$

$$\frac{dt}{dx} = -\operatorname{cosec} x \ \cot x - \operatorname{cosec}^2 x$$

$$= -\operatorname{cosec} x(\cot x + \cos e \, x) \, dx$$

$$-dt = \operatorname{cosec} x(\cot x + \cos e \, cx)$$

$$= \int \frac{-dt}{t}$$

$$= -\log t + c$$

$$\boxed{\int \operatorname{cosec} x \, dx = -\log(\operatorname{cosec} x + \cot x) + c}$$

Example 11:

Evaluate the following integrals.

(i) $\int (5x + 2)^7 \, dx$ (ii) $\int \sin^3 x \cos x \, dx$

(iii) $\int \cos(ax + b) \, dx$ (iv) $\int \frac{(1 + \log x)^2}{x} dx$

(v) $\int \sec^4 x \tan x \, dx$

Solution:

(i) $\int (5x + 2)^7 \, dx$

Put $t = 5x + 2$

$$\frac{dt}{dx} = 5$$

$$dt = 5 \, dx$$

$$\frac{dt}{5} = dx$$

$$= \int t^7 . \frac{dt}{5}$$

$$= \frac{1}{5}\ \frac{t^8}{8} + c$$

$$= \frac{1}{40}\ (5x+2)^8 + c$$

(ii) $\int \sin^3 x \cos x\, dx$

Put $t = \sin x$

$$\frac{dt}{dx} = \cos x$$

$$dt = \cos x\, dx$$

$$= \int t^3\, dt$$

$$= \frac{t^4}{4} + c$$

$$= \frac{\sin^4 x}{4} + c$$

(iii) $\int \cos(ax+b)\, dx$

Put $t = ax + b$

$$\frac{dt}{dx} = a$$

$$dt = a\, dx$$

$$\frac{dt}{a} = dx$$

$$= \int \cos t \,.\, \frac{dt}{a}$$

$$= \frac{1}{a} \int \cos t \,.\, dt$$

$$= \frac{1}{a} \sin t + c$$

$$= \frac{1}{a} \sin(ax+b) + c$$

(iv) $\int \frac{(1+\log x)^2}{x}\, dx$

$$= \int (1+\log x)^2\ \frac{1}{x}\ dx$$

Put $t = 1 + \log x$

$$\frac{dt}{dx} = 0 + \frac{1}{x}$$

$$dt = 0 + \frac{1}{x}\ dx$$

$$= \int t^2 \,.\, dt$$

$$= \frac{t^3}{3} + c$$

$$= \frac{(1+\log x)^3}{3} + c$$

(v) $\int \sec^4 x \tan x \, dx$

$$= \int \sec^3 x \,.\, \sec x \tan x \, dx$$

Put $t = \sec x$

$$\frac{dt}{dx} = \sec x \tan x$$

$$dt = \sec x \tan x \, dx$$

$$= \int t^3 \,.\, dt$$

$$= \frac{t^4}{4} + c$$

$$= \frac{\sec^4 x}{4} + c$$

Note:

When the integrand is of the form F [f (x)] . f' (x). We put f (x) = t & f' (x) dx = dt

Thus $\int F[f(x)] \,.\, f'(x) \, dx = \int F(t) \, dt$

$$= G(t) + c$$

$$= G[f(x)] + c$$

Example 12:

Evaluate the following integrals.

(i) $\int \frac{\sin^2 (\log x)}{x} dx$ (ii) $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

(iii) $\int \frac{\tan x}{\log \sec x} dx$ (iv) $\int \frac{e^x (1+x)}{\cos^2 (xe^x)} dx$

Solution:

(i) $\int \frac{\sin^2 (\log x)}{x} dx$

$$= \int \sin^2 (\log x) \,.\, \frac{1}{x} dx$$

Let $t = \log x$

$$dt = \frac{1}{x} dx$$

$$= \int \sin^2 t \, dt$$

$$= \int \frac{1 - \cos 2t}{2} dt$$

$$= \frac{1}{2} \int (1 - \cos 2t) \, dt$$

$$= \frac{1}{2} \left[t - \frac{\sin 2t}{2} \right] + c$$

$$= \frac{1}{2} \left[\log x - \frac{\sin 2 (\log x)}{2} \right] + c$$

(ii) $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

Put $t = \sqrt{x}$

$$\frac{dt}{dx} = \frac{1}{2\sqrt{x}}$$

$$dt = \frac{1}{2\sqrt{x}}\ dx$$

$$2\ dt = \frac{dx}{\sqrt{x}}\ dx$$

$$= \int \cos t\ .\ 2dt$$

$$= 2 \int \cos t\ .\ dt$$

$$= 2 \sin t + c$$

$$= 2 \sin \sqrt{x} + c$$

(iii) $\int \frac{\tan x}{\log \sec x}\ dx$

Put $t = \sec x$

$$\frac{dt}{dx} = \sec x \tan x$$

$$dt = \sec x \tan x\ dx$$

$$\therefore\ dt = t\ .\ \tan x\ dx$$

$$\frac{dt}{t} = \tan x\ dx$$

$$= \int \frac{\frac{dt}{t}}{\log t}$$

$$= \int \frac{\frac{1}{t}}{\log t}\ dt$$

$$= \log (\log t) + c$$

$$= \log [\log (\sec x)] + c$$

(iv) $\int \frac{e^x (1 + x)}{\cos^2 (xe^x)}\ dx$

Put $t = xe^x$

$$\frac{dt}{dx} = xe^x + 1\ .\ e^x$$

$$dt = e^x (x + 1)\ dx$$

$$= \int \frac{dt}{\cos^2 t}$$

$$= \int \sec^2 t\ dt$$

$$= \tan t + c$$

$$= \tan (xe^x) + c$$

Example 13:

Evaluate the following integrals.

(i) $\int x^2 e^{x^3} \cos (e^{x^3})\ dx$ (ii) $\int e^{\cos^2 x} \sin 2x\ dx$

(iii) $\int \frac{\sin (\tan^{-1} x)}{1 + x^2}\ dx$

Solution:

(i) $\int x^2 e^{x^3} \cos (e^{x^3})\, dx$

Put $t = e^{x^3}$

$\frac{dt}{dx} = e^{x^3} . 3x^2$

$dt = e^{x^3} . 3x^2\, dx$

$\frac{dt}{3} = e^{x^3} . x^2 . dx$

$= \int \cos t . \frac{dt}{3}$

$= \frac{1}{3} \sin t + c$

$= \frac{1}{3} \sin (e^{x^3}) + c$

(ii) $\int e^{\cos^2 x} . \sin 2x\, dx$

Put . $t = \cos^2 x$

$\frac{dt}{dx} = 2 \cos x (- \sin x)$

$dt = -2 \cos x \sin x\, dx$

$dt = - \sin 2x\, dx$

$- dt = \sin 2x\, dx$

$= \int e^t (-dt)$

$= - e^t + c$

$= - e^{\cos^2 x} + c$

(iii) $\int \frac{\sin (\tan^{-1} x)}{1 + x^2} dx$

Put $t = \tan^{-1} x$

$\frac{dt}{dx} = \frac{1}{1 + x^2}$

$dt = \frac{1}{1 + x^2} dx$

$= \int \sin t . dt$

$= - \cos t + c$

$= - \cos (\tan^{-1} x) + c$

Exercise :4.2.1

Evaluate the following problems

1. $\int \frac{x+3}{x^2+6x+4}\,dx$

2. $\int \frac{x}{\sqrt{a^2+x^2}}\,dx$

3. $\int \frac{1}{1-3^{-x}}\,dx$

4. $\int e^x \operatorname{cosec}^2 (e^x)\,dx$

5. $\int \cot x \,.\, \log \sin x\,dx$

6. $\int \frac{dx}{x-\sqrt{x}}$

7. $\int \frac{e^{2x}}{e^{2x}+1}\,dx$

8. $\int x^6 \sin (5x^7)\,dx$

9. $\int \frac{\cot (\log x)}{x}\,dx$

10. $\int \sin x \,.\, \sin (\cos x)\,dx$

11. $\int \frac{dx}{x\cos^2(1+\log x)}\,dx$

12. $\int \frac{1}{x\,(\log x)^{\,n}}\,dx$

13. $\int \frac{10x^9+10^x \log 10}{10^x+x^{10}}\,dx$

14. $\int \frac{e^{m \sin^{-1} x}}{\sqrt{1-x^2}}\,dx$

15. $\int \frac{dx}{\sqrt{x}\,(1+\sqrt{x}}$

16. $\int \frac{\sec^2 x}{(2+3\tan x)^3}\,dx$

17. $\int \tan x \sqrt{\sec x}\,dx$

18. $\int \frac{e^{\tan x}}{\cos^2 x}\,dx$

19. $\int (2e^x-3)^{11} e^x\,dx$

20. $\int e^{x \log x} (1+\log x)\,dx$

Exercise :4.2.1 - Answers

1. $\frac{1}{2}\log (x^2+6x+4)+c$

2. $\sqrt{a^2+x^2}+c$

3. $\log (1+e^{+x})+c$

4. $-\cot (e^x)+c$

5. $\frac{[\log (\sin x)]^2}{2}+c$

6. $2\log (\sqrt{x}-1)+c$

7. $\frac{1}{2}\log (e^{2x}+1)+c$

8. $\frac{-1}{30}\cos (5x^7)+c$

9. $\log \sin (\log x)+c$

10. $\cos (\cos x)+c$

11. $\tan (1+\log x)+c$

12. $\frac{(\log x)^{-n+1}}{-n+1}+c$

13. $\log (10^x+x^{10})+c$

14. $\frac{e^{m \sin^{-1} x}}{m}+c$

15. $2\log (1+\sqrt{x})+c$

16. $\frac{-(2+3\tan x)^{-2}}{6}+c$

17. $2\sqrt{\sec x}+c$

18. $e^{\tan x}+c$

19. $\frac{1}{2}\,\frac{(2e^x-3)^{12}}{12}+c$

20. $e^{x\log x}+c$

Chapter 4.3 STANDARD INTEGRALS

Integration of Rational Algebraic Functions

In this section we are going to discuss how to integrate the rational algebraic functions whose numerator and denominator. Contains some positive integral powers of x with constant co-efficients.

Formulae:

1. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}(x/a) + c$

2. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log[x + \sqrt{x^2 - a^2}] + c$

3. $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log[x + \sqrt{x^2 + a^2}] + c$

4. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}(x/a) + c$

5. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left[\frac{x-a}{x+a}\right] + c$

6. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left[\frac{a+x}{a-x}\right] + c$

Worked Examples:

1. **Evaluate:** $\int \sqrt{4 - 9x^2}\, dx$

Solution:

$$\int \sqrt{4 - 9x^2}\, dx = \int \sqrt{9\left(\frac{4}{9} - x^2\right)}\, dx$$

$$= 3\int \sqrt{\left(\frac{2}{3}\right)^2 - x^2}\, dx$$

$$\boxed{\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c}$$

$$= 3\left[\frac{x}{2}\sqrt{\left(\frac{2}{3}\right)^2 - x^2} + \frac{\left(\frac{2}{3}\right)^2}{2}\sin^{-1}\left(\frac{x}{2/3}\right)\right] + c$$

$$= 3\left[\frac{x}{2}\sqrt{\frac{4}{9} - x^2} + \frac{4}{18}\sin^{-1}\left(\frac{3x}{2}\right)\right] + c$$

$$= \frac{3x}{2}\sqrt{\frac{4}{9} - x^2} + \frac{2}{3}\sin^{-1}\left(\frac{3x}{2}\right) + c$$

2. **Evaluate:** $\int \sqrt{4 - x^2}\, dx$

Solution:

Given $\int \sqrt{4 - x^2}\, dx = \int \sqrt{2^2 - x^2}\, dx$

$$\boxed{\therefore \int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}(x/a) + c} \quad [\therefore \text{Hence } a = 2]$$

$$\therefore \int \sqrt{2^2 - x^2}\, dx = \frac{x}{2}\sqrt{2^2 - x^2} + \frac{2^2}{2}\sin^{-1}(x/2) + c$$

$$= \frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}(x/2) + c$$

3. **Evaluate:** $\int \sqrt{16x^2 - 25}\, dx$

Solution:

$$\int \sqrt{16x^2 - 25}\, dx$$

$$= \int \sqrt{16\left(x^2 - \frac{25}{16}\right)}\, dx$$

$$= 4\int \sqrt{x^2 - \frac{25}{16}}\, dx$$

$$= 4\int \sqrt{x^2 - \left(\frac{5}{4}\right)^2}\, dx$$

$$\boxed{\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\log\left[x + \sqrt{x^2 - a^2}\right] + c}$$

$$= 4\left\{\frac{x}{2}\sqrt{x^2 - \left(\frac{5}{4}\right)^2} - \frac{\left(\frac{5}{4}\right)^2}{2}\log\left[x + \sqrt{x^2 - \left(\frac{5}{4}\right)^2}\right]\right\} + c$$

$$= 4\left\{\frac{x}{2}\sqrt{x^2 - \frac{25}{16}} - \frac{25}{32}\log\left[x + \sqrt{x^2 - \frac{25}{16}}\right]\right\} + c$$

$$= 2x\sqrt{x^2 - \frac{25}{16}} - \frac{25}{8}\log\left[x + \sqrt{x^2 - \frac{25}{16}}\right] + c$$

4. **Evaluate:** $\int \sqrt{9x^2 + 16}\, dx$

Solution:

$$= \int \sqrt{9\left(x^2 + \frac{16}{9}\right)}\, dx$$

$$= 3\int \sqrt{x^2 + \frac{16}{9}}\, dx$$

$$= 3\int \sqrt{x^2 + \left(\frac{4}{3}\right)^2}\, dx$$

$$\boxed{\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\log\left[x + \sqrt{x^2 + a^2}\right] + c}$$

$$= 3\left\{\frac{x}{2}\sqrt{x^2 + \left(\frac{4}{3}\right)^2} + \frac{\left(\frac{4}{3}\right)^2}{2}\log\left[x + \sqrt{x^2 + \left(\frac{4}{3}\right)^2}\right]\right\} + c$$

$$= 3\left[\frac{x}{2}\sqrt{x^2 + \frac{16}{9}} + \frac{16}{8}\log\left(x + \sqrt{x^2 + \frac{16}{9}}\right)\right] + c$$

$$= \frac{3x}{2}\sqrt{x^2 + \frac{16}{9}} + \frac{16}{8}\log\left(x + \sqrt{x^2 + \frac{16}{9}}\right) + c$$

$$= \frac{3x}{2}\sqrt{x^2+\frac{16}{9}} + \frac{8}{3}\log\left(x+\sqrt{x^2+\frac{16}{9}}\right)+c$$

5. **Evaluate:** $\int \frac{dx}{x^2+4}$

Solution:

Given $\int \frac{dx}{x^2+4} = \int \frac{dx}{x^2+2^2}$

$$\boxed{\therefore \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)+c}$$

$$\therefore \int \frac{dx}{x^2+2^2} = \frac{1}{2}\tan^{-1}(x/2)+c$$

6. **Evaluate:** $\int \frac{dx}{4x^2+9}$

Solution:

Given $\int \frac{dx}{4x^2+9} = \frac{1}{4}\left[\int \frac{dx}{x^2+\frac{9}{4}}\right]$

$$\boxed{\therefore \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)+c \quad [\text{Hence } a = 3/2]}$$

$$\therefore \frac{1}{4}\left[\int \frac{dx}{x^2+\left(\frac{3}{2}\right)^2}\right] = \frac{1}{4}\left[\frac{1}{3/2}\tan^{-1}\left(\frac{x}{3/2}\right)\right]+c$$

$$\therefore \frac{1}{4}\left[\int \frac{dx}{x^2+\left(\frac{3}{2}\right)^2}\right] = \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+c$$

7. **Evaluate:** $\int \frac{dx}{(2x+3)^2+9}$

Solution:

$$\int \frac{dx}{(2x+3)^2+9} = \int \frac{dx}{2^2(x+3/2)^2+9}$$

$$= \frac{1}{4}\int \frac{dx}{(x+3/2)^2+9/4}$$

$$= \frac{1}{4}\int \frac{dx}{(x+3/2)^2+\left(\frac{3}{2}\right)^2}$$

$$\boxed{\int \frac{dx}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c}$$

$$= \frac{1}{4}\cdot\frac{1}{3/2}\tan^{-1}\left(\frac{x+3/2}{3/2}\right)+c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{2x+3}{3}\right)+c$$

8. **Evaluate:** $\int \frac{dx}{9-x^2}$

Solution:

Given $\int \frac{dx}{9-x^2} = \int \frac{dx}{3^2-x^2}$

$$\therefore \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left[\frac{a+x}{a-x}\right] + c \quad [\text{Hence } a = 3]$$

$$\therefore \int \frac{dx}{3^2 - x^2} = \frac{1}{2(3)} \log \left[\frac{3+x}{3-x}\right] + c$$

$$\therefore \int \frac{dx}{9 - x^2} = \frac{1}{6} \log \left[\frac{3+x}{3-x}\right] + c$$

9. **Evaluate:** $\int \frac{dx}{x^2 - 36}$

Solution:

$$\text{Given } \int \frac{dx}{x^2 - 36} = \int \frac{dx}{x^2 - 6^2}$$

$$\boxed{\therefore \int \frac{dx}{x^2 - 6^2} = \frac{1}{2a} \log \left[\frac{x-a}{x+a}\right] + c} \quad [\therefore \text{Hence } a = 6]$$

$$\int \frac{dx}{x^2 - 36} = \frac{1}{2 \text{ x } 6} \log \left[\frac{x-6}{x+6}\right] + c$$

$$\int \frac{dx}{x^2 - 36} = \frac{1}{12} \log \left[\frac{x-6}{x+6}\right] + c$$

10. **Evaluate:** $\int \frac{dx}{4x^2 - 81}$

Solution:

$$\int \frac{dx}{4x^2 - 81} = \frac{1}{4} \int \frac{dx}{4x^2 - 81/4}$$

$$= \frac{1}{4} \int \frac{dx}{x^2 - \left(\frac{9}{2}\right)^2}$$

$$\boxed{\int \frac{dx}{x^2 + a^2} = \frac{1}{2a} \log \left[\frac{x-a}{x+a}\right] + c}$$

$$= \frac{1}{4} \cdot \frac{1}{2 \cdot \frac{9}{2}} \log \left[\frac{x - 9/2}{x + 9/2}\right] + c$$

$$= \frac{1}{36} \log \left(\frac{2x-9}{2x+9}\right) + c$$

11. **Evaluate:** $\int \frac{dx}{(x+1)^2 - 9}$

Solution:

$$\text{Given } \int \frac{dx}{(x+1)^2 - 9} = \int \frac{dx}{(x+1)^2 - 3^2}$$

$$\boxed{\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left[\frac{x-a}{x+a}\right] + c} \quad [\therefore \text{Hence } a = 3]$$

$$\int \frac{dx}{(x+1)^2 - 3^2} = \frac{1}{2 \text{ x } 3} \log \left[\frac{x+1-3}{x+1+3}\right] + c$$

$$\int \frac{dx}{(x+1)^2 - 3^2} = \frac{1}{6} \log \left[\frac{x-2}{x+4}\right] + c$$

12. **Evaluate:** $\int \frac{dx}{\sqrt{4-9x^2}}$

Solution:

Given $\int \frac{dx}{\sqrt{4-9x^2}} =$

$\therefore \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}(x/a) + c$

$\therefore \int \frac{dx}{\sqrt{4-9x^2}} = \int \frac{dx}{\sqrt{9\left(\frac{4}{9} - x^2\right)}}$

$= \frac{1}{3} \int \frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2 - x^2}}$

$= \frac{1}{3} \sin^{-1}\left(\frac{x}{2/3}\right) + c$

$\boxed{\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}}$

$= \frac{1}{3} \sin^{-1}\left(\frac{3x}{2}\right) + c$

13. **Evaluate:** $\int \frac{dx}{\sqrt{16-x^2}}$

Solution:

Given $\int \frac{dx}{\sqrt{16-x^2}} = \int \frac{dx}{\sqrt{4^2 - x^2}}$

$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) + c$ $[\therefore$ Hence $a = 4]$

$\therefore \int \frac{dx}{\sqrt{4^2-x^2}} = \sin^{-1}(x/4) + c$

Exercise : 4.3.1

Integrate the following

(1) $\int \frac{dx}{1+x^2}$ (2) $\int \frac{dx}{\sqrt{1-x^2}}$ (3) $\int \frac{dx}{x^2+4}$

(4) $\int \frac{dx}{(3x+2)^2+16}$ (5) $\int \frac{dx}{(x+2)^2-4}$ (6) $\int \frac{dx}{\sqrt{25-(x-1)^2}}$

(7) $\int \frac{dx}{4+9x^2}$ (8) $\int \frac{dx}{9-4x^2}$ (9) $\int \frac{dx}{\sqrt{9-x^2}}$

(10) $\int \frac{dx}{x^2-25}$

Exercise : 4.3.1 - Answers

1. $\tan^{-1}(x) + c$

2. $\sin^{-1}(x) + c$

3. $\frac{1}{3}\tan^{-1}(x/3) + c$

4. $\frac{1}{12}\tan^{-1}\left(\frac{3x+2}{4}\right) + c$

5. $\frac{1}{4}\log\left(\frac{x}{x+4}\right) + c$

6. $\frac{1}{5}\sin^{-1}\left(\frac{x-1}{5}\right) + c$

7. $\frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right) + c$

8. $\frac{1}{12}\log\left(\frac{3+2x}{3-2x}\right) + c$

9. $\sin^{-1}(x/3) + c$

10. $\frac{1}{10}\log\left(\frac{x-5}{x+5}\right) + c$

UNIT - V

INTEGRAL CALCULUS II

5.1 METHODS OF INTEGRATION - INTEGRATION BY PARTS

5.1.1 Introduction

At this point, we have seen the integrals of the functions involving only one function. Integration by parts is applied when the integral is a product of two functions. The evaluation of the integral depends upon the proper choice of u and v.

We have observed that every differentiation rule gives rise to corresponding integration rule. We know the product rule is,

If u and v are functions of x,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides with respect to x, we get

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$\int d(uv) = \int udv + \int vdu$$

$$uv = \int udv + \int vdu$$

$$\Rightarrow \int udv = uv - \int vdu$$

This rule is called integration by parts.

Note :

1) Integration by parts method is generally used to find the integral when the integrand is a product of two different types of functions (or) a single logarithmic function (or) a single inverse trigonometric function (or) a function which is not integrable directly.

2) When both functions of Integrand has direct integral values then the function in the form of polynomial in x is taken as u and other function [Trigonometric or exponential is taken as dv].

3) When any one of the functions of Integrand does not have direct integral value (i.e., functions like log x, $\sin^{-1}x$, $\cos^{-1}x$ …) then that function is taken as u and other function is taken as dv.

Worked Examples:

Part - A

1) Evaluate : $\int xe^x\,dx$

Solution :

Let $u = x$	$dv = e^x\,dx$
$\frac{du}{dx} = 1$	$\int dv = \int e^x\,dx$
$du = dx$	$v = e^x$

$$\int udv = uv - \int vdu$$

$$\int x\,e^x\,dx = x\,e^x - \int e^x\,dx$$

$$= x\,e^x - e^x + c$$

2) Evaluate : $\int \log x \, dx$

Solution :

Let $u = \log x$	$dv = dx$
$\frac{du}{dx} = \frac{1}{x}$	$\int dv = \int dx$
$du = \frac{1}{x}\,dx$	$v = x$

$$\int udv = uv - \int vdu$$

$$\int \log x \, dx = \log x \; x - \int x \frac{1}{x} dx$$

$$= x \log x - \int 1dx$$

$$= x \log x - x + c$$

Part – B and C

1) Evaluate : $\int x \log x \, dx$

Solution :

Let $u = \log x$	$dv = xdx$
$\frac{du}{dx} = \frac{1}{x}$	$\int dv = \int xdx$
$du = \frac{1}{x}dx$	$v = \frac{x^2}{2}$

Integration by parts rule,

$$\int udv = uv - \int vdu$$

$$\int \log x \; x \, dx = (\log x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right) dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\int xdx + c$$

$$= \frac{x^2}{2}\log x - \frac{1}{2}\,\frac{x^2}{2} + c$$

$$= \frac{x^2}{2}\log x - \frac{x^2}{4} + c$$

2) Evaluate : $\int x^n \log x \, dx$

Solution :

Let $u = \log x$	$dv = x^n\,dx$
$\frac{du}{dx} = \frac{1}{x}$	$\int dv = \int x^n\,dx$
$du = \frac{1}{x}dx$	$v = \frac{x^{n+1}}{n+1}$

Integration by parts rule,

$$\int udv = uv - \int vdu$$

$$\int \log x \; x^n \, dx = (\log x)\left(\frac{x^{n+1}}{n+1}\right) - \int \left(\frac{x^{n+1}}{n+1}\right)\left(\frac{1}{x}\right) dx$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^n\,dx + c$$

$$= \frac{x^{n+1}}{n+1} \log x - \frac{1}{n+1}\left(\frac{x^{n+1}}{n+1}\right) + c$$

$$= \frac{x^{n+1}}{n+1} \log x - \frac{x^{n+1}}{(n+1)^2} + c$$

3) Evaluate : $\int x\, e^{2x}\, dx$

Solution :

Let $u = x$	$dv = e^{2x}\, dx$
$du = dx$	$\int dv = \int e^{2x}\, dx$
	$v = \frac{e^{2x}}{2}$

Integration by parts rule,

$$\int udv = uv - \int vdu$$

$$\int x\, e^{2x}\, dx = x\left(\frac{e^{2x}}{2}\right) - \int \frac{e^{2x}}{2} dx$$

$$= \frac{x\, e^{2x}}{2} - \frac{1}{2}\left(\frac{e^{2x}}{2}\right) + c$$

$$= \frac{x\, e^{2x}}{2} - \frac{e^{2x}}{4} + c$$

4) Evaluate : $\int x\, e^{nx}\, dx$

Solution :

Let $u = x$	$dv = e^{nx}\, dx$
$du = dx$	$\int dv = \int e^{nx}\, dx$
	$v = \frac{e^{nx}}{n}$

Integration by parts rule,

$$\int udv = uv - \int vdu$$

$$\int x\, e^{nx}\, dx = x\left(\frac{e^{nx}}{n}\right) - \int \frac{e^{nx}}{n} dx$$

$$= \frac{x\, e^{nx}}{n} - \frac{1}{n} \int e^{nx}\, dx$$

$$= \frac{x\, e^{nx}}{n} - \frac{1}{n}\left(\frac{e^{nx}}{n}\right) + c$$

$$= \frac{x\, e^{nx}}{n} - \frac{e^{nx}}{n^2} + c$$

5) Evaluate : $\int x\, e^{-x}\, dx$

Solution :

Let $u = x$	$dv = e^{-x}\, dx$
$du = dx$	$\int dv = \int e^{-x}\, dx$
	$v = \frac{e^{-x}}{-1} = -e^{-x}$

Integration by parts rule,

$$\int udv = uv - \int vdu$$

$$\int x\, e^{-x}\, dx = x\,(-e^{-x}) - \int (-e^{-x})\, dx$$

$$= -xe^{-x} + \int e^{-x}\,dx$$

$$= -xe^{-x} + \frac{e^{-x}}{-1} + c$$

$$= -xe^{-x} - e^{-x} + c$$

6) Evaluate : $\int x \sin 3x\,dx$

Solution :

Let $u = x$	$dv = \sin 3x\,dx$
$du = dx$	$\int dv = \int \sin 3x\,dx$
	$v = \frac{-\cos 3x}{3}$

Integration by parts rule,

$$\int udv = uv - \int vdu$$

$$\int x \sin 3x\,dx = x\left(\frac{-\cos 3x}{3}\right) - \int \left(\frac{-\cos 3x}{3}\right) dx$$

$$= \frac{-x\cos 3x}{3} + \frac{1}{3}\int \cos 3x\,dx$$

$$= \frac{-x\cos 3x}{3} + \frac{1}{3}\left(\frac{\sin 3x}{3}\right) + c$$

$$= \frac{-x\cos 3x}{3} + \frac{\sin 3x}{9} + c$$

7) Evaluate : $\int x \sin nx\,dx$

Solution :

Let $u = x$	$dv = \sin nx\,dx$
$du = dx$	$\int dv = \int \sin nx\,dx$
	$v = \frac{-\cos nx}{n}$

Integration by parts rule,

$$\int udv = uv - \int vdu$$

$$\int x \sin nx\,dx = x\left(\frac{-\cos nx}{n}\right) - \int \left(\frac{-\cos nx}{n}\right) dx$$

$$= \frac{-x\cos nx}{n} + \frac{1}{n}\int \cos nx\,dx$$

$$= \frac{-x\cos nx}{n} + \frac{1}{n}\left(\frac{\sin nx}{n}\right) + c$$

$$= \frac{-x\cos nx}{n} + \frac{\sin nx}{n^2} + c$$

8) Evaluate : $\int x \cos x\,dx$

Solution :

Let $u = x$	$dv = \cos x\,dx$
$du = dx$	$\int dv = \int \cos x\,dx$
	$v = \sin x$

Integration by parts rule,

$$\int udv = uv - \int vdu$$

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx$$

$$= x \sin x - (-\cos x) + c$$

$$= x \sin x + \cos x + c$$

9) Evaluate : $\int x \cos nx\, dx$

Solution :

Let $u = x$ | $dv = \cos nx\, dx$

$du = dx$ | $\int dv = \int \cos nx\, dx$

$v = \frac{\sin nx}{n}$

Integration by parts rule,

$$\int udv = uv - \int vdu$$

$$\int x \cos nx\, dx = x\left(\frac{\sin nx}{n}\right) - \int \frac{\sin nx}{n} dx$$

$$= \frac{x \sin nx}{n} - \frac{1}{n} \int \sin nx\, dx$$

$$= \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} + c$$

10) Evaluate : $\int x \cos^2 x\, dx$

Solution :

$$\text{Let } I = \int x \cos^2 x\, dx$$

$$= \int x \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{2} \int (x + x \cos 2x)\, dx$$

$$= \frac{1}{2}\left[\int x\, dx + \int x \cos 2x\, dx\right]$$

$$= \frac{1}{2}\left[\frac{x^2}{2} + \int x \cos 2x\, dx\right] \ldots . (1)$$

Consider $\int x \cos 2x\, dx$

Let $u = x$ | $dv = \cos 2x\, dx$

$du = dx$ | $\int dv = \int \cos 2x\, dx$

$v = \frac{\sin 2x}{2}$

Integration by parts rule,

$$\int udv = uv - \int vdu$$

$$\int x \cos 2x\, dx = x\left(\frac{\sin 2x}{2}\right) - \int \frac{\sin 2x}{2} dx$$

$$= \frac{x \sin 2x}{2} - \frac{1}{2} \int \sin 2x\, dx$$

$$= \frac{x \sin 2x}{2} - \frac{1}{2}\left(\frac{-\cos 2x}{2}\right)$$

$$= \frac{x \sin 2x}{2} + \frac{\cos 2x}{4} \quad \ldots . (2)$$

Using (2) in (1) we get

$$I = \frac{1}{2}\left[\frac{x^2}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}\right] + c$$

$$\int x \cos^2 x \, dx = \frac{1}{2}\left[\frac{x^2}{2} + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}\right] + c$$

11) Evaluate : $\int \sin^{-1} x \, dx$

Solution :

Let $u = \sin^{-1} x$	$dv = dx$
$du = \frac{1}{\sqrt{1-x^2}} dx$	$\int dv = \int dx$
	$v = x$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \, dx$$

Take $t = 1 - x^2$

$dt = -2x \, dx$

$\frac{dt}{-2} = x \, dx$

$$\therefore \int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{\frac{dt}{-2}}{\sqrt{t}}$$

$$= x \sin^{-1} x + \frac{1}{2} \int \frac{dt}{\sqrt{t}}$$

$$= x \sin^{-1} x + \frac{1}{2} \, 2\sqrt{t} + c$$

$$= x \sin^{-1} x + \sqrt{1 - x^2} + c$$

Exercise : 5.1.1

Evaluate the following :

1)	$\int x^2 \log x \, dx$	2)	$\int x^3 \log x \, dx$
3)	$\int (\log x)^2 \, dx$	4)	$\int \log 3x \, dx$
5)	$\int x \sin x \, dx$	6)	$\int x \sin 2x \, dx$
7)	$\int x \sec x \tan x \, dx$	8)	$\int x \sin^2 x \, dx$
9)	$\int x \tan^{-1} x \, dx$	10)	$\int x e^{4x} \, dx$
11)	$\int x \cos 4x \, dx$	12)	$\int \cos^{-1} x \, dx$
13)	$\int x e^{5x} \, dx$	14)	$\int x \sin 4x \, dx$
15)	$\int x \sin 6x \, dx$	16)	$\int x \cos 2x \, dx$
17)	$\int x e^{-3x} \, dx$	18)	$\int x e^{-4x} \, dx$

Exercise : 5.1.1 - Answers

1) $\frac{x^3}{3}\log x - \frac{x^3}{9} + c$

2) $\frac{x^4}{4}\log x - \frac{x^4}{16} + c$

3) $x(\log x)^2 - 2x(\log x) + 2x + c$

4) $x\log 3x - x + c$

5) $-x\cos x + \sin x + c$

6) $\frac{-x\cos 2x}{2} + \frac{\sin 2x}{4} + c$

7) $x\sec x - \log(\sec x + \tan x) + c$

8) $\frac{1}{2}\left[\frac{-x\sin 2x}{2} - \frac{\cos 2x}{4}\right] + c$

9) $\frac{1}{2}[x^2\tan^{-1}x + \tan^{-1}x - x] + c$

10) $\frac{xe^{4x}}{4} - \frac{xe^{4x}}{16} + c$

11) $x\frac{\sin 4x}{4} + \frac{\cos 4x}{16} + c$

12) $x\cos^{-1}x - \sqrt{1-x^2} + c$

13) $\frac{xe^{5x}}{5} - \frac{xe^{5x}}{25} + c$

14) $\frac{-x\cos 4x}{4} + \frac{\sin 4x}{16} + c$

15) $\frac{-x\cos 6x}{6} + \frac{\sin 6x}{36} + c$

16) $\frac{x\sin 2x}{2} + \frac{\cos 2x}{4} + c$

17) $\frac{-xe^{-3x}}{3} - \frac{e^{-3x}}{9} + c$

18) $\frac{-xe^{-4x}}{4} - \frac{e^{-4x}}{16} + c$

5.1.2 Integration by parts for Integrand is of the form $e^{mx}\cos nx$ or $e^{mx}\sin nx$

We illustrate that there are some integrals whose integration continues forever. Whenever we integrate function of the form $e^{mx}\cos nx$ or $e^{mx}\sin nx$, we have to use integration by parts twice to get the similar integral on both sides and to solve.

Result

(i) $\int e^{mx}\cos nx\,dx = \frac{e^{mx}}{m^2+n^2}[m\cos nx + n\sin nx] + c$

(ii) $\int e^{mx}\sin nx\,dx = \frac{e^{mx}}{m^2+n^2}[m\sin nx - n\cos nx] + c$

Proof : (i) Let $I = \int e^{mx}\cos nx\,dx$

Take $u = \cos nx$, $\quad du = -n\sin nx\,dx$

$dv = e^{mx}$, $\quad \int dv = \int e^{mx}$

$v = \frac{e^{mx}}{m}$

Applying Integration by parts we get

Formula : $\int u\,dv = uv - \int v\,du$

$$I = \int e^{mx}\cos nx\,dx = \cos nx\left(\frac{e^{mx}}{m}\right) - \int \frac{e^{mx}}{m}(-n\sin nx)\,dx$$

$$I = \frac{e^{mx}}{m}\cos nx + \frac{n}{m}\int e^{mx}\sin nx\,dx$$

Take $u = \sin nx$, $\quad dv = e^{mx}dx$

$du = n\cos nx\,dx$, $\quad \int dv = \int e^{mx}dx$

$v = \frac{e^{mx}}{m}$

Applying Integration by parts, we get

$$I = \frac{e^{mx}}{m}\cos nx + \frac{n}{m}\left[\frac{e^{mx}}{m}\sin nx - \int \frac{e^{mx}}{m}\, n\cos nx\, dx\right]$$

$$= \frac{e^{mx}}{m}\cos nx + \frac{ne^{mx}}{m^2}\sin nx - \frac{n^2}{m^2}\int e^{mx}\cos nx\, dx$$

$$I = \frac{e^{mx}}{m}\cos nx + \frac{n}{m^2}\, e^{mx}\sin nx - \frac{n^2}{m^2} I$$

$$I + \frac{n^2}{m^2} I = \frac{e^{mx}}{m}\cos nx + \frac{n}{m^2}\, e^{mx}\sin nx$$

$$I + \frac{n^2}{m^2} I = \frac{e^{mx}}{m^2}\cos nx + \frac{n}{m^2}\, e^{mx}\sin nx$$

$$\left(1 + \frac{n^2}{m^2}\right) I = \frac{e^{mx}}{m^2}[m\cos nx + n\sin nx]$$

$$\left(\frac{m^2+n^2}{m^2}\right) I = \frac{e^{mx}}{m^2}[m\cos nx + n\sin nx]$$

$$I = \frac{e^{mx}}{m^2+n^2}[m\cos nx + n\sin nx] + c$$

Therefore $\int e^{mx}\cos nx\, dx = \frac{e^{mx}}{m^2+n^2}[m\cos nx + n\sin nx] + c$

Similarly $\int e^{mx}\sin nx\, dx = \frac{e^{mx}}{m^2+n^2}[m\sin nx - n\cos nx] + c$

Worked Examples:

Example

1) Evaluate the integral

$\int e^{2x}\sin x\, dx$

using the formula

$$\int e^{mx}\sin nx\, dx = \frac{e^{mx}}{m^2+n^2}[m\sin nx - n\cos nx\, dx] + c$$

For m = 2, n = 1

$$\int e^{2x}\sin x\, dx = \frac{e^{2x}}{2^2+1^2}[2\sin x - 1\cos x] + c$$

$$= \frac{e^{2x}}{5}[2\sin x - \cos x] + c$$

2) Evaluate the integral

$\int e^{-3x}\cos 2x\, dx$

using the formula

$$\int e^{mx}\cos nx\, dx = \frac{e^{mx}}{m^2+n^2}[m\cos nx + n\sin nx\, dx] + c$$

For m = −3, n = 2

$$\int e^{-3x}\cos 2x\, dx = \frac{e^{-3x}}{(-3)^2+(2)^2}[-3\cos 2x + 2\sin 2x] + c$$

$$= \frac{e^{-3x}}{13}[-3\cos 2x + 2\sin 2x] + c$$

Exercise : 5.1.2

Evaluate the following :

1) $\int e^{mx} \sin nx \, dx$

2) $\int e^{-x} \cos 2x \, dx$

3) $\int e^{-5x} \sin 3x \, dx$

4) $\int e^{3x} \cos 2x \, dx$

5) $\int e^{-4x} \sin 2x \, dx$

6) $\int e^{-3x} \cos x \, dx$

7) $\int x \, e^{-3x} \sin 2x \, dx$

Exercise : 5.1.2 - Answers

1) $\frac{e^{mx}}{m^2+n^2}[m \sin nx - n \cos nx] + c$

2) $\frac{e^{-x}}{5}[2 \sin 2x - \cos 2x] + c$

3) $\frac{-e^{-5x}}{34}[5 \sin 3x + 3 \cos 3x] + c$

4) $\frac{e^{3x}}{13}[3 \cos 2x - 2 \sin 2x] + c$

5) $\frac{e^{-4x}}{10}[2 \sin 2x + \cos 2x] + c$

6) $\frac{e^{-3x}}{10}[\sin x - 3 \cos x] + c$

7) $\frac{-e^{-3x}}{13}[3 \sin 2x + 2 \cos 2x] + c$

5.2 BERNOULLI'S FORMULA

The integration by parts formula will be difficult to apply repeatedly; It takes a lot of space to write down and chances are more to make a distribution error. Fortunately, there is a purely mechanical procedure for performing integration by parts without writing down so much called tabular integration by parts (or) Bernoulli's formula. It is based on the following theorem.

Bernoulli's Formula (Tabular method) :

Suppose that u and v are functions of x and define a sequence of derivatives of u by

$$u' = \frac{du}{dx}, \quad u'' = \left(\frac{du}{dx}\right)', \quad u''' = \left(\frac{du}{dx}\right)''$$

and define a sequence of integrals of v by

$$v_1 = \int v\,dx, \quad v_2 = \int v_1\,dx, \quad v_3 = \int v_2\,dx$$

Then $\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$

Proof :

We know that
$$\begin{aligned}\int u\,dv &= uv - \int v\,du \\ &= uv - \int u'dv_1 \\ &= uv - [u'v_1 - \int v_1\,du'] \\ &= uv - u'v_1 + \int u''dv_2 \\ &= uv - u'v_1 + [u''v_2 - \int v_2\,du''] \\ &= uv - u'v_1 + u''v_2 - u'''v_3 + \ldots\end{aligned}$$

Proceeding like this we get the required formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$$

Note :

This technique is often used when the integrand is a product of two different classes of functions. There are five classes of elementary functions, ILATE : Inverse Logarthmic, Algebraic, Trigonometric, Trigonometric and Exponential.

The success of this method depends on the proper choice of u and v.

To avoid confusion, in the choice of u, give preference in this order ILATE.

Worked Examples:

Part – B

1. Evaluate : $\int x \sin 2x\,dx$

Solution:

Let $I = \int x \sin 2x\,dx$

Put $u = x$ $\qquad dv = \sin 2x\,dx$

$\int dv = \int \sin 2x$

$u' = 1$ $\qquad v = -\frac{\cos 2x}{2}$

$u'' = 0$ $\qquad v_1 = -\frac{\sin 2x}{4}$

$$\therefore I = \int x \sin 2x\, dx \;=\; uv - u'v_1 + u''v_2 - \cdots$$

$$= x\left(\frac{-\cos 2x}{2}\right) - 1\left(\frac{-\sin 2x}{4}\right) + c$$

$$= -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4} + c$$

2. Evaluate : $\int x^2 \sin 3x\, dx$

Solution:

Let $I = \int x^2 \sin 3x\, dx$

Put $u = x^2$ $\qquad dv = \sin 3x\, dx$

$u' = 2x$ $\qquad \int dv = \int \sin 3x\, dx$

$u'' = 2$ $\qquad v = -\frac{\cos 3x}{3}$

$u''' = 0$ $\qquad v_1 = -\frac{\sin 3x}{9}$

$\qquad v_2 = \frac{\cos 3x}{27}$

$\int u\, dv = uv - u'v_1 + u''v_2 \ldots$

$$\int x^2 \sin 3x\, dx = x^2\left(\frac{-\cos 3x}{3}\right) - 2x\left(\frac{-\sin 3x}{9}\right) + 2\left(\frac{\cos 3x}{27}\right) + c$$

$$= -\frac{x^2\cos 3x}{3} + \frac{2x\sin 3x}{9} + \frac{2\cos 3x}{27} + c$$

3. Evaluate : $\int x^2 \sin nx\, dx$

Solution:

Let $I = \int x^2 \sin nx\, dx$

Put $u = x^2$ $\qquad dv = \sin nx\, dx$

$u' = 2x$ $\qquad \int dv = \int \sin nx\, dx$

$u'' = 2$ $\qquad v = -\frac{\cos nx}{n}$

$u''' = 0$ $\qquad v_1 = -\frac{\sin nx}{n^2}$

$\qquad v_2 = \frac{\cos nx}{n^3}$

$\int u\, dv = uv - u'v_1 + u''v_2 + \ldots$

$$\int x^2 \sin nx\, dx = x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right) + c$$

$$= -\frac{x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3} + c$$

4. Evaluate: $\int x^3 \sin x\, dx$

Solution:

Let $I = \int x^3 \sin x\, dx$

Put $u = x^3$ $\qquad dv = \sin x\, dx$

$u' = 3x^2$ $\qquad \int dv = \int \sin x\, dx$

$u'' = 6x$ $\qquad v = -\cos x$

$u''' = 6 \qquad v_1 = -\sin x$

$u'''' = 0 \qquad v_2 = \cos x$

$v_3 = \sin x$

$\int u\,dv = uv - u'v_1 + u''v_2 \ldots..$

$$\int x^3 \sin x\,dx = x^3(-\cos x) - (3x^2)(-\sin x) + 6x(\cos x) - 6(\sin x) + c$$

$$= -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + c$$

5. Evaluate : $\int x \cos 4x\,dx$

Solution:

Let $I = \int x \cos 4x\,dx$

Put $u = x \qquad dv = \cos 4x\,dx$

$u' = 1 \qquad \int dv = \int \cos 4x\,dx$

$u'' = 0 \qquad v = \frac{\sin 4x}{4}$

$v_1 = -\frac{\cos 4x}{16}$

$\int u\,dv = uv - u'v_1 + u''v_2 + \ldots$

$$\int x \cos 4x\,dx = x\left(\frac{\sin 4x}{4}\right) - 1\left(\frac{-\cos 4x}{16}\right) + c$$

$$= \frac{x \sin 4x}{4} + \frac{\cos 4x}{16} + c$$

6. Evaluate : $\int x^2 \cos 2x\,dx$

Solution:

Let $I = \int x^2 \cos 2x\,dx$

Put $u = x^2 \qquad dv = \cos 2x\,dx$

$u' = 2x \qquad \int dv = \int \cos 2x\,dx$

$u'' = 2 \qquad v = \frac{\sin 2x}{2}$

$u''' = 0 \qquad v_1 = -\frac{\cos 2x}{4}$

$v_2 = -\frac{\sin 2x}{8}$

$\int u\,dv = uv - u'v_1 + u''v_2 + \ldots..$

$$\int x^2 \cos 2x\,dx = x^2\left(\frac{\sin 2x}{2}\right) - 2x\left(\frac{-\cos 2x}{4}\right) + 2\left(\frac{-\sin 2x}{8}\right) + c$$

$$= \frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{\sin 2x}{4} + c$$

7. Evaluate : $\int x^2 \cos nx\,dx$

Solution:

Let $I = \int x^2 \cos nx\,dx$

Put $u = x^2 \qquad dv = \cos nx\,dx$

$u' = 2x \qquad \int dv = \int \cos nx\,dx$

$u'' = 2 \qquad v = \frac{\sin nx}{n}$

$u''' = 0 \qquad v_1 = -\frac{\cos nx}{n^2}$

$v_2 = -\frac{\sin nx}{n^3}$

$$\int u\, dv = uv - u'v_1 + u''v_2 + \ldots..$$

$$\int x^2 \cos nx\, dx = x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right) + c$$

$$= x^2\left(\frac{\sin nx}{n}\right) + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3} + c$$

8. Evaluate : $\int x^3 \cos x\, dx$

Solution:

Let $I = \int x^3 \cos x\, dx$

Put $u = x^3 \qquad dv = \cos x\, dx$

$u' = 3x^2 \qquad \int dv = \int \cos x\, dx$

$u'' = 6x \qquad v = \sin x$

$u''' = 6 \qquad v_1 = -\cos x$

$u'''' = 0 \qquad v_2 = -\sin x$

$v_3 = \cos x$

$$\int u\, dv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots..$$

$$\int x^3 \cos x\, dx = x^3 \sin x - (3x^2)(-\cos x) + 6x(-\sin x) - 6(\cos x) + c$$

$$= x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x + c$$

9. Evaluate : $\int x\, e^{5x}\, dx$

Solution:

Let $I = \int x\, e^{5x}\, dx$

Put $u = x \qquad dv = e^{5x}\, dx$

$u' = 1 \qquad \int dv = \int e^{5x}\, dx$

$u'' = 0 \qquad v = \frac{e^{5x}}{5}$

$v_1 = \frac{e^{5x}}{25}$

$$\int u\, dv = uv - u'v_1 + u''v_2 - \ldots.$$

$$\int e^{5x}\, dx = x\left(\frac{e^{5x}}{5}\right) - (1)\left(\frac{e^{5x}}{25}\right) + c$$

$$= \frac{x\, e^{5x}}{5} - \frac{e^{5x}}{25} + c$$

10. Evaluate : $\int x^2 e^{2x}\, dx$

Solution:

Let $I = \int x^2 e^{2x}\, dx \qquad dv = e^{2x}\, dx$

Put $u = x^2 \qquad \int dv = \int e^{2x}\, dx$

$u' = 2x \qquad v = \frac{e^{2x}}{2}$

$u'' = 2 \qquad v_1 = \frac{e^{2x}}{4}$

$u''' = 0 \qquad v_2 = \frac{e^{2x}}{8}$

$\int u\, dv = uv - u'v_1 + u''v_2 - \ldots..$

$$\int x^2 e^{2x}\, dx = x^2\left(\frac{e^{2x}}{2}\right) - 2x\left(\frac{e^{2x}}{4}\right) + 2\left(\frac{e^{2x}}{8}\right) + c$$

$$= \frac{x^2 e^{2x}}{2} - \frac{x\, e^{2x}}{2} + \frac{e^{2x}}{4} + c$$

11. Evaluate : $\int x^2 e^{nx}\, dx$

Solution:

Let $I = \int x^2 e^{nx}\, dx$

Put $u = x^2 \qquad dv = e^{nx}\, dx$

$u' = 2x \qquad \int dv = \int e^{nx}\, dx$

$u'' = 2 \qquad v = \frac{e^{nx}}{n}$

$u''' = 0 \qquad v_1 = \frac{e^{nx}}{n^2}$

$v_2 = \frac{e^{nx}}{n^3}$

$\int u\, dv = uv - u'v_1 + u''v_2 - \ldots..$

$$\int x^2 e^{nx}\, dx = x^2\left(\frac{e^{nx}}{n}\right) - 2x\left(\frac{e^{nx}}{n^2}\right) + 2\left(\frac{e^{nx}}{n^3}\right) + c$$

$$= \frac{x^2 e^{nx}}{n} - \frac{2\, x\, e^{nx}}{n^2} + \frac{2\, e^{nx}}{n^3} + c$$

12. Evaluate : $\int x^2 e^{-x}\, dx$

Solution:

Let $I = \int x^2 e^{-x}\, dx$

Put $u = x^2 \qquad dv = e^{-x}\, dx$

$u' = 2x \qquad \int dv = \int e^{-x}\, dx$

$u'' = 2 \qquad v = \frac{e^{-x}}{-1} = -e^{-x}$

$u''' = 0 \qquad v_1 = \frac{-e^{-2}}{-1} = e^{-x}$

$v_2 = \frac{e^{-x}}{-1} = -\, e^{-x}$

$\int u\, dv = uv - u'v_1 + u''v_2 \ldots..$

$$\int x^2 e^{-x}\, dx = x^2(-e^{-x}) - 2x\,(e^{-x}) + 2(-e^{-x}) + c$$

$$= -x^2 e^{-x} - 2\, x\, e^{-x} - 2\, e^{-x} + c$$

13. Evaluate : $\int x^3 e^x\, dx$

Solution:

Let $I = \int x^3 e^x\, dx$

Put $u = x^3$ $\quad\int dv = \int e^x\, dx$

$u' = 3x^2$ $\quad v = e^x$

$u'' = 6x$ $\quad v_1 = e^x$

$u''' = 6$ $\quad v_2 = e^x$

$u'''' = 0$ $\quad v_3 = e^x$

$\int u\, dv = uv - u'v_1 + u''v_2 - u'''v_3 \ldots..$

$\int x^3 e^x\, dx = x^3(e^x) - 3x^2(e^x) + 6x(e^x) - 6(e^x) + c$

$= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + c$

14. Evaluate : $\int (x^2 + 2) \cos 2x\, dx$

Solution:

Let $I = \int (x^2 + 2) \cos 2x\, dx$

Put $u = x^2 + 2$ $\quad dv = \cos 2x\, dx$

$u' = 2x$ $\quad \int dv = \int \cos 2x\, dx$

$u'' = 2$ $\quad v = \frac{\sin 2x}{2}$

$u''' = 0$ $\quad v_1 = \frac{-\cos 2x}{4}$

$v_2 = \frac{-\sin 2x}{8}$

$\int u\, dv = uv - u'v_1 + u''v_2 - \cdots\cdot$

$$\int (x^2 + 2) \cos 2x\, dx = (x^2 + 2)\left(\frac{\sin 2x}{2}\right) - 2x\left(\frac{-\cos 2x}{4}\right) + 2\left(\frac{-\sin 2x}{8}\right) + c$$

$$= \frac{(x^2+2)(\sin 2x)}{2} + \frac{x \cos 2x}{2} - \frac{\sin 2x}{4} + c$$

15. Evaluate : $\int (x^2 - 4) \sin 3x\, dx$

Solution:

Let $I = \int (x^2 - 4) \sin 3x\, dx$

Put $u = x^2 - 4$ $\quad dv = \sin 3x\, dx$

$u' = 2x$ $\quad \int dv = \int \sin 3x\, dx$

$u'' = 2$ $\quad v = \frac{-\cos 3x}{3}$

$u''' = 0$ $\quad v_1 = \frac{-\sin 3x}{9}$

$v_2 = \frac{\cos 3x}{27}$

$\int u\, dv = uv - u'v_1 + u''v_2 - \cdots\cdot$

$$\int (x^2-4)\sin 3x\,dx = (x^2-4)\left(\frac{-\cos 3x}{3}\right) - 2x\left(\frac{-\sin 3x}{9}\right) + 2\left(\frac{\cos 3x}{27}\right) + c$$

$$= \frac{-(x^2-4)\cos 3x}{3} + \frac{2x\sin 3x}{9} + \frac{\cos 3x}{27} + c$$

16. Evaluate : $\int (x^2-2)\, e^{-2x}\, dx$

Solution:

Let $I = \int (x^2-2)\, e^{-2x}\, dx$

Put $u = x^2 - 2$ $\qquad dv = e^{-2x}\,dx$

$u' = 2x$ $\qquad \int dv = \int e^{-2x}\,dx$

$u'' = 2$ $\qquad v = \frac{e^{-2x}}{-2}$

$u''' = 0$ $\qquad v_1 = \frac{e^{-2x}}{4}$

$\qquad v_2 = \frac{e^{-2x}}{-8}$

$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$

$$\int (x^2-2)\, e^{-2x}\,dx = (x^2-2)\left(\frac{e^{-2x}}{-2}\right) - 2x\left(\frac{e^{-2x}}{4}\right) + 2\left(\frac{e^{-2x}}{-8}\right) + c$$

$$= \frac{-(x^2-2)\,e^{-2x}}{2} - \frac{x\,e^{-2x}}{2} - \frac{e^{-2x}}{4} + c$$

17. Evaluate : $\int x^2 \sin^2 x\, dx$

Solution:

Let $I = \int x^2 \sin^2 x\,dx$

$= \int x^2 \left(\frac{1-\cos 2x}{2}\right) dx$

$= \frac{1}{2}\int (x^2 - x^2\cos 2x)\,dx$

$= \frac{1}{2}\left[\int x^2\,dx - \int x^2 \cos 2x\,dx\right]$

$= \frac{1}{2}\left[\frac{x^3}{3} = \int x^2\cos 2x\,dx\right] \qquad \rightarrow ①$

Consider $\int x^2 \cos 2x\,dx$

Put $u = x^2$ $\qquad dv = \cos 2x\,dx$

$u' = 2x$ $\qquad \int dv = \int \cos 2x\,dx$

$u'' = 2$ $\qquad v = \frac{\sin 2x}{2}$

$u''' = 0$ $\qquad v_1 = \frac{-\cos 2x}{4}$

$\qquad v_2 = \frac{-\sin 2x}{8}$

$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$

$$\int x^2\cos 2x\,dx = x^2\left(\frac{\sin 2x}{2}\right) - 2x\left(\frac{-\cos 2x}{4}\right) + 2\left(\frac{-\sin 2x}{8}\right) + c$$

$$= \frac{x^2\sin 2x}{2} + \frac{x\cos 2x}{2} - \frac{\sin 2x}{4} + c \qquad \rightarrow ②$$

Using ② in ① we get

$$\therefore \quad I = \frac{1}{2}\left[\frac{x^3}{3} - \left(\frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{\sin 2x}{4}\right)\right] + c$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^2 \sin 2x}{2} - \frac{x \cos 2x}{2} + \frac{\sin 2x}{4}\right] + c$$

Exercise : 5.2.1

Part A

1. Write down the Bernoulli's formula for $\int u\, dv$.

Part B and C

Evaluate the following:

1. $\int x \sin x\, dx$
2. $\int x^2 \sin x\, dx$
3. $\int x \sin 4x\, dx$
4. $\int x \sin 6x\, dx$
5. $\int x \sin nx\, dx$
6. $\int x \cos x\, dx$
7. $\int x^2 \cos x\, dx$
8. $\int x^2 \cos 3x\, dx$
9. $\int x^3 \cos 2x\, dx$
10. $\int x \cos nx\, dx$
11. $\int x \sin^2 x\, dx$
12. $\int x \cos^2 x\, dx$
13. $\int x^2 \cos^2 x\, dx$
14. $\int x^3 \cos^2 x\, dx$
15. $\int x^3 \sin^2 x\, dx$
16. $\int x^2 e^x\, dx$
17. $\int x e^{-x}\, dx$
18. $\int x^2 e^{-2x}\, dx$
19. $\int (x^2 + 5) \sin x\, dx$
20. $\int (x^2 - 2) \cos 2x\, dx$

Exercise : 5.2.1 - Answers

Part A

1. $\int u\, dv = uv - u'v_1 + u''v_2 \ldots\ldots$

Part B and C

1. $-x \cos x + \sin x + c$
2. $x^2 \cos x + 2x \sin x + 2 \cos x + c$
3. $\frac{-x \cos 4x}{4} + \frac{\sin 4x}{16} + c$
4. $\frac{-x \cos 6x}{6} + \frac{\sin 4x}{36} + c$
5. $\frac{-x \cos nx}{n} + \frac{\sin nx}{n^2} + c$
6. $x \sin x + \cos x + c$
7. $x^2 \sin x + 2x \cos x - 2 \sin x + c$
8. $\frac{x^2 \sin 3x}{3} + \frac{2}{9} x \cos 3x - \frac{2}{27} \sin 3x + c$

9. $\frac{x^3 \sin 2x}{2} + \frac{3x^2 \cos 2x}{4} - \frac{3x \sin 2x}{4} - \frac{3 \cos 2x}{8} + c$

10. $\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} + c$

11. $\frac{1}{2}\left[\frac{x^2}{2} - \frac{x \sin 2x}{2} - \frac{\cos 2x}{4}\right] + c$

12. $\frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{\cos 2x}{8} + c$

13. $\frac{1}{2}\left[\frac{x^3}{3} + \frac{x^2 \sin 2x}{2} + \frac{x \cos 2x}{2} - \frac{\sin 2x}{4}\right] + c$

14. $\frac{x^4}{8} + \frac{x^3 \sin 2x}{4} + \frac{3x^2 \cos 2x}{8} - \frac{3x \sin 2x}{8} - \frac{3 \cos 2x}{16} + c$

15. $\frac{x^4}{8} - \frac{x^3 \sin 2x}{4} - \frac{3x^2 \cos 2x}{8} + \frac{3x \sin 2x}{8} + \frac{3 \cos 2x}{16} + c$

16. $x^2 e^x - 2x e^x + 2e^x + c$

17. $-x e^{-x} - e^{-x} + c$

18. $\left[\frac{-x^2 e^{-2x}}{2} - \frac{-x e^{-2x}}{2} - \frac{-e^{-2x}}{4}\right] + c$

19. $-(x^2 + 5) \cos x + 2x \sin x + 2 \cos x + c$

20. $(x^3 - 2) \frac{\sin 2x}{2} + \frac{3x^2 \cos 2x}{4} - \frac{3x \sin 2x}{4} - \frac{3 \cos 2x}{8} + c$

5.3 DEFINITE INTEGRALS

5.3.1 Introduction

If we find the value an integral of function in a given range of values of x then the integral is known as a definite integral.

$\int_a^b f(x)\,dx$ means the integral from a to b of f(x) with respect to x. Here 'a' and 'b' are called the lower and upper limits of the definite integral.

Now $\int_a^b f(x)\,dx = [F(x) + c]_a^b$

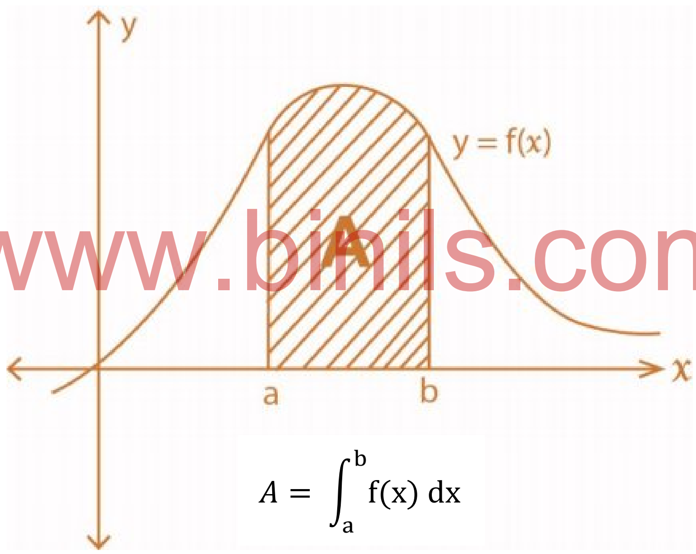
$= [F(b) + c - (F(a) + c)]$

$= F(b) + c - F(a) - c$

$= F(b) - F(a)$

Note :

The constant of integration 'c' disappears in a definite integral.



Worked Examples:

1. Evaluate : $\int_0^1 (x^2 + 2x + 3)\,(x + 1)\,dx$

Solution:

Given $\int_0^1 (x^2 + 2x + 3)\,(x + 1)\,dx$

$= \int_0^1 (x^3 + 2x^2 + 3x + x^2 + 2x + 3)\,dx$

$= \int_0^1 (x^3 + 3x^2 + 5x + 3)\,dx$

$= \left[\left(\frac{x^4}{4}\right)_0^1 + \left(\frac{3\,x^3}{3}\right)_0^1 + 5\left(\frac{x^2}{2}\right)_0^1 + 3(x)_0^1\right]$

$= \frac{1}{4} + 1 + \frac{5}{2} + 3$

$$= \frac{1}{4} + \frac{5}{2} + 4$$

$$= \frac{1+10+16}{4}$$

$$= \frac{27}{4}$$

2. Evaluate : $\int_1^2 x(x - x^2)\, dx$

Solution:

Given $\int_1^2 x(x - x^2)\, dx = \int_1^2 (x^2 - x^3)\, dx$

$$= \int_1^2 x^2 dx - \int_1^2 x^3 dx$$

$$= \left[\left(\frac{x^3}{3}\right)_1^2 - \left(\frac{x^4}{4}\right)_1^2\right]$$

$$= \left[\left(\frac{8}{3} - \frac{1}{3}\right) - \left(\frac{16}{4} - \frac{1}{4}\right)\right]$$

$$= \left(\frac{8-1}{3}\right) - \left(\frac{16-1}{4}\right)$$

$$= \frac{7}{3} - \frac{15}{4}$$

$$= \frac{28-45}{12}$$

$$= \frac{-17}{12}$$

3. Evaluate : $\int_0^{\frac{\pi}{2}} \sin x\, dx$

Solution:

Given $\int_0^{\frac{\pi}{2}} \sin x\, dx$ $\left[\because \cos\frac{\pi}{2} = \cos 90^o = 0, \cos 0 = 1\right]$

$$= [-\cos x]_0^{\frac{\pi}{2}} = \left[-\cos\frac{\pi}{2} - (-\cos 0)\right]$$

$$= -0 + 1$$

$$= 1$$

4. Evaluate : $\int_0^4 \left(\frac{x^2+2}{3}\right) dx$

Solution:

Given $\int_0^4 \left(\frac{x^2+2}{3}\right) dx = \frac{1}{3}\left[\int_0^4 (x^2 + 2)\right] dx$

$$= \frac{1}{3}\left[\left(\frac{x^3}{3} + 2x\right)_0^4\right]$$

$$= \frac{1}{3}\left[\left(\frac{4^3}{3} + 2(4)\right) - \left(\frac{0^3}{3} + 2(0)\right)\right]$$

$$= \frac{1}{3}\left[\frac{64}{3} + 8\right]$$

$$= \frac{1}{3}\left[\frac{64+24}{3}\right]$$

$$= \frac{88}{9}$$

5. Evaluate: $\int_0^1 (x^2+2x+3)(x+1)\,dx$

Solution:

Given $\int_0^1 (x^2+2x+3)(x+1)\,dx$

Put $u = x^2+2x+3$ | When $x = 0$, $u = 3$

$du = (2x+2)\,dx$ | When $x = 1$, $u = 6$

$du = 2(x+1)\,dx$

$\frac{1}{2}du = (x+1)\,dx$

$$\int_0^1 (x^2+2x+3)(x+1)\,dx = \int_3^6 u\frac{1}{2}\,du$$

$$= \frac{1}{2}\left[\int_3^6 u\,du\right]$$

$$= \frac{1}{2}\left[\frac{u^2}{2}\right]_3^6$$

$$= \frac{1}{4}[6^2-3^2]$$

$$= \frac{1}{4}[36\text{-}9]$$

$$= \frac{27}{4}$$

6. Evaluate : $\int_0^{\frac{\pi}{2}} \left(\frac{\cos^2 x}{1+\sin x}\right) dx$

Solution:

Given $\int_0^{\frac{\pi}{2}} \left(\frac{\cos^2 x}{1+\sin x}\right) dx = \int_0^{\frac{\pi}{2}} \frac{1-\sin^2 x}{1+\sin x}\,dx$

$$= \int_0^{\frac{\pi}{2}} \frac{(1+\sin x)\,(1-\sin x)}{(1+\sin x)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} (1-\sin x)\,dx$$

$$= [x-(-\cos x)]_0^{\frac{\pi}{2}}$$

$$= [x+\cos x]_0^{\frac{\pi}{2}}$$

$$= \left[\frac{\pi}{2}+\cos\frac{\pi}{2}-(0+\cos 0)\right]$$

$$= \left[\frac{\pi}{2}+0-(0+1)\right]$$

$$= \frac{\pi}{2} - 1$$

7. Evaluate : $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$

Solution:

Given $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$ $\left[\because \cos^2 x = \frac{1+\cos 2x}{2}\right]$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\sin 2\left(\frac{\pi}{2}\right)}{2} - \left(0 + \frac{\sin 2\,(0)}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\sin \pi}{2} - (0 + 0)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + 0\right]$$

$$= \frac{\pi}{4}$$

8. Evaluate : $\int_0^{\frac{\pi}{2}} \sin^3 x\, dx$

Solution:

Given $\int_0^{\frac{\pi}{2}} \sin^3 x\, dx$

$$\sin 3x = 3 \sin x - 4 \sin^3 x$$

$$4\sin^3 x = 3 \sin x - \sin 3x$$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

$$\int_0^{\frac{\pi}{2}} \sin^3 x\, dx = \int_0^{\frac{\pi}{2}} \left[\frac{1}{4}(3 \sin x - \sin 3x)\right] dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4}(3 \sin x - \sin 3x) dx$$

$$= \frac{1}{4}\left[\int_0^{\frac{\pi}{2}} 3 \sin x\, dx - \int_0^{\frac{\pi}{2}} \sin 3x\, dx\right]$$

$$= \frac{1}{4}\left[3(-\cos x)_0^{\frac{\pi}{2}} - \left(\frac{-\cos 3x}{3}\right)_0^{\frac{\pi}{2}}\right]$$

$$= \frac{1}{4}\left[3\left(-\cos \frac{\pi}{2}\right) - 3\,(-\cos 0)\right] - \frac{1}{4}\left[\frac{-\cos 3\frac{\pi}{2}}{3} - \left(\frac{-\cos 3(0)}{3}\right)\right]$$

$$= \frac{1}{4}\left[3(0 + 1) - \left(-0 + \frac{1}{3}\right)\right] \qquad \left[\because \cos 3\frac{\pi}{2} = 0\right]$$

$$= \frac{1}{4}\left[3 - \frac{1}{3}\right]$$

$$= \frac{1}{4}\left[\frac{9-1}{3}\right]$$

$$= \frac{1}{4}\left[\frac{8}{3}\right] = \frac{8}{12}$$

$$= \frac{2}{3}$$

9. Evaluate : $\int_1^2 x \log x\, dx$

Solution:

Given $\int_1^2 x \log x\, dx$

Put u	=	$\log x$		dv	=	$x\,dx$
du	=	$\frac{1}{x}dx$		$\int dv$	=	$\int x\,dx$
				v	=	$\frac{x^2}{2}$

$$\therefore \int_1^2 x\log x\,dx = \int u\,dv$$

$$= [uv] - \int v\,du$$

$$= \left[\log x\left(\frac{x^2}{2}\right)\right]_1^2 - \int_1^2 \frac{x^2}{2}\left(\frac{1}{x}\right)dx$$

$$= \left[\frac{x^2\log x}{2}\right]_1^2 - \frac{1}{2}\left[\int x\,dx\right]$$

$$= \left(\frac{x^2\log x}{2}\right)_1^2 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2$$

$$= \frac{1}{2}[(4\log 2 - \log 1)] - \frac{1}{4}[4-1]$$

$$= 2\log 2 - \frac{3}{4}$$

10. Evaluate : $\int_1^2 \log x\,dx$

Solution:

Given $\int_1^2 \log x\,dx$

Put u	=	$\log x$		dv	=	dx
$\frac{du}{dx}$	=	$\frac{1}{x}$		$\int dv$	=	$\int dx$
du	=	$\frac{1}{x}dx$		v	=	x

$$\therefore \int_1^2 \log x\,dx = [\log x\,(x)]_1^2 - \int_1^2 x\left(\frac{1}{x}\right)dx$$

$$= [x\log x]_1^2 - \int_1^2 dx$$

$$= (x\log x)_1^2 - [x]_1^2$$

$$= (2\log 2 - 1\log 1) - (2-1)$$

$$= (2\log 2 - 0) - (1) \qquad [\because \log 1 = 0]$$

$$\int_1^2 \log x\,dx = 2\log 2 - 1$$

11. Evaluate : $\int_1^2 \frac{1}{x}\,dx$

Solution:

Given $\int_1^2 \frac{1}{x}\,dx$

$$= [\log x]_1^2$$

$$= \log 2 - \log 1 \qquad [\because \log 1 = 0]$$

$$= \log 2$$

Exercise: 5.3.1

1. $\int_0^2 (1+x+x^2)\,dx$
2. $\int_1^2 (x-x^2)\,dx$
3. $\int_0^{\frac{\pi}{2}} \cos x\,dx$
4. $\int_0^{\pi} \sin x\,dx$
5. $\int_0^{\frac{\pi}{4}} \tan^2 x\,dx$
6. $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$
7. $\int_0^{\frac{\pi}{2}} \cos^3 x\,dx$
8. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\cos x}\,dx$
9. $\int_0^{\frac{\pi}{2}} (2+\sin x)^2\,dx$
10. $\int_0^2 3(x-1)^2\,dx$
11. $\int_0^{\frac{\pi}{4}} \sec^2 x\,dx$
12. $\int_0^{\frac{\pi}{4}} \operatorname{cosec}^2 x\,dx$

Exercise: 5.3.1 - Answers:

Part -A

1. $\frac{20}{3}$
2. $\frac{-5}{6}$
3. 1
4. 0
5. 1- $\frac{\pi}{4}$
6. $\frac{\pi}{4}$
7. $\frac{2}{3}$
8. $\frac{\pi}{2}$ -1
9. $\frac{19}{3}$
10. 2
11. 1
12. ∞ (infinite)

5.3.2 Properties of Definite Integrals

Properties :

1. $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy$
2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
4. $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$, Where f(x) is Even function
5. If f(x) is odd function ,

   then, $\int_{-a}^a f(x)\,dx = 0$
6. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Note : 1. f (x) is Even function when f (-x) = f (x)

2. f (x) is odd function when f (-x) = - f (x)

Part –B & C

1. Evaluate : $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx \quad \rightarrow ①$$

By the property, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}\,dx \quad \rightarrow ②$$

$$① + ② \Rightarrow I + I = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}\,dx$$

$$2\,I = \int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sin x+\cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} dx$$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} dx$$

$$= \frac{1}{2}(x)_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}-0\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{2}\right)$$

$$I = \frac{\pi}{4}$$

Hence, $\boxed{\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx = \frac{\pi}{4}}$

2. Evaluate : $\int_0^{\frac{\pi}{2}} \log(\tan x)\,dx$

Solution:

$$I = \int_0^{\frac{\pi}{2}} \log(\tan x)\,dx \quad \rightarrow ①$$

By the property, $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^{\frac{\pi}{2}} \log\left[\tan\left(\frac{\pi}{2}-x\right)\right]dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\cot x)\,dx \quad \rightarrow ②$$

$$① + ② \Rightarrow 2I = \int_0^{\frac{\pi}{2}} \log(\tan x)\,dx + \int_0^{\frac{\pi}{2}} \log(\cot x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} [\log(\tan x)\,dx + \log(\cot x)\,dx\,]$$

$$= \int_0^{\frac{\pi}{2}} \log[\tan x\,.\cot x]\,dx$$

$$= \int_0^{\frac{\pi}{2}} \log(1)dx \qquad \because \log 1 = 0$$

$$2I = \int_0^{\frac{\pi}{2}} 0\, dx$$

$$2I = 0$$

$$I = 0$$

3. Evaluate : $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x \sin x\, dx$

Solution:

$$f(x) = x \sin x$$

$$f(-x) = -x \sin(-x)$$

$$= -x(-\sin x)$$

$$= x \sin x = f(x)$$

$$\therefore f(-x) = f(x)$$

$\therefore$ $f(-x)$ is a Even function

Hence, $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x \sin x\, dx = 2\int_0^{\frac{\pi}{2}} x \sin x\, dx \rightarrow$ ① $\quad \left[\because \int_{-a}^{a} f(x)dx = 2\int_0^{a} f(x)dx\right]$

$$= 2I$$

$$I = \int_0^{\frac{\pi}{2}} x \sin x\, dx$$

$u = x$	$\int dv = \int \sin x\, dx$
$u' = 1$	$v = -\cos x$
$u'' = 0$	$v_1 = -\sin x$

$$\int u\, dv = uv - u'v_1 + u''v_2 + \ldots$$

$$\int_0^{\frac{\pi}{2}} x \sin dx = [x(-\cos x) - (1)(-\sin x)]_0^{\frac{\pi}{2}}$$

$$= [-x\cos x + \sin x)]_0^{\frac{\pi}{2}}$$

$$= \left[-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right] - [-0\cos 0 + \sin 0]$$

$$= [0 + 1] - 0$$

$$I = 1$$

Substitute in equation ①

$$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x \sin x\, dx = 2\int_0^{\frac{\pi}{2}} x \sin x\, dx$$

$$= 2(1)$$

$$= 2$$

4. Evaluate : $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} x^3 \sin^2 x\, dx$

Solution:

$$f(x) = x^3 \sin^2 x = x^3 (\sin x)^2$$

$$f(-x) = (-x)^3 [\sin(-x)]^2 = -x^3 (-\sin x)^2$$

$$= -x^3 \sin^2 x$$

$$f(-x) = -f(x)$$

$\therefore$ f (x) is odd function,

$$\therefore \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} x^3 \sin^2 x\, dx = 0$$

5. Evaluate : $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin x \cos^4 x\, dx$

Solution:

$$f(x) = \sin x \cos^4 x$$

$$= \sin x [\cos x]^4$$

$$f(-x) = \sin(-x)[\cos(-x)]^4 = -\sin x \cos^4 x$$

$$f(-x) = -f(x)$$

$\therefore$ f(x) is a odd function,

$$\therefore \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin x \cos^4 x\, dx = 0$$

6. Evaluate : $\int_{-1}^{1} \log\left(\frac{3-x}{3+x}\right) dx$

Solution:

$$f(x) = \log\left(\frac{3-x}{3+x}\right) = \log(3\text{-}x) - \log(3+2)$$

$$f(-x) = \log(3+x) - \log(3-x)$$

$$= -[\log(3-x) - \log(3+x)]$$

$$= \log\left(\frac{3-x}{3+x}\right) = \text{-}f(x)$$

$\therefore$ f(x) is a odd function,

$$\therefore \int_{-1}^{1} \log\left(\frac{3-x}{3+x}\right) dx = 0$$

7. Evaluate : $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\, dx$

Solution:

$$f(x) = \sin^2 x$$

$$f(-x) = [\sin(-x)]^2 = (-\sin x)^2 = \sin^2 x = f(x)$$

$$f(-x) = f(x)$$

$\therefore$ f(x) is a even function,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x\,dx = 2\int_0^{\frac{\pi}{2}} \sin^2 x\,dx \qquad \left[\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx\right]$$

$$= 2\int_0^{\frac{\pi}{2}} \left(\frac{1-\cos 2x}{2}\right) dx$$

$$= 2\,\frac{1}{2}\int_0^{\frac{\pi}{2}} (1-\cos 2x)dx$$

$$= \int_0^{\frac{\pi}{2}} dx - \int_0^{\frac{\pi}{2}} \cos 2x\,dx$$

$$= (x)_0^{\frac{\pi}{2}} - \left(\frac{\sin 2x}{2}\right)_0^{\frac{\pi}{2}}$$

$$= \left(\frac{\pi}{2} - 0\right) - \frac{1}{2}\left[\sin 2\left(\frac{\pi}{2}\right) - \sin 0\right]$$

$$= \frac{\pi}{2} - \frac{1}{2}[0-0] \qquad [\because \sin\pi = 0, \sin 0 = 0]$$

$$= \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2}$$

Exercise: 5.3.2

1. Evaluate : $\int_0^{\frac{\pi}{2}} \frac{(\sin x)^{\frac{3}{2}}}{(\sin x)^{\frac{3}{2}}+(\cos x)^{\frac{3}{2}}} dx$
2. Evaluate : $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} dx$
3. Evaluate : $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^3 x\,dx$
4. Evaluate : $\int_{-1}^{1} \sin^3 x \cos^4 x\,dx$
5. Evaluate : $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x^3 \cos^2 x\,dx$
6. Evaluate : $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 \cos x\,dx$
7. Evaluate : $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x \sin^2 x\,dx$
8. Evaluate : $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sqrt{\cot x}}$
9. Evaluate : $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sqrt{\tan x}}$

Exercise: 5.3.2 - Answers

1. $\frac{\pi}{4}$
2. $\frac{\pi}{4}$
3. $\frac{4}{3}$
4. 0
5. 0
6. $\frac{2}{3}$
7. 0
8. $\frac{\pi}{4}$
9. $\frac{\pi}{4}$

Notes: