2.2 MEASURES OF PERFORMANCE OF THE STANDARD FIRST ORDER SYSTEM

Consider the following block diagram of the closed loop control system. Here, an open loop transfer function, — is connected with a unity negative feedback.

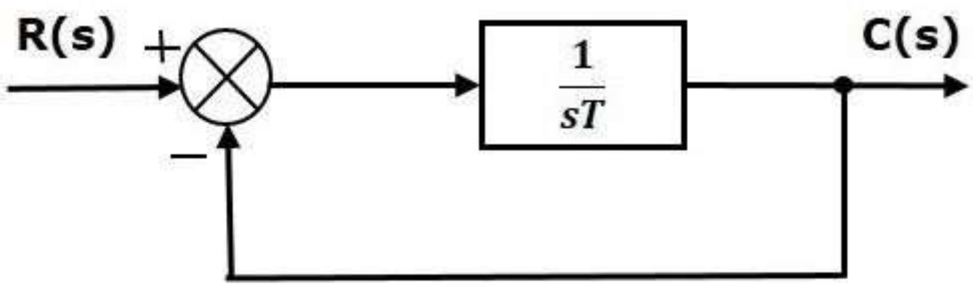

Figure 2.2.1: Blok diagram of closed Control System

[Source: "Control System Engineering" by Nagoor Kani, page-2.7]

We know that the transfer function of the closed loop control system has unity negative feedback as,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Substitute, $G(s) = \frac{1}{sT}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{sT}}{1+\frac{1}{sT}} = \frac{1}{sT+1}$$

The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system**.

We can re-write the above equation as

$$C(s) = \left(\frac{1}{sT+1}\right)R(s)$$

Where,

- **C(s)** is the Laplace transform of the output signal c(t),
- **R(s)** is the Laplace transform of the input signal r(t), and

- **T** is the time constant.

Follow these steps to get the response (output) of the first order system in the time domain.

- Take the Laplace transform of the input signal r(t).
- Consider the equation, $C_{(s)} = \left(\frac{1}{sT+1}\right) R(s)$
- Substitute R(s) value in the above equation.
- Do partial fractions of C(s) if required.
- Apply inverse Laplace transform to C(s).

We have standard test signals like impulse, step, ramp and parabolic. Let us now find out the responses of the first order system for each input, one by one. The name of the response is given as per the name of the input signal. For example, the response of the system for an impulse input is called as impulse response.

Impulse Response of First Order System

Consider the **unit impulse signal** as an input to the first order system.

So, r(t)=δ(t)

Apply Laplace transform on both the sides.

R(s)=1

Consider the equation,

$$C(s) = \left(\frac{1}{sT+1}\right)R(s)$$

Substitute, R(s)=1 in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)(1) = \frac{1}{sT+1}$$

Rearrange the above equation in one of the standard forms of Laplace transforms.

C(s)=1/ T(s+1T) ⇒C(s)=(1/T) * (1/s+1T)

Apply inverse Laplace transform on both

sides.

The unit impulse response is shown in the following figure 2.2.1.

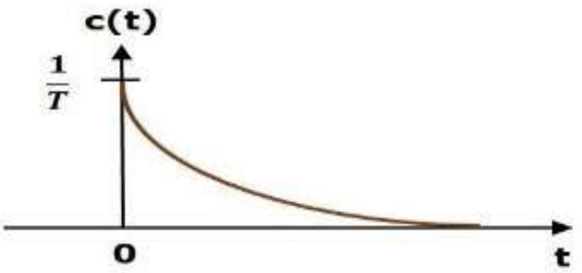

Figure 2.2.1: unit impulse response

[Source: "Control System Engineering" by Nagoor Kani, page-2.8]

The **unit impulse response**, c(t) is an exponential decaying signal for positive values of 't' and it is zero for negative values of 't'.

Step Response of First Order System

Consider the

unit step signal

as an input to

first order

system. So,

r(t)=u(t)

Apply Laplace transform on both the sides.

R(s)=1s

Consider the equation

$$C(s) = \left(\frac{1}{sT+1}\right)R(s)$$

Substitute, R(s)=1/s in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\frac{1}{s} = \frac{1}{s(sT+1)}$$

Do partial fractions of C(s).

$C(s)=1/s(sT+1)=A/s+B/(sT+1)$

$\Rightarrow 1/s(sT+1)=(A(sT+1)+Bs) / s(sT+1)$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$1=A(sT+1)+Bs$

By equating the constant terms on both the sides, you will get A = 1.

Substitute, A = 1 and equate the coefficient of the **s** terms on both the sides.

$0=T+B \Rightarrow B=-T$

Substitute, A = 1 and B = −T in partial fraction expansion of C(s).

$C(s)=1/s-T/(sT+1)=1/s-T/(T(s+1/T))$

Apply inverse Laplace transform on both the sides.

$c(t)=(1-e^{-(t/T)})u(t)$

The **unit step response**, c(t) has both the transient and the steady state terms.

The transient term in the unit step response is -

$c_{tr}(t)=-e^{-(t/T)}u(t)$

The steady state term in the unit step response is -

$c_{ss}(t)=u(t)$

The following figure 2.2.3 shows the unit step response.

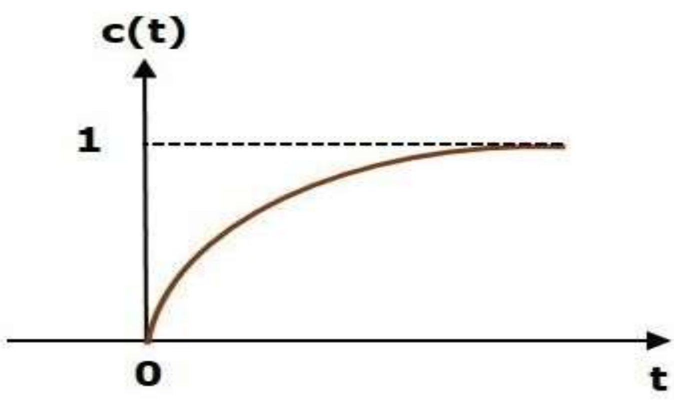

Figure 2.2.3: unit step response

[Source: "Control System Engineering" by Nagoor Kani, page-2.8]

The value of the **unit step response, c(t)** is zero at t = 0 and for all negative values of t. It is gradually increasing from zero value and finally reaches to one in steady state. So, the steady state value depends on the magnitude of the input.

Ramp Response of First Order System

Consider the **unit ramp signal** as an input to the first order system.

So, $r(t)=t\,u(t)$

Apply Laplace transform on both the sides.

$R(s)=1/s^2$

$$C(s)=\left(\frac{1}{sT+1}\right)R(s)$$

Substitute, $R(s)=1/s^2$ in the above equation.

$$C(s)=\left(\frac{1}{sT+1}\right)\frac{1}{s^2}=\frac{1}{s^2(sT+1)}$$

Do partial fractions of C(s).

$$C(s)=\frac{1}{s^2(sT+1)}=\frac{A}{s^2}+\frac{B}{s}+\frac{C}{sT+1}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

By equating the constant terms on both the sides, you will get A = 1.

Substitute, A = 1 and equate the coefficient of the s terms on both the sides.

$0=T+B \Rightarrow B=-T$

Similarly, substitute $B = -T$ and equate the coefficient of s^2 terms on both the sides. You will get $C=T^2$.

Substitute $A = 1$, $B = -T$ and $C=T^2$ in the partial fraction expansion of C(s).

$$C(s)=1/s^2-T/s+T^2/sT+1 =1/s^2-T/s+T^2/T(s+1T)$$

$$\Rightarrow C(s)=1/s^2-T/s+T/(s+1/T)$$

Apply inverse Laplace transform on both the sides.

$$c(t)=(t-T+Te^{-(tT)})$$

The **unit ramp response**, c(t) has both the transient and the steady state terms.

The transient term in the unit ramp response is -

$$c_{tr}(t)=Te^{-(t/T)}u(t)$$

The steady state term in the unit ramp response is -

$$c_{ss}(t)=(t-T)u(t)$$

The following figure 2.2.4 shows the unit ramp response.

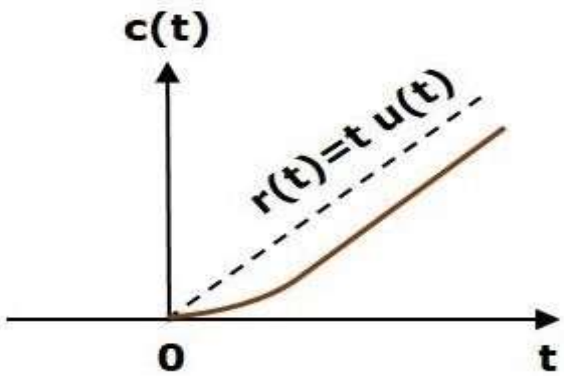

Figure 2.2.4: unit ramp response

[Source: "Control System Engineering" by Nagoor Kani, page-2.8]

The **unit ramp response**, c(t) follows the unit ramp input signal for all positive values of t. But, there is a deviation of T units from the input signal.

Parabolic Response of First Order System

Consider the **unit parabolic signal** as an input to the first order system.

So, $r(t)=(t^2/2)u(t)$

Apply Laplace transform on both the sides.

$R(s)=1/s^3$

Consider the equation,

$$C(s) = \left(\frac{1}{sT+1}\right)R(s)$$

Substitute $R(s)=1/s^3$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^3}\right) = \frac{1}{s^3(sT+1)}$$

Do partial fractions of C(s).

$$C(s) = \frac{1}{s^3(sT+1)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{B}{s} + \frac{D}{sT+1}$$

After simplifying, you will get the values of A, B, C and D as 1, $-T$, T^2 and $-T^3$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^2}{s + \frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(\frac{t^2}{2} - Tt + T^2 - T^2 e^{-\frac{t}{T}}\right) u(t)$$

The **unit parabolic response**, c(t) has both the transient and the steady state terms.

The transient term in the unit parabolic response is

$$C_{tr}(t) = -T^2 e^{-(t/T)} u(t)$$

The steady state term in the unit parabolic response is

$$C_{ss}(t) = (t^2/2 - Tt + T^2) u(t)$$

From these responses, we can conclude that the first order control systems are not stable with the ramp and parabolic inputs because these responses go on increasing even at infinite amount of time. The first order control systems are stable with impulse and step inputs because these responses have bounded output. But, the impulse response doesn't have steady state term. So, the step signal is widely used in the time domain for analyzing the control systems from their responses.

2.4 STEADY STATE ERROR CONSTANT

The deviation of the output of control system from desired response during steady state is known as **steady state error**. It is represented as e_{ss}. We can find steady state error using the final value theorem as follows.

$$e_{ss} = \lim_{s \to 0} e(t) = \lim_{s \to 0} sE(s)$$

Where,

E(s) is the Laplace transform of the error signal, e(t)

Let us discuss how to find steady state errors for unity feedback and non-unity feedback control systems one by one.

Steady State Errors for Unity Feedback Systems

Consider the following block diagram of closed loop control system, which is having unity negative feedback.

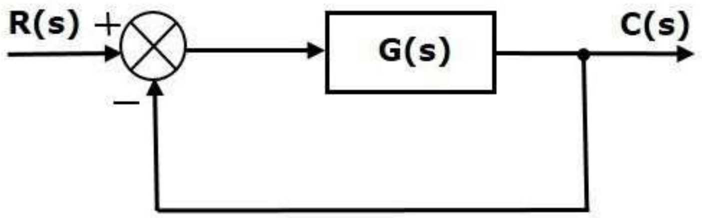

Figure 2.4.1: unit closed loop system

[Source: "Control System Engineering" by Nagoor Kani, page-2.33]

Where,

- R(s) is the Laplace transform of the reference Input signal r(t)
- C(s) is the Laplace transform of the output signal c(t)

We know the transfer function of the unity negative feedback closed loop control system as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$C(s) = \frac{R(s)G(s)}{1 + G(s)}$$

The output of the summing point is -

E(s)=R(s)−C(s)

Substitute C(s) value in the above equation.

$$E(s) = R(s) - \frac{R(s)G(s)}{1 + G(s)}$$

$$E(s) = \frac{R(s) + R(s)G(s) - R(s)G(s)}{1 + G(s)}$$

$$E(s) = \frac{R(s)}{1 + G(s)}$$

Substitute E(s)E(s) value in the steady state error formula

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

The following table shows the steady state errors and the error constants for standard input signals like unit step, unit ramp & unit parabolic signals.

Input signal	Steady state error essess	Error constant
unit step signal	$\frac{1}{1 + k_p}$	$k_p = \lim_{s \to 0} G(s)$
unit ramp signal	$\frac{1}{k_v}$	$k_v = \lim_{s \to 0} sG(s)$
unit parabolic signal	$\frac{1}{k_a}$	$k_a = \lim_{s \to 0} s^2 G(s)$

Where, , and are position error constant, velocity error constant and acceleration error constant respectively.

Note − If any of the above input signals has the amplitude other than unity, then

multiply corresponding steady state error with that amplitude.

Note − We can't define the steady state error for the unit impulse signal because, it exists only at origin. So, we can't compare the impulse response with the unit impulse input as **t** denotes infinity.

Example

Let us find the steady state error for an input signal $r(t)=(5+2t+t^2/2)u(t)$ of unity negative feedback control system with ( ) $\frac{(\ )}{(\ )(\ )}$

The given input signal is a combination of three signals step, ramp and parabolic. The following table shows the error constants and steady state error values for these three signals.

Input signal	Error constant	Steady state error
$r_1(t)=5u(t)$	( )	____
$R_2(t)=2tu(t)$	( )	__
$r_3(t)=t^2/2\ \ u(t)$	( )	__

We will get the overall steady state error, by adding the above three steady state errors.

$$e_{ss}=e_{ss1}+e_{ss2}+e_{ss3}$$

$$\Rightarrow ess=0+0+1=1$$

Therefore, we got the steady state error e_{ss} as **1** for this example.

Steady State Errors for Non-Unity Feedback Systems

Consider the following fig.2.4.2 block diagram of closed loop control system, which is having nonunity negative feedback.

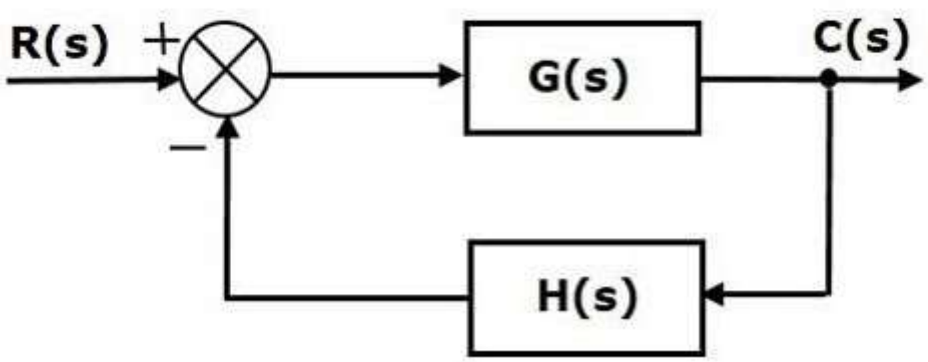

Figure 2.4.2: closed loop system

[Source: "Control System Engineering" by Nagoor Kani, page-2.33]

We can find the steady state errors only for the unity feedback systems. So, we have to convert the non-unity feedback system into unity feedback system. For this, include one unity positive feedback path and one unity negative feedback path in the above block diagram. The new block diagram looks like as shown below fig 2.4.3.

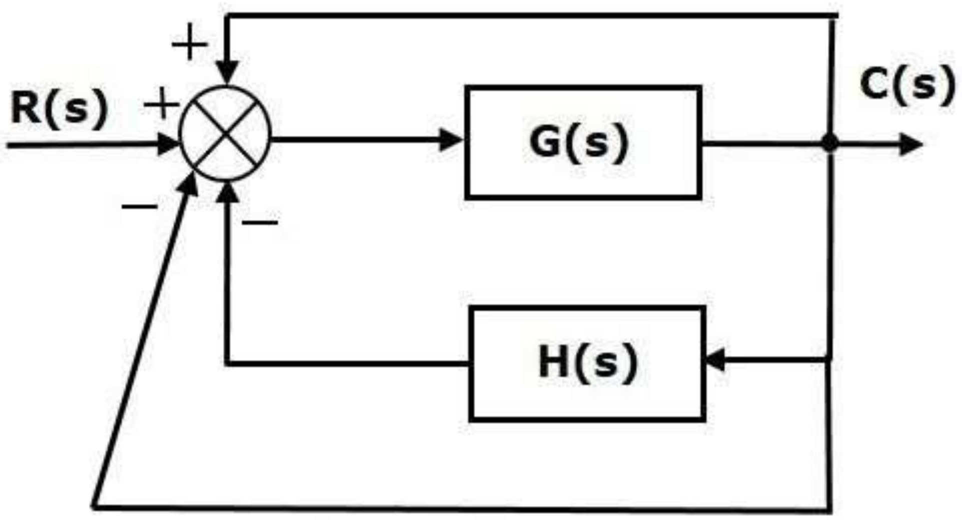

Figure 2.4.3: simplified block diagram

[Source: "Control System Engineering" by Nagoor Kani, page-2.33]

Simplify the above block diagram by keeping the unity negative feedback as it is. The following is the simplified block diagram in fig 2.4.4.

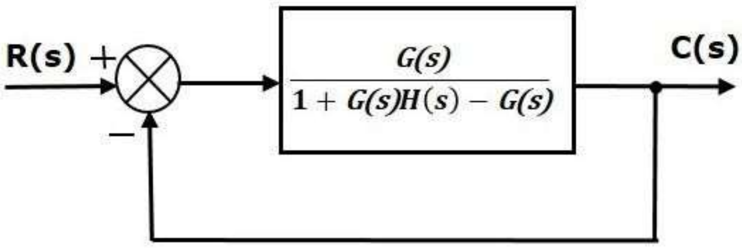

Figure 2.4.4: simplified unity feedback system

[Source: "Control System Engineering" by Nagoor Kani, page-2.34]

This block diagram resembles the block diagram of the unity negative feedback closed loop control system. Here, the single block is having the transfer function G(s)/1+G(s)H(s)−G(s) instead of G(s). You can now calculate the steady state errors by using steady state error formula given for the unity negative feedback systems.

2.1 TRANSIENT AND STEADY STATE RESPONSE

If the output of control system for an input varies with respect to time, then it is called the **time response** of the control system. The time response consists of two parts.

- Transient response
- Steady state response

The response of control system in time domain is shown in the following figure 2.1.1.

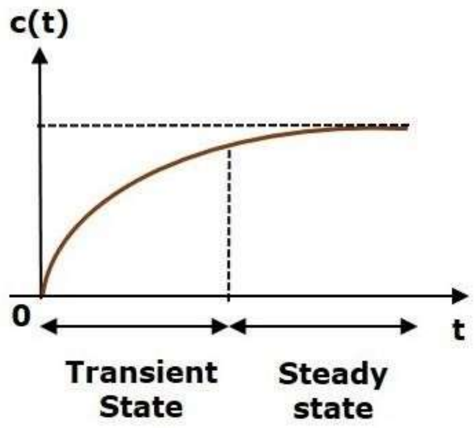

Figure 2.1.1: time domain response

[Source: "Control System Engineering" by Nagoor Kani, page-2.1]

Here, both the transient and the steady states are indicated in the figure. The responses corresponding to these states are known as transient and steady state responses.

Mathematically, we can write the time response c(t) as

$c(t)=c_{tr}(t)+c_{ss}(t)$

Where,

- $c_{tr}(t)$ is the transient response
- $c_{ss}(t)$ is the steady state response

Transient Response

After applying input to the control system, output takes certain time to reach steady state. So, the output will be in transient state till it goes to a steady state. Therefore, the response of the control system during the transient state is known as **transient response**.

The transient response will be zero for large values of ‘t’. Ideally, this value of ‘t’ is infinity and practically, it is five times constant.

Mathematically, we can write it as

$$\lim_{t\to\infty} C_{tr}(t) = 0$$

Steady state Response

The part of the time response that remains even after the transient response has zero value for large values of ‘t’ is known as **steady state response**. This means, the transient response will be zero even during the steady state.

Example

Let us find the transient and steady state terms of the time response of the control system $c(t)=10+5e^{-t}$

Here, the second term $5e^{-t}$ will be zero as **t** denotes infinity. So, this is the **transient term**. And the first term 10 remains even as **t** approaches infinity. So, this is the **steady state term**.

Standard Test Signals

The standard test signals are impulse, step, ramp and parabolic. These signals are used to know the performance of the control systems using time response of the output.

Unit Impulse Signal

A unit impulse signal, δ(t) is defined as

$$\delta(t)=0 \text{ for } t\neq 0$$

and $$\int_{0-}^{0+} \delta(t)dt = 1$$

The following figure 2.1.2 shows unit impulse signal.

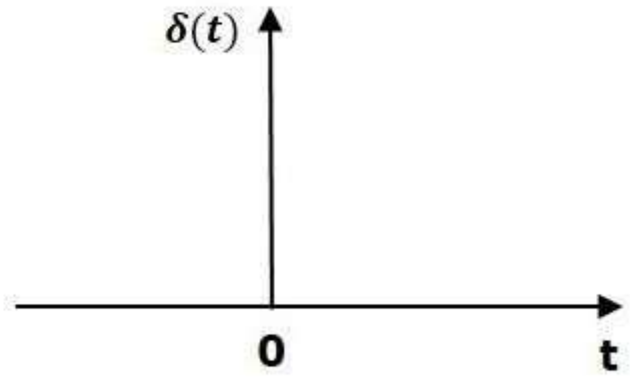

Figure 2.1.2: Unit impulse response

[Source: "Control System Engineering" by Nagoor Kani, page-2.2]

So, the unit impulse signal exists only at't' is equal to zero. The area of this signal under small interval of time around 't' is equal to zero is one. The value of unit impulse signal is zero for all other values of 't'.

Unit Step Signal

A unit step signal, u(t) is defined as

$$u(t)=1;t\geq 0$$
$$=0;t<0$$

Following figure 2.1.3 shows unit step signal.

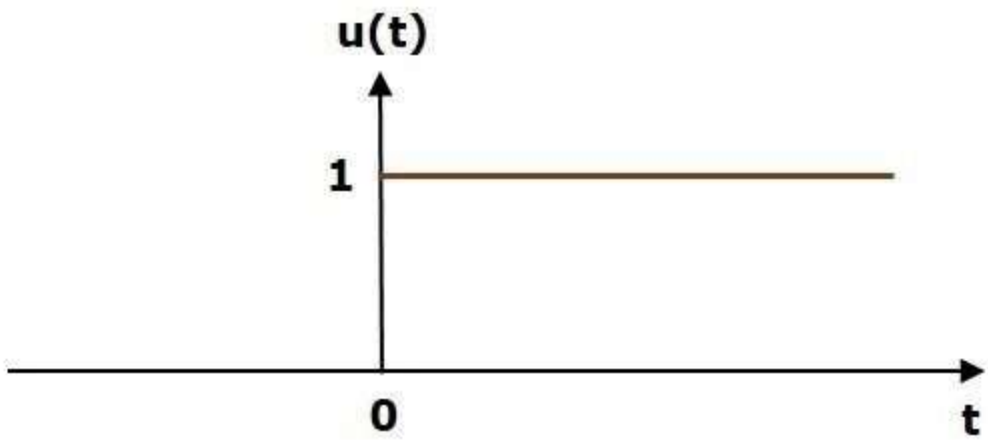

Figure 2.1.3: unit step response

[Source: "Control System Engineering" by Nagoor Kani, page-2.2]

So, the unit step signal exists for all positive values of 't' including zero. And its value is one during this interval. The value of the unit step signal is zero for all negative values of 't'.

Unit Ramp Signal

A unit ramp signal, r(t) is defined as

$$r(t)=t;t\geq 0$$

$$=0;t<0$$

We can write unit ramp signal, r(t) in terms of unit step signal, u(t) as

$r(t)=tu(t)$

Following figure 2.1.4 shows unit ramp signal.

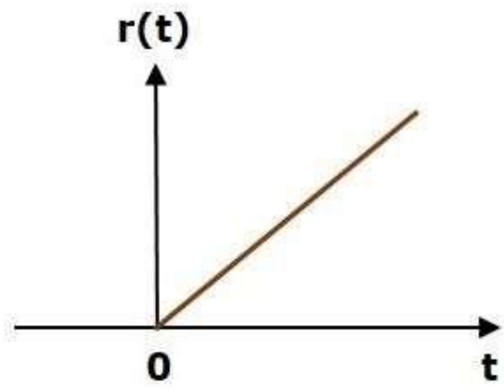

Figure 2.1.4: unit ramp response

[Source: "Control System Engineering" by Nagoor Kani, page-2.2]

So, the unit ramp signal exists for all positive values of 't' including zero. And its value increases linearly with respect to 't' during this interval. The value of unit ramp signal is zero for all negative values of 't'.

Unit Parabolic Signal

A unit parabolic signal, p(t) is defined as,

$$p(t)=\frac{t^2}{2};t\geq 0$$

$$=0;t<0$$

We can write unit parabolic signal, p(t) in terms of the unit step signal, u(t) as,

$$p(t)=\frac{t^2}{2}u(t)$$

The following figure 2.1.4 shows the unit parabolic signal.

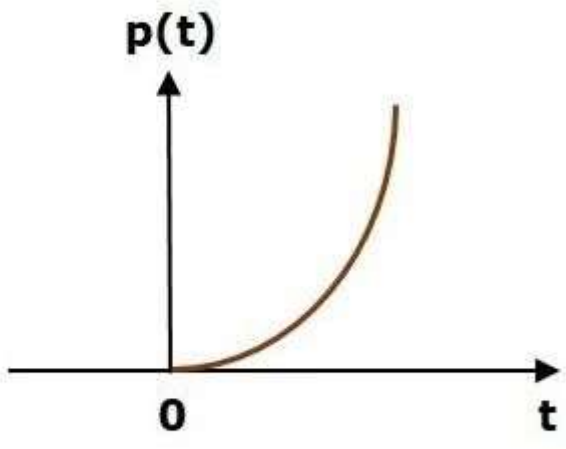

Figure 2.1.4: unit parabolic response

[Source: "Control System Engineering" by Nagoor Kani, page-2.2]

So, the unit parabolic signal exists for all the positive values of **'t'** including zero. And its value increases non-linearly with respect to 't' during this interval. The value of the unit parabolic signal is zero for all the negative values of 't'.

Test Signals

Input	r(t)	R(s)
Step Input	A	A/s
Ramp Input	At	A/s^2
Parabolic Input	$At^2 / 2$	A/s^3
Impulse Input	δ(t)	1

Transfer Function

- One of the types of Modeling a system
- Using first principle, differential equation is obtained
- Laplace Transform is applied to the equation assuming zero initial conditions
- Ratio of LT (output) to LT (input) is expressed as a ratio of polynomial in *s* in the transfer function

Order of a system

The Order of a system is given by the order of the differential equation governing the system. Alternatively, order can be obtained from the transfer function. In the transfer function; the maximum power of s in the denominator polynomial gives the order of the system

2.5 CONTROLLERS

A controller is a device introduced in the system to modify the error signal and to produce a control signal. The manner which the controller produces the Control signal is called the control action. The controllers modify the transient response of the system.

Depending on Control action controllers can be classified as.

1. Two-position or ON-OFF controllers
2. Proportional controllers
3. Integral controllers
4. Proportional plus integral controllers
5. Proportional plus derivative controllers
6. Proportional plus integral plus derivative controllers

Proportional Controller

The proportional controller produces an output, which is proportional to error signal.

$$u(t) \propto e(t)$$

$$\Rightarrow u(t) = K_P e(t)$$

Apply Laplace transform on both the sides -

$$U(s) = K_P E(s)$$

$$U(s)/E(s) = K_P$$

Therefore, the transfer function of the proportional controller is K_P.

Where,

U(s) is the Laplace transform of the actuating signal u(t)

E(s) is the Laplace transform of the error signal e(t)

K_P is the proportionality constant

The block diagram of the unity negative feedback closed loop control system along with the proportional controller is shown in the following figure 2.5.1.

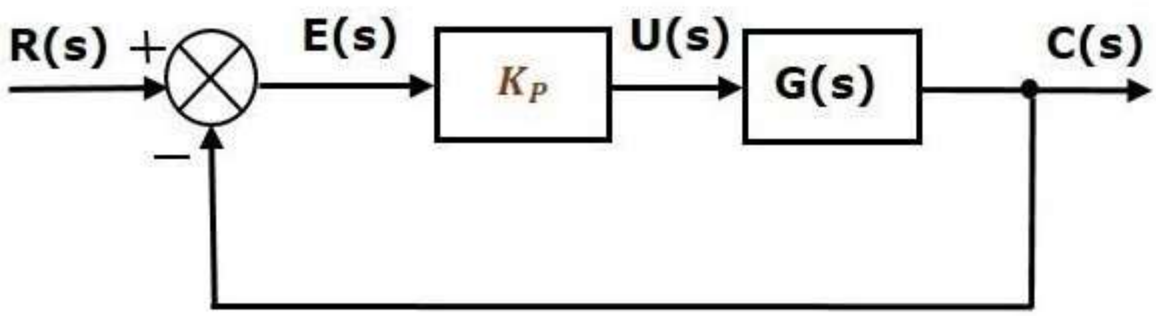

Figure 2.5.1: block diagram of proportional controller

[Source: "Control System Engineering" by Nagoor Kani, page-5.64]

The proportional controller is used to change the transient response as per the requirement.

Derivative Controller

The derivative controller produces an output, which is derivative of the error signal.

$$u(t)=K_D\ de(t)/dt$$

Apply Laplace transform on both sides.

$$U(s)=K_D sE(s)$$

$$U(s)\ /\ E(s)=K_D s$$

Therefore, the transfer function of the derivative controller is KDsKDs.

Where, K_D is the derivative constant.

The block diagram of the unity negative feedback closed loop control system along with the derivative controller is shown in the following figure 2.5.2.

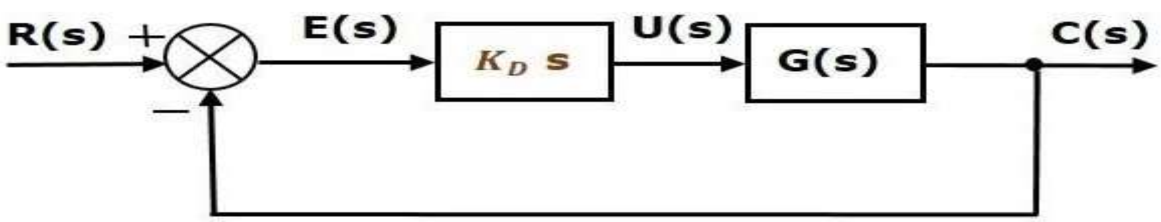

Figure 2.5.2: block diagram of derivative controller

[Source: "Control System Engineering" by Nagoor Kani, page-5.64]

The derivative controller is used to make the unstable control system into a stable one.

Integral Controller

The integral controller produces an output, which is integral of the error signal.

$$( ) \int ( )$$

Apply Laplace transform on both the sides -

$$U(s)=K_I E(s)/s$$

$$U(s)/E(s)=K_I/s$$

Therefore, the transfer function of the integral controller is K_I/s.

Where, K_I is the integral constant.

The block diagram of the unity negative feedback closed loop control system along with the integral controller is shown in the following figure 2.5.3.

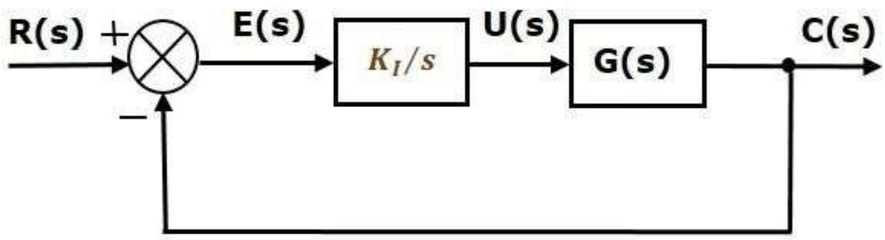

Figure 2.5.3: block diagram of integral controller

[Source: "Control System Engineering" by Nagoor Kani, page-5.64]

The integral controller is used to decrease the steady state error.

Let us now discuss about the combination of basic controllers.

Proportional Derivative (PD) Controller

The proportional derivative controller produces an output, which is the combination of the outputs of proportional and derivative controllers.

$$u(t)=K_{Pe}(t)+K_D\, de(t)/dt$$

Apply Laplace transform on both sides -

$$U(s)=(K_P+K_D s)E(s)$$

$$U(s) / E(s)=K_P+K_D s$$

Therefore, the transfer function of the proportional derivative controller is $K_P+K_D s$.

The block diagram of the unity negative feedback closed loop control system along with the proportional derivative controller is shown in the following figure 2.5.4.

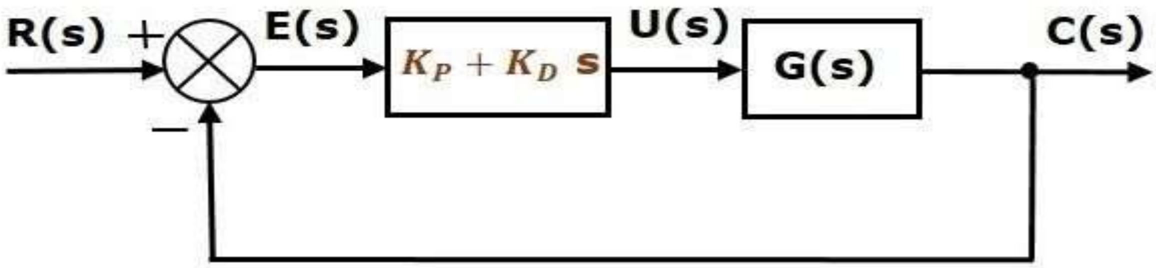

Figure 2.5.4: block diagram of PD controller

[Source: "Control System Engineering" by Nagoor Kani, page-5.64]

The proportional derivative controller is used to improve the stability of control system without affecting the steady state error.

Proportional Integral (PI) Controller

The proportional integral controller produces an output, which is the combination of outputs of the proportional and integral controllers.

$$u(t)=K_P e(t)+ \quad \int ()$$

Apply Laplace transform on both sides -

$$U(s)=(K_P+K_I /s)E(s)$$

$$U(s)/E(s)=K_P+K_I /s$$

Therefore, the transfer function of proportional integral controller is K_P+K_I / s.

The block diagram of the unity negative feedback closed loop control system along with the proportional integral controller is shown in the following figure2.5.5.

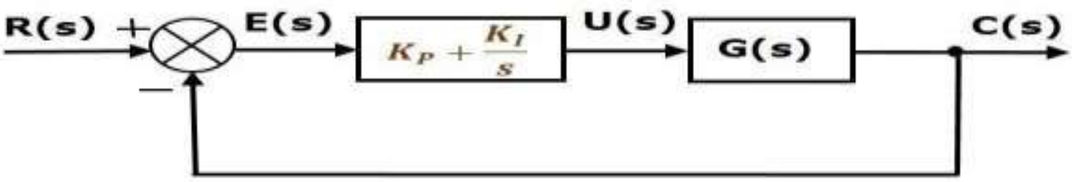

Figure 2.5.5: block diagram of PI controller

[Source: "Control System Engineering" by Nagoor Kani, page-5.64]

The proportional integral controller is used to decrease the steady state error without affecting the stability of the control system.

Proportional Integral Derivative (PID) Controller

The proportional integral derivative controller produces an output, which is the combination of the outputs of proportional, integral and derivative controllers.

$u(t)=K_Pe(t)+ \int ( ) +K_D\, de(t)/dt$ Apply Laplace transform on both sides -

$$U(s)=(K_P+K_I /s+K_Ds)E(s)$$

$$U(s)/E(s)=K_P+K_I / s+K_Ds$$

Therefore, the transfer function of the proportional integral derivative controller is $K_P+K_I /s +K_Ds$.

The block diagram of the unity negative feedback closed loop control system along with the proportional integral derivative controller is shown in the following figure.

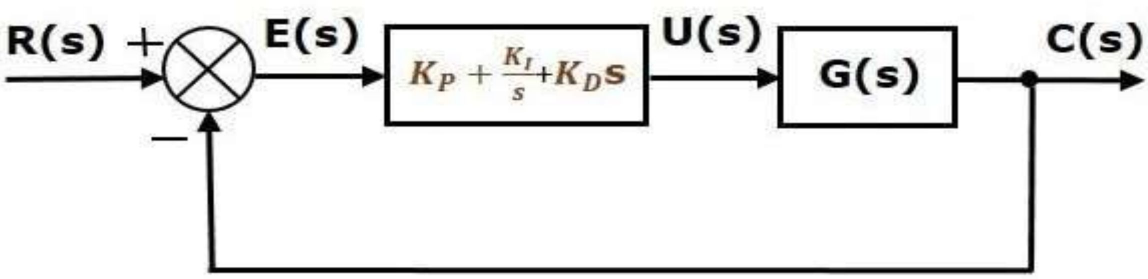

Figure 2.5.6: block diagram of PID controller

[Source: "Control System Engineering" by Nagoor Kani, page-5.64]

The proportional integral derivative controller is used to improve the stability of the control system and to decrease steady state error