4.1 CHARACTERISTIC EQUATION

The characteristic equation is nothing more than setting the denominator of the closed-loop transfer function to zero. In control theory, there are two main methods of analyzing feedback systems: the transfer function (or frequency domain) method and the state space method. The characteristic equation is the equation which is solved to find a matrix's eigenvalues, also called the characteristic polynomial. Characteristic equation is used to solve linear differential equations. Characteristic equations of auxiliary differential equations are used to solve a partial differential equation.

The properties of transfer function are given below:

- The ratio of Laplace transform of output to Laplace transform of input assuming all initial conditions to be zero.

- The transfer function of a system does not depend on the inputs to the system.

- The system poles and zeros can be determined from its transfer function.

Closed loop transfer function:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

Characteristic Equation:

$$1 \pm G(s)H(s) = 0$$

4.5 DESIGN OF LAG, LEAD AND LAG LEAD COMPENSATOR USING BODE PLOTS

DESIGN PROCEDURE OF LAG COMPENSATOR USING BODE PLOTS

1. Determine the compensator gain K to meet the steady state error requirement.

2. Draw the Bode plots of KG(s).

3. From the Bode plots, find the frequency ω_g at which the phase of KG(s) is

$$\angle KG(\omega_g) = PM - 180^o + 5^o \sim 10^o$$

4. Calculate β to make ω_g the gain crossover frequency,

$$20 \log \beta = 20 \log K + 20 \log|G(j\omega_g)|$$

5. Choose T to be much greater than 1/ ω_g, for example, T=10/ ω_g.

6. Verify the results using MATLAB.

DESIGN PROCEDURE OF LEAD COMPENSATOR USING BODE PLOTS

1. Draw the Bode plot for the uncompensated system and obtain the current phase margin available.

2. Calculate the phase margin required to meet the damping coefficient or percent overshoot requirement. Don't forget to add some extra phase margin to compensate for imperfections in the controller design (approximately 10 degrees of phase is good).

$$PM = \tan^{-1} \frac{2\zeta}{\left(\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}\right)}$$

$$\zeta \cong \frac{PM}{100}$$

3. Calculate the value of alpha from the following equation. Use the phase margin obtained in Step (4) as the maximum phase value:

$$\alpha = \frac{1 - \sin \phi_{max}}{1 + \sin \phi_{max}}$$

4. Calculate the gain corresponding to the maximum phase frequency using the equation below. We are going to look for the new phase margin frequency that we want to design for by looking for places where this gain is present on the Bode plot.

$$|G(j\omega_{max})| = \frac{1}{\sqrt{\beta}}$$

5. Find the new maximum phase margin frequency by looking for the point where the uncompensated system's magnitude curve is the negative value of the gain calculated in Step (4).

6. Select the break frequencies, T and beta*T using the maximum frequency equation given below:

$$\omega_{max} = \frac{1}{T\sqrt{\beta}}$$

7. Reset the system gain to adjust for the compensator's gain.

8. Check that the bandwidth still meets design requirements. Simulate the system and repeat the design as necessary.

DESIGN PROCEDURE OF LAG-LEAD COMPENSATOR USING BODE PLOTS

The lag-lead compensator is the analog to the PID controller. The lag-lead compensator can meet multiple design requirements: the lag component reduces high frequency gain, stabilizes the system and meets steady state requirements, while the lead component is used to meet transient response design requirements. The general equation for this kind of compensator is given below:

$$G_{lag-lead}(s) = G_{lead}(s)G_{lag}(s) = \left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}}\right)\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\gamma T_2}}\right), \gamma > 1$$

(1) Calculate the required bandwidth to meet the transient performance requirement (usually expressed in terms of the settling time, rise time or peak time). Use the equation provided above.

(2) Set the DC gain of the uncompensated system to meet the steady state requirements (this requires use of the Final Value Theorem).

(3) Draw the Bode plot for the uncompensated system and obtain the current phase margin available.

(4) Calculate the phase margin required to meet the damping coefficient or percent overshoot requirement. Don't forget to add some extra phase margin to compensate

for imperfections in the controller design (approximately 10 degrees of phase is good).

(5) Select a new phase margin frequency that is slightly less than the bandwidth.

(6) At this new phase margin frequency, calculate the phase lead required to obtained the phase margin from Step (4). Add some additional phase to adjust for the lag compensator's effects, if you have not already done so in Step (4).

(7) Design the lag compensator. Choose the higher breakpoint frequency as the phase margin frequency divided by 10. A plot of the interaction between beta and the phase margin is used to select beta, but for our purposes, I think that spacing the pole and zero of the lag compensator apart by a factor of 10 is sufficient for our design purposes.

(8) We can calculate gamma as the inverse of beta, so now we are ready to design the lead compensator. Use the equation given below to find T.

(9) Check that the bandwidth still meets design requirements. Simulate the system and repeat the design as necessary.

4.4 EFFECT OF LAG, LEAD AND LAG-LEAD COMPENSATION ON FREQUENCY RESPONSE

Every control system which has been designed for a specific application should meet certain performance specification. There are always some constraints which are imposed on the control system design in addition to the performance specification. The choice of a plant is not only dependent on the performance specification but also on the size, weight & cost. Although the designer of the control system is free to choose a new plant, it is generally not advised due to the cost & other constraints. Under this circumstance, it is possible to introduce some kind of corrective sub-systems in order to force the chosen plant to meet the given specification. We refer to these sub-systems as compensator whose job is to compensate for the deficiency in the performance of the plant.

Necessary of Compensation

1. In order to obtain the desired performance of the system, we use compensating networks. Compensating networks are applied to the system in the form of feed forward path gain adjustment.

2. Compensate a unstable system to make it stable.

3. A compensating network is used to minimize overshoot.

4. These compensating networks increase the steady state accuracy of the system. An important point to be noted here is that the increase in the steady state accuracy brings instability to the system.

5. Compensating networks also introduces poles and zeros in the system thereby causes changes in the transfer function of the system. Due to this, performance specifications of the system change.

EFFECT OF LAG COMPENSATION ON FREQUENCY RESPONSE

The Lag Compensator is an electrical network which produces a sinusoidal output having the phase lag when a sinusoidal input is applied. The lag compensator circuit in the 's' domain is shown in the following figure.
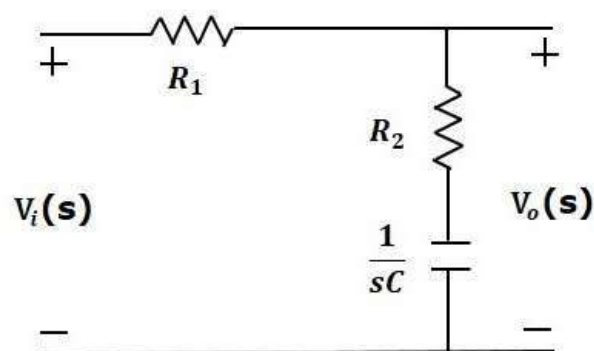

Figure 4.4.1 Electrical lag compensator

[Source: "Control Systems" by A Nagoor Kani, Page: 4.65]

Here, the capacitor is in series with the resistor R_2 and the output is measured across this combination. The transfer function of this lag compensator is

$$\frac{V_0(s)}{V_i(s)} = \frac{1}{\beta}\left(\frac{s + \frac{1}{\tau}}{s + \frac{1}{\beta\tau}}\right)$$

$$\tau = R_2C$$

$$\beta = \frac{R_1 + R_2}{R_2}$$

$$\beta > 1$$

Pole, $s = -\frac{1}{\beta\tau}$

Zero, $s = -\frac{1}{\tau}$

Let s = jω,

$$\frac{V_0(j\omega)}{V_i(j\omega)} = \frac{1}{\beta}\left(\frac{j\omega + \frac{1}{\tau}}{j\omega + \frac{1}{\beta\tau}}\right)$$

Phase angle,
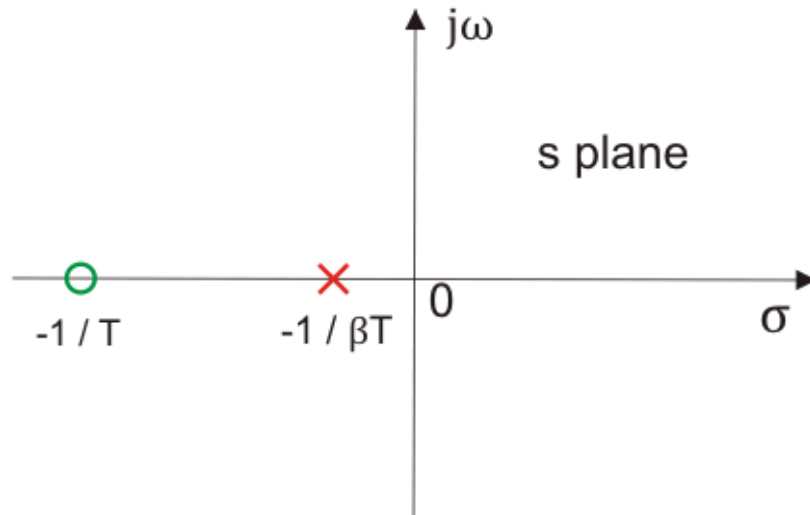
$$\phi = \tan^{-1}\omega\tau - \tan^{-1}\beta\omega\tau$$

Figure 4.4.2 Pole-zero plot of lag compensator

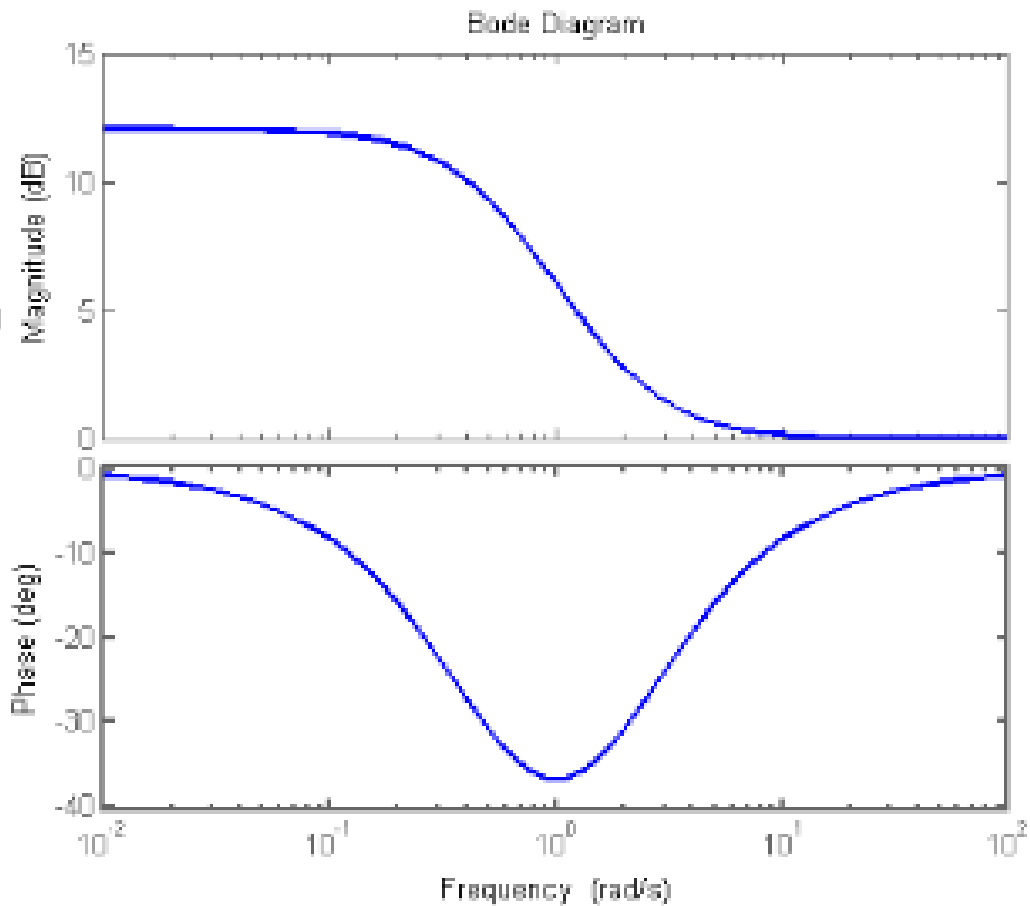*[Source: "Control Systems" by A Nagoor Kani, Page: 4.65]*

Figure 4.4.3 Bode plot of lag compensator

[Source: "Control Systems" by A Nagoor Kani, Page: 4.67]

The phase of the output sinusoidal signal is equal to the sum of the phase angles of input sinusoidal signal and the transfer function. So, in order to produce the phase lag at the output of this compensator, the phase angle of the transfer function should be negative. This will happen when $\beta > 1$.

Effect of Phase Lag Compensation

1.   Gain crossover frequency increases.

2.   Bandwidth decreases.

3.   Phase margin will be increase.

4.   Response will be slower before due to decreasing bandwidth, the rise time and the settling time become larger.

Advantages of Phase Lag Compensation

1.   Phase lag network allows low frequencies and high frequencies are attenuated.

2.   Due to the presence of phase lag compensation the steady state accuracy increases.

Disadvantages of Phase Lag Compensation

1.   Due to the presence of phase lag compensation the speed of the system decreases.

EFFECT OF LEAD COMPENSATION ON FREQUENCY RESPONSE

The lead compensator is an electrical network which produces a sinusoidal output having phase lead when a sinusoidal input is applied. The lead compensator circuit in the 's' domain is shown in the following figure. Lead compensator are used to improve the transient response of a system.
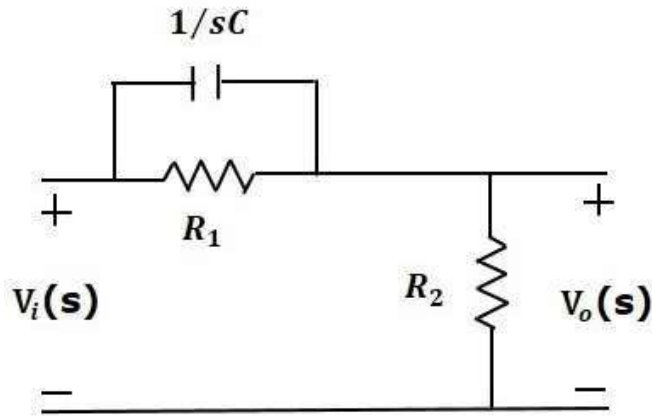

Figure 4.4.4 Electrical lead compensator

[Source: "Control Systems" by A Nagoor Kani, Page: 4.70]

Taking $i_2 = 0$ and applying Laplace Transform, we get,

$$\frac{V_o(s)}{V_i(s)} = \frac{R_2(R_1Cs + 1)}{R_1 + R_2 + R_2R_1Cs}$$

$$\frac{V_o(s)}{V_i(s)} = \alpha \left(\frac{\tau s + 1}{\alpha \tau s + 1} \right)$$

$$\tau = R_1 C$$

$$\alpha = \frac{R_2}{R_1 + R_2}$$

$$\alpha < 1$$

Pole, $s = -\dfrac{1}{\alpha\tau}$

Zero, $s = -\dfrac{1}{\tau}$

Let $s = j\omega$,

$$\frac{V_o(j\omega)}{V_i(j\omega)} = \alpha \left(\frac{\tau j\omega + 1}{\alpha \tau j\omega + 1} \right)$$

Phase angle,

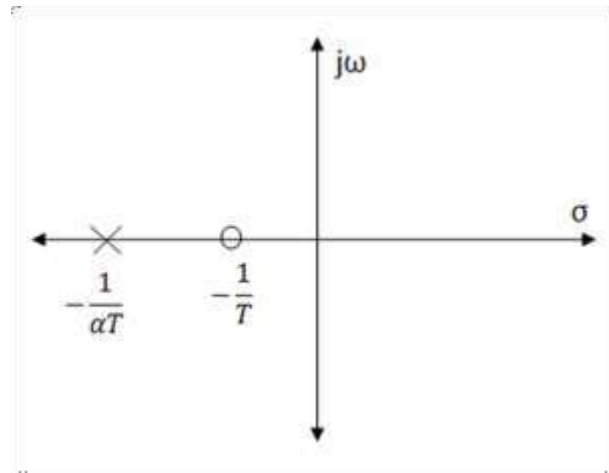$$\phi = \tan^{-1} \omega\tau - \tan^{-1} \alpha\omega\tau$$

Figure 4.4.5 Pole-zero plot of lead compensator

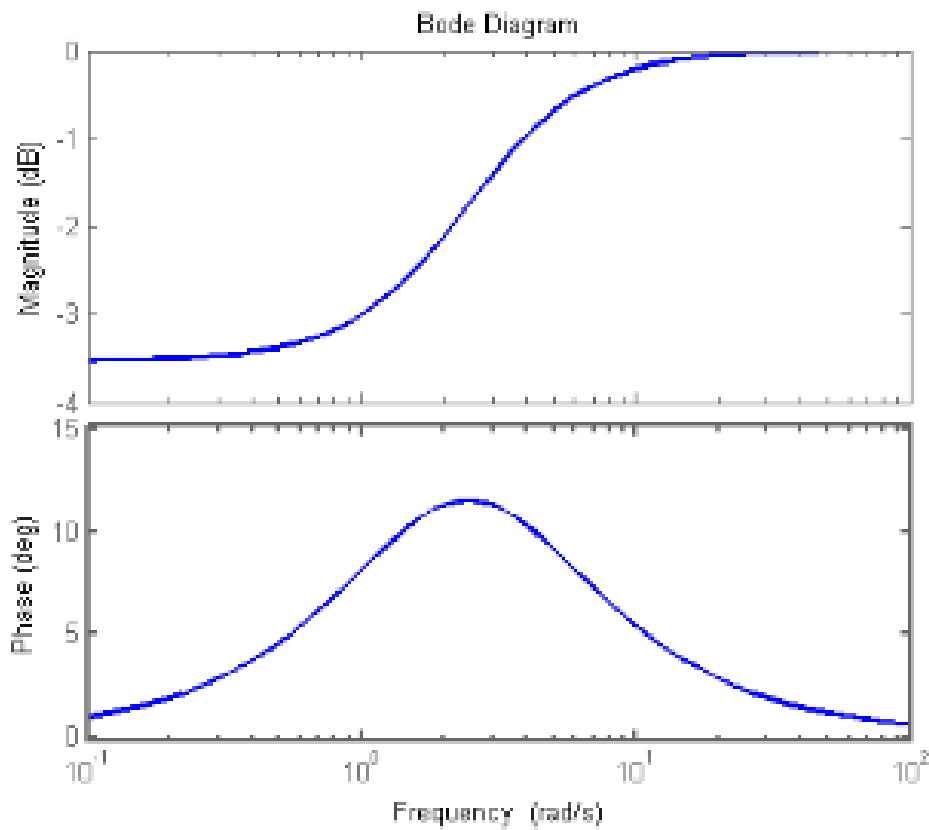*[Source: "Control Systems" by A Nagoor Kani, Page: 4.69]*

Figure 4.4.6 Bode plot of lead compensator

[Source: "Control Systems" by A Nagoor Kani, Page: 4.71]

The phase of the output sinusoidal signal is equal to the sum of the phase angles of input sinusoidal signal and the transfer function. So, in order to produce the phase lead at the output of this compensator, the phase angle of the transfer function should be positive.

This will happen when $0<\alpha<1$. Therefore, zero will be nearer to origin in pole-zero configuration of the lead compensator.

Bode plot of lead compensator

Maximum phase lead occurs at

$$\omega_m = \frac{1}{\tau\sqrt{\alpha}}$$

Let Φ_m = maximum phase lead

$$\phi_m = \frac{1-\alpha \sin}{1+\alpha}$$

$$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$$

Magnitude at maximum phase lead

$$|G_c(j\omega)| = \frac{1}{\sqrt{\alpha}}$$

Effect of Phase Lead Compensation

1. The velocity constant K_v increases.
2. The slope of the magnitude plot reduces at the gain crossover frequency so that relative stability improves and error decrease due to error is directly proportional to the slope.
3. Phase margin increases.
4. Response becomes faster.

Advantages of Phase Lead Compensation

1. Due to the presence of phase lead network the speed of the system increases because it shifts gain crossover frequency to a higher value.
2. Due to the presence of phase lead compensation maximum overshoot of the system decreases.

Disadvantages of Phase Lead Compensation

1. Steady state error is not improved.

EFFECT OF LAG-LEAD COMPENSATION ON FREQUENCY RESPONSE

Lag-Lead compensator is an electrical network which produces phase lag at one frequency region and phase lead at other frequency region. It is a combination of both the lag and the lead compensators. The lag-lead compensator circuit in the 's' domain is shown in the following figure.
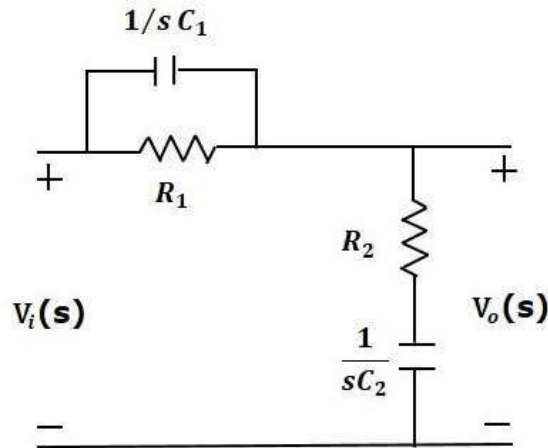

Figure 4.4.7 Electrical lag-lead compensator

[Source: "Control Systems" by A Nagoor Kani, Page: 4.73]

This circuit looks like both the compensators are cascaded. So, the transfer function of this circuit will be the product of transfer functions of the lead and the lag compensators.

$$\frac{V_o(s)}{V_i(s)} = \beta \left(\frac{\tau_1 s+1}{\beta\tau_1 s+1} \right) \frac{1}{\alpha} \left(\frac{s+\frac{1}{\tau_2}}{s+\frac{1}{\alpha\tau_2}} \right)$$

We know, $\alpha\beta=1$

$$\frac{V_o(s)}{V_i(s)} = \left(\frac{s+\frac{1}{\tau_1}}{s+\frac{1}{\beta\tau_1}} \right) \left(\frac{s+\frac{1}{\tau_2}}{s+\frac{1}{\alpha\tau_2}} \right)$$

where,

$$\tau_1 = R_1 C_1$$
$$\tau_2 = R_2 C_2$$

Advantages of Phase Lag Lead Compensation

1.   Due to the presence of phase lag-lead network the speed of the system increases because it shifts gain crossover frequency to a higher value.

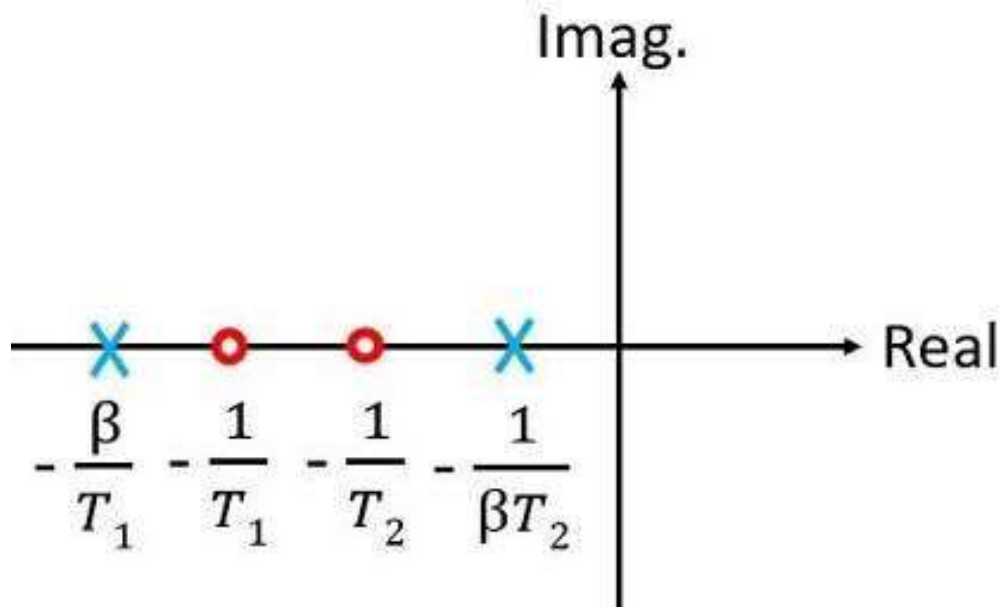2.   Due to the presence of phase lag-lead network accuracy is improved.

Figure 4.4.8 Pole-zero plot of lag-lead compensator

[Source: "Control Systems" by A Nagoor Kani, Page: 4.73]

4.3 NYQUIST STABILITY CRITERION

Nyquist criterion is a graphical method of determining stability of feedback control systems by using the Nyquist plot of their open-loop transfer functions.

Feedback transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Poles and zeros of the open loop transfer function

$$G(s)H(s) = \frac{K(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)}$$

$$1 + G(s)H(s) = \frac{(s - p_1)(s - p_2) \dots (s - p_n) + K(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)}$$

Number of closed loop poles – Number of zeros of 1+GH = Number of open loop poles

$$1 + G(s)H(s) = \frac{(s - z_{c1})(s - z_{c2}) \dots (s - z_{cm})}{(s - p_1)(s - p_2) \dots (s - p_n)}$$

where, z_{c1}, z_{c2}, ….., z_{cm} = zeros of 1+G(s)H(s)

These are also poles of the closed loop transfer function

$$Magnitude, |1 + G(s)H(s)| = \frac{|(s - z_{c1})||(s - z_{c2})| \dots |(s - z_{cm})|}{|(s - p_1)||(s - p_2)| \dots |(s - p_n)|}$$

$$Angle, \angle 1 + G(s)H(s) = \frac{\angle(s - z_{c1})\angle(s - z_{c2}) \dots \angle(s - z_{cm})}{\angle(s - p_1)\angle(s - p_2) \dots \angle(s - p_n)}$$

The s-plane to 1+GH plane mapping phase angle of the 1+G(s)H(s) vector, corresponding to a point on the s-plane is the difference between the sum of the phase of all vectors drawn from zeros of 1+GH (closed loop poles) and open loops on the s plane. If this point s is moved along a closed contour enclosing any or all of the above zeros and poles, only the phase of the vector of each of the enclosed zeros or open-loop poles will change by 3600. The direction will be in the same sense of the contour enclosing zeros and in the opposite sense for the contour enclosing open-loop poles. A stability test for time invariant linear systems can also be derived in the frequency domain. It is known as Nyquist stability criterion. It is based on the complex analysis result known as *Cauchy's principle of argument*. Note that the system transfer function is a complex function. By applying Cauchy's principle of argument to the *open-loop system* transfer function, we will get information about stability of the closed-loop

system transfer function and arrive at the Nyquist stability criterion (Nyquist, 1932). The importance of Nyquist stability lies in the fact that it can also be used to determine the relative degree of system stability by producing the so-called phase and gain stability margins. These stability margins are needed for frequency domain controller design techniques. Only the essence of the Nyquist stability criterion is presented and the phase and gain stability margins are defined. The Nyquist method is used for studying the stability of linear systems with pure time delay.

For a SISO feedback system the closed-loop transfer function is given by,

$$M(s) = \frac{G(s)}{1 + G(s)H(s)}$$

where, G(s) represents the system and H(s) is the feedback element. Since the system poles are determined as those values at which its transfer function becomes infinity, it follows that the closed-loop system poles are obtained by solving the following equation.

$$1 + G(s)H(s) = 0 = \Delta(s) \text{ which, in fact,}$$

represents the *system characteristic equation*. **Principles of Argument**

When a closed contour in the s-plane encloses a certain number of poles and zeros of 1+G(s)H(s) in the clockwise direction, the number of encirclements of the origin by the corresponding contour in the G(s)H(s) plane will encircle the point (-1,0) a number of times given by the difference between the number of its zeros and poles of 1+G(s)H(s) it enclosed on the s-plane. Let F(s) be an analytic function in a closed region of the complex plane given in figure 4.3.1 except at a finite number of points (namely, the poles of F(s)). It is also assumed that F(s) is analytic at everypoint on the contour. Then, *as s travels around the contour in the s - plane in the clockwise direction, the function encircles the origin in the (Re{F(s)}, Im{F(s)}) - plane in the same directiontimes (see figure 4.3.1), with given by,*

$$N = Z - P$$

where Z and P stand for the number of zeros and poles (including their multiplicities) of the function F(s) inside the contour.
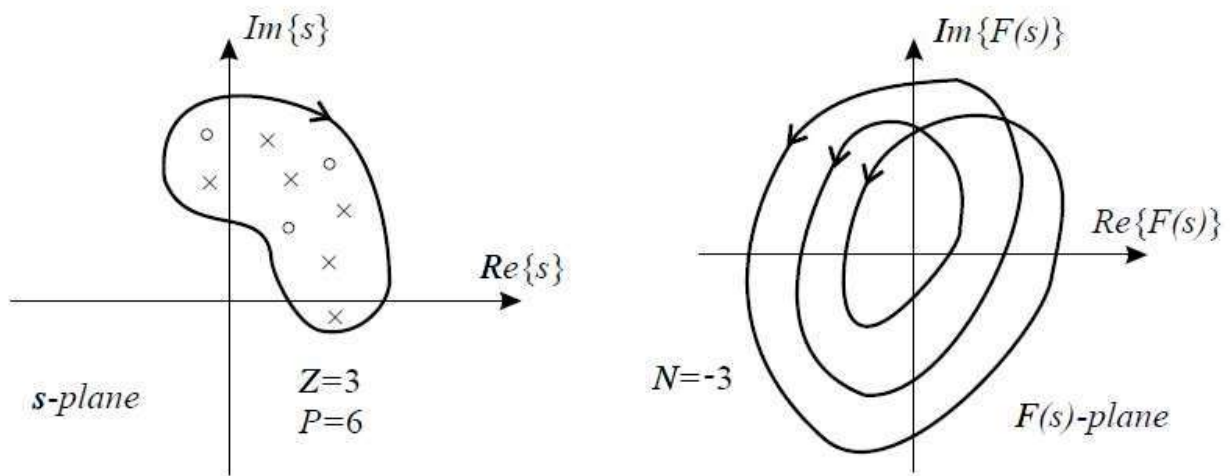
$$arg\{F(s)\} = (Z - P)2\pi = 2\pi N$$

Figure 4.3.1 s-plane and F(s) plane contours

[Source: "Control Systems" by A Nagoor Kani, Page: 4.27]

Contour in the s-plane

The Nyquist plot is a polar plot of the function $D(s) = 1+G(s)H(s)$ when 's' travels around the contour given in figure 4.3.2.
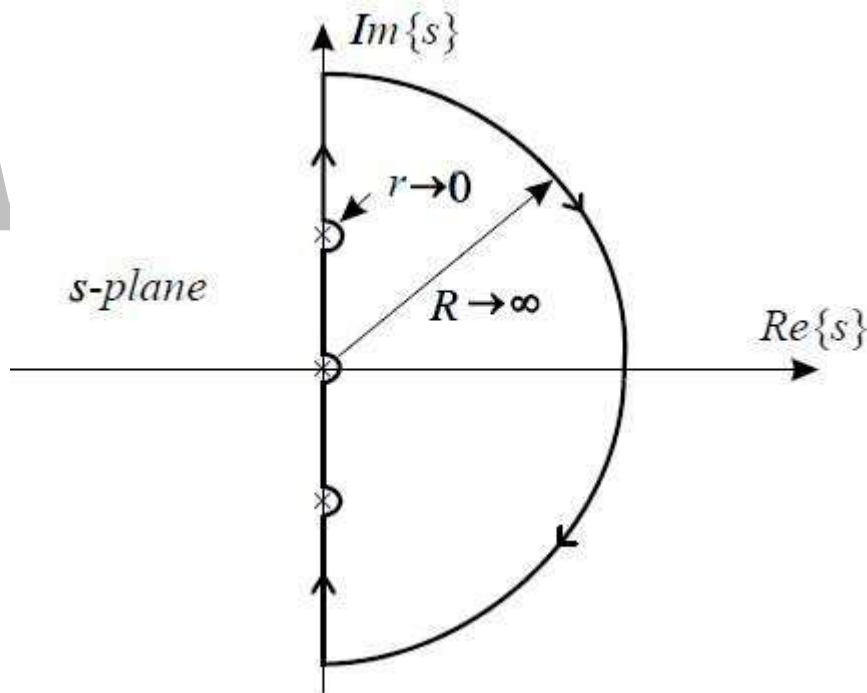

Figure 4.3.2 Nyquist contour when the poles are on imaginary axis and at origin

[Source: "Control Systems" by A Nagoor Kani, Page: 4.33]

Phase and Gain Stability Margins

Two important notions can be derived from the Nyquist diagram: *phase and gain stability margins*. The phase and gain stability margins are presented in figure 4.3.3.
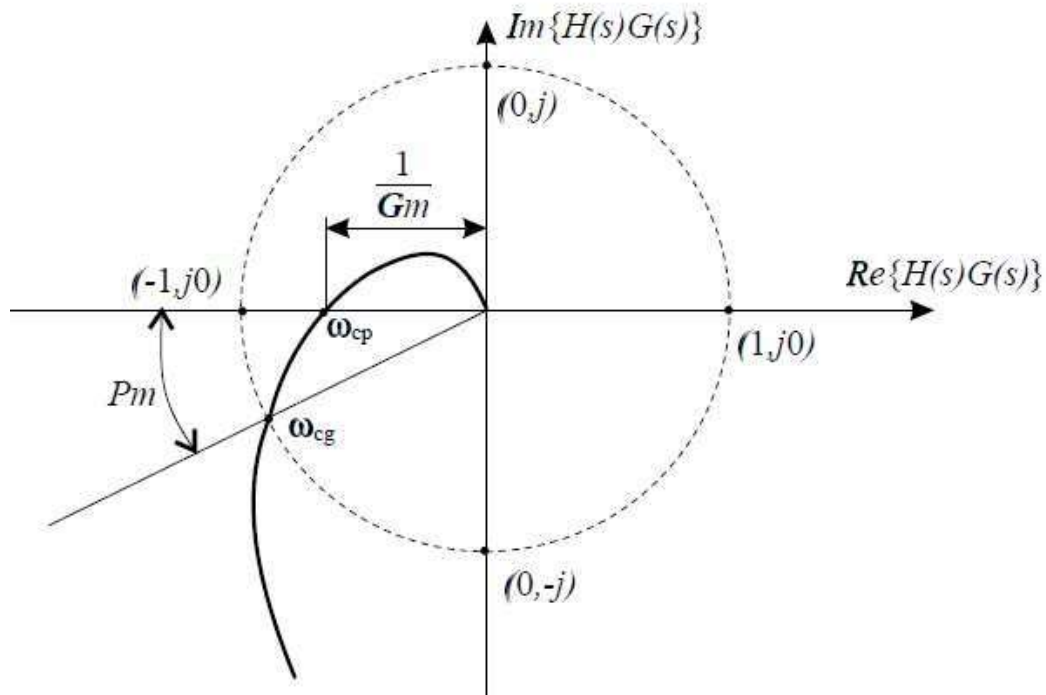
Figure 4.3.3 Gain and Phase margin

[Source: "Control Systems" by A Nagoor Kani, Page: 4.33]

They give the degree of relative stability; in other words, they tell how far the given system is from the instability region. Their formal definitions are given by

$$PM = 180^o + arg\{G(j\omega_{gc})H(j\omega_{gc})\}$$

$$GM(dB) = 20 \log \frac{1}{|G(j\omega_{pc})H(j\omega_{pc})|}, (dB)$$

where, ω_{gc} and ω_{pc} stand for gain and phase crossover frequency respectively.

$$|G(j\omega_{gc})H(j\omega_{gc})| = 1 \Rightarrow \omega_{gc}$$

$$arg\{G(j\omega_{pc})H(j\omega_{pc})\} = 180^o \Rightarrow \omega_{pc}$$

PROCEDURE FOR INVESTIGATING STABILITY USING NYQUIST CRITERION

The following procedure can be followed to investigate the stability of closed loop system from the knowledge of open loop system, using Nyquist stability criterion.

1. Choose a Nyquist contour as shown in figure, which encloses the entire right half s-plane except the singular points. The Nyquist contour encloses all the right half s-plane poles and zeros of G(s)H(s). [The poles on imaginary axis are singular points and so they are avoided by taking a detour around it as shown in figures.

2. The Nyquist contour should be mapped in the G(s)H(s)-plane using the function G(s)H(s) to determine the encirclement -1 + j0 point in the G(s)H(s)-plane. The

Nyquist contour of the figure can be divided into four sections $C_1.C_2.C_3$ and C_4. The mapping of the four sections in the G(s)H(s)-plane can be carried sectionwise and then combined together to get entire G(s)H(s)-contour.

3. In section C_1, the value of ω varies from 0 to + infinite. The mapping of section C_1 is obtained by letting s = jω in G(s)H(s) and varying ω from 0 to + infinite.

The locus of G(jω)H(jω) as ω is varied from 0 to + infinite will be the G(s)H(s)-contour in G(s)H(s)-plane corresponding to section C_1 in s-plane. This locus is the plot of G(jω)H(jω). There are three ways of mapping this section of G(s)H(s)-contour, they are,

(i) Calculate the values of G(jω)H(jω) for various values of ω and sketch the actual locus of G(jω)H(jω).

(or)

(ii) Separate the real part and imaginary part of G(jω)H(jω). Equate the imaginary part to zero, to find the frequency at which the G(jω)H(jω) locus crosses real axis (to find phase crossover frequency). Substitute this frequency on real part and find the crossing point of the locus on real axis. Sketch the approximate locus of G(jω)H(jω) from the knowledge of type number and order of the system (or from the value of G(jω)H(jω) at ω = 0 and ω = infinite).

(or)

(iii) Separate the magnitude and phase of G(jω)H(jω). Equate the phase of G(jω)H(jω) to -180º and solve for ω. This value of ω is the phase crossover frequency and the magnitude at this frequency is the crossing point on real axis. Sketch the approximate root locus as mentioned in method (ii).

4. The section C_2 of Nyquist contour has a semicircle of infinite radius. Therefore, every point on section C_2 has infinite magnitude but the argument varies from +π/2 to - π/2. Consider the loop transfer function in time constant form and with y number of poles at origin, as shown below. Let G(s)H(s) has m zeros & n poles including poles at origin. For practical systems, n>m. From the above two equations we can conclude that the section C_2 of Nyquist contour in s-plane is mapped as circles/circular are around origin with radius tending to zero in the G(s)H(s)-plane.

5. In section C3, the value of ω varies from -∞ to 0. The mapping of section C3 is obtained by letting s=jω in G(s)H(s) and varying ω from -∞ to 0. The locus of G(jω)H(jω) as ω is varied from -∞ to 0 will be the G(s)H(s)-contour in G(s)H(s)-plane corresponding to section C3 in s-plane. This locus is the inverse polar plot of G(jω)H(jω). The inverse polar plot is given by the mirror image of polar plot with respect to real axis.

6. The section C4 of Nyquist contour has a semicircle of zero radius. Therefore, every point on semicircle has zero magnitude but the argument varies from -π/2 to π/2. Hence the mapping of section C4 from s-plane to G(s)H(s)-plane can be obtained by letting in G(s)H(s) and varying θ from -π/2 to π/2.

PERFORMANCE CRITERIA

For ordinary random inputs (i.e. inputs such that the error E is a stationary random function of time t), it is usual to adopt the mean -square- error as the performance criterion. This is the analogue of integral- square-error for simple transient inputs.

4.2 ROUTH HURWITZ CRITERION

Consider a closed-loop transfer function

$$H(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 \; s^n + a_1 \quad s^{n-1} + \cdots + a_{n-1} \quad s + a_n} = \frac{B(s)}{A(s)}$$

where the a_i's and b_i's are real constants and m ≤ n. An alternative to factoring the denominator polynomial, Routh's stability criterion, determines the number of closed-loop poles in the right-half s-plane.

Algorithm for applying Routh's stability criterion

The algorithm described below, like the stability criterion, requires the order of A(s) to be finite.

1. Factor out any roots at the origin to obtain the polynomial, and multiply by −1 if necessary, to obtain

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0$$

$$where, a_0 \neq 0 \text{ and } a_n > 0$$

2. If the order of the resulting polynomial is at least two and any coefficient a_i is zero or negative, the polynomial has at least one root with nonnegative real part. To obtain the precise number of roots with nonnegative real part, proceed as follows. Arrange the coefficients of the polynomial, and values subsequently calculated from them as shown below:

s^n	a_0	a_2	a_4	a_6	...
s^{n-1}	a_1	a_3	a_5	a_7	...
s^{n-2}	b_1	b_2	b_3	b_4	...
s^{n-3}	c_1	c_2	c_3	c_4	...
s^{n-4}	d_1	d_2	d_3	d_4	...
$\vdots$	$\vdots$	$\vdots$		
s^2	e_1	e_2		
s^1	f_1			
s^0	g_0			

The array is generated until all subsequent coefficients are zero. Similarly, cross multiply the coefficients of the two previous rows to obtain the c_i, d_i, etc. Until the nth row of the array has been completed. Missing coefficients are replaced by zeros. The resulting array is called the Routh array. The powers of s are not considered to be part of the array. We can think of them as labels. The column beginning with a0 is considered to be the first column of the array. The Routh array is seen to be triangular. It can be shown that multiplying a row by a positive number to simplify the calculation of the next row does not affect the outcome of the application of the Routh criterion. where, the coefficients b_i are,

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$\vdots$$

3. Count the number of sign changes in the first column of the array. It can be shown that a necessary and sufficient condition for all roots of (2) to be located in the left-half plane is that all the a_i are positive and all of the coefficients in the first column be positive.

Example: Generic Cubic Polynomial

Consider the generic cubic polynomial:

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

where all the ai are positive. The Routh array is

s^3	a_0	a_2
s^2	a_1	a_3
s^1	$\dfrac{a_1 a_2 - a_0 a_3}{a_0}$	
s^0	a_3	

So, the condition that all roots have negative real parts is

$$a_1 a_2 > a_0 a_3$$

Example: A Quadratic Polynomial.

Next, we consider the fourth-order polynomial:

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Here we illustrate the fact that multiplying a row by a positive constant does not change the result. One possible Routh array is given at left, and an alternative is given at right,

s^4	1	3	5
s^3	2	4	0
s^2	1	5	
s^1	-6		
s^0	5		

Also,

s^4	1	3	5
s^3	2	4	0
	1	2	0
s^2	1	5	
s^1	-3		
s^0	5		

In this example, the sign changes twice in the first column so the polynomial equation A(s) = 0 has two roots with positive real parts.

Necessity of all coefficients being positive

In stating the algorithm above, we did not justify the stated conditions. Here we show that all coefficients being positive is necessary for all roots to be located in the left half-plane. It can be shown that any polynomial ins, all of whose coefficients are real, can be factored into a product of a maximal number linear and quadratic factors also having real coefficients. Clearly a linear factor (s+a) has nonnegative real root if a is positive. For both roots of a quadratic factor (s2+bs+c) to have negative real parts both b and c must

be positive. (If c is negative, the square root of $b2-4c$ is real and the quadratic factor can be factored into two linear factors so the number of factors was not maximal.) It is easy to see that if all coefficients of the factors are positive, those of the original polynomial must be as well. To see that the condition is not sufficient, we can refer to several examples above.

Example: Determining Acceptable Gain Values

Consider a system whose closed-loop transfer function is

$$H(s) = \frac{K}{s(s^2 + s + 1)(s + 2) + K}$$

Characteristic equation

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

Route array is

s^4	1	3	K
s^3	3	2	
s^2	7/3	K	
s^1	(14-9K)/7		
s^0	K		

For the system to be stable, the elements of the first column of the Routh array should be positive. Based on that condition, the s^1 row yields the condition that, for stability,

$$\frac{(14 - 9K)}{7} > 0$$

$$(14 - 9K) > 0$$

$$14 > 9K$$

$$\frac{14}{9} > K$$

The s^0 row yields the condition that, for stability,

$$K > 0$$

Hence, the system is stable when the value of K lies in the range of

$$0 < K < 14/9$$

Special Case: Zero First-Column Element.

If the first term in a row is zero, but the remaining terms are not, the zero is replaced by a small, positive value of ϵ and the calculation continues as described above. Here's an example:

$$s^3 + 2s^2 + s + 2 = 0$$

Routh array is

s^3	1	1
s^2	2	2
s^1	$0 \cong \epsilon$	
s^0	2	

Special Case: Zero Row

If all the coefficients in a row are zero, a pair of roots of equal magnitude and opposite sign is indicated. These could be two real roots with equal magnitudes and opposite signs or two conjugate imaginary roots. The zero row is replaced by taking the coefficients of dP(s)/ds, where P(s), called the auxiliary polynomial, is obtained from the values in the row above the zero row. The pair of roots can be found by solving dP(s)/ds= 0. Note that the auxiliary polynomial always has even degree. It can be shown that an auxiliary polynomial of degree 2n has n pairs of roots of equal magnitude and opposite sign.