Boundary Conditions for Electromagnetic fields

The differential forms of Maxwell's equations are used to solve for the field vectors provided the field quantities are single valued, bounded and continuous. At the media boundaries, the field vectors are discontinuous and their behaviors across the boundaries are governed by boundary conditions.

The integral equations are assumed to hold for regions containing discontinuous media as shown in figure 1.1.Boundary conditions can be derived by applying the Maxwell's equations in the integral form to small regions at the interface of the two media. The procedure is similar to those used for obtaining boundary conditions for static electric fields (chapter 2) and static magnetic fields (chapter 4). The boundary conditions are summarized as follows

With reference to fig 4.3

$$\hat{a}_n \times \left(\vec{E}_1 - \vec{E}_2\right) = 0 \qquad 4.27(a)$$

$$\hat{a}_n \cdot \left(\vec{D}_1 - \vec{D}_2\right) = \rho_S \qquad 4.27(b)$$

$$\hat{a}_n \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{J}_S \qquad 4.27(c)$$

$$\hat{a}_n \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0 \qquad 4.27(d)$$

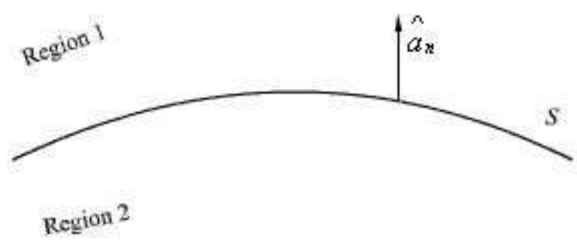

Fig 1.1 Boundary conditions of electromagnetic fields
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

Equation 4.27 (a) says that tangential component of electric field is continuous across the interface while from 4.27 (c) we note that tangential component of the magnetic field is discontinuous by an amount equal to the surface current density. Similarly 4.27(b) states that normal component of electric flux density vector $\vec{D}$ is discontinuous across the interface by an amount equal to the surface current density while normal component of the magnetic flux density is continuous.

If one side of the interface, as shown in fig 4.4, is a perfect electric conductor, say region 2, a surface current $\vec{J_S}$ can exist even though $\vec{E}$ is zero as $\sigma = \infty$.

Thus eqn 4.27(a) and (c) reduces to

$$\widehat{a_n} \times \vec{H} = \vec{J_S} \quad (4.28(a))$$

$$\widehat{a_n} \times \vec{E} = 0 \quad (4.28(b))$$

Plane waves in Lossless medium

In a lossless medium, ε and μ are real numbers, so k is real.

In Cartesian coordinates each of the equations 6.1(a) and 6.1(b) are equivalent tothree scalar Helmholtz's equations, one each in the components Ex, Ey and Ez or Hx, Hy, Hz.

For example if we consider Ex component we can write

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0 \quad \text{...(6.2)}$$

A uniform plane wave is a particular solution of Maxwell's equation assuming electric field (and magnetic field) has same magnitude and phase in infinite planes perpendicular to the direction of propagation. It may be noted that in the strict sense a uniform plane wave doesn't exist in practice as creation of such waves are possible with sources of infinite extent. However, at large distances from the source, the wavefront or the surface of the constant phase becomes almost spherical and a small portion of this large sphere can be considered to plane. The characteristics of plane waves are simple and useful for studying many practical scenarios.

Let us consider a plane wave which has only Ex component and propagating along z .Since the plane wave will have no variation along the plane perpendicular to z i.e., xy plane, $\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = 0$. The Helmholtz's equation (6.2) reduces to,

$$\frac{d^2 E_x}{dz^2} + k^2 E_x = 0 \quad \text{...(6.3)}$$

The solution to this equation can be written as

$$\begin{aligned} E_x(z) &= E_x^{+}(z) + E_x^{-}(z) \\ &= E_0^{+} e^{-jkz} + E_0^{-} e^{jkz} \end{aligned} \quad \text{..(6.4)}$$

E_0^+ & E_0^- are the amplitude constants (can be determined from boundary conditions). In the time domain, $\mathcal{E}_x(z,t) = \mathrm{Re}(E_x(z) e^{j\omega t})$

$$\mathcal{E}_x(z,t) = E_0^{+} \cos(\omega t - kz) + E_0^{-} \cos(\omega t + kz) \quad \text{............................(6.5)}$$

assuming E_0^+ & E_0^- are real constants.

Here, $\varepsilon_X^{+}(z,t) = E_0^{+}\cos(\omega t - \beta z)$ represents the forward traveling wave. The $\varepsilon_X^{+}(z,t)$ plot offor several values of t is shown in the Figure2.1.

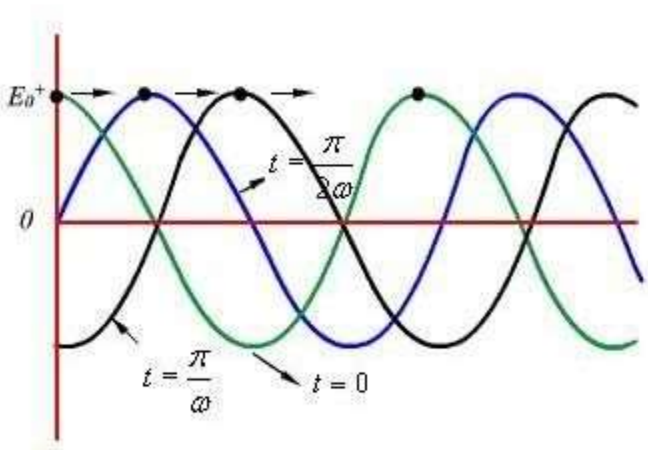

Figure 2.1: Plane wave traveling in the + *z* direction
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

As can be seen from the figure, at successive times, the wave travels in the +z direction. If we fix our attention on a particular point or phase on the wave (as shown by the dot)
i.e. , $\omega t - kz$ = constant

Then we see that as *t* is increased to $t + \Delta t$, z also should increase to $z + \Delta z$ so that

$$\omega(t + \Delta t) - k(z + \Delta z) = \text{constant} = \omega t - \beta z$$

Or, $$\omega \Delta t = k \Delta z$$

Or, $$\frac{\Delta z}{\Delta t} = \frac{\omega}{k}$$

When $\Delta t \rightarrow 0$,

we write $\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt}$ = phase velocity v_p .

$$\therefore v_p = \frac{\omega}{k}$$ ....(6.6)

If the medium in which the wave is propagating is free space i.e., $\varepsilon = \varepsilon_0$, $\mu = \mu_0$

Then $$v_p = \frac{\omega}{\omega\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = C$$

Where 'C' is the speed of light. That is plane EM wave travels in free space with thespeed of light.

The wavelength λ is defined as the distance between two successive maxima (or minimaor any other reference points).

i.e., $$(\omega t - kz) - \left[\omega t - k(z + \lambda)\right] = 2\pi$$

or, $$k\lambda = 2\pi$$

or, $$\lambda = \frac{2\pi}{k}$$

Substituting $k = \dfrac{\omega}{v_p}$,

$$\lambda = \frac{2\pi v_p}{2\pi f} = \frac{v_p}{f}$$

or, $\lambda f = v_p$ ....(6.7)

Thus wavelength λ also represents the distance covered in one oscillation of the wave. Similarly, $\varepsilon^-(z,t) = E_0^- \cos(\omega t + kz)$ represents a plane wave traveling in the -z direction.

The associated magnetic field can be found as follows:From (6.4),

$$\vec{E}_x^{\,+}(z) = E_0^{\,+} e^{-jkz}\hat{a}_x$$

$$\vec{H} = -\frac{1}{j\omega\mu}\nabla\times\vec{E}$$

$$= -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_0^{\,+} e^{-jkz} & 0 & 0 \end{vmatrix}$$

$$= \frac{k}{\omega\mu} E_0^{+} e^{-jkz} \hat{a}_y$$

$$= \frac{E_0^{+}}{\eta} e^{-jkz} \hat{a}_y = H_0^{+} e^{-jkz} \hat{a}_y \qquad ....(6.8)$$

where $\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$ is the intrinsic impedance

of themedium. When the wave travels in free space

i$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi = 377\Omega$ of the free space.

In the time domain,

$$\vec{H}^{+}(z,t) = \hat{a}_y \frac{E_0^{+}}{\eta} \cos(\omega t - \beta z) \qquad .... (6.9)$$

Which represents the magnetic field of the wave traveling in the +z

direction. For the negative traveling wave,

$$\vec{H}^{-}(z,t) = -a_y \frac{E_0^{+}}{\eta} \cos(\omega t + \beta z) \qquad ....(6.10)$$

For the plane waves described, both the E & H fields are perpendicular to the direction of propagation, and these waves are called TEM (transverse electromagnetic) waves.

The *E* & *H* field components of a TEM wave is shown in Fig 2.2.

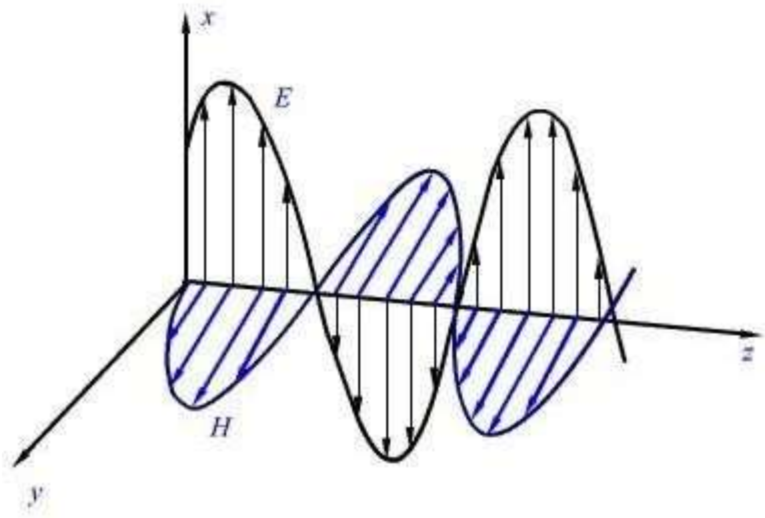

Figure 2.2 : E & H fields of a particular plane wave at time t.
(www.brainkart.com/subject/Electromagnetic-Theory_206/)