Absolute Electric Potential and potential differences and its calculation

In the previous sections we have seen how the electric field intensity due to a charge or a charge distribution can be found using Coulomb's law or Gauss's law. Since a charge placed in the vicinity of another charge (or in other words in the field of other charge) experiences a force, the movement of the charge represents energy exchange. Electrostatic potential is related to the work done in carrying a charge from one point to the other in the presence of an electric field.

Let us suppose that we wish to move a positive test charge Δq from a point *P* toanother point *Q* as shown in the Fig. 4.1.

The force at any point along its path would cause the particle to accelerate $d\vec{l}$ and move it outof the region if unconstrained. Since we dealing with Δq an electrostatic case, a force equal to the negative of that acting on the charge is to be applied while moves from *P* to *Q*. The work done by this external agent in moving the charge by a distance is givenby:

$$dW = -\Delta q \vec{E}.d\vec{l} \qquad \text{..........................(2.23)}$$

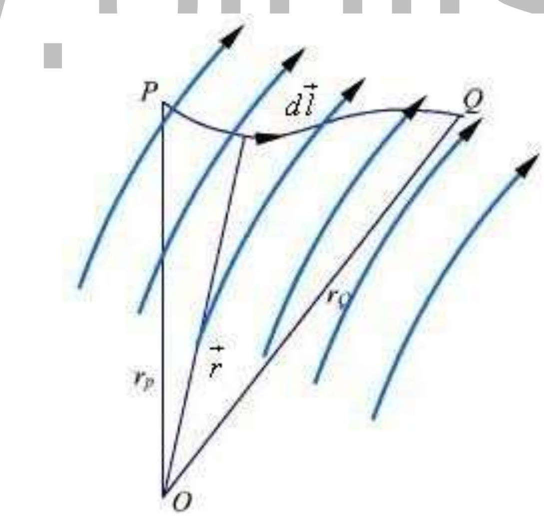


Fig 4.1: Movement of Test Charge in Electric Field
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

The negative sign accounts for the fact that work is done on the system by theexternal agent.

$$W = -\Delta q \int_P^Q \vec{E} \cdot d\vec{l} \qquad \text{....................................(2.24)}$$

The potential difference between two points *P* and *Q* , *VPQ*, is defined as the workdone per unit charge, i.e.

$$V_{PQ} = \frac{W}{\Delta Q} = -\int_{P}^{Q} \vec{E} \cdot d\vec{l}$$ ....(2.25)

It may be noted that in moving a charge from the initial point to the final point if the potential difference is positive, there is a gain in potential energy in the movement, external agent performs the work against the field. If the sign of the potential difference is negative, work is done by the field.

We will see that the electrostatic system is conservative in that no net energy is exchanged if the test charge is moved about a closed path, i.e. returning to its initial position. Further, the potential difference between two points in an electrostatic field is a point function; it is independent of the path taken. The potential difference is measured in

Joules/Coulomb which is referred to as **Volts**.

Let us consider a point charge *Q* as shown in the Fig. *4.2*.

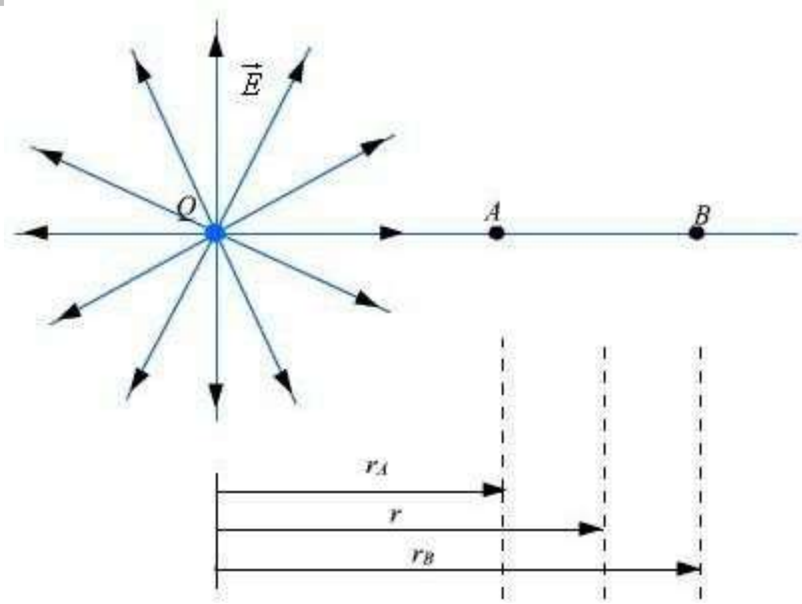


Fig 4.2: Electric Potential calculation for a point charge
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

Further consider the two points *A* and *B* as shown in the Fig. *2*.9.

Considering the movement of a unit positive test charge from B to A , we can write an expression for the potential difference as:

$$V_{BA} = -\int_B^A \vec{E}\cdot d\vec{l} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2}\hat{a}_r \cdot dr\hat{a}_r = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right] = V_A - V_B$$

.............................*(2.26)*

It is customary to choose the potential to be zero at infinity. Thus potential at any point ($rA = r$) due to a point charge Q can be written as the amount of work done in bringing a unit positive charge from infinity to that point (i.e. $rB =$ ($V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$

..............................*(2.27)*

Or, in other words,

$$V = -\int_{\infty}^{r} E.dl$$

.................................(2.28)

Let us now consider a situation where the point charge Q is not located at the origin as shown in Fig. *4.3*.

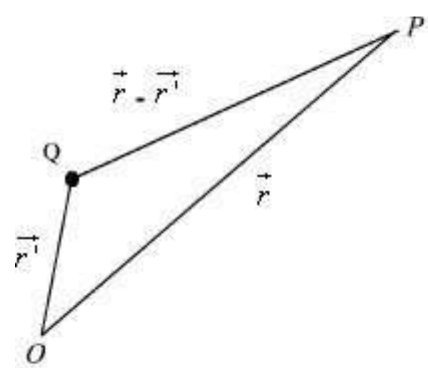


Fig *4.3*: Electric Potential due a Displaced Charge
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

The potential at a point P becomes

$$V(r) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\left|\vec{r} - \vec{r}'\right|}$$

..............................*(2.29)*

So far we have considered the potential due to point charges only. As any other type of charge distribution can be considered to be consisting of point charges, the same basic ideas now can be extended to other types of charge distribution also.

Let us first consider *N* point charges *Q1*, *Q2*,.....*QN* located at points with position vectors $\vec{r}_1$, $\vec{r}_2$,...... $\vec{r}_N$. The potential at a point having position vector $\vec{r}$ can be written as:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q_1}{\left|\vec{r}-\vec{r}_1\right|} + \frac{Q_2}{\left|\vec{r}-\vec{r}_2\right|} + \cdots \frac{Q_N}{\left|\vec{r}-\vec{r}_N\right|}\right) \quad \text{...............................(2.30a)}$$

or,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=n}^{N}\frac{Q_n}{\left|\vec{r}-\vec{r}_n\right|}$$

For continuous charge distribution, we replace point charges *Qn* by corresponding charge elements $\rho_L dl$ or $\rho_S ds$ or $\rho_V dv$ depending on whether the charge distribution is linear, surface or a volume charge distribution and the summation is replaced by an integral. With these modifications we can write:

For line charge,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_L \frac{\rho_L(\vec{r}')dl'}{\left|\vec{r}-\vec{r}_n\right|} \quad \text{.................................(2.31)}$$

For surface charge,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_S(\vec{r}')ds'}{\left|\vec{r}-\vec{r}_n\right|} \quad \text{.. (2.32)}$$

For volume charge,

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho_V(\vec{r}')dv'}{\left|\vec{r}-\vec{r}_n\right|} \quad \text{..(2.33)}$$

It may be noted here that the primed coordinates represent the source coordinates andthe unprimed coordinates represent field point.

Further, in our discussion so far we have used the reference or zero potential atinfinity. If any other point is chosen as reference, we can write:

$$V = \frac{Q}{4\pi\varepsilon_0 r} + C \quad \text{..............................(2.34)}$$

where *C* is a constant. In the same manner when potential is computed from a knownelectric field we can write:

$$V = -\int \vec{E} \cdot d\vec{l} + C \quad \text{.............................(2.35)}$$

The potential difference is however independent of the choice of reference.

$$V_{AB} = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l} = \frac{W}{Q} \quad \text{.....................(2.36)}$$

We have mentioned that electrostatic field is a conservative field; the work done in moving a charge from one point to the other is independent of the path. Let us consider moving a charge from point $P1$ to $P2$ in one path and then from point $P2$ back to $P1$ over a different path.
If the work done on the two paths were different, a net positive or negative amount of work would have been done when the body returns to its original position $P1$. In a conservative field there is no mechanism for dissipating energy corresponding to any positive work neither any source is present from which energy could be absorbed in the case of negative work. Hence the question of different works in two paths is untenable, the work must have to be independent of path and depends on the initial and final positions.

Since the potential difference is independent of the paths taken, $VAB = - VBA$, andover a closed path,

$$V_{BA} + V_{AB} = \oint \vec{E} \cdot d\vec{l} = 0 \quad \text{.............................(2.37)}$$

Applying Stokes's theorem, we can write:

$$\oint \vec{E} \cdot d\vec{l} = \int_s \left(\nabla \times \vec{E}\right) \cdot d\vec{s} = 0$$

.........................(2.38)

from which it follows that for electrostatic field,

$$\nabla \times \vec{E} = 0 \qquad \text{..................................(2.39)}$$

Any vector field $\vec{A}$ that satisfies $\nabla \times \vec{A} = 0$ is called an

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial x}dz = -\vec{E} \cdot d\vec{l}$$

$$\left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right) \cdot \left(dx\hat{a}_x + dy\hat{a}_y + dz\hat{a}_z\right) = -\vec{E} \cdot d\vec{l}$$

irrotationalfield. From our definition of potential, we

can write

$$\nabla V \cdot d\vec{l} = -\vec{E} \cdot d\vec{l} \qquad \text{.............................(2.40)}$$

from which we obtain,

$$\vec{E} = -\nabla V \qquad \text{.....................................(2.41)}$$

From the foregoing discussions we observe that the electric field $\vec{E}$ strength at any point isthe negative of the potential gradient at any $\vec{V}$ point.

Behavior of dielectrics in static electric field: Polarization of dielectric

Here we briefly describe the behavior of dielectrics or insulators when placed in static electric field. Ideal dielectrics do not contain free charges. As we know, all material media are composed of atoms where a positively charged nucleus (diameter ~ 10^{-15}m) is surrounded by negatively charged electrons (electron cloud has radius ~ 10^{-10}m) moving around the nucleus. Molecules of dielectrics are neutral macroscopically; an externally applied field causes small displacement of the charge particles creating small electric dipoles.These induced dipole moments modify electric fields both inside and outside dielectric material.

Molecules of some dielectric materials posses permanent dipole moments even in the absence of an external applied field. Usually such molecules consist of two or more dissimilar atoms and are called *polar* molecules. A common example of such molecule is water molecule H_2O. In polar molecules the atoms do not arrange themselves to make the net dipole moment zero. However, in the absence of an external field, the molecules arrange themselves in a random manner so that net dipole moment over a volume becomes zero.
Under the influence of an applied electric field, these dipoles tend to align themselves along the field as shown in figure *5.1*. There are some materials that can exhibit net permanent dipole moment even in the absence of applied field. These materials are called *electrets* that made by heating certain waxes or plastics in the presence of electric field. The applied field aligns the polarized molecules when the material is in the heated state and they are frozen to their new position when after the temperature is brought down to its normal temperatures. Permanent polarization remains without an externally applied field.

As a measure of intensity of polarization, polarization $\vec{P}$ vector (in C/m^2) is defined as:

$$\vec{P} = \lim_{\Delta v \to 0} \frac{\sum_{i=1}^{n\Delta v} \vec{P_k}}{\Delta v} \qquad \text{.......................}(2.59)$$

n being the number of molecules per unit volume i.e. $\vec{P}$ is the dipole moment per unit volume. Let us now consider a dielectric material having polarization $\vec{P}$ and computethe potential at an external point O due to an elementary dipole $\vec{P}$ dv'.

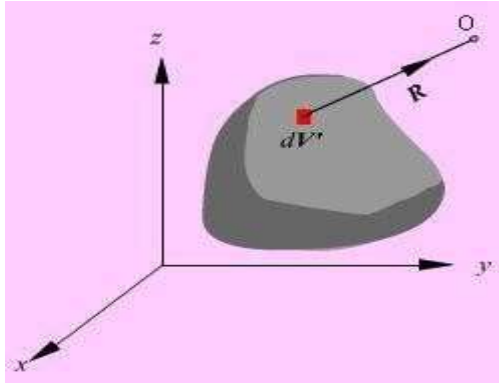


Fig *5.1*: Potential at an External Point due to an Elementary $\vec{P}$ Dipole
***dv'*.**

(www.brainkart.com/subject/Electromagnetic-Theory_206/)

With reference to the figure *2*.16, we can write:

$$dV = \frac{\vec{P}dv' \cdot \hat{a}_R}{4\pi\varepsilon_0 R^2}$$

Therefore .... *(2.60)*

$$R = \left\{(x-x')^2 + (y-y')^2 + (z-z')^2\right\}^{1/2}$$

. ...*(2.61)*

where x,y,z represent the coordinates of the external point O and x',y',z' are thecoordinates of the source point.

From the expression of *R*, we can verify that

$$\nabla'\left(\frac{1}{R}\right) = \frac{\hat{a}_R}{R^2}$$ . ...*(2.63)*

$$V = \frac{1}{4\pi\varepsilon_0}\int_{V'} \vec{P}\cdot\nabla'\left(\frac{1}{R}\right)dv'$$ ....*(2.64)*

Using the vector identity, $\nabla'\cdot(f\vec{A}) = f\nabla'\cdot\vec{A} + \vec{A}\cdot\nabla' f$,where *f* is a scalar quantity , wehave,

$$V = \frac{1}{4\pi\varepsilon_0}\left[\int_{v'} \nabla'\cdot\left(\frac{\vec{P}}{R}\right)dv' - \int_{v'}\frac{\nabla'\cdot\vec{P}}{R}dv'\right] \quad \text{.....................}(2.65)$$

Converting the first volume integral of the above expression to surface integral, we canwrite

$$V = \frac{1}{4\pi\varepsilon_0}\oint_{S'}\frac{\vec{P}\cdot\hat{a}'_n}{R}ds' + \frac{1}{4\pi\varepsilon_0}\int_{v'}\frac{\left(-\nabla'\cdot\vec{P}\right)}{R}dv' \quad \text{.................}(2.66)$$

where $\hat{a}'_n$ is the outward normal from the surface element *ds'* of the dielectric. From the above expression we find that the electric potential of a polarized dielectric may be found from the contribution of volume and surface charge distributions having densities

$$\rho_{ps} = \vec{P}\cdot\hat{a}_n \quad \text{..}(2.67)$$

$$\rho_{pv} = -\nabla\cdot\vec{P} \quad \text{......................}(2.68)$$

These are referred to as polarisation or bound charge densities. Therefore we may replace a polarized dielectric by an equivalent polarization surface charge density and a polarization volume charge density. We recall that bound charges are those charges that are not free to move within the dielectric material, such charges are result of displacement that occurs on a molecular scale during polarization. The total bound charge on th $\oint_S \rho_{ps}ds = \oint_S \vec{P}\cdot d\vec{s}$

$$\text{..................}(2.69)$$

The charge that remains inside the surface is

$$\int_v \rho_{pv}dv = \int_v -\nabla\cdot\vec{P}dv \quad \text{....................}(2.70)$$

The total charge in the dielectric material is zero as

$$\oint_S \rho_{ps}ds + \int_v \rho_{pv} = \oint_S \vec{P}\cdot d\vec{s} + \int_v -\nabla\cdot\vec{P}dv = \int_v \nabla\cdot\vec{P} - \int_v \nabla\cdot\vec{P} = 0$$

....................*(2.71)*

If we now consider that the dielectric region containing charge ρ_v density

$$\rho_t = \rho_v + \rho_{pv}$$ ....*(2.72)*

Since we have taken into account the effect of the bound charge density, we canwrite

$$\nabla \cdot \vec{E} = \frac{(\rho_v + \rho_{pv})}{\varepsilon_0}$$ ....*(2.73)*

Using the definition of ρ_{pv} wehave

$$\nabla . (\varepsilon_0 \vec{E} + \vec{P}) = \rho_v$$ ....*(2.74)*

Therefore the electric flux density $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

When the dielectric properties of the medium are linear and isotropic, polarisation is directly proportional to the applied field strength and

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$ ....*(2.75)*

is the electric susceptibility of the dielectric.
Therefore,

$$\vec{D} = \varepsilon_0 (1 + \chi_e) \vec{E} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}$$ ....(2.76)

$\varepsilon_r = 1 + \chi_e$ is called relative permeability or the dielectric constant of $\varepsilon_0 \varepsilon_r$ the medium.is called the absolute permittivity.

A dielectric medium is said to be linear when is χ_e independent of the $\vec{E}$ medium is ε_r homogeneous if is χ_e also independent of space coordinates. A linear homogeneous and isotropic medium is called a **simple medium** and for such medium the relative permittivity is a constant.Dielectric const may be a function of space coordinates. For anistropic materials, the dielectric constant is different in different directions of the electric field, D and E are related by a permittivity tensor which may be written as:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \text{.......................(2.77)}$$

For crystals, the reference coordinates can be chosen along the principal axes, which make off diagonal elements of the permittivity matrix zero. Therefore, we have

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \text{.......................(2.78)}$$

Media exhibiting such characteristics are called **biaxial**. Further, if $\varepsilon_1 = \varepsilon_2$ then the medium is called **uniaxial**. It may be $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$ that for isotropic media, .

Lossy dielectric materials are represented by a complex dielectric constant, the imaginary part of which provides the power loss in the medium and this is in general dependant on frequency.

Another phenomenon is of importance is **dielectric breakdown**. We observed that the applied electric field causes small displacement of bound charges in a dielectric material that results into polarization. Strong field can pull electrons completely out of the molecules. These electrons being accelerated under influence of electric field will collide with molecular lattice structure causing damage or distortion of material. For very strong fields, avalanche breakdown may also occur. The dielectric under such condition will become conducting.

The maximum electric field intensity a dielectric can withstand without breakdown is referred to as the **dielectric strength** of the material.

Boundary conditions for Electrostatic fields

In our discussions so far we have considered the existence of electric field in the homogeneous medium. Practical electromagnetic problems often involve media with different physical properties. Determination of electric field for such problems requires the knowledge of the relations of field quantities at an interface between two media. The conditions that the fields must satisfy at the interface of two different media are referred to as ***boundary conditions*** .

In order to discuss the boundary conditions, we first consider the field behavior insome common material media.

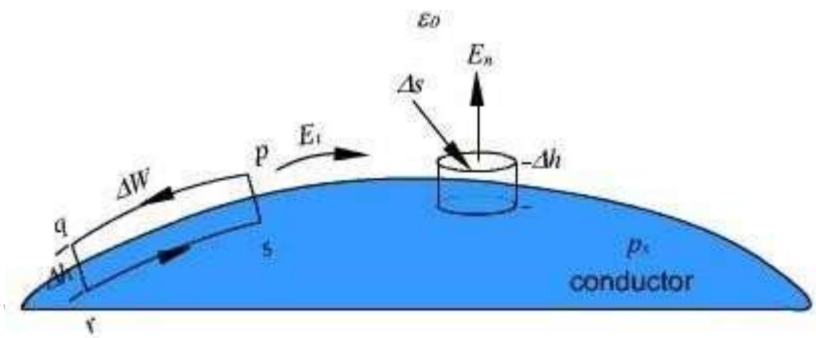


Fig 6.1: Boundary Conditions for at the surface of a Conductor
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

In general, based on the electric properties, materials can be classified into three categories: conductors, semiconductors and insulators (dielectrics). In *conductor* , electrons in the outermost shells of the atoms are very loosely held and they migrate easily from one atom to the other. Most metals belong to this group. The electrons in the atoms of *insulators* or *dielectrics* remain confined to their orbits and under normal circumstances they are not liberated under the influence of an externally applied field. The electrical properties of *semiconductors* fall between those of conductors and insulators since semiconductors have very few numbers of free charges.

The parameter *conductivity* is used characterizes the macroscopic electrical property of a material medium. The notion of conductivity is more important in dealing with the currentflow and hence the same will be considered in detail later on.

If some free charge is introduced inside a conductor, the charges will experience a force due
to mutual repulsion and owing to the fact that they are free to move, the

charges will appear on the surface. The charges will redistribute themselves in such a manner that the field within the conductor is zero. Therefore, under steady condition, inside a conductor

.

From Gauss's theorem it follows that

$$\rho_v = 0 \quad ....(2.51)$$

The surface charge distribution on a conductor depends on the shape of the conductor. The charges on the surface of the conductor will not be in equilibrium if there is a tangential component of the electric field is present, which would produce movement of the charges. Hence under static field conditions, tangential component of the electric field on the conductor surface is zero. The electric field on $\vec{E}$ the surface of the conductor is normal everywhere to the surface . Since the tangential component of electric field is

zero, the conductor surface is an equipotential surface. As = 0 inside the conductor, the conductor as a whole has the same potential. We may further note that charges require a $\sim 10^{-19}$

finite time to redistribute in a conductor. However, this time is very small like copper.

Let us now consider an interface between a conductor and free space as

shown in the figure 6.1

Let us consider the closed path *pqrsp* for which we can write,

$$\vec{E} \quad \oint \vec{E} \cdot d\vec{l} = 0 \quad ....(2.52)$$

For $\Delta h \to 0$ and noting that inside the conductor is zero, we can write

$$E_t \Delta w = 0 ....(2.53)$$

Et is the tangential component of the field. Therefore we find that

$$E_t = 0 \; .... (2.54)$$

In order to determine the normal component E_n, the normal component $\vec{E}$ of , at the surface of the conductor, we consider a small cylindrical Gaussian surface as shown in the Fig.12. Let Δs represent the area of the top and bottom faces and Δh represents the height of the cylinder. Once again, $\Delta h \rightarrow 0$ as , we approach the surface of the conductor. Since $\vec{E}$ = 0 inside the conductor is zero,

$$\varepsilon_0 \oint_s \vec{E} \cdot d\vec{s} = \varepsilon_0 E_n \Delta s = \rho_s \Delta s$$

............(2.55)

$$E_n = \frac{\rho_s}{\varepsilon_0}$$ ....(2.56)

Therefore, we can summarize the boundary conditions at the surface of a conductor as:

$E_t = 0$ ....(2.57)

$$E_n = \frac{\rho_s}{\varepsilon_0}$$ ....(2.58)

Capacitance and Capacitors

We have already stated that a conductor in an electrostatic field is an Equipotential body and any charge given to such conductor will distribute themselves in such a manner that electric field inside the conductor vanishes. If an additional amount of charge is supplied to an isolated conductor at a given potential, this additional charge will increase the surface

charge density ρ_s. Since the potential of the conductor is given by, the potential

$$V = \frac{1}{4\pi\varepsilon_0}\int_{S'} \frac{\rho_s ds'}{r}$$

of the conductor will also increase maintaining the ratio $\frac{Q}{V}$ same. Thus we can write

$$C = \frac{Q}{V}$$

where the constant of proportionality C is called the capacitance of the isolated conductor.SI unit of capacitance is Coulomb/ Volt also called Farad denoted by F. It can It can be seen that if V=1, C = Q. Thus capacity of an isolated conductor can also be defined as the amount of charge in Coulomb required to raise the potential of the conductor by 1 Volt.

Of considerable interest in practice is a capacitor that consists of two (or more) conductors carrying equal and opposite charges and separated by some dielectric media or free space. The conductors may have arbitrary shapes. A two-conductor capacitor is shown in figure 7.1.

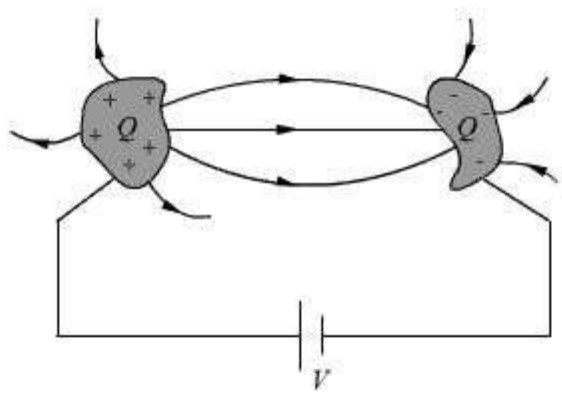


*Fig **7.1**:Capacitance and Capacitors*
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

When a d-c voltage source is connected between the conductors, a charge transfer occurs which results into a positive charge on one conductor and negative charge on the other conductor. The conductors are equipotential surfaces and the field lines are perpendicular to the conductor surface. If *V* is the mean potential difference between the conductors, the

capacitance is given by $C = \frac{Q}{V}$. Capacitance of a capacitor depends on the geometry of the conductor and the permittivity of the medium between them and does not depend on the charge or potential difference between conductors. The capacitance can be computed

by assuming *Q*(at the same time -*Q* on the other conductor), first determining $\vec{E}$ using Gauss's theorem and then determining $V = -\int \vec{E}.d\vec{l}$. We illustrate this procedure by taking the example of a parallel plate capacitor.

Parallel plate capacitor

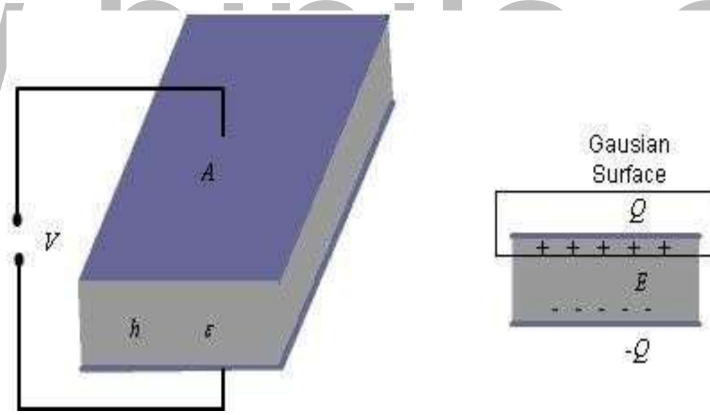


Fig 7.2: Parallel Plate Capacitor
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

For the parallel plate capacitor shown in the figure 7.2, let each plate has area A and a distance h separates the plates. A dielectric of permittivity ε fills the region between the plates. The electric field lines are confined between the plates. We ignore the flux fringing at the edges of the plates and charges are assumed to be uniformly distributed over the

$$\rho_s = \frac{Q}{A}$$

conducting plates with $E = \frac{\rho_s}{\varepsilon} = \frac{Q}{A\varepsilon}$ *(2.85)*

By Gauss's theorem we can write,

As we have ρ to be uniform and fringing of field is neglected, we see that E is constant in the region between the plates and therefore, we can write .Thus, $V = Eh = \frac{hQ}{\varepsilon A}$

for a parallel plate capacitor we have,. $C = \frac{Q}{V} = \varepsilon \frac{A}{h}$ ....*(2.86)*

Series and parallel Connection of capacitors

Capacitors are connected in various manners in electrical circuits; series and parallel connections are the two basic ways of connecting capacitors. We compute the equivalent capacitance for such connections.

Series Case: Series connection of two capacitors is shown in the figure 7.3. For this case we can write,

$$V = V_1 + V_2 = \frac{Q}{C_1} + \frac{Q}{C_2}$$

$$\frac{V}{Q} = \frac{1}{C_{eqs}} = \frac{1}{C_1} + \frac{1}{C_2} \quad \text{.....................(2.87)}$$

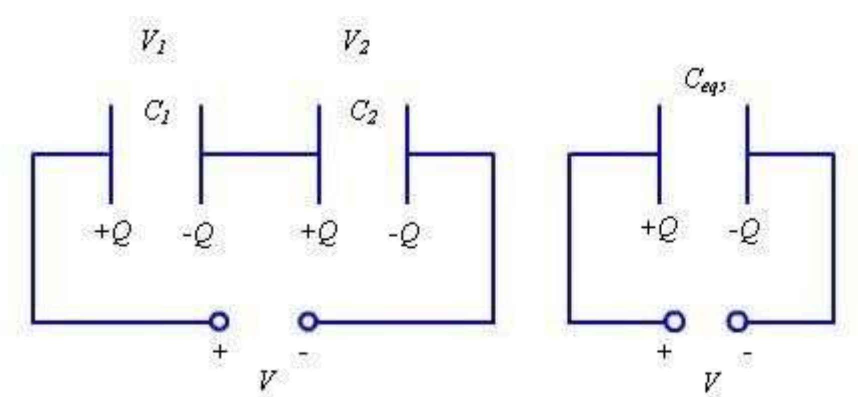
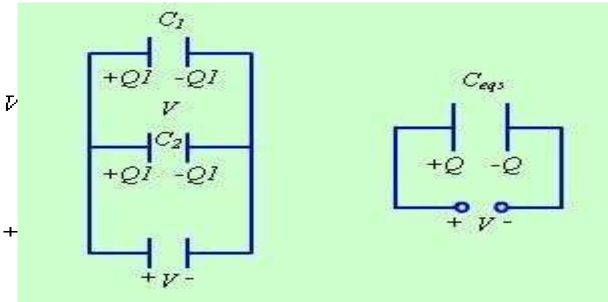


Fig 7.3: Series Connection of Capacitors

(www.brainkart.com/subject/Electromagnetic-Theory_206/)

$W = (V_{1N}Q_1 + \ldots\ldots + V$ $\ldots_{-2)N}Q_{N-2}) + V_{(N-1)N}Q_{N-1}$



$2W = (V_{1N} + V_{1(N-1)} +$ $\ldots_1\ldots\ldots$

$\ldots\ldots V_{N2} + V_{N(N-1)})Q_N$

Fig 7.4:Parallel Connection of Capacitors
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

The same approach may be extended to more than two capacitors connected inseries. **Parallel Case:** For the parallel case, the voltages across the capacitors are the

same. The total charge $Q = Q_1 + Q_2 = C_1V + C_2V$

Therefore, $C_{eq} = \frac{Q}{V} = C_1 + C_2$ ....(2.88)

Poisson's and Laplace's Equations

For electrostatic field, we have seen that

$$\nabla \cdot \vec{D} = \rho_v$$
$$\vec{E} = -\nabla V$$

....(2.97)

Form the above two equations we can write

$$\nabla \cdot (\varepsilon \vec{E}) = \nabla \cdot (-\varepsilon \nabla V) = \rho_v$$ ....(2.98)

Using vector identity we can write, $\varepsilon \nabla \cdot \nabla V + \nabla V \cdot \nabla \varepsilon = -\rho_v$ ....(2.99)

For a simple homogeneous medium, is constant and $\nabla \varepsilon = 0$. Therefore,

$$\nabla \cdot \nabla V = \nabla^2 V = -\frac{\rho_v}{\varepsilon}$$ ....(2.100)

This equation is known as **Poisson's equation**. Here we have introduced a ∇^2 (del square), called the Laplacian operator. In Cartesian coordinates,

$$\nabla^2 V = \nabla \cdot \nabla V = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) \cdot \left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right)$$ ....(2.101)

Therefore, in Cartesian coordinates, Poisson equation can be written as:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} = -\frac{\rho_v}{\varepsilon}$$ ....(2.102)

In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$ ....(2.103)

In spherical polar coordinate system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$ ....*(2.104)*

At points in simple media, where no free charge is present, Poisson's equation reduces to

$$\nabla^2 V = 0$$ ....*(2.105)*

which is known as Laplace's equation.

Application of poisons and Laplace's equations:

Laplace’s and Poisson’s equation are very useful for solving many practical electrostaticfield problems where only the electrostatic conditions (potential and charge) at some boundaries are known and solution of electric field and potential is to be found throughoutthe volume.

<u>Coulomb's Law</u>

Coulomb's Law states that the force between two point charges $Q1$and $Q2$ is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.Point charge is a hypothetical charge located at a single point in space. It is an idealised model of a particle having an electric charge.

Mathematically, $F = \frac{kQ_1Q_2}{R^2}$,where k is the proportionality constant.

In SI units, $Q1$ and $Q2$ are expressed in Coulombs(C) and R is in meters.

Force F is in Newtons (N) and $k = \frac{1}{4\pi\varepsilon_0}$ is called the permittivity of free space.

(We are assuming the charges are in free space. If the charges are any other dielectric medium, we will $\varepsilon = \varepsilon_0 \varepsilon_r$ instead where ε_r is called the relativepermittivity or the dielectric constant of the medium).

Therefore $F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}$ ....(2.1)

As shown in the Figure 1.1 let the position vectors of the point

charges

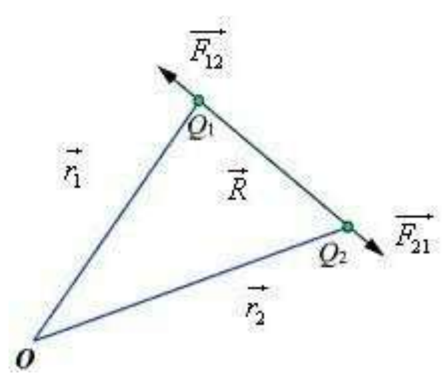


Fig 1.1: Coulomb's Law
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

$\vec{r}_1$ $\vec{r}_2$ $\vec{F}_{12}$

$Q1$and $Q2$ aregiven byand . Let represent the force on *Q1* due to charge $Q2$.

The charges are separated by a distance of $R = \left|\vec{r}_1 - \vec{r}_2\right| = \left|\vec{r}_2 - \vec{r}_1\right|$. We define the unit vectors as

$$\widehat{a_{12}} = \frac{\left(\vec{r}_2 - \vec{r}_1\right)}{R} \qquad \widehat{a_{21}} = \frac{\left(\vec{r}_1 - \vec{r}_2\right)}{R}$$

.... (2.2)

$\vec{F}_{12}$ can be defined as .

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\widehat{a_{12}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\frac{\left(\vec{r}_2 - \vec{r}_1\right)}{\left|\vec{r}_2 - \vec{r}_1\right|^3}$$

due to charge $Q2$ can be calculated and if $\vec{F}_{21}$ represents this force then we can write

$$\vec{F}_{21} = -\vec{F}_{12}$$

When we have a number of point charges, to determine the force on a particular charge due to all other charges, *we apply principle of superposition*. If we have *N* number of charges

$Q1$,$Q2$,.........QN located respectively at the points represented by the position vectors $\vec{r}_N$ $\vec{r}_1$,

,...... , the force experienced by a charge Q located at $\vec{r}$ is given by,

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{Q_i\left(\vec{r} - \vec{r}_i\right)}{\left|\vec{r} - \vec{r}_i\right|^3}$$

.................................(2.3)

Electric Dipole

An electric dipole consists of two point charges of equal magnitude but of opposite signand separated by a small distance.

As shown in figure *9.1*, the dipole is formed by the two point charges *Q* separated by a distance *d* , the charges being placed symmetrically about the origin. Let usconsider a point *P* at a distance *r*, where we are interested to find the field.

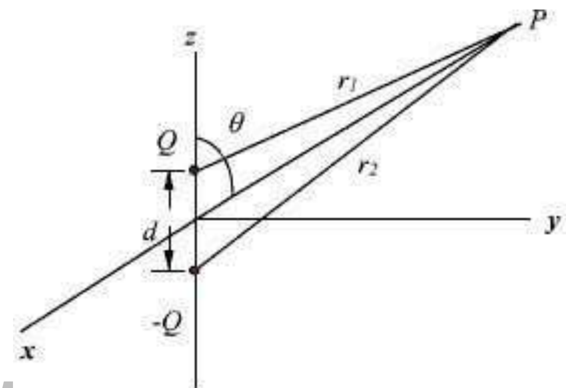


Fig *9.1* : Electric Dipole
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

The potential at P due to the dipole can be written as:

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r_1} - \frac{Q}{r_2}\right] = \frac{Q}{4\pi\varepsilon_0}\left[\frac{r_2 - r_1}{r_1 r_2}\right] \qquad \text{..........................(2.42)}$$

When $r1$ and $r2 >> d$, we can write $r_2 - r_1 = 2\times\frac{d}{2}\cos\theta = d\cos\theta$ and r . Therefore,

$$V = \frac{Q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2} \qquad \text{..(2.43)}$$

We can write,

$$Qd\cos\theta = Qd\hat{a}_z \cdot \hat{a}_r \quad ....(2.44)$$

The quantity $\vec{P} = Q\vec{d}$ is called the **dipole moment** of the electric dipole. Hence the expression for the electric potential can now be written as:

$$V = \frac{\vec{P}\cdot\hat{a}_r}{4\pi\varepsilon_0 r^2} \quad ....(2.45)$$

It may be noted that while potential of an isolated charge varies with distance as $1/r$ that of an electric dipole varies as $1/r^2$ with distance.

$$V = \frac{\vec{P}\cdot(\vec{r}-\vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}\,'\right|^3} \quad ....(2.46)$$

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{a}_\theta\right] \quad ....(2.47)$$

$$= \frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3}\hat{a}_r + \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3}\hat{a}_\theta$$

$$= \frac{Qd}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right)$$

$$\vec{E} = \frac{\vec{P}}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{a}_r + \sin\theta\hat{a}_\theta\right)$$

Electric Field

The electric field intensity or the electric field strength at a point is defined as the forceper unit charge. That is

$$\vec{E} = \lim_{Q \to 0} \frac{\vec{F}}{Q} \quad \text{or,} \quad \vec{E} = \frac{\vec{F}}{Q} \qquad \text{.. (2.4)}$$

The electric field intensity *E* at a point *r* (observation point) due a point charge *Q* located at $\vec{r}\,'$ (source point) is given by:

$$\vec{E} = \frac{Q(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} \qquad \text{..}(2.5)$$

For a collection of *N* point charges *Q*1 ,*Q*2 ,.........*Q*N located at $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_N$ electric field intensity at point $\vec{r}$ is obtained as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_k(\vec{r} - \vec{r}_i)}{\left|\vec{r} - \vec{r}_i\right|^3} \qquad \text{......................................(2.6)}$$

The expression *(2.6)* can be modified suitably to compute the Electric filed dueto a continuous distribution of charges.

In figure 2.1 we consider a continuous volume distribution of charge d*(t)* in theregion denoted as the source region.

For an elementary charge $dQ = \rho(\vec{r}\,')dv'$, i.e. considering this charge as point charge,we can write the field expression as:

$$d\vec{E} = \frac{dQ(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} = \frac{\rho(\vec{r}\,')dv'(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} \qquad \text{............}(2.7)$$

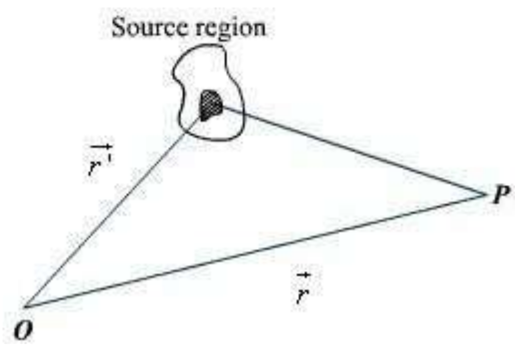


Fig 2.1: Continuous Volume Distribution of Charge
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

When this expression is integrated over the source region, we get the electric field at thepoint *P* due to this distribution of charges. Thus the expression for the electric field at *P* can be written as:

$$\overline{E(r)} = \int_V \frac{\rho(\vec{r}\,')(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} dv' \quad \text{...} (2.8)$$

Similar technique can be adopted when the charge distribution is in the form of a line charge density or a surface charge density.

$$\overline{E(r)} = \int_L \frac{\rho_L(\vec{r}\,')(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} dl' \quad \text{..} (2.9)$$

$$\overline{E(r)} = \int_S \frac{\rho_s(\vec{r}\,')(\vec{r} - \vec{r}\,')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}\,'\right|^3} ds' \quad \text{..}(2.10)$$

Electric flux density:

As stated earlier electric field intensity or simply 'Electric field' gives the strength of the field at a particular point. The electric field depends on the material media in which the field is being considered. The flux density vector is defined to be independent of the material media (as we'll see that it relates to the charge that is producing it).For a linear isotropic medium under consideration; the flux densityvector is defined as:

$$\vec{D} = \varepsilon\vec{E} \quad \text{..}(2.11)$$

We define the electric flux _ as

$$\psi = \int_S \vec{D}.d\vec{s} \qquad \text{.....................................(2.12)}$$

Electrostatic Energy and Energy Density

We have stated that the electric potential at a point in an electric field is the amount of work required to bring a unit positive charge from infinity (reference of zero potential) to that point. To determine the energy that is present in an assembly of charges, let us first determine the amount of work required to assemble them. Let us consider a number of discrete charges Q_1, Q_2,......., Q_N are brought from infinity to their present position one by one. Since initially there is no field present, the amount of work done in bring Q1 is zero. Q2 is brought in the presence of the field of Q1, the work done $W_1 = Q_2 V_{21}$ where V_{21} is the potential at the location of Q2 due to Q1. Proceeding in this manner, we can write, the total work done

$$W = V_{21}Q_2 + (V_{31}Q_3 + V_{32}Q_3) + \ldots + (V_{N1}Q_N + \ldots + V_{N(N-1)}Q_N)$$

....(2.89)

V_{IJ} represent voltage at the I^{th} charge location due to J^{th} charge. Therefore,

$$2W = V_1Q_1 + \ldots + V_NQ_N = \sum_{I=1}^{N} V_I Q_I$$

…..(2.91)

Or,

$$W = \frac{1}{2}\sum_{I=1}^{N} V_I Q_I$$

................*(2.92)*

If instead of discrete charges, we now have a distribution of charges over a volume *v* thenwe can write,

$$W = \frac{1}{2}\int_v V \rho_v \, dv$$

..............*(2.93)*

where ρ_v is the volume charge density and *V* represents the potential function.

$$\rho_v = \nabla \cdot \vec{D} \qquad W = \frac{1}{2}\int_v \left(\nabla \cdot \vec{D}\right) V \, dv$$

....*(2.94)*

Using the vector identity,

$\nabla.(V\vec{D}) = \vec{D}\cdot\nabla V + V\nabla\cdot\vec{D}$, we can write

$$W = \frac{1}{2}\int_v\left(\nabla.(V\vec{D}) - \vec{D}\cdot\nabla V\right)dv$$

$$= \frac{1}{2}\oint_s\left(V\vec{D}\right).d\vec{s} - \frac{1}{2}\int_v\left(\vec{D}\cdot\nabla V\right)dv$$

................*(2.95)*

In the expression $\frac{1}{2}\oint_s\left(V\vec{D}\right).d\vec{s}$, for point charges, since V varies as $\frac{1}{r}$ and D varies as $\frac{1}{r^2}$, $r \to \infty$, $\vec{D}$ varies as $\frac{1}{r^3}$, term tends to zero. Thus the equation for W reduces to

$$W = -\frac{1}{2}\int_v\left(\vec{D}.\nabla V\right)dv = \frac{1}{2}\int_v\left(\vec{D}.\vec{E}\right)dv = \frac{1}{2}\int_v\left(\varepsilon E^2\right)dv = \int_v w_e dv$$

$$w_e = \frac{1}{2}\varepsilon E^2$$

................(2.96)

Which is called the energy density in the electrostatic field.

Gauss's Law

Gauss's law is one of the fundamental laws of electromagnetism and it states that the total electric flux through a closed surface is equal to the total charge enclosed by the surface.

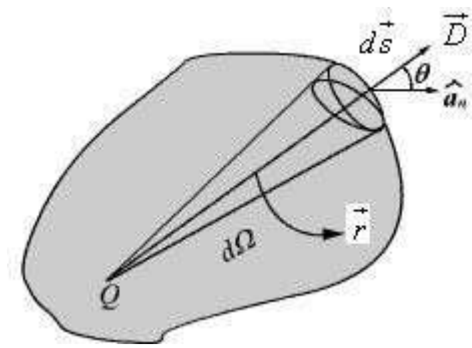


Fig 3.1: Gauss's Law
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

Let us consider a point charge Q located in an isotropic homogeneous medium of dielectric constant as shown in fig 3.1. The flux density at a distance r on a surface enclosing the charge is given by

$$\vec{D} = \varepsilon \vec{E} = \frac{Q}{4\pi r^2}\hat{a}_r \qquad \text{...}(2.13)$$

If we consider an elementary area ds, the amount of flux passing through theelementary area is given by

$$d\psi = \vec{D}.ds = \frac{Q}{4\pi r^2} ds\cos\theta \qquad \text{.....................................}(2.14)$$

But $\frac{ds\cos\theta}{r^2} = d\Omega$, is the elementary solid angle subtended by the area $d\vec{s}$ at the location of Q. Therefore we can write $d\psi = \frac{Q}{4\pi} d\Omega$

For a closed surface enclosing the charge, we can write $\psi = \oint_s d\psi = \frac{Q}{4\pi}\oint_s d\Omega = Q$ which can seen to be same as what we have stated in the definition of Gauss'sLaw. Application of Gauss's Law

Gauss's law is particularly useful in computing $\vec{E}$ or $\vec{D}$ where the charge distribution has some symmetry. We shall illustrate the application of Gauss's Law with some examples.

An infinite line charge

As the first example of illustration of use of Gauss's law, let consider the problem of determination of the electric field produced by an infinite line charge of density LC/m. Let us consider a line charge positioned along the *z*-axis as shown in Fig. 3.2

If we consider a close cylindrical surface as shown in Fig. 3.2 using Gauss's theormwe can write,

$$\rho_L l = Q = \oint_S \varepsilon_0 \vec{E}.d\vec{s} = \int_{S_1} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_2} \varepsilon_0 \vec{E}.d\vec{s} + \int_{S_3} \varepsilon_0 \vec{E}.d\vec{s}$$

.....................................(2.15)

Considering the fact that the unit normal vector to areas $S1$ and $S3$ are perpendicular to

the
electric field, the surface integrals for the top and bottom surfaces evaluates to zero. Hencewe can write, $\rho_L l = \varepsilon_0 E.2\pi\rho l$

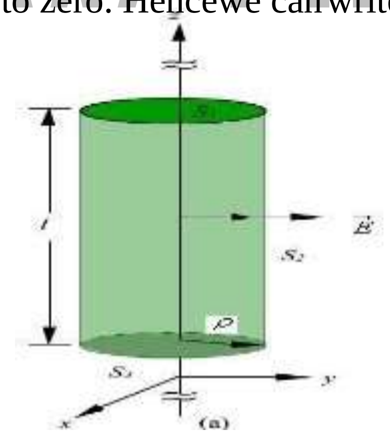


Fig 3.2: Infinite Line Charge
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\hat{a}_\rho \qquad \text{.....................................(2.16)}$$

Infinite Sheet of Charge

As a second example of application of Gauss's theorem, we consider an infinitecharged sheet covering the *x*-*z* plane as shown in figure *2*.5.

Assuming a surface charge density ρ_s of for the infinite surface charge, if we consider a cylindrical volume having Δs sides placed symmetrically as shown in figure 3.3, we can write:

$$\oint_s \vec{D}\cdot d\vec{s} = 2D\Delta s = \rho_s\Delta s$$

$$\therefore \quad \vec{E} = \frac{\rho_s}{2\varepsilon_0}\hat{a}_y \qquad \text{..............(2.17)}$$

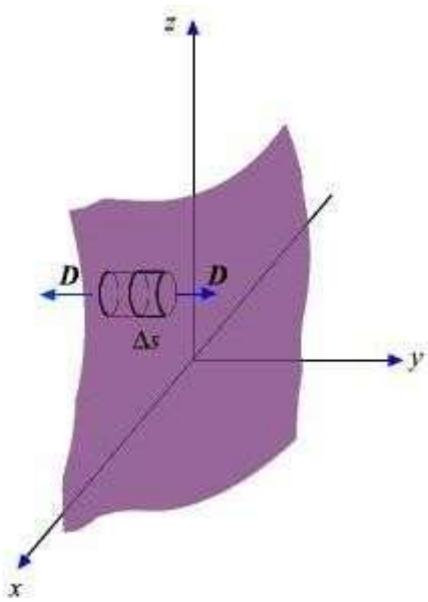


Fig 3.3: Infinite Sheet of Charge

(www.brainkart.com/subject/Electromagnetic-Theory_206/)

It may be noted that the electric field strength is independent of distance. This is true for the infinite plane of charge; electric lines of force on either side of the charge will be perpendicular to the sheet and extend to infinity as parallel lines. As number of lines of force per unit area gives the strength of the field, the field becomes independent of distance. For a finite charge sheet, the field will be a function of distance.

Uniformly Charged Sphere

Let us consider a sphere of radius r_0 having a uniform volume charge density of ρ_v To determine $\vec{D}$ everywhere, inside and outside the sphere, we construct Gaussian surfaces of radius $r < r_0$ and $r > r_0$ as shown in Fig. 3.4 (a) and (b)

For the $r \le r_0$ region ; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r^3 \qquad ....(2.18)$$

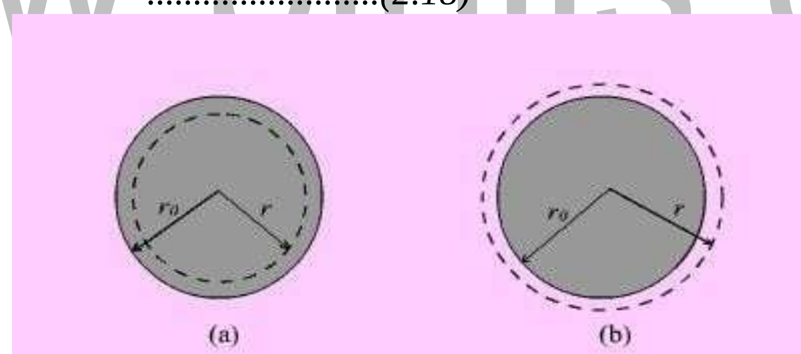


Fig 3.4:Uniformly Charged Sphere
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

By applying Gauss's theorem,

$$\oint_s \vec{D}\cdot d\vec{s} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} D_r r^2 \sin\theta d\theta d\phi = 4\pi r^2 D_r = Q_{en} \qquad ....(2.19)$$

Therefore

$$\vec{D} = \frac{r}{3}\rho_v \hat{a}_r \qquad 0 \le r \le r_0$$ ....*(2.20)*

For the region $r \ge r_0$; the total enclosed charge will be

$$Q_{en} = \rho_v \frac{4}{3}\pi r_0^3$$ ....(2.21

) By applying Gauss's theorem,

$$\vec{D} = \frac{r_0^3}{3r^2}\rho_v \hat{a}_r \qquad r \ge r_0$$ ....(2.22)

Gauss divergence theorem:

The gauss law can be stated in the point form by the divergence of electric flux density is equal to the volume charge density.

Method Of Images

The replacement of the actual problem with boundaries by an enlarged region or withimage charges but no boundaries is called the method of images.
Method of images is used in solving problems of one or more point charges in thepresence of boundary surfaces.

Continuity of equation:

The relation between density and the volume charge density at a point called continuity ofequation

$\nabla \bullet J = - \rho / t\ v$

Boundary Conditions for perfect Electric Fields:

Let us consider the relationship among the field components that exist at the interface between two dielectrics as shown in the figure *8.1*. The permittivity of the medium 1

and medium 2 are ε_1 and ε_2 respectively and the interface may also have a

net charge density ρ_s Coulomb/m.

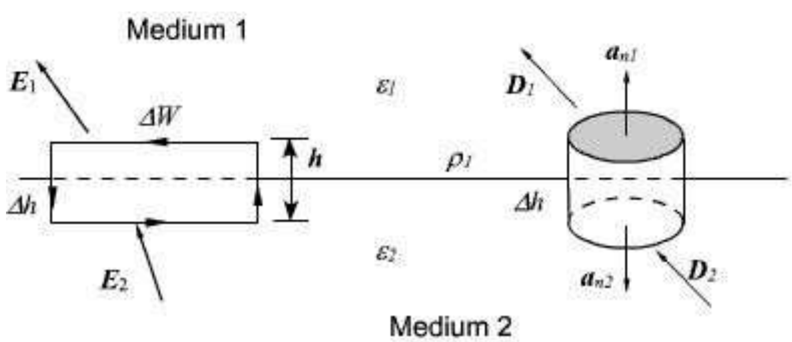


Fig *8.1*: Boundary Conditions at the interface between two dielectrics
(www.brainkart.com/subject/Electromagnetic-Theory_206/)

We can express the electric field in terms of the

$$\vec{E_1} = \vec{E_{1t}} + \vec{E_{1n}}$$

tangential andnormal components $\vec{E_2} = \vec{E_{2t}} + \vec{E_{2n}}$
..........*(2.79*

where E_t and E_n are the tangential and normal components of the electric field

respectively. Let us assume that the closed path is very small so that

over the elementalpath length the
variation of E can be neglected. Moreover very near to the interface, $\Delta h \rightarrow 0$. Therefore

$$\oint \vec{E}.d\vec{l} = E_{1t}\Delta w - E_{2t}\Delta w + \frac{h}{2}\left(E_{1n} + E_{2n}\right) - \frac{h}{2}\left(E_{1n} + E_{2n}\right) = 0 \quad \text{.......................(2.80)}$$

Thus, we have,

$E_{1t} = E_{2t}$ or $\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$ i.e. the **tangential component of an electric field iscontinuous across the interface.**

For relating the flux density vectors on two sides of the interface we apply Gauss's law to asmall pillbox volume as shown in the figure. Once again as $\Delta h \rightarrow 0$, we can write

$$\oint \vec{D}\cdot d\vec{s} = (\vec{D_1}.\hat{a}_{n2} + \vec{D_2}\cdot \hat{a}_{n1})\Delta s = \rho_s \Delta s \quad \text{..................(2.81a)}$$

i.e.,

$$D_{1n} - D_{2n} = \rho_s \quad \text{...(2.81b)}$$

.e., $\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$ ....(2.81c)

Thus we find that the **normal component of the flux density vector *D* is discontinuous across an interface by an amount of discontinuity equal to the surface charge density at the interface.**

Example

Two further illustrate these points; let us consider an example, which involves therefraction of D or E at a charge free dielectric interface as shown in the figure 8.2.

Using the relationships we have just derived, we can write

$$D_{1n} = D_1 \cos\theta_1 = D_{2n} = D_2 Cos\theta_2$$

....
…………………………………………………………(2.82a)

$$E_{1t} = E_1 \sin\theta_1 = \frac{D_1}{\varepsilon_1}\sin\theta_1 = E_{2t} = E_2 \sin\theta_2 = \frac{D_2}{\varepsilon_2}\sin\theta_2$$

....(2.82b)

$$\frac{D_1}{\varepsilon_1}\sin\theta_1 = \frac{D_2}{\varepsilon_2}\sin\theta_2$$
.......................(2.83a)

$$D_1 \cos\theta_1 = D_2 Cos\theta_2$$
.....................*(2.83b)*

Therefore,

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$
........................*(2.84)*

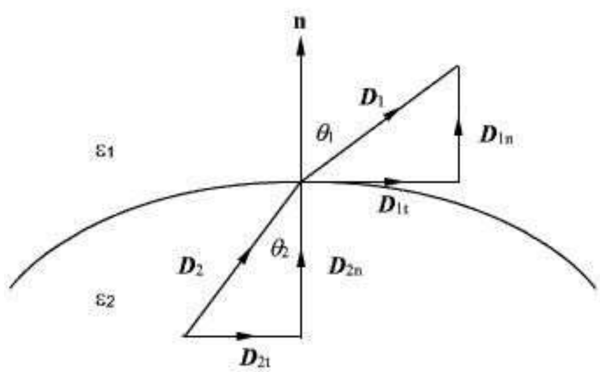


Fig *8.2*: Refraction of D or E at a Charge Free DielectricInterface

(www.brainkart.com/subject/Electromagnetic-Theory_206/)