2.1 FOURIER SERIES ANALYSIS

The Fourier representation of signals can be used to perform frequency domain analysis of signals in which we can study the various frequency components present in the signal, magnitude and phase of various frequency components.

Conditions for existence of Fourier series:

The Fourier series exist only if the following Dirichlet's conditions are satisfied.

- The signal $x(t)$ must be single valued function.
- The signal $x(t)$ must possess only a finite number of discontinuous in the period T.
- The signal must have a finite number of maxima and minima in the period T.
- $x(t)$ must be absolutely integrable.

$$\int_0^T |x(t)|dt < \infty$$

Types of Fourier series:

- Trigonometric Fourier series
- Exponential Fourier series
- Cosine Fourier series

TRIGONOMETRIC FOURIER SERIES

The trigonometric form of Fourier series of a periodic signal, $x(t)$ with period T is defined as

$$x(t) = a_o + \sum_{n=1}^{\infty} a_n \cos n\Omega_o t + \sum_{n=1}^{\infty} b_n \sin n\Omega_o t \text{ ------- (1)}$$

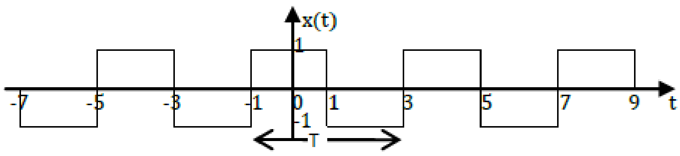
$where\ a_o,\ a_n,\ b_n$ → Fourier coefficients of trigonometric form of Fourier series

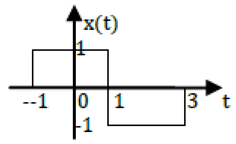
$$a_o = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)\,dt$$

$$a_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)\cos n\,\Omega_o t\,dt$$

$$b_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)\sin n\,\Omega_o t\,dt$$

EXAMPLE 1: Find the trigonometric Fourier series for the periodic signal x(t) as shown in Figure





Solution:

$$T = 3-(-1) = 4 \;and\; \Omega_o = \frac{2\pi}{T} = \frac{\pi}{2}$$

To find a_o

$$a_o = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)dt = \frac{1}{4}\left[\int_{-1}^{1} 1dt + \int_{1}^{3} -1dt\right]$$

$$= \frac{1}{4}\left[[t]_{-1}^{1} - 1[t]_{1}^{3}\right]$$

$$= \frac{1}{4}[2-2] = 0$$

To find a_n

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\Omega_0 t\,dt = \frac{2}{4}\left[\int_{-1}^{1}\cos n\Omega_0 t\,dt + \int_{1}^{3}(-1)\cos n\Omega_0 t\,dt\right]$$

$$= \frac{1}{2}\left[\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_{-1}^{1} - \left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_{1}^{3}\right]$$

$$= \frac{1}{2}\left[\left[\frac{\sin n\frac{\pi t}{2}}{n\frac{\pi t}{2}}\right]_{-1}^{1} - \left[\frac{\sin n\frac{\pi t}{2}}{n\frac{\pi t}{2}}\right]_{1}^{3}\right]$$

$$= \frac{1}{2}\left(\frac{2}{n\pi}\right)\left[\sin n\frac{\pi}{2} - \left(\sin n\frac{\pi}{2}(-1)\right) - \left(\sin n\frac{\pi}{2}(3) - \sin n\frac{\pi}{2}\right)\right]$$

$$= \left[\frac{1}{n\pi}\right]\left[\sin n\frac{\pi}{2} + \sin n\frac{\pi}{2} - \sin 3n\frac{\pi}{2} + \sin n\frac{\pi}{2}\right]$$

$$= \frac{1}{n\pi}\left[3\sin\frac{n\pi}{2} - \left(-\sin n\frac{\pi}{2}\right)\right] = \frac{4}{n\pi}\left[\sin n\frac{\pi}{2}\right]$$

To find b_n

$$b_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)\ \sin n\,\Omega_o dt$$

$$= \frac{2}{4}\Big[\int_{-1}^{1} \sin n\,\Omega_o t dt + \int_{1}^{3} -\sin n\,\Omega_o t dt\Big]$$

$$= \frac{1}{2}\Big[\Big[\frac{-\cos n\,\Omega_o t}{n\,\Omega_o}\Big]_{-1}^{1} - \Big[\frac{-\cos n\,\Omega_o t}{n\,\Omega_o}\Big]_{1}^{3}\Big] = \frac{1}{2}\Big[\Big[\frac{-\cos n\frac{\pi}{2}t}{n\frac{\pi}{2}}\Big]_{-1}^{1} + \Big[\frac{\cos n\frac{\pi}{2}t}{n\frac{\pi}{2}}\Big]_{1}^{3}\Big]$$

$$= \frac{1}{2}\Big[\frac{-2}{n\pi}\Big(\cos n\frac{\pi}{2} - \cos n\frac{\pi}{2}(-1)\Big) + \frac{2}{n\pi}\Big(\cos n\frac{\pi}{2}(3) - \cos n\frac{\pi}{2}\Big)\Big]$$

$$= \frac{1}{2}\Big[0 + \frac{2}{n\pi}\Big(\cos\Big(2n\pi - \frac{n\pi}{2}\Big) - \cos n\frac{\pi}{2}\Big)\Big]$$

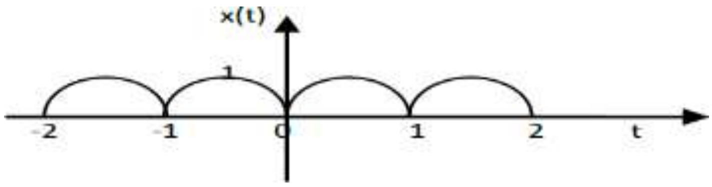
$$= \Big[\frac{1}{n\pi}\Big(\cos n\frac{\pi}{2} - \cos n\frac{\pi}{2}\Big)\Big] = 0$$

Trigonometric Fourier Series

$$\boldsymbol{x(t) = a_o + \sum_{n=1}^{\infty} a_n \cos n\,\Omega_o t + \sum_{n=1}^{\infty} b_n \sin n\Omega_o t}$$

$$\boldsymbol{= \sum_{n=1}^{\infty} \frac{4}{n\pi} sin\Big(\frac{n\pi}{2}\Big)\cos n\Omega_o t = \sum_{n=1}^{\infty} \frac{4}{n\pi} sin\Big(\frac{n\pi}{2}\Big)\cos n\frac{\pi}{2}}$$

EXAMPLE:2 Obtain Fourier series of the following full wave rectified sine wave shown in figure



Solution:

$$\boldsymbol{x(t) = x(-t);\ Given\ signal\ is\ even\ signal,\ so\ b_n = 0}$$

$$T = 1\ and\ \Omega_o = \frac{2\pi}{1} = 2\pi$$

To find a_o

$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} x(t)\,dt = \frac{2}{1}\int_0^{\frac{1}{2}} x(t)\,dt = \left[2\int_0^{\frac{1}{2}} \sin \pi t\ dt\right]$$

$$= 2\left[-\frac{\cos \pi t}{\pi}\right]_0^{\frac{1}{2}} = -\frac{2}{\pi}\left[\cos\frac{\pi}{2} - \cos 0\right] = \frac{2}{\pi}$$

To find a_n

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos n\Omega_o t\,dt = \frac{4}{1}\int_0^{\frac{1}{2}} \sin \pi t \cos n2\pi t\ dt$$

$$= 2\int_0^{\frac{1}{2}} [sin((1+2n)\pi t) + sin((1-2n)\pi t)]\,dt$$

$$= 2\left[-\frac{\cos((1+2n)\pi t)}{(1+2n)\pi} - \frac{\cos((1-2n)\pi t)}{(1-2n)\pi}\right]_0^{\frac{1}{2}}$$

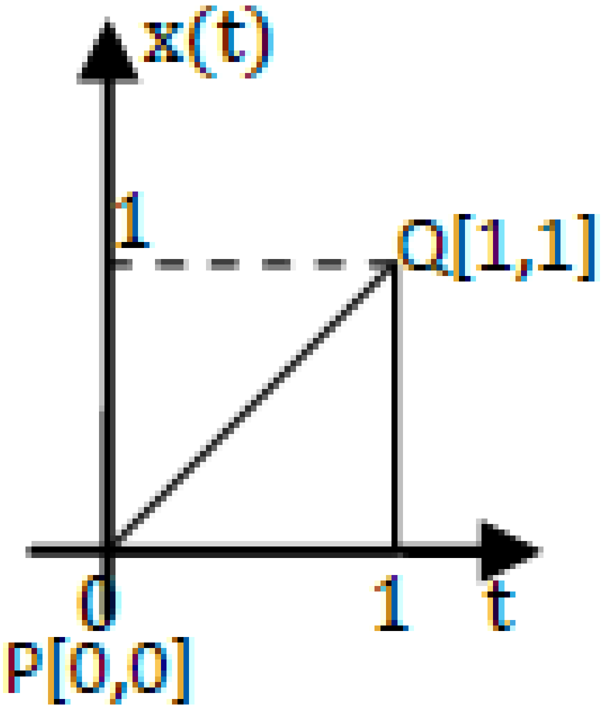
$$= \frac{2}{\pi}\left[-\frac{\cos\left((1+2n)\frac{\pi}{2}\right)}{1+2n} - \frac{\cos\left((1-2n)\frac{\pi}{2}\right)}{1-2n} + \frac{1}{1+2n} + \frac{1}{1-2n}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{1+2n} + \frac{1}{1-2n}\right] = \frac{2}{\pi}\left[\frac{1-2n+1+2n}{1-4n^2}\right] = \frac{4}{\pi(1-4n^2)}$$

Trigonometric Fourier Series

$$x(t) = a_o + \sum_{n=1}^{\infty} a_n \cos n\,\Omega_o t + \sum_{n=1}^{\infty} b_n \sin n\Omega_o t$$

$$x(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4}{\pi(1-4n^2)} \cos n2\pi t$$



EXPONENTIAL FOURIER SERIES

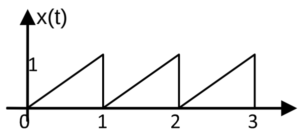
The exponential form of Fourier series of a periodic signal $x(t)$ with period T is defined as,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\Omega_0 t}$$

The Fourier coefficient C_n can be evaluated using the following formulae

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\Omega_0 t}dt$$

EXAMPLE 3: Find exponential series for the signal shown in figure

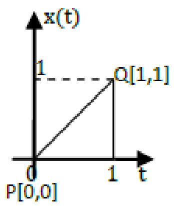


Solution:

$$T = 1, \Omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi$$

Consider the equation of a straight line

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$



Consider points P,Q as shown in figure

Coordinates of point P = [0,0]

Coordinates of point Q = [1,1]

On substituting the coordinates of points P and Q in equation

$$\frac{x(t)-0}{1-0}=\frac{t-0}{1-0}\Rightarrow x(t)=t$$

$$x=t,\ y=x(t)$$

To find C_o

$$c_0=\frac{1}{T}\int_0^T x(t)dt=\frac{1}{1}\int_0^1 (t)dt$$

$$=\left[\frac{t^2}{2}\right]_0^1=\frac{1}{2}$$

To find C_n

$$c_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\Omega_0 t}dt$$

$$=\frac{1}{1}\int_0^1 te^{-jn2\pi t}dt=\left[t\frac{e^{-jn2\pi t}}{-jn2\pi}\right]_0^1-\int_0^1\frac{e^{-jn2\pi t}}{-jn2\pi}dt$$

$$=\frac{e^{-j2\pi n}}{-j2\pi n}+0+\left[\frac{e^{-j2\pi nt}}{-j^2(2\pi n)^2}\right]_0^1$$

$$=j\frac{e^{-jn2\pi}}{n2\pi}+\frac{e^{-jn2\pi}}{n^2 4\pi^2}-\frac{1}{n^2 4\pi^2}$$

$$=\frac{j}{n2\pi}+\frac{1}{n^2 4\pi^2}-\frac{1}{n^2 4\pi^2}=\frac{j}{n2\pi}$$

$$c_n=\frac{j}{n2\pi}$$

$$C_1=\frac{j}{2\pi},\qquad C_2=\frac{j}{4\pi}$$

$$c_{-1}=\frac{j}{-2\pi},\qquad c_{-2}=\frac{j}{-4\pi},\qquad c_{-3}=\frac{j}{-6\pi}$$

Exponential Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\Omega_0 t}$$

$$\therefore\ x(t) = +\cdots - \frac{j}{6\pi}e^{-j6\pi t} - \frac{j}{4\pi}e^{-j4\pi t} - \frac{j}{2\pi}e^{-j2\pi t} + \frac{1}{2} + \frac{j}{2\pi}e^{j2\pi t} + \frac{j}{4\pi}e^{j4\pi t} + \frac{j}{6\pi}e^{j6\pi t} + \cdots$$

$$= \frac{1}{2} + \frac{j}{2\pi}\left[e^{j2\pi t} - e^{-j2\pi t}\right] + \frac{j}{4\pi}\left[e^{j4\pi t} - e^{-j4\pi t}\right] + \frac{j}{6\pi}\left[e^{j6\pi t} - e^{-j6\pi t}\right] + \cdots$$

$$= \frac{1}{2} + \frac{1}{\pi}\left[\frac{e^{j2\pi t} - e^{-j2\pi t}}{(-1)2j}\right] + \frac{1}{2\pi}\left[\frac{e^{j4\pi t} - e^{-j4\pi t}}{(-1)2j}\right] + \frac{1}{3\pi}\left[\frac{e^{j6\pi t} - e^{-j6\pi t}}{(-1)2j}\right]$$

$$= \frac{1}{2} + \left(\frac{-1}{\pi}\right) sin\ 2\pi t - \frac{1}{2\pi} sin\ 4\pi t - \frac{1}{3\pi}\sin 6\pi t$$

$$= \frac{1}{2} - \frac{1}{\pi}\left[sin\ 2\pi t + \frac{1}{2} sin\ 4\pi t + \frac{1}{3} sin\ 6\pi t + \cdots\right]$$

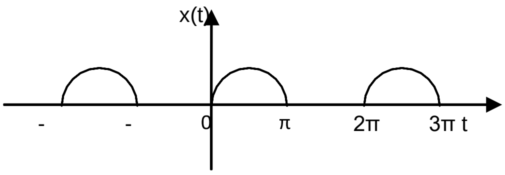
COSINE FOURIER SERIES

Cosine representation of $x(t)$ is

$$x(t) = A_o + \sum_{n=1}^{\infty} A_n \cos(n\Omega_o t + \theta_n)$$

Where A_0 is dc component, A_n is harmonic amplitude or spectral amplitude and θ_n is phase coefficient or phase angle *or spectral angle*

EXAMPLE 4: Determine the cosine Fourier series of the signal shown in Figure



Solution:

The signal shown in is periodic with period

$$T = 2\pi \ and \ \Omega_0 = \frac{2\pi}{2\pi} = 1$$

The given signal is sinusoidal signal,

$$\therefore x(t) = A \sin \Omega t$$

Here, $\Omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1, \ A = 1$

$$\therefore x(t) = \sin t$$

To find a_o

$$a_o = \frac{1}{T}\int_0^T x(t)\, dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} \sin t\, dt = \frac{1}{2\pi}[-\cos t\,]_0^{\pi} = \frac{1}{2\pi}[-\cos\pi + \cos 0] = \frac{1}{2\pi}[2] = \frac{1}{\pi}$$

To find a_n

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\, dt = \frac{2}{2\pi}\int_0^{\pi} \sin t\ \cos nt\, dt = \frac{1}{2\pi}\int_0^{\pi} [\sin(1+n)t + \sin(1-n)t]\, dt$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(1+n)t}{(1+n)} - \frac{\cos(1-n)t}{(1-n)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(1+n)\pi}{(1+n)} - \frac{\cos(1-n)\pi}{(1-n)} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$for\ n = odd:\ a_n = \frac{1}{2\pi}\left[-\frac{1}{1+n} - \frac{1}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right] = 0$$

$$for\ n = even:\ a_n = \frac{1}{2\pi}\left[\frac{1}{1+n} + \frac{1}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-n+1+n}{1-n^2}\right] = \frac{2}{\pi(1-n^2)}$$

$$a_n = \begin{cases} 0 & for\ n = odd \\ \dfrac{2}{\pi(1-n^2)} & for\ n = even \end{cases}$$

To find b_n

$$\boldsymbol{b_n} = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\ dt = \frac{2}{2\pi}\int_0^\pi \sin t\ \sin nt\ dt$$

$$= \frac{1}{2\pi}\int_0^\pi \left(\cos(1-n)t - \cos(1+n)t\right) dt$$

$$= \frac{1}{2\pi}\left[\frac{\sin(1-n)t}{(1-n)} - \frac{\sin(1+n)t}{(1+n)}\right]_0^\pi$$

$$= \frac{1}{2\pi}\left[\frac{\sin(1-n)\pi}{(1-n)} - \frac{\sin(1+n)\pi}{(1+n)} - 0\right] = 0$$

calculate the Fourier coefficients of Cosine Fourier series from Trigonometric Fourier series:

$$A_0 = a_0 = \frac{1}{\pi}$$

$$A_n = \sqrt{a_n^2 + b_n^2} = \frac{2}{\pi(1-n^2)},\ for\ n\,even$$

$$\theta_n = -tan^{-1}\frac{b_n}{a_n} = 0$$

Cosine Fourier Series

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\Omega_0 t + \theta_n)$$

$$x(t) = \frac{1}{\pi} + \sum_{\substack{n=1 \\ (n=even)}}^{\infty} \frac{2}{\pi(1-n^2)} \cos nt$$

$$= \frac{1}{\pi} + \frac{2}{\pi(1-4)} \cos 2t + \frac{2}{\pi(1-16)} \cos 4t + \cdots$$

$$= \frac{1}{\pi} - \frac{2}{3\pi} \cos 2t - \frac{2}{15\pi} \cos 4t + \cdots = \frac{1}{\pi} - \frac{2}{\pi} [\frac{1}{3} \cos 2t + \frac{1}{15} \cos 4t + \cdots]$$

2.2 FOURIER TRANSFORM

The Fourier representation of periodic signals has been extended to non-periodic signals byletting the fundamental period T tend to infinity and this Fourier method of representing non- periodic signals as a function of frequency is called Fourier transform.

Definition of Continuous time Fourier Transform

The Fourier transform (FT) of Continuous time signals is called Continuous Time Fourier Transform

Let $x(t) = continuous\ time\ signal$

$$X(j\omega) = F\{x(t)\}$$

The Fourier transform of continuous time signal, x(t) is defined as

$$X(j\omega) = F\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$$

Conditions for existence of Fourier transform

The Fourier transform $x(t)$ exist if it satisfies the following Dirichlet's condition

1. $x(t)$ should be absolutely integrable

$$ie\ ,\quad \int_{-\infty}^{\infty} x(t)dt < \infty$$

2. $x(t)$should have a finite number of maxima and minima with in any finite interval.
3. $x(t)$ should have a finite number of discontinuities with in any interval.

Definition of Inverse Fourier Transform

The inverse Fourier Transform of $X(j\omega)$ is defined as,

$$x(t) = F^{-1}\{X(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)^{jt}\ d\Omega$$

EXAMPLE 1: Find Fourier transform of impulse signal.

Solution:

$$\mathrm{F}\{x(t)\} = X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt$$

$$\left[\because \text{ Impulse signal } \delta(t) = \begin{cases} 1 & for\ t = 0 \\ 0 & for\ t \neq 0 \end{cases}\right]$$

$$F[\delta(t)] = \delta(0)e^{-j\omega(0)} = 1$$

EXAMPLE 2: Find Fourier transform of double sided exponential signal.

Solution:

$$F[e^{-a|t|}] = \begin{cases} e^{-at} & : t \geq 0 \\ e^{at} & : t \leq 0 \end{cases}$$

$$F[e^{-a|t|}] = \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt$$

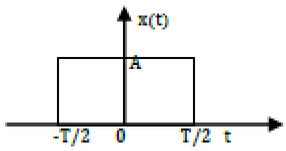
$$= \int_{-\infty}^{0} e^{(a-j\omega)t}dt + \int_{0}^{\infty} e^{-(a+j\omega)t}dt$$

$$= \left[\frac{e^{(a-j\omega)t}}{(a-j\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)}\right]_{0}^{\infty}$$

$$= \frac{1}{(a-j\omega)} + \frac{1}{(a+j\omega)} = \frac{a+j\omega+a-j\omega}{a^2+\omega^2}$$

$$= \frac{2a}{a^2+\omega^2}$$

EXAMPLE 3: Find Fourier transform of rectangular pulse function shown in figure



Solution:

$$x(t) = \pi(t) = A\ ; \quad \frac{-T}{2} \le t \le \frac{T}{2}$$

$$F[\pi(t)] = \int_{-\frac{T}{2}}^{\frac{T}{2}} A e^{-j\omega t}\, dt = A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_{\frac{-T}{2}}^{\frac{T}{2}}$$

$$= \frac{A}{-j\omega}\left[e^{-j\omega\frac{T}{2}} - e^{j\omega\frac{T}{2}}\right] = \frac{2A}{j\omega}\left[\frac{e^{j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}}}{2}\right] = \frac{2A}{\omega}\sin\omega\frac{T}{2}$$

$$= \frac{2A}{\omega T} T \sin\omega\frac{T}{2} = AT\frac{\sin\omega\frac{T}{2}}{\omega\frac{T}{2}} = AT\,sinc\,\omega\frac{T}{2}$$

EXAMPLE 4: Find inverse Fourier transform $X(j\,\omega) = \delta(\omega)$.

Solution:

$$F^{-1}[X(j\omega)] = F^{-1}[\delta(\omega)]$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t}\, d\omega = \frac{1}{2\pi}[1]$$

$$\delta(\omega) = \begin{cases} 1 & for\ \omega = 0 \\ 0 & for\ \omega \neq 0 \end{cases}$$

$$F^{-1}[\delta(\omega)] = \frac{1}{2\pi}$$