

PH8151 ENGINEERING PHYSICS

UNIT-IV QUANTUM PHYSICS

13-Mark

1. a) What is Compton effect? Give the theory of Compton effect and show that the Compton shift. $\lambda' - \lambda = \frac{h}{m_0 c} (1 - \cos \theta)$

[OR]

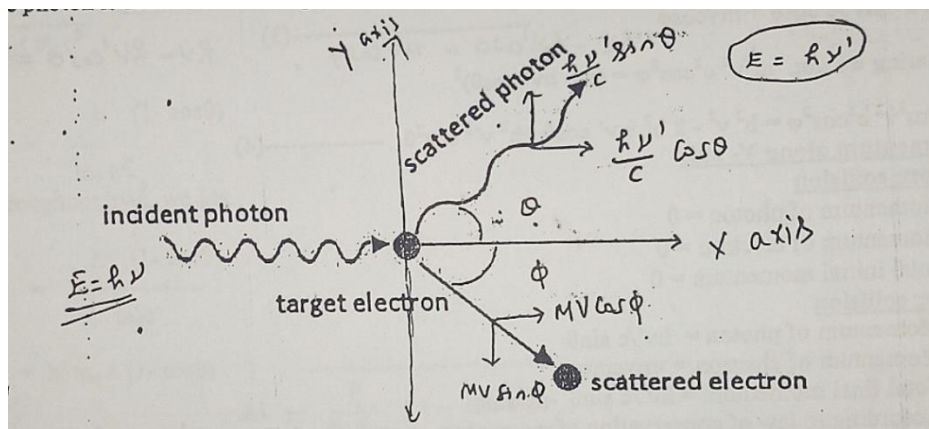
- b) Derive an expression for the change in wavelength suffered by an X – ray photon when it collides with an electron.

When a beam of monochromatic X rays of high frequency passes through matter the scattered beam consists of two types of radiation.

- Radiation having same wavelength as that of incident radiation.
- Modified radiation of slightly longer wavelength.
- The phenomenon of scattering of incident radiation into scattered radiations is called Compton effect. The change in wavelength between the modified and unmodified radiation is called Compton shift.

Expression for Compton shift:

Let a photon of energy $h \nu$ collide with an electron at rest. During the collision the photon gives a fraction of its energy to the free electron. The electron gains kinetic energy and recoils with a velocity 'v' and the photon is scattered at an angle θ with less energy $h \nu'$.



Let $h \nu$ and $h \nu/c$ be the energy and momentum of the incident photon respectively and $h \nu'$ and $h \nu'/c$ be that of scattered photon which makes an angle θ with the x – axis. Let m_0 be the rest mass of electron during collision it makes an angle ϕ to the x – axis from the direction of the incident photon.

Energy before collision:

$$\text{Energy of incident photon} = h \nu$$

$$\text{Energy of electron (at rest)} = m_0 c^2$$

$$\text{Total initial energy} = h \nu + m_0 c^2$$

After collision:

$$\text{Energy of scattered photon} = h \nu'$$

$$\text{Energy of scattered electron} = m c^2$$

$$\text{Total final energy} = h \nu' + m c^2$$

According to the law of conservation of energy, energy before collision is equal to energy after collision. Therefore

$$h\nu + m_0 c^2 = h\nu' + m c^2 \text{ ----- (1)}$$

$$m c^2 = h\nu - h\nu' + m_0 c^2 = h(\nu - \nu') + m_0 c^2$$

Squaring we get,

$$m^2 c^4 = [h(\nu - \nu') + m_0 c^2]^2 = h^2(\nu - \nu')^2 + 2hm_0 c^2(\nu - \nu') + m_0^2 c^4$$

$$m^2 c^4 = h^2 \nu^2 - 2h^2 \nu \nu' + h^2 \nu'^2 + 2hm_0 c^2(\nu - \nu') + m_0^2 c^4 \text{ ----- (2)}$$

Since momentum is a vector quantity,

Momentum along x – axis:

Before collision:

$$\text{Momentum of photon} = h \nu / c$$

$$\text{Momentum of electron} = 0$$

$$\text{Total initial momentum} = h \nu / c$$

After collision:

$$\text{Momentum of photon} = h \nu' / c \cos \theta$$

$$\text{Momentum of electron} = m v \cos \phi$$

$$\text{Total final momentum} = \frac{h\nu'}{c} \cos \theta + m v \cos \phi$$

According to law of conservation of momentum, total momentum before collision is equal to total momentum after collision.

$$h\nu' / c = h\nu' / c \cos \theta + m v \cos \phi \text{ ----- (3)}$$

$$\frac{hv'}{c} - \frac{hv'}{c} \cos \theta = mv \cos \phi$$

$$hv - hv' \cos \theta = mvc \cos \phi$$

Squaring we get, $m^2 v^2 c^2 \cos^2 \phi = (hv - hv' \cos \theta)^2$

$$m^2 v^2 c^2 \cos^2 \phi = h^2 v^2 - 2h^2 v v' \cos \theta + h^2 v'^2 \cos^2 \theta \text{ ----- (4)}$$

Momentum along Y-axis

Before collision

Momentum of photon = 0

Momentum of electron = 0

Total initial momentum = 0

After collision:

Momentum of photon = $hv'/c \sin \theta$

Momentum of electron = $-mv \sin \phi$

Total final momentum = $hv'/c \sin \theta - mv \sin \phi$

According to law of conservation of momentum, momentum before collision is equal to momentum after collision.

$$0 = hv'/c \sin \theta - mv \sin \phi \text{ ----- (5)}$$

$$mv \sin \phi = hv'/c \sin \theta, mvc \sin \phi = hv' \sin \theta$$

Squaring we get,

$$m^2 v^2 c^2 \sin^2 \phi = h^2 v'^2 \sin^2 \theta \text{ ----- (6)}$$

Adding equation (6) and (4) we get

$$m^2 v^2 c^2 \sin^2 \phi + m^2 v^2 c^2 \cos^2 \phi = h^2 v'^2 \sin^2 \theta + [h^2 v^2 - 2h^2 v v' \cos \theta + h^2 v'^2 \cos^2 \theta]$$

$$m^2 v^2 c^2 = h^2 v^2 + h^2 v'^2 - 2h^2 v v' \cos \theta \text{ ----- (7)}$$

Subtracting eq (7) from (2) we get,

$$\begin{aligned} m^2 c^4 - m^2 v^2 c^2 &= h^2 v^2 - 2h^2 v v' + h^2 v'^2 + 2hm_0 c^2 (v - v') + m_0^2 c^4 \\ &- [h^2 v^2 + h^2 v'^2 - 2h^2 v v' \cos \theta] \end{aligned}$$

$$m^2 c^2 (c^2 - v^2) = 2hm_0 c^2 (v - v') + m_0^2 c^4 - 2h^2 v v' (1 - \cos \theta) \text{ ----- (8)}$$

According to the theory of relativity the variation of mass with the velocity of electron is given by $m = m_0/\sqrt{1 - v^2/c^2}$

Squaring this we get, $m^2 = \frac{m_0^2 c^2}{c^2 - v^2}$

$m^2(c^2 - v^2) = m_0^2 c^2$

Multiply throughout by c^2 we get,

$m^2 c^2 (c^2 - v^2) = m_0^2 c^4$

Substituting this in equation (8) we get

$m_0^2 c^4 = m_0^2 c^4 + 2hm_0 c^2(v - v') - 2h^2 vv'(1 - \cos \theta)$

$2hm_0 c^2(v - v') = 2h^2 vv'(1 - \cos \theta)$

Divided by 2h

$m_0 c^2(v - v') = hvv'(1 - \cos \theta)$

Divided by $v v'$

$\frac{v-v'}{vv'} = \frac{h(1-\cos \theta)vv'}{vv'm_0c^2}$

$\frac{1}{v'} - \frac{1}{v} = \frac{h(1-\cos \theta)}{m_0c^2}$

Multiply throughout by c we get

$\frac{c}{v'} - \frac{c}{v} = \frac{h(1-\cos \theta)}{m_0c}$

$\lambda - \lambda' = h/m_0c(1 - \cos \theta)$ ----- (9)

$d\lambda = \frac{h}{m_0c}(1 - \cos \theta)$

This is called Compton equation.

2. a) **Solve Schrodinger wave function for a particle in one dimension and obtain the energy Eigen value.**

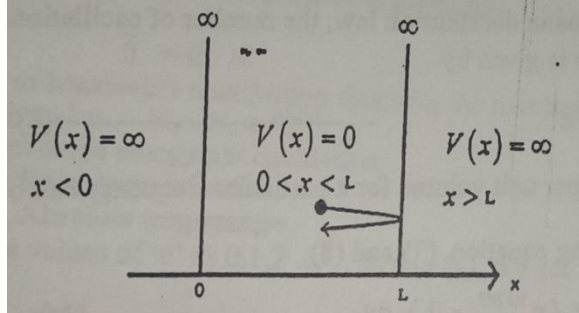
[OR]

- b) **Explain the application of Schrodinger's wave equation to one dimensional potential value.**

[OR]

- c) **Obtain the Eigen values and Eigen functions for an electron enclosed in a one-dimensional potential box.**

Consider a particle of mass 'm' moving along X-axis is one dimensional box of infinite height and width 'L'. The particle moves freely inside the box and is confined between the width 'L' of the box. It makes elastic collision with the walls.



Boundary conditions:

(i) Outside and at the wall of the box

$$V(x) = \infty \text{ and hence } |\Psi|^2 = 0 \text{ in the region } 0 > x > L, \text{ therefore } \Psi = 0 \text{ at } x = 0 \text{ and } x = L$$

(ii) Inside the box

$$V(x) = 0 \text{ and } |\Psi|^2 \neq 0 \text{ in the region } 0 < x < L$$

The Schrodinger's time independent wave equation for one dimension is

$$\frac{d^2\Psi}{dx^2} + \frac{2m}{\hbar^2}(E - V)\Psi = 0 \text{ ----- (1)}$$

Now, inside the box, $V = 0$

Therefore equation (1) becomes

$$\frac{d^2\Psi}{dx^2} + \frac{2mE}{\hbar^2}\Psi = 0$$

$$\text{or } \frac{d^2\Psi}{dx^2} + K^2\Psi = 0 \text{ ----- (2)}$$

$$\text{Where, } K^2 = \frac{2mE}{\hbar^2} \text{ ----- (3)}$$

Equation (2) is similar to the equation of harmonic motion.

A possible solution of this equation is

$$\Psi(x) = A \sin Kx + B \cos Kx \text{ ----- (4)}$$

Where A and B are constants.

The constants A and B are found by applying the boundary conditions $\Psi = 0$ at $x = 0$ and $x = L$.

When $x = 0$, we get,

$$\Psi = 0 = A \sin(0) + B \cos(0)$$

i.e., $B = 0$

When $x = L$, we get

$$\Psi = 0 = A \sin KL$$

or $A \sin KL = 0$

$\rightarrow A = 0$, but $\sin KL = 0$

Or $KL = n\pi$

Where, $K = \frac{n\pi}{L}$ $n = 1, 2, 3, \dots$

Substituting the values of B and K in equation (4) we get

$$\Psi_n(x) = A \sin \frac{n\pi x}{L} \text{ ----- (5)}$$

This is called Eigen function or characteristics function.

The value of A can be obtained by applying normalization condition.

Since the particle is inside the well of length 'L', the probability of that particle inside the well is unity.

Therefore, $\int_0^L |\Psi(x)|^2 dx = 1$

$$\int_0^L A^2 \sin^2 \frac{n\pi x}{L} dx = 1$$

$$A^2 \int_0^L \left(\frac{1 - \cos 2 \frac{n\pi x}{L}}{2} \right) dx = 1$$

$$\frac{A^2}{2} \left[\int_0^L dx - \int_0^L \cos 2 \frac{n\pi x}{L} dx \right] = 1$$

$$\frac{A^2}{2}$$

$$\frac{A^2}{2} L = 1 \Rightarrow A = \sqrt{\frac{2}{L}}$$

Therefore, the wave function becomes

$$\Psi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \text{ ----- (6)}$$

This is known as normalized Eigen function.

Eigen energy:

$$K = \frac{n\pi}{L}$$

Substituting the value of 'K' in equation (3), we get

$$\frac{n^2\pi^2}{L^2} = \frac{2mE}{h^2}$$

$$\frac{n^2\pi^2}{L^2} = \frac{2mE \times 4\pi^2}{h^2}$$

Therefore, $E_n = \frac{n^2h^2}{8mL^2}$ ----- (7)

These energies defined by the above equation with n = 1, 2, 3... Are called Eigen energies of the system.

For each value of the quantum number 'n', there is an energy value called Eigen value and wave function called Eigen function.

The first three energy levels and the corresponding wave functions are shown below.

For n = 1, $E_1 = 1^2 \left[\frac{h^2}{8mL^2} \right]$ $\psi_1 = \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L}$

For n = 2, $E_2 = 2^2 \left[\frac{h^2}{8mL^2} \right] = 4E_1$, $\psi_2 = \sqrt{\frac{2}{L}} \sin \frac{2\pi x}{L}$

For n = 3, $E_3 = 3^2 \left[\frac{h^2}{8mL^2} \right] = 9E_1$, $\psi_3 = \sqrt{\frac{2}{L}} \sin \frac{3\pi x}{L}$

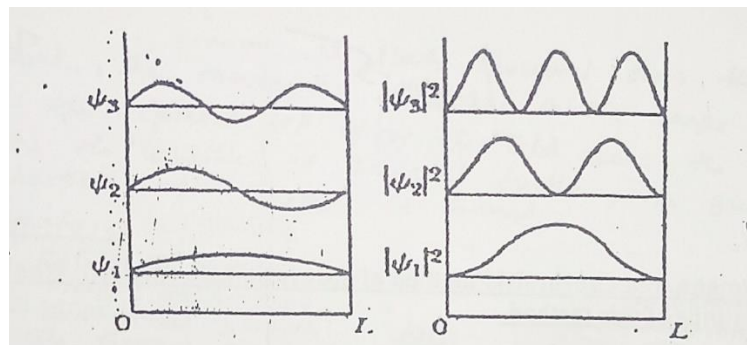


Figure shows the electron energy levels and the corresponding wave functions. In the first energy level, the probability of finding the electron is maximum at the centre of the well. In the second energy level, the probability of finding the electron at the centre of the well is zero and vice versa.

3. Demonstrate the working of scanning tunnelling microscope?

A scanning tunnelling microscope is an instrument used for imaging surfaces at the atomic level.

Principle:

The basic principle used in scanning tunnelling microscope is the tunnelling of electron between the sharp metallic tip of the probe and the surface of the sample.

Construction:

The components of a STM include scanning tip, piezoelectric controlled scanner, distance control and scanning unit, vibration isolation system and computer.

Scanning Tip:

Electrons tunnel from the scanning tip to the sample and creates the tunnelling current.

Piezoelectric controlled scanner:

Piezoelectric crystals expand and contract very slightly. It is used to control the horizontal position X, Y and the height z of the scanning tip.

Distance control and scanning unit:

It provides a coarse control over the distance between the tip and the sample.

Vibration Isolation system:

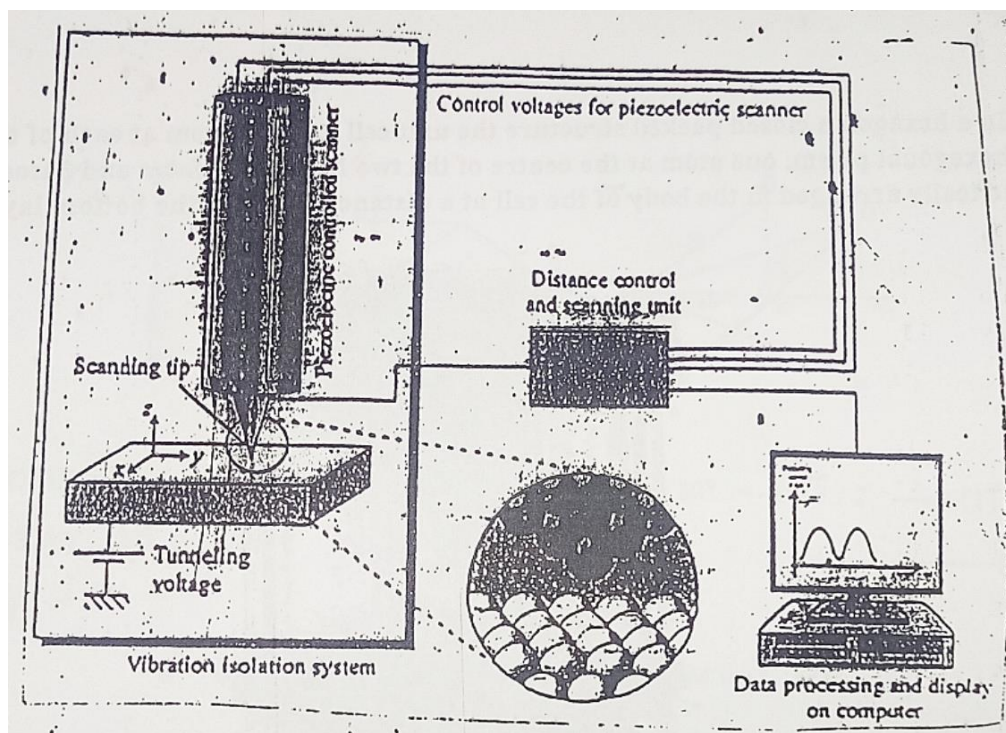
It isolates STM from vibrations, since it deals with extremely fine position measurement.

Computer:

Computer records the tunnelling current to produce a 3-dimensional map of the sample surface.

Working:

- The circuit is switched ON and the necessary bias voltage is given to the probe.
- A small air gap should always be maintained between the tip of the probe and the sample.
- Due to biasing, the electron will tunnel between the tip of the probe and the sample and produce a small electric current called tunnelling current.
- Based on the distance between the tip of the probe and the sample the current increases or decreases.
- For maintaining constant current, the distance between the tip and the sample should be adjusted in such a way that at tip passes very close to the sample.
- The tip scans the atom line by line and the topography of the sample is recorded in the computer.



Advantages:

- It gives a three-dimensional profile of a surface.
- STM's are versatile. They can be used in ultra-high vacuum, air, water and other liquids and gases.
- They can be operated at low temperatures.

Disadvantages:

- It requires a lot of skill and precision.
- STM requires very stable and clean surfaces, excellent vibration control and sharp tips.
- STM use highly specialized equipment that is fragile and expensive.

4. a) Based on quantum concept derive the Planck's radiation formula.

[OR]

b) What are the basic postulates of quantum theory of light? Derive Planck's law of radiation.

c) Write the postulates of Planck's quantum theory of radiation. Using quantum theory derive an expression for the average energy emitted by a black body and arrive at Planck's radiation law in terms of frequency.

(i) Planck's Hypothesis:

- The blackbody is made up of number of vibrating particles. The particles can vibrate in all possible frequencies.
- The frequency of radiation emitted by an oscillator is the same as the frequency of its vibration.
- An oscillator cannot emit energy in a continuous manner. It can emit energy in the multiples of a small unit called quanta or photon.
- Each photon has an energy $\epsilon = hv$, where 'v' is the frequency of radiation and 'h' is the Planck's constant.
- The vibrating particles radiate energy when they move from one energy state to another.

(ii) Derivation:

Let us consider the number of vibrating particles in the body N. Let the particles $N_0, N_1, N_2, \dots, N_n$ having energies $0, \epsilon, 2\epsilon, 3\epsilon, \dots, n\epsilon$

Hence the total number of vibrating particles, $N = N_0 + N_1 + N_2 + \dots + N_n$ --- (1)

Total energy of the body, $E = N_0 \times 0 + N_1 \times \epsilon + N_2 \times 2\epsilon + \dots + N_n \times n\epsilon$ --- (2)

Therefore, average energy of the particle is given by

$$= E/N \text{ ----- (3)}$$

According to Maxwell's distribution formula, the number of vibrating particles having energy $n\varepsilon$ given by $N_n = N_0 e^{-n\varepsilon/KT}$ ----- (4)

Where,

$\varepsilon \rightarrow$ average energy per oscillation

$K \rightarrow$ Boltzmann constant

$T \rightarrow$ Absolute temperature

Substituting the values of 'n' as 0, 1, 2in eq (4), we get

$$N_0 = N_0 e^{-0\varepsilon/KT} = N_0$$

$$N_1 = N_0 e^{-1\varepsilon/KT}$$

$$N_2 = N_0 e^{-2\varepsilon/KT} \text{ etc}$$

Substituting these values in equation (1) we get

$$N = N_0 + N_0 e^{-1\varepsilon/KT} + N_0 e^{-2\varepsilon/KT} + \dots$$

$$= N_0 [1 + e^{-1\varepsilon/KT} + e^{-2\varepsilon/KT} + \dots]$$

$[1 + x + x^2 + x^3 + \dots + x^n = 1/(1 - x)]$ using this, the previous equation be

$$N = N_0 / [1 - e^{-\varepsilon/KT}] \text{ ----- (5)}$$

The total energy E is given by

$$E = N_0 \times 0 + N_1 \times \varepsilon + N_2 \times 2\varepsilon + \dots$$

$$= 0 + \varepsilon N_0 e^{-1\varepsilon/KT} + 2\varepsilon N_0 e^{-2\varepsilon/KT} + \dots$$

$$= \varepsilon N_0 e^{-\varepsilon/KT} [1 + 2e^{-1\varepsilon/KT} + 3e^{-2\varepsilon/KT} + \dots]$$

Simplify using Binomial series

$$[1 + 2x + 3x^2 + 4x^3 + \dots + nx^{n-1} = 1/(1 - x)^2]$$

$$E = \frac{\varepsilon N_0 e^{-\varepsilon/KT} x}{(1 - e^{-\varepsilon/KT})^2}$$

$$E = \frac{\varepsilon N_0 e^{-\varepsilon/KT}}{(1 - e^{-\varepsilon/KT})^2} \text{ ----- (6)}$$

Substituting eq (5) and eq (6) in eq (3) we get,

$$E_\nu d\nu = (8\pi\nu^2/c^3) \times (h\nu/e^{h\nu/KT} - 1)d\nu$$

$$E_\nu d\nu = (8\pi h\nu^3/c^3) \times (1/e^{h\nu/KT} - 1)d\nu \text{ ----- (9)}$$

This is known as Planck's radiation law in terms of frequency.

This law can be expressed in terms of wavelength as

$$\text{We know, } c = v\lambda, v = c/\lambda, dv = c/\lambda^3 d\lambda$$

$$|dv| = c/\lambda^3 d\lambda$$

Substituting these in eq (9), we get

$$E_\lambda d\lambda = (8\pi h(c^3/\lambda^3)/c^3) \times (1/e^{hv/KT} - 1) \times c/\lambda^2 d\lambda$$

This equation represents Planck's radiation law in terms of wavelength.

5. i) a) **what are the drawbacks of classical free electron theory? Derive Schrodinger time independent wave equations.**

[OR]

b) Derive the time – independent and time dependent Schrodinger wave equations.

ii) A photon of frequency ν_0 scatters from an electron at rest and moves in a direction making an angle of 60° with the incident direction. If the frequency of the scattered photon is half that of incident photon, calculate the frequency of the incident photon.

Definition:

The equation that describes the wave nature of a particle in mathematical form is known as Schrodinger wave equation.

There are two forms of Schrodinger wave equation. They are

- (i) Time independent wave equation
- (ii) Time dependent wave equation

Consider a system of stationary waves associated with a particle. Let x, y, z be the coordinates of the particle and be the wave displacement for de-Broglie's waves at any time t.

The classical differential equation of a wave motion is given by

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \text{ ----- (1)}$$

Here v is the wave velocity,

Equation (1) is written as

$$\nabla^2 \psi = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \text{ ----- (2)}$$

Where, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ and ∇ is the Laplacian operator.

The solution of equation (2) gives Ψ as a periodic displacement in terms of time t .

$$\Psi(x, y, z, t) = \Psi_0(x, y, z)e^{-i\omega t} \text{ ----- (3)}$$

$\Psi_0(x, y, z)$ is a function of x, y, z only and gives amplitude at the point consider?

Differentiating equation (3) with respect to t , we get

$$\frac{\partial \Psi}{\partial t} = -i\omega \Psi_0 e^{-i\omega t}$$

$$\frac{\partial^2 \Psi}{\partial t^2} = (i\omega)^2 \Psi_0 e^{-i\omega t}$$

$$\frac{\partial^2 \Psi}{\partial t^2} = -\omega^2 \Psi \text{ ----- (4)}$$

Substituting (4) in (2), we have

$$\nabla^2 \Psi = \frac{1}{v^2} (-\omega^2 \Psi)$$

$$\nabla^2 \Psi + \frac{\omega^2}{v^2} \Psi = 0 \text{ ----- (5)}$$

$$\text{Angular frequency } \omega = 2\pi\nu = 2\pi \frac{v}{\lambda} \text{ ----- (6)}$$

Substituting (6) in (5)

$$\nabla^2 \Psi + \frac{4\pi^2}{\lambda^2} \Psi = 0 \text{ ----- (7)}$$

We know that de-Broglie wavelength $\lambda = \frac{h}{mv}$ on substituting the value of λ in equation (7), we get

$$\nabla^2 \Psi + \frac{4\pi^2}{\left(\frac{h}{mv}\right)^2} \Psi = 0$$

$$\nabla^2 \Psi + \frac{4\pi^2 m^2 v^2}{h^2} \Psi = 0 \text{ ----- (8)}$$

If E is the total energy of the particle, V is the potential energy and $\frac{1}{2}mv^2$ is the Kinetic energy, then

Total energy (E) = potential Energy (V) + Kinetic Energy ($\frac{1}{2}mv^2$)

$$E = V + \frac{1}{2}mv^2$$

$$E - V = \frac{1}{2}mv^2$$

$$2(E - V) = mv^2$$

$$2m(E - V) = m^2v^2$$

$$m^2v^2 = 2m(E - V) \text{ ----- (9)}$$

Substituting equation (9) in equation (8)

$$\nabla^2\psi + \frac{4\pi^2 2m(E-V)}{h^2} \psi = 0$$

$$\nabla^2\psi + \frac{8\pi^2 m(E-V)}{h^2} \psi = 0 \text{ ----- (10)}$$

This equation (10) is known as Schrodinger time independent wave equation

Let us now substitute $h = \frac{h}{2\pi}$ the equation (10) is modified as,

$$\nabla^2\psi + \frac{2m(E-V)}{h^2} \psi = 0 \text{ ----- (11)}$$

Schrodinger time dependent wave equation:

Schrodinger time independent wave equation is derived from Schrodinger time independent wave equation. The solution of classical differential equation of the wave motion is given by

$$\Psi(x, y, z, t) = \Psi_0(x, y, z)e^{-i\omega t} \text{ ----- (1)}$$

Differentiating equation (1) with respect to time t we get

$$\frac{\partial \Psi}{\partial t} = -i\omega \Psi_0 e^{-i\omega t} \text{ ----- (2)}$$

$$\frac{\partial \Psi}{\partial t} = -i(2\pi v)\Psi_0 e^{-i\omega t}$$

$$\frac{\partial \Psi}{\partial t} = -i(2\pi v)\Psi \quad \therefore v = \frac{E}{h}$$

$$\frac{\partial \Psi}{\partial t} = -i \left(2\pi \frac{E}{h} \right) \Psi$$

$$\frac{\partial \Psi}{\partial t} = -i \left(\frac{E}{h} \right) \Psi \quad \text{----- (3)}$$

Multiplying by I on both the sides

$$i \frac{\partial \Psi}{\partial t} = -i^2 \left(\frac{E}{h} \right) \Psi$$

$$i \frac{\partial \Psi}{\partial t} = \left(\frac{E}{h} \right) \Psi$$

$$\frac{2}{v_0} - \frac{1}{v_0} = \frac{h}{m_0 c^2} (1 - \cos \theta)$$

$$\frac{1}{v_0} = \frac{(6.625 \times 10^{-34})}{9.11 \times 10^{-31} \times (3 \times 10^8)^2} (1 - \cos 60^\circ)$$

$$\frac{1}{v_0} = 0.0404 \times 10^{-19}$$

$$v_0 = 24.75 \times 10^{19}$$

$$v_0 = 2.475 \times$$