UNIT I

STATIC ELECTRIC FIELD

Vector Algebra, Coordinate Systems, Vector differential operator, Gradient, Divergence, Curl, Divergence theorem, Stokes theorem, Coulombs law, Electric field intensity, Point, Line, Surface and Volume charge distributions, Electric flux density, Gauss law and its applications, Gauss divergence theorem, Absolute Electric potential, Potential difference, Calculation of potential differences for different configurations. Electric dipole, Electrostatic Energy and Energy density.

1. INTRODUCTION

Electromagnetics (EM) is a branch of physics or electrical engineering in which electric and magnetic phenomena are studied. Field is a function that specifies a quantity everywhere in a region or a space. If at each point of a region or space there is a corresponding value of some physical function then region is called a field. If the field produced due to the magnetic effect it is called magnetic field. There are two types of electric charges, positive and negative. Such an electric charge produces a field around it which is called an electric field.

Moving charges produces a current and a current carrying conductor produce a magnetic field. In such a case electric and magnetic fields are related to each other. Such a field is called electromagnetic field. An electromagnetic field, sometimes referred to as an EM field, is generated when charged particles, such as electrons, are accelerated. All electrically charged particles are surrounded by electric fields. Charged particles in motion produce magnetic fields. When the velocity of a charged particle changes, an EM field is produced. Electromagnetic (EM) may be regarded as the study of the interactions between electric charges at rest and in motion. It entails the analysis, synthesis, physical interpretation, and application of electric and magnetic fields.

Natural sources of electromagnetic field: Electromagnetic fields are present everywhere in our environment but are invisible to the human eye. Electric fields are produced by the local build-up of electric charges in the atmosphere associated with thunderstorms. The earth's magnetic field causes a compass needle to orient in a North-South direction and is used by birds and fish for navigation

Some of the branches of study where electromagnetic principles find application are RF communication, Microwave Engineering ,Antennas ,Electrical Machines, Satellite Communication, Atomic and nuclear research, Radar Technology, Remote sensing, EMI EMC, Quantum Electronics etc.

1.1. VECTOR ALGEBRA

Vector algebra is a mathematical shorthand. Any physical quantity can be represented either as a scalar or a vector.

A scalar is a quantity that has only magnitude. Quantities such as time, mass, distance, temperature, entropy, electric potential, and population are scalars.

A **vector** is a quantity that has both magnitude and direction. Vector quantities include velocity, force, displacement, and electric field intensity. Another class of physical quantities is called *tensors,* of which scalars and vectors are special cases. For most of the time, we shall be concerned with scalars and vectors

To distinguish between a scalar and a vector it is customary to represent a vector by a letter with an arrow on top of it, such as $\vec{A}$ and $\vec{B}$ or by a letter in boldface type such as **A** and **B**. A scalar is represented simply by a letter—e.g., A and B. EM theory is essentially a study of some particular fields.

A **field** is a function that specifies a particular quantity everywhere in a region. If the quantity is scalar (or vector), the field is said to be a scalar (or vector) field. Examples of scalar fields are temperature distribution in a building, sound intensity in a theater, electric potential in a region, and refractive index of a stratified medium. The gravitational force on a body in space and the velocity of raindrops in the atmosphere are examples of vector fields.

Unit Vector

A vector A has both magnitude and direction. The magnitude of A is a scalar written as A or |A|. A unit vector $\overline{a_A}$ along A is defined as a vector whose magnitude is unity (i.e., 1) and its direction is along A, that is,

$$\text{Unit vector } a_A = \frac{\bar{A}}{|A|}$$

Note that $|a_A| = 1$. Thus we may write A as $A = A\bar{a}_A$ which completely specifies A in terms of its magnitude A and its direction $\bar{a}_A$.

1.1.1. Sum and Difference of two Vector

Vector Addition

Vector addition has a very simple geometrical interpretation. To add vector $\bar{B}$ to vector $\bar{A}$, we simply place the tail of $\bar{B}$ at the head of $\bar{A}$. The sum is a vector $\bar{C}$ from the tail of $\bar{A}$ to the head of $\bar{B}$. Thus, we write $\bar{C} = \bar{A} + \bar{B}$. The same result is obtained if the roles of $\bar{A}$ are reversed $\bar{B}$. That is, $\bar{C} = \bar{A} + \bar{B} = \bar{B} + \bar{A}$. This commutative property is illustrated below with the parallelogram construction.

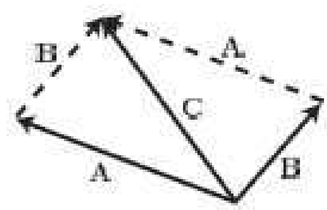


Fig 1.1. Vector addition

Since the result of adding two vectors is also a vector, we can consider the sum of multiple vectors. It can easily be verified that vector sum has the property of association, that is, $(\bar{A} + \bar{B}) + \bar{C} = \bar{A} + (\bar{B} + \bar{C})$.

Vector subtraction

Since $\bar{A} - \bar{B} = \bar{A} + (-\bar{B})$, in order to subtract $\bar{B}$ from A, we simply multiply B by −1 and then add.

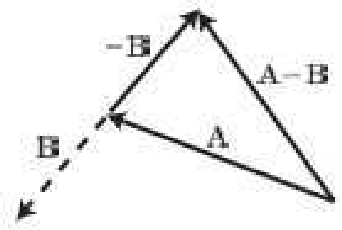


Fig 1.2. Vector Subtraction

1.1.2. Multiplication of Scalar and Vector

Vector Multiplication

When two vectors $\overline{A}$ and $\overline{B}$ are multiplied, the result is either a scalar or a vector depending on how they are multiplied. Thus there are two types of vector multiplication:

i. Scalar (or dot) product: $\overline{A} \bullet \overline{B}$
ii. Vector (or cross) product: $\overline{A} \times \overline{B}$.

i. Scalar product ("Dot" product)

The **dot product** of two vectors $\overline{A}$and $\overline{B}$, written as $\overline{A} \bullet \overline{B}$. is defined geometrically as the product of the magnitudes of $\overline{A}$ and $\overline{B}$ and the cosine of the angle between them. Thus:

$$\overline{A} \bullet \overline{B} = AB \cos \theta_{AB}$$

where θ_{AB} is the smaller angle between A and B. The result of $\overline{A} \bullet \overline{B}$ is called either the scalar product because it is scalar, or the dot product due to the dot sign. Here θ, is the angle between the vectors A and B when they are drawn with a common origin.

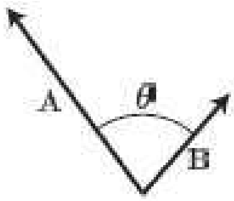


Fig 1.3. Vector Multiplication

- Commutative law: $\overline{A} - \overline{B} = \overline{B} . A$
- Distributive law: $\overline{A} (\overline{B} + \overline{C}) = \overline{A}\,\overline{B} + \overline{A}\,\overline{C}$

  $\overline{A}. \overline{A} = |A|^2 = A^2$

ii. Vector (or cross) product

The **cross product** of two vectors $\overline{A}$ and B. written as $\overline{A} \times \overline{B}$. is a vector quantity whose magnitude is ihe area of the parallopiped formed by $\overline{A}$ and $\overline{B}$, and is in the direction of advance of a right-handed screw as $\overline{A}$ is turned into B.

Thus $$\overline{A} \times \overline{B} = AB \sin \theta_{AB} \overrightarrow{a_n}$$

Where $\overrightarrow{a_n}$ is a unit vector normal to the plane containing A and B. The direction of an is taken as the direction of the right thumb when the fingers of the right hand rotate from $\overline{A}$ to $\overline{B}$. The vector multiplication of $\overline{A}$and $\overline{B}$ is called cross product due to the cross sign; it is also called vector product.

$$\mathbf{\bar{A}} \text{ X } \mathbf{\bar{B}} = \begin{vmatrix} \overrightarrow{\mathbf{a_x}} & \overrightarrow{\mathbf{a_y}} & \overrightarrow{\mathbf{a_z}} \\ \mathbf{A_x} & \mathbf{A_y} & \mathbf{A_z} \\ \mathbf{B_x} & \mathbf{B_y} & \mathbf{B_z} \end{vmatrix}$$

$$\bar{A} \text{ X } (\bar{B} + \bar{C}) = \bar{A} \text{ X } \bar{B} + \bar{A} \text{ X } \bar{C}$$

$$\bar{A} \text{ X } \bar{A} = 0$$

Also note that

$$\overrightarrow{a_x} \text{ X } \overrightarrow{a_y} = \overrightarrow{a_z}$$
$$\overrightarrow{a_y} \text{ X } \overrightarrow{a_z} = \overrightarrow{a_x}$$
$$\overrightarrow{a_z} \text{ X } \overrightarrow{a_x} = \overrightarrow{a_z}$$
$$\overrightarrow{a_z} \text{ X } \overrightarrow{a_x} = \overrightarrow{a_z}$$

S.NO	LAWS	ADDITION	MULTIPLICATION
1.	Commutative	$\bar{A} + \bar{B} = \bar{B} + \bar{A}$	$k\bar{A} = (\bar{A}\ k$
2.	Associative	$(\bar{A} + (\bar{B} + \bar{C}) = (\bar{A} + \bar{B}) + \bar{C}$	$k(l(\bar{A}\ ) = (kl)\ (\bar{A}$
3.	Distributive	$k(\bar{A} + \bar{B}) = k\bar{A} + k\bar{B}$	

where *k* and *l* are scalars.

1.1.3. Differentiation

The differential vector operator ∇ is called del or nelba defined as

$$\nabla = \frac{\partial}{\partial \mathbf{x}} \overline{\mathbf{ax}} + \frac{\partial}{\partial \mathbf{y}} \overline{\mathbf{ay}} + \frac{\partial}{\partial \mathbf{z}} \overline{\mathbf{az}}$$

There are three possible operations with ∇.

1. Gradient
2. Divergence
3. Curl

1.1.4. Solenoidal and irrotational vector

A vector $\bar{A}$ is said to be solenoidal if its divergence is zero. i.e, $\nabla . \bar{A} = 0$ then $\bar{A}$ is said to be Solenoidal.

A vector $\bar{A}$ is said to be irrotational if the curl is zero. $\nabla . \bar{A} = 0$. Then $\bar{A}$ is said to be irrotational.

Identical vector:

Two vectors are identical if there difference is zero. Thus $\bar{A}$ and $\bar{B}$ are identical if $\bar{A} - \bar{B} = 0$

1.2. **COORDINATE SYSTEMS**

1.2.1. rectangular co ordinate system

In rectangular co ordinate system three co ordinates axes are at right angle to each other and call it x,y,z axes. These three axes are intersect at a common point is an origin of the system. At any point P(x,y,z) is specified as the intersection of three planes.

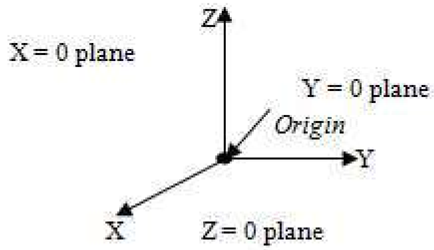


Fig 1.4 Co ordinate Systems

The x ,y ,z values are constant being the co ordinates value of the point. The unit vector along the three co ordinate axis are given as $\hat{a}_x$,$\hat{a}_y$,$\hat{a}_z$. The components vectors have unit magnitude and direction. The unit vector have unit magnitude and directed along the co ordinate axis. A unit vector is given direction is a vector in that direction divided by its magnitude. It is given by,

$$\overline{a_r} = \frac{\overline{r}}{|r|}$$

$$\bar{a}_r = \frac{x\bar{a}_x + y\bar{a}_y + z\bar{a}_z}{\sqrt{x^2 + y^2 + z^2}}$$

Consider a point P(x,y,z) in Cartesian coordinate system as shown in fig. Then the position vector of point P is represented by the direction of point P from the origin. If the three coordinate axes with the magnitude x,y,z. Thus the position vector of point P can be represented as.

$$\overline{r_{op}} = x_1\overline{a_x} + y_1\overline{a_y} + z_1\overline{a_y}$$

The magnitude of the three vectors are,

$$|r_{op}| = \sqrt{x^2 + y^2 + z^2}$$

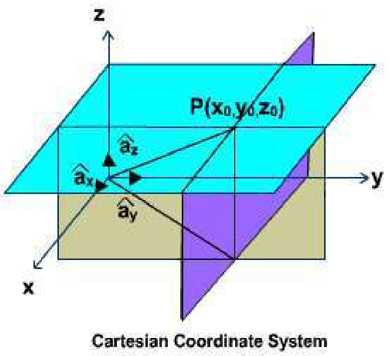


Fig 1.5 Rectangular co ordinate system

Consider the points P(x ,y ,z) and Q(x+dx ,y+dy ,z+dz) in rectangular co ordinate system.

Displacement vector $\overline{dl} = dx\bar{a}_x + dy\bar{a}_y + dz\bar{a}_z$

Differential Volume dv = dxdydz

Figure1.5 shows the six planes define a rectangular parallel piped. The differential length dl from P to Q is the diagonal of the parallel piped is given by,

$$\mathbf{dl} = \sqrt{\mathbf{dx^2 + dy^2 + dz^2}}$$

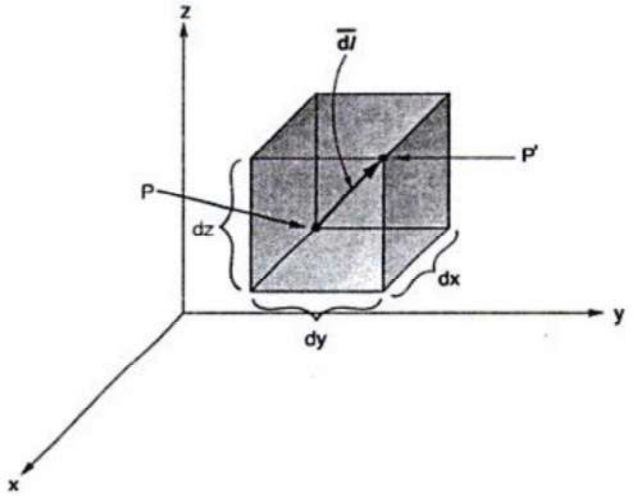


Fig.1.6. differential length and differential elements in Cartesian co ordinate system

The differential area ds_1 = dxdy (normal to z direction)

ds_2 = dydz (normal to x direction)

ds_3 = dzdx (normal to y direction)

The differential volume dv = dxdydz

1.2.2. Cylindrical Coordinate System

One coordinate system that we work in is the standard Cartesian (x,y,z) system. But, if you are doing a problem with either spherical symmetry (going out the same distance in the x, y, and z directions is the same) or cylindrical symmetry (this means symmetry about one axis), using different coordinate systems may make the problem easier. For example, suppose you are trying to calculate the electric field due to a line of charge lying on the z-axis. The electric field at (1, 0, 0) is the same as the electric field at (0, 1, 0). The only thing that matters is the distance from the z-axis. For this problem, it may be Easier to work in cylindrical coordinates.

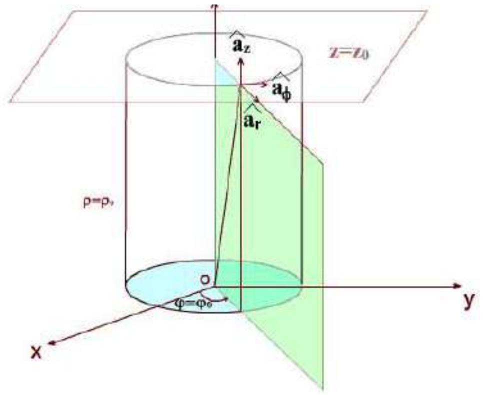


Fig 1.7 Cylindrical co ordinate system

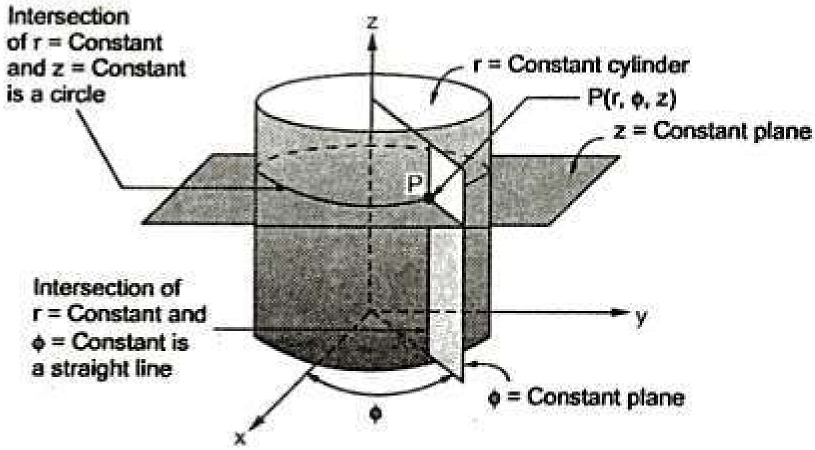


Fig 1.8.Representing point P in cylindrical system

The cylindrical co ordinate system is the three dimensional version of the polar co ordinate of analytical geometry. In this system consider any points as the intersection of three mutually perpendicular surfaces.

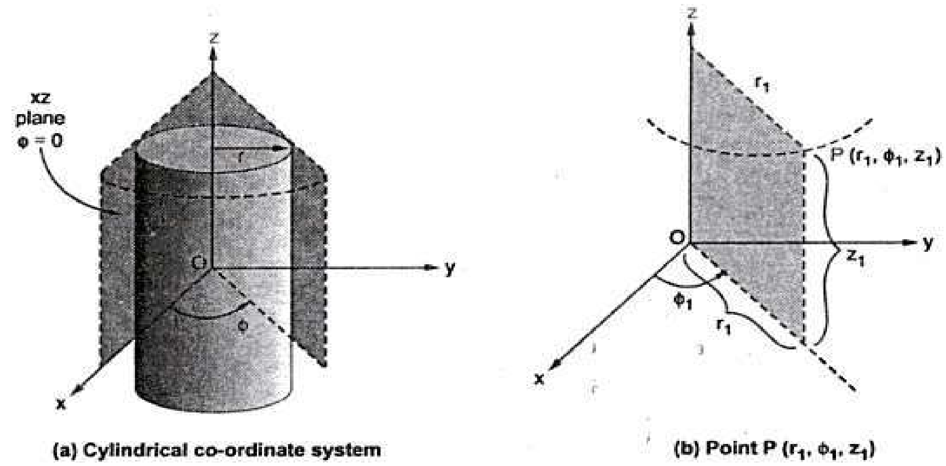


Fig 1.9. Analytical geometry of cylindrical co ordinate system

The surface used to defined the cylindrical co ordinate system are,

i. Plane of constant z which parallel to xy plane.
ii. A cylinder of radious r with z axis as the axis of the cylinder.
iii. A half perpendicular to xy plane and at an angle Φ with respect to xy plane.

The angle Φ is called azimuth angle,

The ranges of variable,

$$0 \leq r \leq \infty$$
$$0 \leq \Phi \leq 2\pi$$
$$-\infty \leq Z \leq \infty$$

They are a circular cylinder (ρ= constant) a plane (Φ = constant) and another plane (z = constant). The co ordinates are ρ,Φ,z. A differential volume element in cylindrical co ordinates may be obtained by increasing ρ,Φ and z by the differential increments dρ, dΦ and dz.

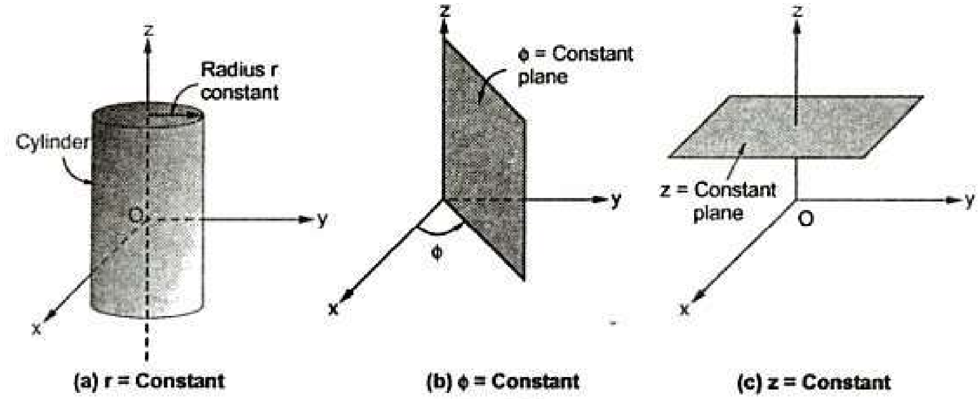


Fig 1.10. differential increments dρ, dΦ and dz

The differential length dl = $\sqrt{(d\rho)^2 + (\rho d\varphi)^2 + (dz)^2}$

The shape of this small volume is truncated. As the volume element becomes very small, its shape approaches that of a rectangular parallelo piped. It has side of the length dρ, ρdΦ,dz.

Unit Vectors:

The unit vectors in the cylindrical coordinate system are functions of position. It is convenient to express them in terms of the cylindrical coordinates and the unit vectors of the rectangular coordinate system which are not themselves functions of position.

$$\overline{a_\rho} = \frac{\rho}{\bar{\rho}} = \frac{x\bar{x} + y\bar{y}}{\rho} = \bar{x}cos\varphi + \bar{y}sin\varphi$$

$$\varphi = \bar{Z} \text{ x } \bar{\rho} = -\bar{x}\sin\varphi + \bar{y}\cos\varphi$$

$$\bar{Z} = \bar{Z}$$

The differential length in ρ and z direction are dρ and dz respectively. In φ direction dφ there exists as differential are length in φ direction. This differential length due to dφ in φdirection is $\rho d\varphi$ as shown in fig.

Thus the differential length are,

$d\rho$= differential length in ρ direction

$\rho d\varphi$=differential length in φ direction

Dz= differential length in z direction.

The differential length $dl = \sqrt{(d\rho)^2 + (\rho d\varphi)^2 + (dz)^2}$

The differential area ds = dρdz (ρ,z plane)

ds = ρdρdΦ (ρ,Φ plane)

ds = ρdΦdz (Φ,z plane)

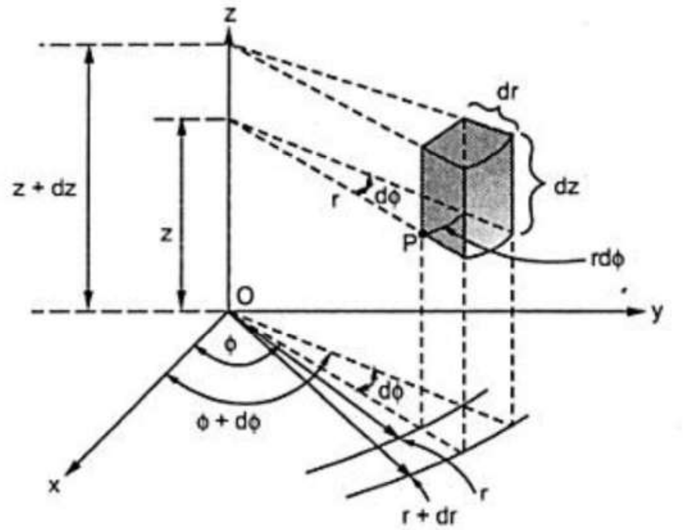


Fig 1.11 Differential Volume element in cylindrical co ordinate system

The differential volume dv = $\rho d\rho d\Phi dz$

1.2.3. Spherical co ordinate system

In this system consider any point as the point of intersection of the spherical surface (radious r= constant) (θ = angle between r and z) (Φ = constant).

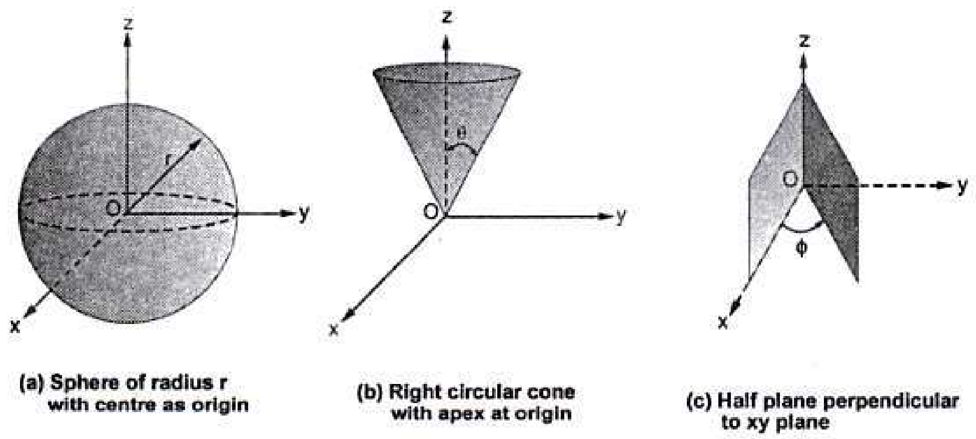


Fig 1.12. Spherical co ordinate system

i. The sphere radius r origin as the center of the sphere.
ii. A right circular cone with its apex at the origin and its axis as z axis. Its half angle is θ. It rotate about z axis and θ varies from 0 to 180.
iii. A half plane perpendicular to xy plane containing z axis, making an angle Φ with the xz plane.

The co ordinate of this systems are r, θ, Φ.

The ranges of variable are,

$$0 \leq r \leq \infty$$

$0 \le \Phi \le 2\pi$

$0 \le \theta \le \pi$ as half angle.

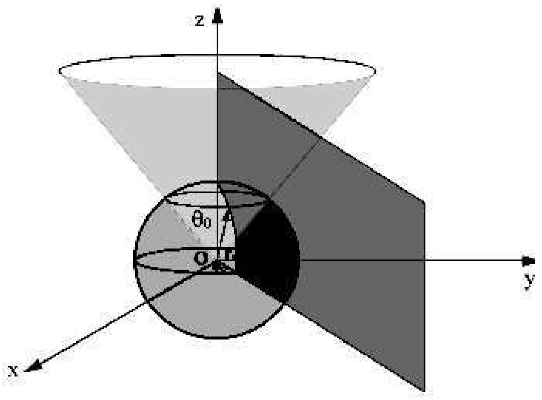


Fig 1.14 Spherical co ordinate system

The differential volume element may be obtained in spherical co ordinate by increasing r, θ, Φ by dr,dθ and dΦ. The side of this volume elements are dr, rdΦ, rsinθ dΦ.

The differential length $\mathbf{dl} = \sqrt{(\mathbf{dr})^2 + (\mathbf{rd\theta})^2 + (\mathbf{rsin\theta d\varphi})^2}$

The differential area $ds = r\, dr\, d\theta$ (r,θ plane)

$ds = r\sin\theta\, d\Phi\, dr$ (r,Φ plane)

$ds = r^2\sin\theta\, d\theta\, d\Phi$ (θ,Φ plane)

The differential volume $dv = r^2 sin\theta dr\, d\theta d\Phi$

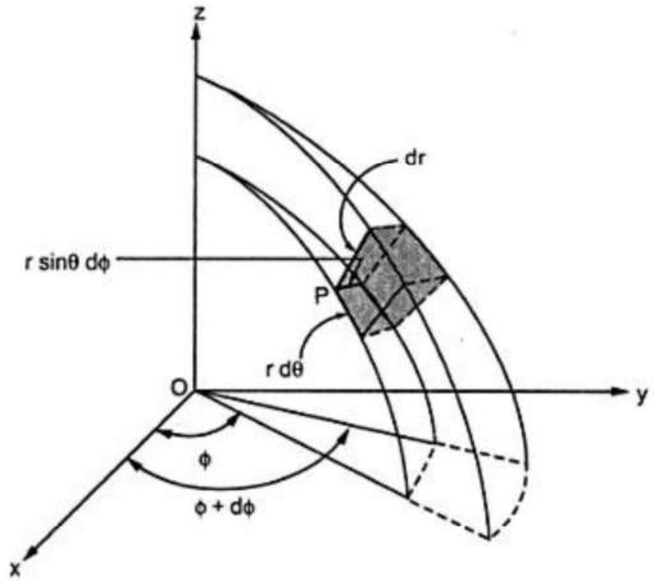


Fig 1.15 Differential Volume in Spherical co ordinate system

Table : Differential lengths, surface area, and volume elements for each geometry. The surface element is subscripted by the coordinate perpendicular to the surface.

	Cartesian coordinate system	*Cylindrical coordinate system*	*Spherical co ordinate system*
The differential length	dl= $\sqrt{dx^2+dy^2+dz^2}$	dl= $\sqrt{(d\rho)^2+(\rho d\varphi)^2+(dz}$	dl= $\sqrt{(dr)^2+(rd\theta)^2+(rsin\theta d\varphi)}$
The differential area	ds_1=dxdy(x,yplane) ds_2=dydz(y,zplane) ds_3=dzdx(z,xplane)	ds = dρdz (ρ,z plane) ds = ρdρdΦ (ρ,Φ plane) ds = ρdΦdz (Φ,z plane)	ds = r dr dθ (r,θ plane) ds = rsinθ dΦ dr (r,Φ plane) ds = r^2sinθ dθ dΦ (θ,Φ plane)
The differential volume	dv = dxdydz	dv = ρdρdΦdz	dv = r^2sinθ dr dθ dΦ

1.2.4. VECTOR TRANSFORMATION

It is necessary to transform a vector from one co ordinate system to another co ordinate system. Transformation of a vector between Cartesian and cylindrical co ordinate system, Cartesian and spherical system are carried out.

(i)Transformation between Cartesian and Cylindrical coordinates:

Let us consider A = Aρ$\overline{a\rho}$ + AΦ$\overline{a\Phi}$ + Az $\overline{az}$ is to be expressed in Cartesian co ordinate as A = Ax$\overline{a_x}$+Ay$\overline{a_y}$+Az$\overline{a_z}$ in doing so that

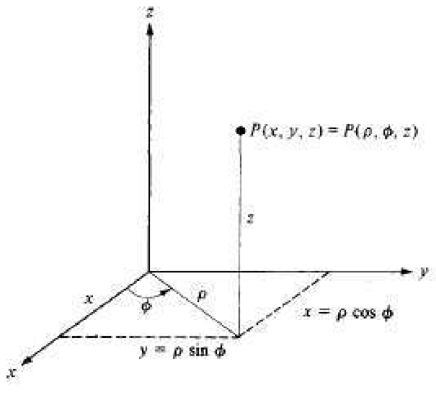


Fig 1.16: Relationship between *(x, y, z)* and *(ρ, Φ,*z).

$$x = \rho\cos\theta,\ y = \rho\sin\theta,\ z = z$$

$$\rho=\sqrt{x^2+y^2},\ \Phi=tan^{-1}\frac{y}{x},\ z = z$$

In matrix form, the transformation of vector A from *(Ax,Ay,Az)* to*(Aρ,* AΦ, Az) as

$$\begin{bmatrix} Ax \\ Ay \\ Az \end{bmatrix} = \begin{bmatrix} \cos \Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos \Phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A\rho \\ A\Phi \\ Az \end{bmatrix}$$

The inverse of the transformation *(Aρ,* AΦ, Az) to *(Ax,Ay,Az)*

$$\begin{bmatrix} A\rho \\ A\Phi \\ Az \end{bmatrix} = \begin{bmatrix} \cos \Phi & -\sin\Phi & 0 \\ \sin\Phi & \cos \Phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Ax \\ Ay \\ Az \end{bmatrix}$$

(*ii)Transformation between Cartesian and Spherical coordinates:*

The space variables (x, *y, z)* in Cartesian coordinates can be related to variables (r, θ, *Φ)* of a spherical coordinate system. Given a vector A = Ar $\overline{ar}$+$A\theta\overline{a_\theta} + A\varphi\overline{a_\varphi}$ in the spherical polar coordinate system, its component in the Cartesian coordinate system can be found out as follows

$$r=\sqrt{x^2+y^2+z^2},\ \theta=\tan^{-1}\frac{\sqrt{x^2+y^2}}{z},\quad \Phi=\tan^{-1}\frac{y}{x}$$

(or)

$$x = r \sin\theta \cos\Phi,\ y = r \sin\theta \sin\Phi,\ z = r \cos\theta$$

In matrix form, the *(Ax, Ay, Az)* → (Ar, *Aθ, AΦ)* vector transformation is performed according to

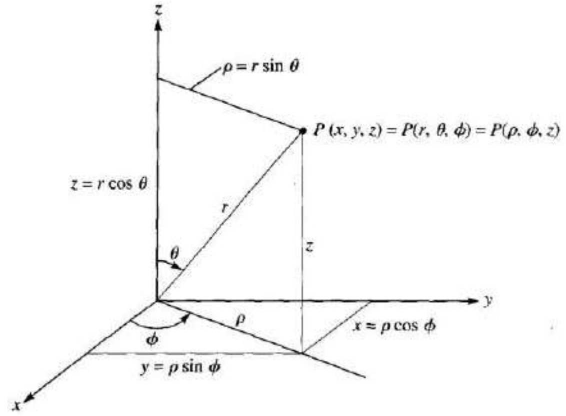


Fig1.17 Relationships between space variables *(x, y, z), (r,θ,Φ)* and (ρ, *Φ, z).*

$$\begin{bmatrix} Ar \\ A\theta \\ A\Phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos \Phi & \sin\theta\sin\Phi & \cos\theta \\ \cos\theta\cos\Phi & \cos\theta\sin\Phi & -\sin\theta \\ -\sin\Phi & \cos \Phi & 0 \end{bmatrix} \begin{bmatrix} Ax \\ Ay \\ Az \end{bmatrix}$$

The inverse transformation (Ar, *Aθ, AΦ)* → (Ax, Ay, Az) is similarly obtained,

$$\begin{bmatrix} Ax \\ Ay \\ Az \end{bmatrix} = \begin{bmatrix} \sin\theta\cos \Phi & \cos\theta\cos\Phi & -\sin\Phi \\ \sin\theta\sin\Phi & \cos\theta\sin\Phi & \cos \Phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} Ar \\ A\theta \\ A\Phi \end{bmatrix}$$

Thus we see that a vector in one coordinate system is transformed to another coordinate system through two-step process: i. Finding the component vectors ii. variable transformation.

(*iii)Transformation between cylindrical and Spherical coordinates:*

Point transformation between cylindrical and spherical coordinates is obtained using

r= $r=\sqrt{\rho^2+z^2}$, $\theta=\tan^{-1}\frac{\rho}{z}$, $\Phi=\Phi$

$\rho = r\sin\theta,\ z = r\cos\theta,\ \Phi = \Phi$

In matrix form, the (Ar, $A\theta$, $A\Phi$) to ($A\rho$, $A\Phi$, Az) vector transformation is performed according to

$$\begin{bmatrix} Ar \\ A\theta \\ A\Phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A\rho \\ A\Phi \\ Az \end{bmatrix}$$

In matrix form, the ($A\rho$, $A\Phi$, Az) to (Ar, $A\theta$, $A\Phi$) vector transformation is performed according to

$$\begin{bmatrix} A\rho \\ A\Phi \\ Az \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} Ar \\ A\theta \\ A\Phi \end{bmatrix}$$

1.3. VECTOR DIFFERENTIAL OPERATOR

The del operator, written ∇, is the vector differential operator.

In Cartesian coordinates,

$$\nabla = \frac{\partial}{\partial x}\overline{a_x} + \frac{\partial}{\partial y}\overline{a_y} + \frac{\partial}{\partial z}\overline{a_z}$$

This vector differential operator, otherwise known as the *gradient operator,* is not a vector in itself, but when it operates on a scalar function, for example, a vector ensues. The operator is useful in

1. The gradient of a scalar ∇, written, as W
2. The divergence of a vector A, written as ∇ • A
3. The curl of a vector A, written as ∇ X A
4. The Laplacian of a scalar *V,* written as ∇^2 V

Each of these will be defined in detail in the subsequent sections. Before do that, it is appropriate to obtain expressions for the del operator V in cylindrical and spherical coordinates.

∇ in cylindrical coordinates:

To obtain ∇ in terms of ρ, Φ, z and

$$\rho = \sqrt{x^2 + y^2},\ \Phi = \tan^{-1}\frac{y}{x}$$

Hence

$$\frac{\partial}{\partial x} = \cos\Phi\frac{\partial}{\partial \rho} - \frac{\sin\Phi}{\rho}\frac{\partial}{\partial \Phi}$$

$$\frac{\partial}{\partial y} = \sin\Phi\frac{\partial}{\partial \rho} + \frac{\cos\Phi}{\rho}\frac{\partial}{\partial \Phi}$$

$$\nabla = \overline{a}_\rho\frac{\partial}{\partial \rho} + \overline{a}_\varphi\frac{1}{\rho}\frac{\partial}{\partial \varphi} + \overline{a}_z\frac{\partial}{\partial z}$$

∇ in spherical coordinates:

$$r = \sqrt{x^2 + y^2 + z^2},\ \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z},\quad \Phi = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial}{\partial x} = \sin\theta\cos\Phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\varphi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\Phi}{\rho}\frac{\partial}{\partial \Phi}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\Phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\varphi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\Phi}{\rho}\frac{\partial}{\partial \Phi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\nabla = \frac{\partial}{\partial r}\bar{a}_r + \bar{a}_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \bar{a}_\Phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

The unit vectors are placed to the right of the differential operators because the unit vectors depend on the angles.

1.4. CURL , DIVERGENCE, GRADIENT

1.4.1. Definition Of Curl

The curl of A is an axial (or rotational) vector whose magnitude is the maximum circulation of A per unit area as the area lends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.

The circulation of a vector field around a closed path is given by curl of a vector. Mathematically it is defined as,

$$(curl\ of\ H)N = \lim_{\Delta S_N \to 0}\left(\frac{\oint H.dl}{\Delta S_N}\right)$$

Where ΔS_N – Planar area enclosed by the closed line integral

The subscript N indicates that the components of the curl is that component which is normal to the surface enclosed by the closed path. The maximum circulation of h per unit area tends to zero whose direction is normal to the surface is called curl of H.

Symbolically represented as Curl H = ∇ x H

In rectangular co ordinate system x,y and z of the curl H are given by,

$$\text{Curl } H \Rightarrow \nabla \times H = \left[\frac{\partial Hz}{\partial y} - \frac{\partial Hy}{\partial z}\right]\overline{a_x} + \left[\frac{\partial Hx}{\partial z} - \frac{\partial Hz}{\partial x}\right]\overline{a_y} + \left[\frac{\partial Hy}{\partial x} - \frac{\partial Hx}{\partial y}\right]\overline{a_z}$$

$$\nabla \times \bar{H} = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

In cylindrical co ordinates

$$\text{Curl } H \Rightarrow \nabla \times H = \left[\frac{1}{\rho}\frac{\partial Hz}{\partial \varphi} - \frac{\partial H\varphi}{\partial z}\right]\overline{a_\rho} + \left[\frac{\partial H\rho}{\partial z} - \frac{\partial Hz}{\partial \rho}\right]\overline{a_\varphi} + \left[\frac{1}{\rho}\frac{\partial \rho H\varphi}{\partial \rho} - \frac{1}{\rho}\frac{\partial H\rho}{\partial \varphi}\right]\overline{a_z}$$

$$\nabla \times \bar{H} = \frac{1}{r}\begin{vmatrix} a_r & ra_\varphi & a_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ H_r & H_\varphi & H_z \end{vmatrix}$$

In spherical co ordinates

$$\text{Curl } H \Rightarrow \nabla \times H = \frac{1}{r\, sin\theta}\left[\frac{\partial H\varphi\, sin\theta}{\partial \theta} - \frac{\partial H\theta}{\partial \varphi}\right]\overline{a_r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial Hr}{\partial \varphi} - \frac{\partial rH\varphi}{\partial r}\right]\overline{a_\theta} + \frac{1}{r}\left[\frac{\partial rH\theta}{\partial r} - \frac{\partial Hr}{\partial \theta}\right]\overline{a_\theta}$$

$$\nabla \times \bar{H} = \frac{1}{r^2\, sin\theta}\begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

Thus maximum circulation of $\bar{F}$ per unit area tends to zero whose direction is normal to the surface is called curl of $\bar{F}$.

Note the following properties of the curl:

- The curl of a vector field is another vector field.
- The curl of a scalar field *V*, ∇ X *V*, makes no sense.

- ∇X(A + B) = ∇ X A + ∇ X B
- ∇ X (A X B) = A(∇ • B) - B(∇ • A) + (B • ∇A - (A • ∇)B
- ∇ X (VA) = V∇ X A + ∇V X A
- The divergence of the curl of a vector field vanishes, that is, ∇ • (∇ X A) = 0.
- The curl of the gradient of a scalar field vanishes, that is, ∇ X ∇V = 0.

1.4.2. Divergence

Divergence of vector field D at a point P is the outward flux per unit volume as the volume shrinks about point P. i.e, $\lim \quad \Delta V \to 0$ representing differential volume element at point P.

Divergence of D = div D = $\lim_{\Delta V \to 0} \oint_S \frac{A.ds}{\Delta V}$

Div A = ∇.D – divergence of D

∇= vector operator = $\frac{\partial}{\partial x} D_x + \frac{\partial}{\partial y} D_y + \frac{\partial}{\partial z} D_z$

The divergence of vector flux density A is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero. i.e, ∇. A = 0. The positive divergence for any vector quantity indicates a source of that vector quantity at a point. Similarly a negative divergence indicates a sink.

∴ ∇.D = $\frac{\partial}{\partial x} D_x + \frac{\partial}{\partial y} D_y + \frac{\partial}{\partial z} D_z$ This is the divergence of D in Rectangular system.

In cylindrical system

$$\nabla . D(\text{div } D) = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D\rho) + \frac{1}{\rho}\frac{\partial D\varphi}{\partial \varphi} + \frac{\partial Dz}{\partial z}$$

In spherical System

$$\nabla . D(\text{div } D) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 Dr) + \frac{1}{r \sin\theta}\frac{\partial(\sin\theta\, D_\theta)}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial D_\varphi}{\partial \varphi}$$

The vector field having its divergence zero is called solenoidal field. ∇. $\bar{A}$ = 0 for $\bar{A}$ to be solenoidal

Note the following properties of the divergence of a vector field:

- It produces a scalar field (because scalar product is involved).
- The divergence of a scalar V, div V, makes no sense.
- ∇ • (A + B) = ∇ • A + ∇ • B
- ∇ • (VA) = V∇ • A + A • ∇V

1.4.3. Gradient

The gradient of a scalar is a vector. Consider V be the unique function of x,y,z coordinates in rectangular system. This is the scalar function and denoted as V(x,y,z). The vector operates in Cartesian system denoted ∇ called del.

The vector operator, ∇ = $\frac{\partial}{\partial x}\overline{a_x} + \frac{\partial}{\partial y}\overline{a_y} + \frac{\partial}{\partial z}\overline{a_z}$

The scalar operator, ∇.V = $\frac{\partial V}{\partial x}\overline{a_x} + \frac{\partial v}{\partial y}\overline{a_y} + \frac{\partial v}{\partial z}\overline{a_z}$ i.e,

∇.V = grad V

In rectangular co ordinate system,

$$\nabla V = \frac{\partial V}{\partial x}\overline{a_x} + \frac{\partial v}{\partial y}\overline{a_y} + \frac{\partial v}{\partial z}\overline{a_z}$$

In cylindrical system,

$$\nabla V = \frac{\partial v}{\partial \rho}\overline{a_\rho} + \frac{1}{\rho}\frac{\partial v}{\partial \varphi}\overline{a_\varphi} + \frac{\partial v}{\partial z}\overline{a_z}$$

In spherical system,

$$\nabla V = \frac{\partial v}{\partial r}\overline{a_r} + \frac{1}{r}\frac{\partial v}{\partial \theta}\overline{a_\theta} + \frac{1}{r\sin\theta}\frac{\partial v}{\partial \varphi}\overline{a_\varphi}$$

Properties of Gradient:

- The gradient ∇V gives the maximum rate of change of V per unit distance.
- The gradient ∇V always indicates the direction of the maximum rate of change of V

1.5. DIVERGENCE THEOREM

Statement

The volume integral of the divergence of a vector field over a volume is equal to the surface integral of the normal component of this vector over the surface bounding this volume.

$$\iiint_V \nabla.\vec{A}\,dv = \oint\!\!\oint \vec{A}.ds$$

Proof: Let us consider a volume V enclosed by a surface S . Let us subdivide the volume in large number of cells. Let the k^{th} cell has a volume V and the corresponding surface is denoted by Sk. Interior to the volume, cells have common surfaces. Outward flux through these common surfaces from one cell becomes the inward flux for the neighboring cells. Therefore when the total flux from these cells is considered, get the net outward flux through the surface surrounding the volume. The divergence theorem can be applied to any field but partial derivatives of that vector must exist. The divergence theorem is applied to the flux density. Both sides of the divergence theorem give the net charge enclosed by the closed surface. It converts the surface integral into volume integral provided that the closed surface encloses certain volume. It is applied with Gauss law.

The divergence of any vector A is given by

$$\nabla.\vec{A} = \frac{\partial Ax}{\partial x} + \frac{\partial Ay}{\partial y} + \frac{\partial Az}{\partial z},$$

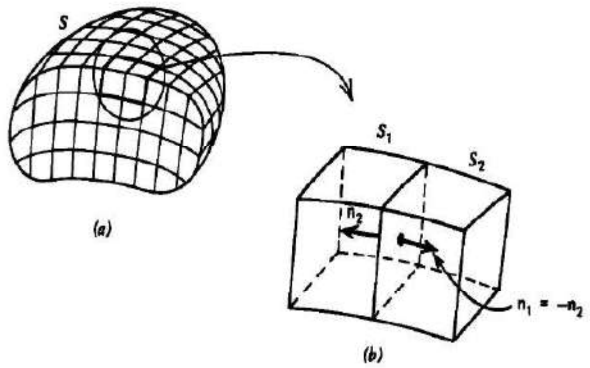


Fig 1.18 Divergence theorem

Take volume Integral on both sides,

$$\iiint_V \nabla.\vec{A}\, dv = \iiint_V \left[\frac{\partial Ax}{\partial x} + \frac{\partial Ay}{\partial y} + \frac{\partial Az}{\partial z}\right] dxdydz$$

Consider an element volume in x direction,

$$\iiint_V \frac{\partial Ax}{\partial x}\, dx\, dy\, dz = \iint_V \left[\int \frac{\partial Ax}{\partial x}.dx\right] dy\, dz$$

But,

$$\int_{x1}^{x2} \frac{\partial Ax}{\partial x} dx = Ax_2 - Ax_1 = Ax$$

Then,

$$\iiint_V \frac{\partial Ax}{\partial x}\, dx\, dy\, dz = \iint_s Ax\, dy\, dz = \iint_s Ax\, ds_x$$

Where dsx = x component of surface area ds.

$$\iiint_V \frac{\partial Ay}{\partial y}\, dx\, dy\, dz = \iint_s Ay\, dsy$$

$$\iiint_V \frac{\partial Az}{\partial z}\, dx\, dy\, dz = \iint_s Az\, dsz$$

$$\therefore \iiint_V \nabla.\vec{A}\, dv = \iiint_V \left[\frac{\partial Ax}{\partial x} + \frac{\partial Ay}{\partial y} + \frac{\partial Az}{\partial z}\right] dxdydz$$

$$= \iint_s (As\, dsx + Aydsy + Az\, dsz)$$

$$= \oiint \vec{A}.ds$$

$$\therefore \iiint_V \nabla.\vec{A}\, dv = \oiint \vec{A}.ds$$

Hence proved.

1.6. STOKES THEOREM

Statement

Stokes's theorem states that the circulation of a vector field A around a (closed) path is equal to the surface integral of the curl of A over the open surface *S* bounded by L provided that A and ∇ X A are continuous S.

The line integral of a vector around a closed path is equal to surface integral of the normal component of its equal to the integral of the normal component of its curl ever any closed surface.

$$\oint H.dl = \iint_s \nabla \times H\, ds$$

Proof Consider an arbitrary surface. This is broken up into incremental surfaces of areas ∇s as shown in Fig.

If H is any field vector, then by definition of the curl to one of these incremental surfaces. The Stokes theorem relates the line integral. It states that, The line integral of H around a closed path L is equal to the integral of curl of H over the open surface S enclosed by the closed path L. Mathematically it is expressed as,

$$\oint_L \boldsymbol{H.dl} = \int_s (\nabla \boldsymbol{x H})\ \boldsymbol{ds}$$ dl- Perimeter of total surface S

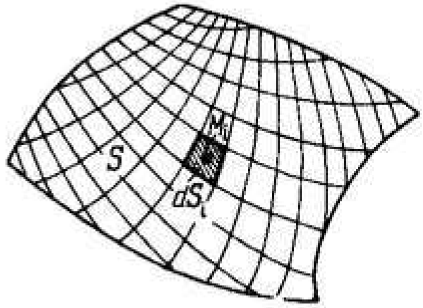


Fig 1.19 Stokes theorem

Stokes theorem is applicable only when H and ∇xH are continuous on the surface S. The path L and open surface S enclosed by the path L.

$$\frac{\oint H.dl \nabla s}{\nabla s} = (\nabla\, x\, H) N$$

Where N indicates normal to the surface and dl∇s indicate that the closed path of an incremental area ∇s.

The curl of H normal to the surface can be written as

$$\frac{\oint H.dl \nabla s}{\nabla s} = (\nabla\, x\, H).aN,$$

$$\frac{\oint H.dl \nabla s}{\nabla s} = (\nabla\, x\, H).aN \nabla s$$

$$=(\nabla xH).\ \nabla s.$$

Where a_N is a unit vector normal to ∇s.

The closed integral for whole surface S is given by the surface s integral of the normal component of curl H.

$$\oint H.dl = \iint_s \nabla \times \mathrm{H.ds}$$

∴ Hence proved

1.7. COULOMBS LAW

Coulomb's law is an experimental law formulated in 1785 by the French colonel, Charles Augustin de Coulomb. It deals with the force a point charge exerts on another point charge. By *a point charge* we mean a charge that is located on a body whose dimensions are much smaller than other relevant dimensions. For example, a collection of electric charges on a pinhead may be regarded as a point charge. Charges are generally measured in coulombs (C).

Coulomb's law states that the force f between two point charges (Q1 and Q2 is):

- Along the line joining them
- Directly proportional to the product Q1Q2of the charges
- Inversely proportional to the square of the distance *R* between them.

Point charge is a hypothetical charge located at a single point in space. It is an idealized model of a particle having an electric charge.

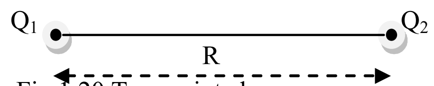


Fig 1.20 Two point charges

Consider two point charges Q_1 and Q_2 as shown in figure separated by a distance R. The force acting along the line joining Q_1 and Q_2. The force exerted between them is repulsive if the charges a same polarity. While it is attractive if the charges are of different polarity.

Mathematically,

$$\mathbf{F} \propto \frac{\mathbf{Q_1 Q_2}}{\mathbf{R^2}}$$

Q_1 and Q_2 are expressed in Coulombs (C) and R is in meters. Force F is in Newton's (N)

$$F = K\frac{Q_1Q_2}{R^2}$$

Where k is the proportionality constant.

$$K = \frac{1}{4\pi\varepsilon}$$

$$F = \frac{1}{4\pi\varepsilon}\frac{Q_1Q_2}{R^2}$$

ε is called the permittivity of free space. $\varepsilon = \varepsilon_0 \varepsilon_r$

ε_0 – permitivity in free space = 8.854 $\times 10^{-12}$ = $\frac{1}{36\pi}\mathbf{x10^{-19}F/m}$

ε_r – Relative permitivity

(If the charges are any other dielectric medium, we will use instead where is called the relative permittivity or the dielectric constant of the medium).

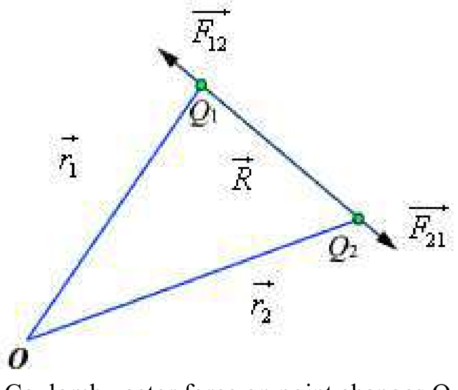


Fig 1.21 Coulomb vector force on point changes Qy and Q2.

Consider the two point charges Q_1 and Q_2 located at the points having r_1 and r_2. The force exerted by Q_1 and Q_2 act along the direction R_{12}. Where a_{12} is the unit vector form can be expressed as,

$$\overline{F_{12}} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{{R_{12}}^2}\overline{a_{12}}$$

$\overline{a_{12}}$ - unit vector along

$$R_{12} = \frac{vector}{magnitude\ of\ vector}$$

$$\overline{a_{12}} = \frac{\overline{R_{12}}}{|R_{12}|} = \frac{\overline{r_2 - r_1}}{|R_{12}|} = \frac{\overline{r_2 - r_1}}{|\overline{r_2 - r_1}|}$$

It is worthwhile to note that

The force expressed by coulombs law is a mutual force for each of the two charges experiences a force of the same magnitude but opposite direction

$$\overline{F_1} = -\overline{F}_2 = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{{R_{12}}^2}\overline{a_{12}} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{{R_{21}}^2}\overline{a_{21}}$$

Coulombs law is linear if any one charge is increased n times then the force exerted also increased by n times. Then the force exerted also increased by n times $\overline{F_1} = -\overline{F}_2$ **then** $n\overline{F_1} = -n\overline{F}_2$ n- scalar

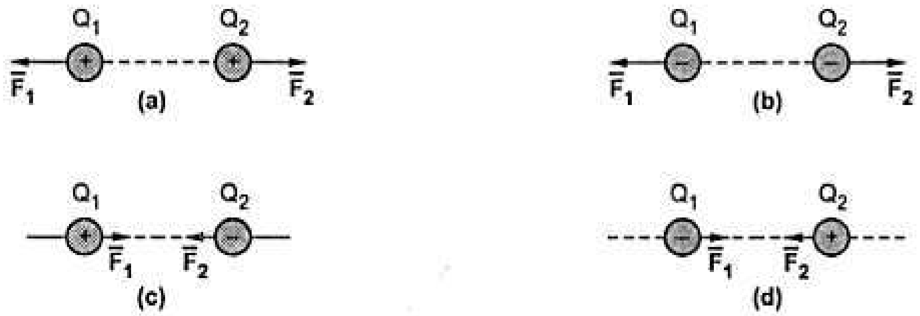


Fig.1.22. Signs of Q1 and Q2

- Like charges (charges of the same sign) repel each other while unlike charges attract.
- The distance *R* between the charged bodies Q1 and Q2 must be large compared with the linear dimensions of the bodies; that is, Q1 and Q2 must be point charges.
- Q1 and Q2 must be static (at rest).
- The signs of Q1 and Q2 must be taken into account.

Steps to Solve Problems on Coulomb's Law

Step 1 :	Obtain the position vectors of the points where the charges are located.
Step 2 :	Obtain the unit vector along the straight line joining the charges. The direction is towards the charge on which the force exerted is to be calculated.
Step 3 :	Using Coulomb's law, express the force exerted in the vector form.
Step 4 :	If there are more charges, repeat steps 1 to 3 for each charge exerting a force on the charge under consideration.
Step 5 :	Using the principle of superposition, the vector sum of all the forces calculated earlier is the resultant force, exerted on the charge under consideration.

1.8. ELECTRIC FIELD INTENSITY

The electric field intensity (or electric field strength) K is the force per unit charge when placed in the electric field.

Thus $$E=\lim_{Q\to 0}\frac{F}{Q}$$

or simply

$$E=\frac{F}{Q}$$

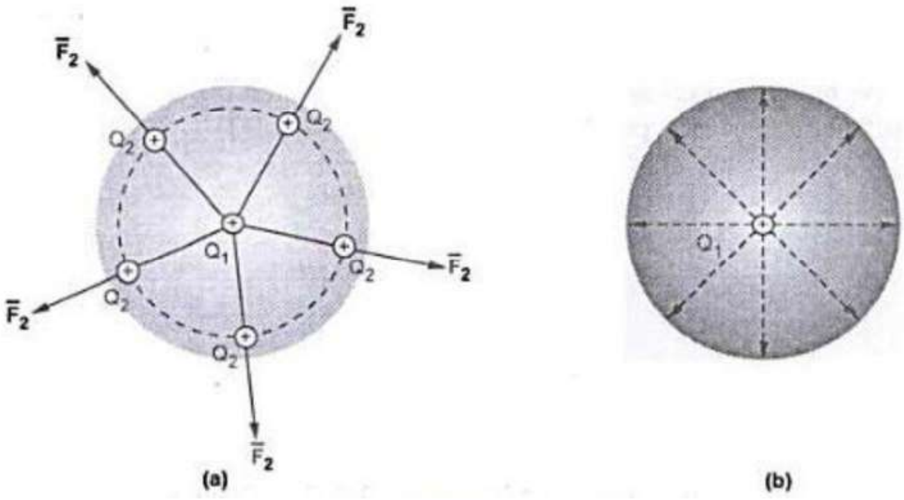


Fig1.23. Electric field intensity

The electric field intensity E is obviously in the direction of the force F and is measured in Newton/coulomb or volts/meter. The electric field intensity at point r due to a point charge located at r1 is readily obtained using

$$\bar{E}=\frac{\mathbf{Q}}{4\pi\varepsilon_0\mathbf{R_{12}}^2}\overline{\mathbf{a}_{12}}=\frac{\mathbf{Q}}{4\pi\varepsilon_0\mathbf{R_{12}}^2}\frac{\bar{\mathbf{r}}-\bar{\mathbf{r}}_1}{|\bar{\mathbf{r}}-\bar{\mathbf{r}}_1|}$$

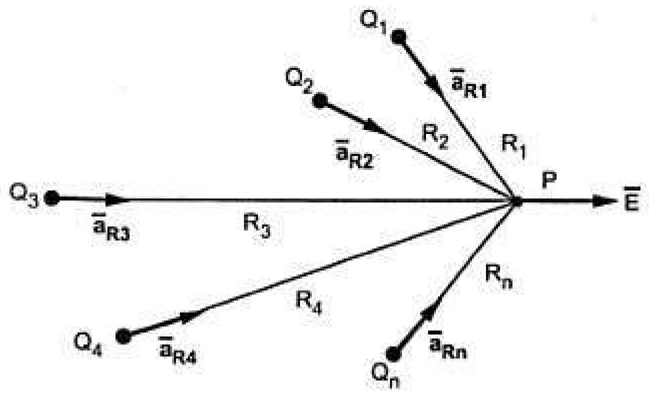


Fig.1.24. $\bar{E}$ due to n number of charges

For *N* point charges Q1 Q2,. . ., QN located at r1, r2,. . . , *rN,* the electric field intensity at point r is obtained from

$$\bar{\mathbf{E}} = \frac{\mathbf{Q1}}{\mathbf{4\pi\varepsilon_0}}\frac{\bar{\mathbf{r}}-\bar{\mathbf{r}}_1}{|\bar{\mathbf{r}}-\bar{\mathbf{r}}_1|^3} + \frac{\mathbf{Q1}}{\mathbf{4\pi\varepsilon_0}}\frac{\bar{\mathbf{r}}-\bar{\mathbf{r}}_2}{|\bar{\mathbf{r}}-\bar{\mathbf{r}}_2|^3} + \cdots + \frac{\mathbf{Q1}}{\mathbf{4\pi\varepsilon_0}}\frac{\bar{\mathbf{r}}-\bar{\mathbf{r}}_N}{|\bar{\mathbf{r}}-\bar{\mathbf{r}}_N|^3}$$

or

$$\bar{E} = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{N}\frac{Q_k(\bar{r}-\bar{r}_k)}{|\bar{r}-\bar{r}_k|^3}$$

1.9. POINT, LINE, SURFACE AND VOLUME CHARGE DISTRIBUTIONS

The forces and electric fields due to point charges, which are essentially charges occupying very small physical space. It is also possible to have continuous charge distribution along a line, on a surface, or in a volume as illustrated in Figure. It is customary to denote the line charge density, surface charge density, and volume charge density by ρL (in C/m), ρs (in C/m2), and ρv (in C/m3), respectively. These must not be confused with ρ (without subscript) used for radial distance in cylindrical coordinates. The charge element dQ and the total charge *Q* due to these charge distributions are obtained.

Line charge density is denoted as ρ_L. it is the ratio of total charge in coulomb to total length in meters

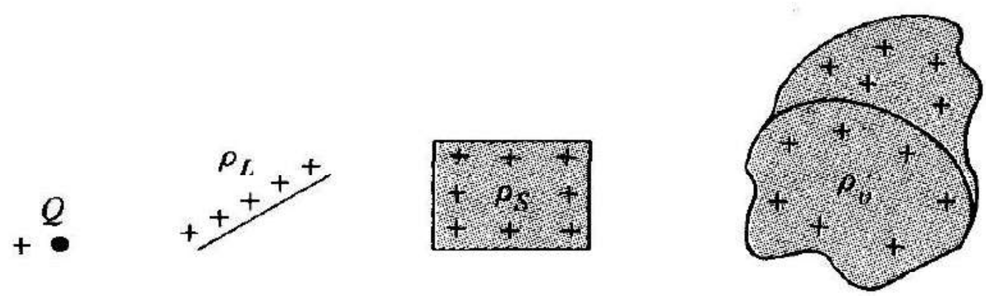
$$\rho_l = \frac{TOTAL\ CHARGE\ IN\ COLUME}{TOTAL\ LENGTH\ OF\ THE\ LINE}(C/M)$$

Surface charge density is denoted as ρs. it is the ratio of total charge in coulomb to total Surface area in meter^2

$$\rho_s = \frac{TOTAL\ CHARGE\ IN\ COLUME}{TOTAL\ \text{SURFACE AREA}}(C/M^2)$$

Volume charge density is denoted as ρ_L. it is the ratio of total charge in coulomb to total Volume in meter^3

$$\rho_v = \frac{TOTAL\ CHARGE\ IN\ COLUME}{TOTAL\ \text{VOLUME}}(C/M^3)$$



Point charge Line charge Surface charge Volume charge

Fig 1.25. Various charge distributions and charge elements.

$$dQ = \rho L dL \rightarrow Q = \int_L \rho_L dL \qquad \text{(line charge)}$$

$$dQ = \rho S dS \rightarrow Q = \int_S \rho_S dS \qquad \text{(surface charge)}$$

$dQ = \rho V dS \rightarrow Q = \int_V \rho_V dV$ (volume charge)

The electric field intensity due to each of the charge distributions ρL, ρs, and ρv may be regarded as the summation of the field contributed by the numerous point charges making up the charge distribution. Thus by replacing Q , with charge element

$dQ = \rho L\, dl, \rho s\, dS, or\ \rho v\, dv\ and$

Integrating,

$$E = \int_L \frac{\rho_L dL}{4\pi\varepsilon_0 {R_{12}}^2}\,\overline{a_{12}}(line\ charge)$$

$$E = \int_S \frac{\int_S \rho_S dS}{4\pi\varepsilon_0 {R_{12}}^2}\,\overline{a_{12}}\ (surface\ charge)$$

$$E = \int_V \frac{\rho_V dV}{4\pi\varepsilon_0 {R_{12}}^2}\,\overline{a_{12}}(volume\ charge)$$

It should be noted that R^2 and $\overline{a_{12}}$ vary as the integrals are evaluated.

i. Point Charge

The point is very small compared to the surrounding surface area.

$$E = \frac{F}{Q}$$

$$E = \frac{Q}{4\pi\varepsilon_0 {R_{12}}^2}\overline{\boldsymbol{a_{12}}}$$

ii. A Line Charge

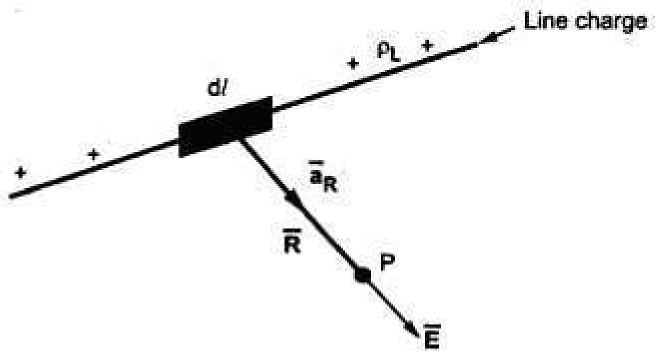


Fig.1.26 line charge

Consider a line charge with uniform charge density *pL* extending from *A* to *B* along the z-axis as shown in Figure. The charge element *dQ* associated with element *dl* = *dz* of the line is

$$dQ = \rho_L dL = \rho_L dZ$$

And hence the total charge *Q* is

$$Q = \int_{ZA}^{ZB} \rho_L dL$$

$$E = \int_S \frac{\rho_L dL}{4\pi\varepsilon_0 {R_{12}}^2} a_{R12}$$

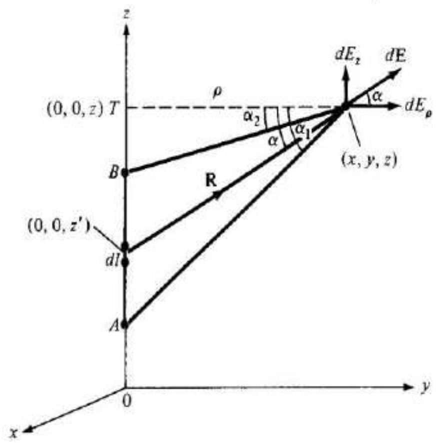


Fig 1.27. Evaluation of the E field due to line Charge

iii. A Surface Charge

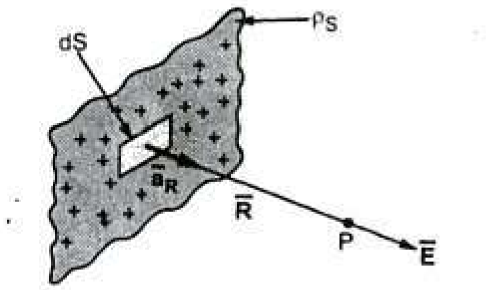


Fig.1.28 surface charge

Consider an infinite sheet of charge in the xy-plane with uniform charge density *ps.* The charge associated with an elemental area *dS* is

$$dQ = \rho_S dS = \rho s\ \rho\ d\Phi\ d\rho$$

and hence the total charge is

$$Q = \int_{S1}^{S2} \rho_S dS$$

The contribution to the E field at point P(0, 0, *h)* by the elemental surface 1 shown in Figure,

$$E = \int_S \frac{\rho_S dS}{4\pi\varepsilon_0 {R_{12}}^2} \overline{a_{12}}$$

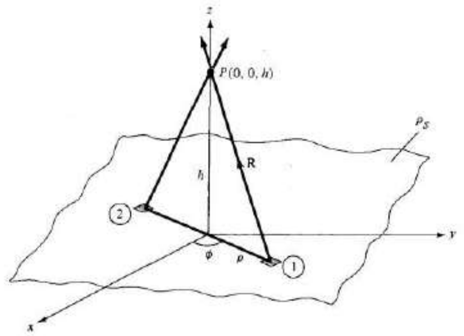


Fig1.29. Evaluation of the E field due to an infinite sheet of charge

iv. A Volume Charge

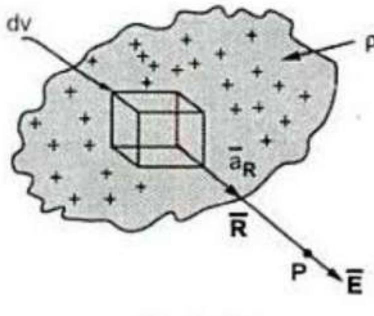


Fig.1.30. volume charge

Let the volume charge distribution with uniform charge density *pv* be as shown in Figure. The charge *dQ* associated with the elemental volume *dv* is

$$dQ = \rho v\ dv$$

and hence the total charge in a sphere of radius *a* is

$$Q = \int_{V1} \rho_V \mathrm{dV} = \rho_V \int_{V1} \mathrm{dV}$$

The electric field *dE* at P(0, 0, *z)* due to the elementary volume charge is,

$$\bar{E} = \int_V \frac{\rho_V \mathrm{dV}}{4\pi\varepsilon_0 R_{12}{}^2}\overline{a_{12}}$$

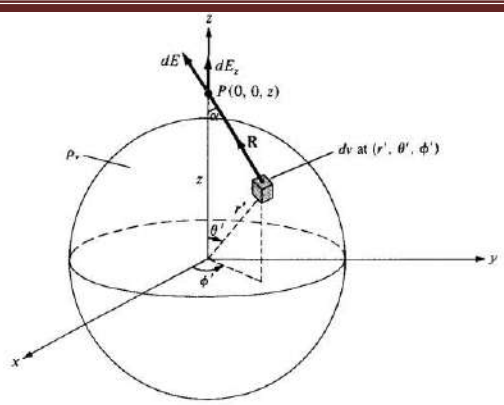


Fig1.31. Evaluation of the E field due to a volume charge distribution.

1.10. ELECTRIC FLUX DENSITY

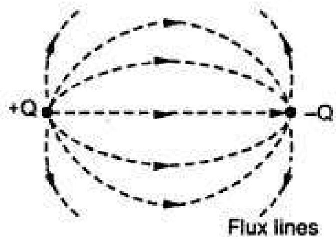


Fig.1.32 flux line

The flux due to the electric field E can be calculated using the general definition of flux.

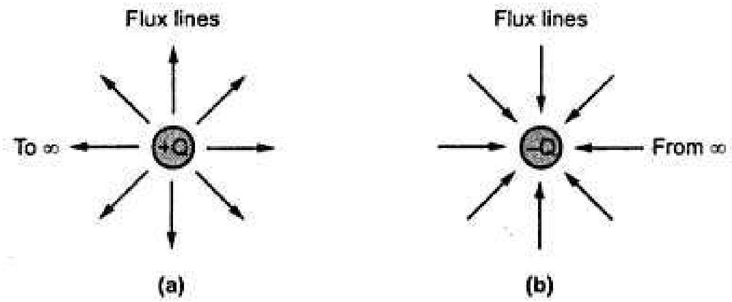


Fig.1.33 flux line

For practical reasons, however, this quantity is not usually considered as the most useful flux in electrostatics. Also the electric field intensity is dependent on the medium in which the charge is placed. Suppose a new vector field D independent of the medium is defined by

$$D = \varepsilon E$$

$$D = \frac{Q}{4\pi {R_{12}}^2}$$

Electric flux f in terms of Ψ,

$$\Psi = \int D.dS$$

In SI units, one line of electric flux emanates from +1 C and terminates on - 1 C. Therefore, the electric flux is measured in coulombs. Hence, the vector field D is called the *electric flux density* and is measured in coulombs per square meter. For historical reasons, the electric flux density is also called *electric displacement.*

A volume charge distribution

$$\overline{D} = \int_V \frac{\rho_V dV}{4\pi {R_{12}}^2}\overline{a_{12}}$$

D is a function of charge and position only; it is independent of the medium.

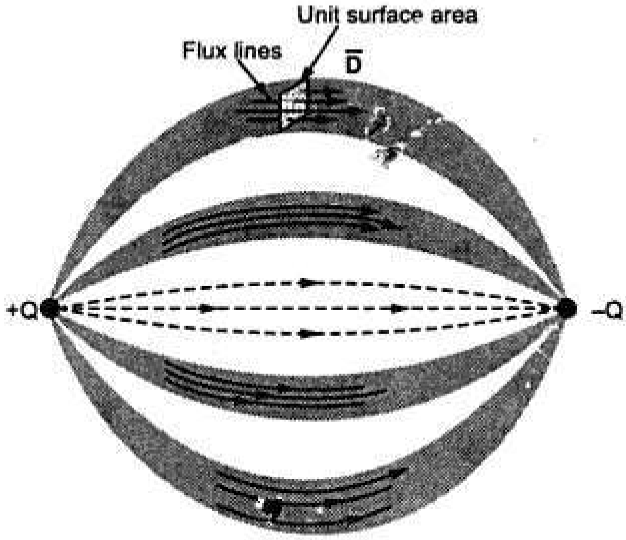


Fig.1.34 Electric Flux density

Consider a unit surface area as shown in the Fig. The number of flux lines are passing through this surface area.

The net flux passing normal through the unit surface area is called the **electric flux density**. It is denoted as $\overline{\mathbf{D}}$. It has a specific direction which is normal to the surface area under consideration hence it is a vector field.

Consider a sphere with a charge Q placed at its centre. There are no other charges present around. The total flux distributes radially around the charge is $\psi = Q$. This flux distributes uniformly over the surface of the sphere.

Now, ψ = Total flux

While, S = Total surface area of sphere

Then electric flux density is defined as,

$$D = \frac{\psi}{S} \text{ in magnitude}$$

As ψ is measured in coulombs and S in square metres, the units of D are C/m^2. This is also called **displacement flux density** or **displacement density.**

Vector Form of Electric Flux Density

Consider the flux distribution, due to a certain charge in the free space as shown in the Fig.

Consider the differential surface area dS at point P. The flux crossing through this differential area is $d\psi$. The direction of $\overline{D}$ is same as that of direction of flux lines at that point. The differential area and flux lines are at right angles to each other at point P. Hence the direction of $\overline{D}$ is also normal to the surface area, in the direction of unit vector $\overline{a}_n$ which is normal to the surface area dS. Near point P, all the lines of flux $d\psi$ are having direction of that of $\overline{a}_n$ as the differential area dS is very small. Hence the flux density $\overline{D}$ at the point P can be represented in the vector form as,

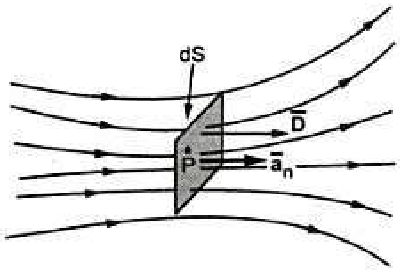


Fig. Flux through dS

$$\overline{D} = \frac{d\psi}{dS}\,\overline{a}_n \quad C/m^2$$

where $d\psi$ = Total flux lines crossing normal through the differential area dS

dS = Differential surface area

$\overline{a}_n$ = Unit vector in the direction normal to the differential surface area

1.11. GAUSS LAW AND ITS APPLICATIONS

Gauss Law

Gauss's law states that the total electric flux Ψ through any *closed* surface is equal to the total charge enclosed by that surface. Ψ = Qenc

$$\Psi = \int d\Psi = \int D.dS$$

Total charge enclosed $Q = \int_{V1} \rho_V \, dV$

$$Q = \int D.dS = \int_V \rho_V dV$$

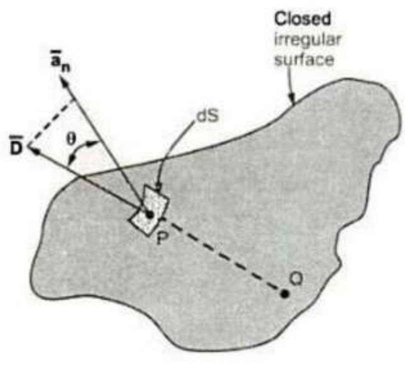


Fig 1.35. Flux through irregular surface area

Gauss's law provides an easy means of finding E or D for symmetrical charge distributions such as a point charge, an infinite line charge, an infinite cylindrical surface charge, and a spherical distribution of charge. A continuous charge distribution has rectangular symmetry if it depends only on *x* (or *y* or z), cylindrical symmetry if it depends only on ρ or, spherical symmetry if it depends only on r (independent of θ and Φ). It must be stressed that whether the charge distribution is symmetric or not, Gauss's law always holds.

Applications

The procedure for applying Gauss's law to calculate the electric field involves first knowing whether symmetry exists. Once symmetric charge distribution exists, we construct a mathematical closed surface (known as a *Gaussian surface).* The surface is chosen such that D is normal or tangential to the Gaussian surface. When D is normal to the surface, D • *dS* = *D dS* because D is constant on the surface. When D is tangential to the surface, D • *dS* = 0. Thus we must choose a surface that has some of the symmetry exhibited by the charge distribution.

i. ***Point Charge (Proof of gauss law)***

Suppose a point charge *Q* is located at the origin. To determine D at a point *P,* it is easy to see that choosing a spherical surface containing *P* will satisfy symmetry conditions.

Thus, a spherical surface centered at the origin is the Gaussian surface in this case and is shown in Figure1.18. Since D is everywhere normal to the Gaussian surface, that is, D = $D_r a_r$ applying Gauss's law (Ψ = Q enclosed) gives

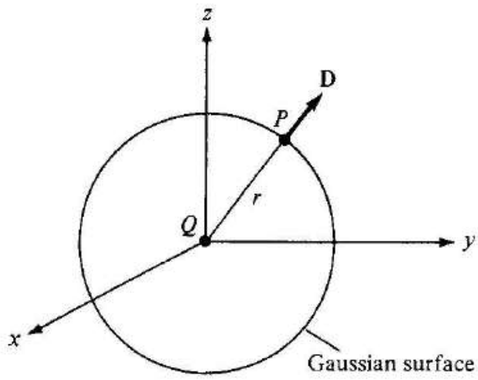


Fig1.36. Gaussian surface about a point charge.

$$Q = \oint D_r . dS = D_r . \oint dS$$

where $\S\, dS = \int_0^{2\pi} \int_0^{\pi} r2 \sin\theta \, d\theta \, d\varphi = 4\pi r^2$ is the surface area of the Gaussian surface. Thus

$$D = \frac{Q}{4\pi R^2} \overline{a_R}$$

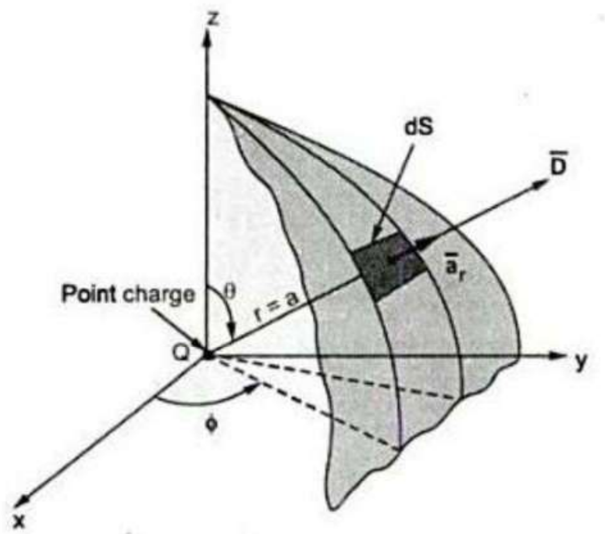


Fig 1.37 Proof of gauss law

ii. ***Infinite Line Charge***

Suppose the infinite line of uniform charge *pL* C/m lies along the z-axis. To determine D at a point *P,* choose a cylindrical surface containing *P* to satisfy symmetry condition as shown in Figure. D is constant on and normal to the cylindrical Gaussian surface; that is, D = *Dρaρ.* If we apply Gauss's law to an arbitrary length *l* of the line

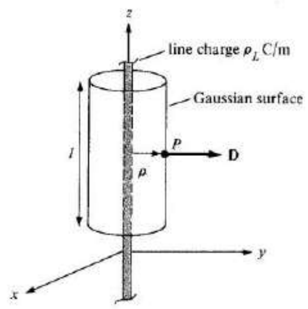


Fig 1.38. Gaussian surface about an infinite line charge

$$Q = \oint D.dS = D_r.\oint dS$$

$\oint dS = 2\pi\rho l$ is the surface area of the Gaussian surface. Note that $\int$ D $\bullet$ dS evaluated on the top and bottom surfaces of the cylinder is zero since D has no z-component; that means that D is tangential to those surfaces. Thus

$$D = \frac{\rho l}{2\pi\rho}\overline{a_\rho}$$

iii. ***Infinite Sheet of Charge***

Consider the infinite sheet of uniform charge ρ*s* C/m^2 lying on the *z* = 0 plane. To determine D at point *P,* we choose a rectangular box that is cut symmetrically by the sheet of charge and has two of its faces parallel to the sheet as shown in Figure. As D is normal to the sheet, D = $D_z a_z$, and applying Gauss's law gives

$$\rho s \int dS = Q = \oint D . dS = D_Z . [\int ds_{top} + \int dS_{\text{bottom}}]$$

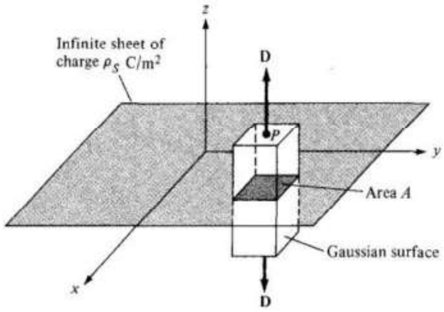


Fig 1.38. Gaussian surface about an infinite line sheet of charge.

Note that D • *dS* evaluated on the sides of the box is zero because D has no components along a_x and a_y. If the top and bottom area of the box each has area *A,*

$$\rho_s A = D_Z(A + A)$$

and thus

$$\bar{\mathbf{D}} = \frac{\boldsymbol{\rho s}}{\mathbf{2}} \overline{\mathbf{a_Z}}$$

$$\textbf{And } \bar{\mathbf{E}} = \frac{\bar{\mathbf{D}}}{\boldsymbol{\varepsilon_0}} = \frac{\boldsymbol{\rho s}}{\mathbf{2}\boldsymbol{\varepsilon_0}} \overline{\mathbf{a_Z}}$$

iv. ***Uniformly Charged Sphere***

Consider a sphere of radius *a* with a uniform charge ρv C/m^3. To determine D everywhere, construct Gaussian surfaces for cases $r \geq a$ and $r \leq a$ separately. Since the charge has spherical symmetry, it is obvious that a spherical surface is an appropriate Gaussian surface.

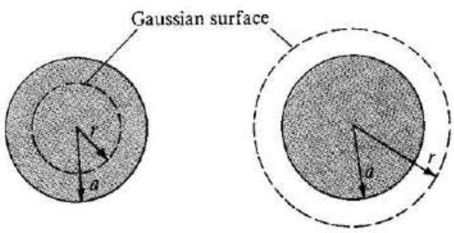


Fig 1.39. Gaussian surface for a uniformly charged sphere when: (a) $r \geq a$ and (b) $r \leq a$.

For $r \leq a$, the total charge enclosed by the spherical surface of radius r,

$$\text{Qenc} = \int \rho_V dV = \rho_V \int dV = \rho_V \int_0^{2\pi} \int_0^{\pi} \int_0^{r} r^2 \sin\theta dr d\theta d\varphi$$

$$= \rho_V \frac{4}{3} \pi r^3$$

And $$\Psi = \int D . dS = D_r . \oint dS = D_r \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta d\theta d\varphi$$

$$= D_r 4\pi r^2$$

Hence Ψ = Qenc gives

$$D_r 4\pi r^2 = \rho_V \frac{4}{3}\pi r^3$$

Or $\quad D_r = \rho_V \frac{r}{3} ar \quad\quad 0 < r \leq a$

For $r \geq a$, the Gaussian surface is shown in Figure. The charge enclosed by the surface is the entire charge in this case, that is,

$$Qenc = \int \rho_V dV = \rho_V \int dV = \rho_V \int_0^{2\pi}\int_0^{\pi}\int_0^{a} r^2 \sin\theta dr d\theta d\varphi$$

$$= \rho_V \frac{4}{3}\pi a^3$$

While $\quad \Psi = \int D.dS = D_r.\oint dS = D_r \int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta d\theta d\varphi$

$$= D_r 4\pi r^2$$

$$\Psi = Qenc$$

$$D_r 4\pi r^2 = \rho_V \frac{4}{3}\pi a^3$$

$$D_r = \rho_V \frac{a^3}{3r^2} ar$$

Thus from above equations D everywhere is given by,

$$D = \begin{cases} \rho_V \frac{r}{3} ar & 0 < r \leq a \\ \rho_V \frac{a^3}{3r^2} ar & r \geq a \end{cases}$$

and |D| is as sketched

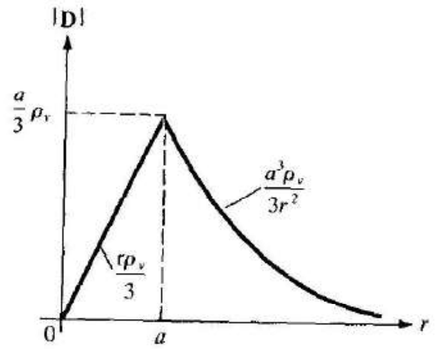


Fig 1.40. Sketch of |D| against *r* for a uniformly charged sphere.

1.12. GAUSS DIVERGENCE THEOREM

According to Gauss law,

$$Q = \int D.dS = \int_{V1} \rho_V dV$$

$$= \text{Total charge enclosed } Q = \int_{V1} \rho_V dV$$

$$Q = \int D.dS = \int_{V1} \rho_V dV$$

By applying divergence theorem to the middle term,

$$\oint D.dS = \int \nabla.D\, dV$$

Comparing the two volume integrals,

$$\rho_V = \nabla . D$$

This is the first of the four Maxwell's equations to be derived. Above equation states that the volume charge density is the same as the divergence of the electric flux density. This should not be surprising to us from the way we defined the divergence of a vector. Gauss's law is an alternative statement of Coulomb's law; proper application of the divergence theorem to Coulomb's law results in Gauss's law.

1.13. ABSOLUTE ELECTRIC POTENTIAL

$$E = \frac{Q}{4\pi\varepsilon_0 {R_{12}}^2}\bar{\mathbf{a_r}}$$

$$V_{AB} = -Q\int_A^B \frac{Q}{4\pi\varepsilon_0 {R_{12}}^2}\bar{\mathbf{a_r}} \bullet \text{drar}$$

$$V_{AB} = \frac{Q}{4\pi\varepsilon_0 {R_{12}}^2}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

$$V_{AB} = V_B - V_A$$

where V_B and V_A are the potentials (or absolute potentials) at B and A, respectively. Thus the potential difference V_{AB} may be regarded as the potential at B with reference to A. In problems involving point charges, it is customary to choose infinity as reference; that is, assume the potential at infinity is zero. Thus if $V_A = 0$ as $r_A \to \infty$, the potential at any point ($r_B \to r$) due to a point charge Q located at the origin is

$$V = \frac{Q}{4\pi\varepsilon_0 R}a_R$$

E points in the radial direction, any contribution from a displacement in the θ or π direction is wiped out by the dot product $E \bullet dl = E \cos\theta\, dl = E\, dr$. Hence the potential difference V_{AB} is independent of the path as asserted earlier.

The potential at any point is the potential difference that point and a chosen point at which the potential is zero

In other words, by assuming zero potential at infinity, the potential at a distance r from the point charge is the work done per unit charge by an external agent in transferring a test charge from infinity to that point. Thus

$$V = -\int_\infty^r E \bullet dl$$

Considered the electric potential due to a point charge. The same basic ideas apply to other types of charge distribution because any charge distribution can be regarded as consisting of point charges. The superposition principle, which we applied to electric fields, applies to potentials. For n point charges Q1, Q2,… ,Qn located at points with position vectors r b r2,. . ., rn, the potential at r is

$$V = \frac{Q1}{4\pi\varepsilon_0 R1}a_{R1} + \frac{Q2}{4\pi\varepsilon_0 R2}a_{R2} + \ldots + \frac{Q3}{4\pi\varepsilon_0 Rn}a_{Rn}$$

$$V = \frac{1}{4\pi\varepsilon_0}\sum_{k=1}^{n}\frac{Qk}{4\pi\varepsilon_0 Rk}a_{Rk}$$

$$E = -\nabla V$$

The electric field intensity is the gradient of V. The negative sign shows that the direction of E is opposite to the direction in which V increases; E is directed from higher to lower levels of V.

Since the curl of the gradient of a scalar function is always zero (∇ X ∇V = 0), E must be a gradient of some scalar function.

1.14. POTENTIAL DIFFERENCE

To move a point charge *Q* from point *A* to point *B* in an electric field E as shown in Figure. From Coulomb's law, the force on *Q* is F = *QE* so that the *work done* in displacing the charge by *dl* is

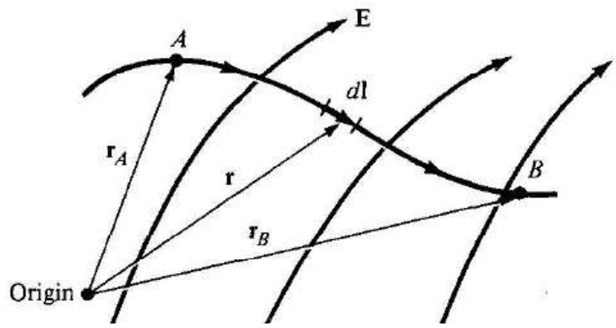


Fig 1.41. Displacement of point charge *Q* in an electrostatic field E.

$$dW = - F \bullet dl = -QE \bullet l$$

The negative sign indicates that the work is being done by an external agent. Thus the total work done, or the potential energy required, in moving *Q* from A to *B* is

$$W = -Q \int F \bullet dl = -Q\int E \bullet dl$$

Dividing *W* by Q gives the potential energy per unit charge. This quantity, denoted by V_{AB}, is known as the *potential difference* between points *A* and *B*. Thus

$$V_{AB} = \frac{W}{Q} = -Q \int_A^B E \bullet dl$$

- In determining V_{AB}, *A* is the initial point while *B* is the final point.
- If *VAB* is negative, there is a loss in potential energy in moving *Q* from *A* to *B;* this implies that the work is being done by the field. However, if *VAB* is positive, there is a gain in potential energy in the movement; an external agent performs the work.
- *VAB* is independent of the path taken (to be shown a little later).
- *VAB* is measured in joules per coulomb, commonly referred to as volts (V).

The potential difference between points *A* and *B* is independent of the path taken. Hence,

$$V_{AB} = - V_{BA}$$

1.15. CALCULATION OF POTENTIAL DIFFERENCES FOR DIFFERENT CONFIGURATIONS

For continuous charge distributions, Consider a general system composed of point charges q1, q2, q3 linear charge with density ρL coulomb /m, surface charge with density ρs coulomb /m^2 and volume charge distribution with ρv coulomb /m^3. The potential at any point is then given by summation of contribution of each of the above.

Potential at ρ due to point charges Q1, Q2, Q3 is given by,

$$V\rho = \frac{1}{4\pi\varepsilon 0}\left[\frac{Q1}{r1} + \frac{Q2}{r2} + \frac{Q3}{r3}\right]$$

$$= \frac{1}{4\pi\varepsilon 0}\sum_{N=1}^{n}\frac{QN}{rN}$$

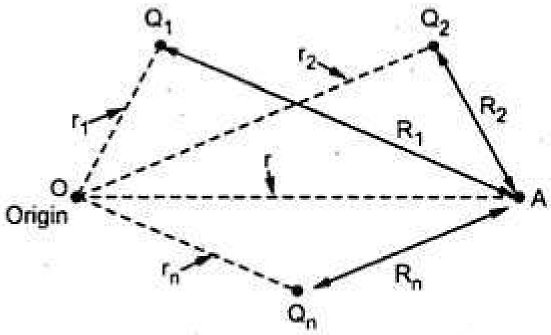


Fig.1.42. Potential due to several points

Potential at ρ due to line charge of density ρL , differential length dL is given by

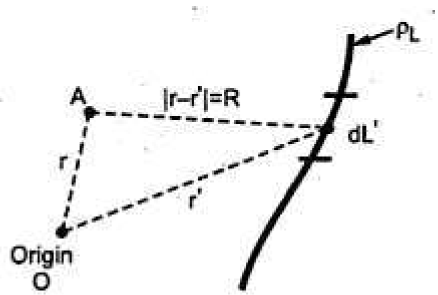


Fig.1.43. Potential due to line

$$V_L = \frac{1}{4\pi\varepsilon 0}\int\frac{\rho L}{r}\,dL$$

Potential at ρ due to surface charge of density ρs, differential surface dS is given by

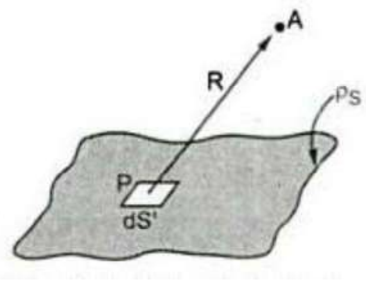


Fig.1.44 Potential due to surface

$$Vs = \frac{1}{4\pi\varepsilon 0}\iint\frac{\rho s}{r}\,ds$$

Potential at is ρ due to volume charge of density ρv, differential volume dV is given by

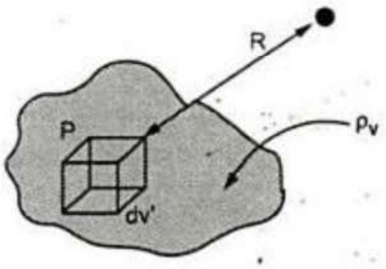


Fig.1.45 Potential due to volume

$$V_v = \frac{1}{4\pi\varepsilon 0} \iiint \frac{\partial v}{r} dv$$

By principle of superstition

Total Electric Potential is sum of all the potential due to point, line, surface and volume charge.

$$V = \frac{1}{4\pi\varepsilon 0}\left[\sum_{n=1}^{3} \frac{\theta n}{rn} + \int \frac{\rho L}{r} dL + \int\int \frac{\rho s}{r} ds + \int\int\int \frac{\partial v\, dv}{r}\right]$$

Potential difference	$V_{AB} = -\int_B^A \bar{E} \bullet d\bar{L}$
Absolute potential due to point charge	$V_{AB} = \frac{Q}{4\pi\varepsilon_0 R} V$
Absolute potential due to line charge	$V_{AB} = \int_L \frac{\rho_L dL'}{4\pi\varepsilon_0 R} V$
Absolute potential due to surface charge	$V_A = \int_S \frac{\rho_s dS'}{4\pi\varepsilon_0 R} V$
Absolute potential due to volume charge	$V_A = \int_v \frac{\rho_v dv'}{4\pi\varepsilon_0 R} V$
If the reference is other than infinity	$V_A = \frac{Q}{4\pi\varepsilon_0 R} + C\ V$
In all the expressions, R is the distance of point A from the charge Q or differential charge dQ.	

1.16. ELECTRIC DIPOLE

An electric dipole is two equal and opposite charges separated by small distance.

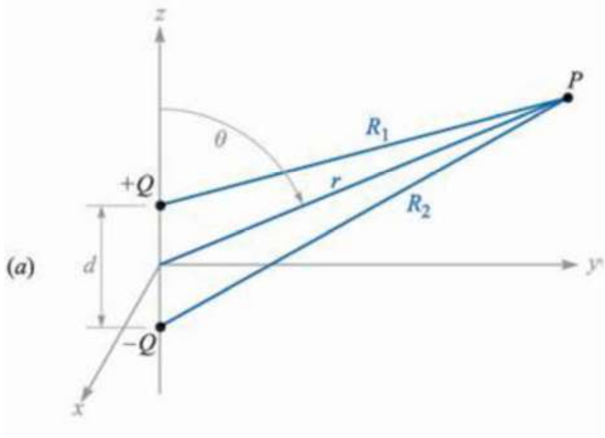


Fig1.46. An electric dipole.

Consider a dipole with θ and - θ Let point ρ be at a distance r1, and r2 from + θ and -θ and r from origin. The potential at ρ is zero when +θ and -θ are superposed. So r1=0.

But as r1 ≠ r2, the potential be non zero being the sum of the contribution by the two charges taken separately.

Potential at ρ due to +θ= $\frac{Q}{4\pi\Sigma 0 r_1}$ Potential at ρ due to -θ= $\frac{-Q}{4\pi\Sigma 0 r_2}$

Resultant Potential at ρ is

$$V = \frac{Q}{4\pi\Sigma 0}\left[\frac{1}{r1} - \frac{1}{r2}\right]$$

To find value of r_1 and r_2

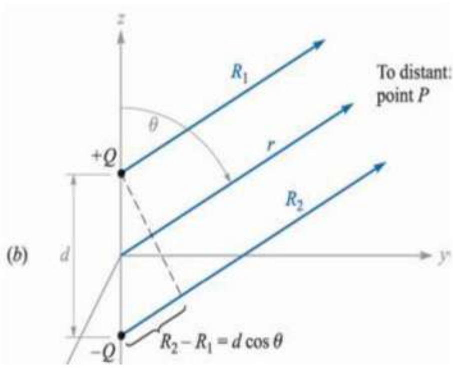


Fig1.47. An electric dipole to distant point P.

$$r_1 = r - \frac{d}{2} Cos\,\theta$$

$$r_2 = r + \frac{d}{2} Cos\,\theta$$

$$\therefore V = \frac{Q}{4\pi\Sigma 0}\left[\frac{1}{r-\frac{d}{2}\cos\theta} - \frac{1}{r+\frac{d}{2}\cos\theta}\right]$$

$$= \frac{Q}{4\pi\Sigma 0}\left[\frac{d\cos\theta}{\left(r-\frac{d}{2}\cos\theta\right)\left(r+\frac{d}{2}\cos\theta\right)}\right]$$

$$= \frac{Q}{4\pi\Sigma 0}\left[\frac{d\cos\theta}{r^2}\right] \quad as \quad r >> \frac{d}{2}\,Cos\,\theta$$

$$V = \frac{Qd\,Cos\,\theta}{4\pi\varepsilon 0 r^2}$$

$$V = \frac{\rho\,Cos\,\theta}{4\pi\varepsilon 0 r^2}\ Volts$$

Where ρ - Dipole moment = Qd.

1.17. ELECTROSTATIC ENERGY AND ENERGY DENSITY

To determine the energy present in an assembly of charges, first determine the amount of work necessary to assemble them. Suppose to position three point charges Q_1, Q_2, and Q_3 in an initially empty space shown shaded in Figure. No work is required to transfer Q_1 from infinity to *Pl* because the space is initially charge free and there is no electric field. The work done in transferring Q_2 from infinity to P_2 is equal to the product of Q_2 and the potential V_{21} at P_2 due to Q_1. Similarly, the work done in positioning Q_3 at P_3 is equal to $Q_3(V_{32} + V_{31})$, where V_{32} and V_{31} are the potentials at P_3 due to Q_2 and Q_1 respectively. Hence the total work done in positioning the three charges is

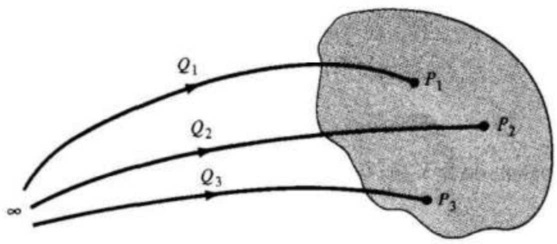


Figure 1.48 Assembling of charges.

$$W_E = W_1 + W_2 + W_3$$
$$= 0 + Q_2V_{21} + Q_3(V_{32} + V_{31})$$

Where V_{23} is the potential at P_2 due to Q_3, V_{12} and V_{13} are, respectively, the potentials at P_1 due to Q_2 and Q_3. Adding gives

$$2W_E = Q_1(V_{12} + V_{13}) + Q_2(V_{2l} + V_{23}) + Q_3(V_{31} + V_{32})$$
$$= Q_1V_1 + Q_2V_2 + Q_3V_3$$

where V_1, V_2, and V_3 are total potentials at P_1. P_2, and P_3, respectively. In general, if there are *n* point charges,

$$WE = \frac{1}{2}\sum_{k=1}^{n} Q_k V_k \quad (in\ joules)$$

If, instead of point charges, the region has a continuous charge distribution, the summation becomes integration;

$$WE = \frac{1}{2}\int \rho_l\, Vdl\ (line\ charge)$$

$$WE = \frac{1}{2}\int \rho_s\, V\, dS\ (surface\ charge)$$

$$WE = \frac{1}{2}\int \rho_V\, V\, dv\ (volume\ charge)$$

Since $\rho v = \nabla \bullet D$,

$$W_E = \frac{1}{2}\int (\nabla \bullet D)\, V\, dv$$

But for any vector A and scalar V, the identity

$$V \bullet \nabla A = A \bullet \nabla V + V(\nabla \bullet A)$$

or

$$(\nabla \bullet A)V = \nabla \bullet VA - A \bullet \nabla V$$

holds. Applying the identity we get

$$W_E = -(V \bullet VD)\, dv\, (D \bullet VV)\, dv$$

$$W_E = \frac{1}{2}\int (\nabla \bullet VD)\, dv - \frac{1}{2}\int (D \bullet \nabla V)\, dV$$

By applying divergence theorem to the first term on the right-hand side of this equation, we Have

$$W_E = \frac{1}{2}\oint_S (VD) \bullet dS - \frac{1}{2}\oint_V (D \bullet \nabla V) dV$$

$$W_E = \frac{1}{2}\oint_V (D \bullet \nabla V) dV = \frac{1}{2}\int (D \bullet E)\, dV$$

and since $E = -\nabla V$ and $D = \varepsilon o E$

$$W_E = \frac{1}{2}\int (D \bullet E) dV = \frac{1}{2}\int \varepsilon_0 E^2 dv$$

From this, define electrostatic energy density W_E (in J/m^3) as

$$W_E = \frac{dW_E}{dV} = \frac{1}{2} D \bullet E = \frac{1}{2}\varepsilon_0 E^2 = \frac{D^2}{2\varepsilon_0}$$

$$W_E = \int W_E\, dV$$

PROBLEMS

Example:1.1

Find the gradient of the following scalar fields:

(a) $V = e^{-z} \sin 2x \cosh y$

(b) $U = \rho^2 z \cos 2\phi$

(c) $W = 10r \sin^2 \theta \cos \phi$

Solution:

(a) $$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$
$$= 2e^{-z}\cos 2x \cosh y\, \mathbf{a}_x + e^{-z}\sin 2x \sinh y\, \mathbf{a}_y - e^{-z}\sin 2x \cosh y\, \mathbf{a}_z$$

(b) $$\nabla U = \frac{\partial U}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\mathbf{a}_\phi + \frac{\partial U}{\partial z}\mathbf{a}_z$$
$$= 2\rho z \cos 2\phi\, \mathbf{a}_\rho - 2\rho z \sin 2\phi\, \mathbf{a}_\phi + \rho^2 \cos 2\phi\, \mathbf{a}_z$$

(c) $$\nabla W = \frac{\partial W}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial W}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\mathbf{a}_\phi$$
$$= 10 \sin^2\theta \cos\phi\, \mathbf{a}_r + 10 \sin 2\theta \cos\phi\, \mathbf{a}_\theta - 10 \sin\theta \sin\phi\, \mathbf{a}_\phi$$

Example:1.2

Calculate the volume of a sphere of radius R using integration.

Solution : The differential volume of a sphere is,

$$dv = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

The limits for r are 0 to R, as sphere is of radius R.

The θ varies from 0 to π while φ varies from 0 to 2π.

$$\therefore \quad v = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi}\left[\frac{r^3}{3}\right]_0^R \sin\theta\, d\theta\, d\phi = \frac{R^3}{3}\int_0^{2\pi}\left[-\cos\theta\right]_0^{\pi} d\phi$$

$$= \frac{R^3}{3}\left[-\cos\pi - (-\cos 0)\right]\int_0^{2\pi} d\phi = \frac{R^3}{3}\left[-(-1)-(-1)\right]\left[\phi\right]_0^{2\pi}$$

$$= \frac{R^3}{3}\times 2\times 2\pi = \frac{4}{3}\pi R^3$$

Example 1.3:

Calculate the surface area of a sphere of radius R, by integration.

Solution : Consider the differential surface area normal to the r direction which is,

$$dS_r = r^2 \sin\theta \, d\theta \, d\phi$$

Now the limits of ϕ are 0 to 2π while θ varies from 0 to π.

$$\therefore \quad S_r = \int_0^{2\pi}\int_0^{\pi} r^2 \sin\theta \, d\theta \, d\phi$$

But note that **radius of sphere is constant,** given as $r = R$

$$S_r = R^2 \int_0^{2\pi}\int_0^{\pi} \sin\theta \, d\theta \, d\phi = R^2[-\cos\theta]_0^{\pi} [\phi]_0^{2\pi}$$

$$= R^2 \times [-\cos\pi - (-\cos 0)] \times 2\pi = R^2[-(-1)-(-1)]2\pi = 4\pi R^2$$

Example1.4:

Use spherical coordinates and integrate to find the area of the region $0 \le \phi \le \alpha$ on the spherical shell of radius a. What is the area if $\alpha = 2\pi$?

Solution : Consider the spherical shell of radius a hence $r = a$ is constant.

Consider differential surface area normal to r direction which is radially outward.

$$dS_r = r^2 \sin\theta \, d\theta \, d\phi = a^2 \sin\theta \, d\theta \, d\phi \quad \text{... as } r = a$$

But ϕ is varying between 0 to α while for spherical shell θ varies from 0 to π.

$$\therefore \quad S_r = a^2 \int_0^{\alpha}\int_0^{\pi} \sin\theta \, d\theta \, d\phi = a^2[-\cos\theta]_0^{\pi} [\phi]_0^{\alpha}$$

$$= a^2 \cdot [-\cos\pi - (-\cos 0)]\alpha = 2a^2\alpha$$

So area of the region is $\mathbf{2a^2\alpha}$.

If $\alpha = 2\pi$, the area of the region becomes $4\pi a^2$, as the shell becomes complete sphere of radius a when ϕ varies from 0 to 2π.

Tips: Distances in all Co-ordinate Systems

Consider two points A and B with the position vectors as,

$\overline{A} = x_1 \bar{a}_x + y_1 \bar{a}_y + z_1 \bar{a}_z$ and $\overline{B} = x_2 \bar{a}_x + y_2 \bar{a}_y + z_2 \bar{a}_z$

then the distance d between the two points in all the three co-ordinate systems are given by,

$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$	... Cartesian
$d = \sqrt{r_2^2 + r_1^2 - 2 r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2}$	... Cylindrical
$d = \sqrt{r_2^2 + r_1^2 - 2 r_1 r_2 \cos\theta_2 \cos\theta_1 - 2 r_1 r_2 \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1)}$	... Spherical

Example1.5:

Three point charges −1 nC, 4 nC, and 3 nC are located at (0, 0, 0), (0, 0, 1), and (1, 0, 0), respectively. Find the energy in the system.

Solution:

$$
\begin{aligned}
W &= W_1 + W_2 + W_3 \\
&= 0 + Q_2V_{21} + Q_3(V_{31} + V_{32}) \\
&= Q_2 \cdot \frac{Q_1}{4\pi\varepsilon_o\,|(0,0,1) - (0,0,0)|} \\
&\quad + \frac{Q_3}{4\pi\varepsilon_o}\left[\frac{Q_1}{|(1,0,0) - (0,0,0)|} + \frac{Q_2}{|(1,0,0) - (0,0,1)|}\right] \\
&= \frac{1}{4\pi\varepsilon_o}\left(Q_1Q_2 + Q_1Q_3 + \frac{Q_2Q_3}{\sqrt{2}}\right) \\
&= \frac{1}{4\pi \cdot \frac{10^{-9}}{36\pi}}\left(-4 - 3 + \frac{12}{\sqrt{2}}\right) \cdot 10^{-18} \\
&= 9\left(\frac{12}{\sqrt{2}} - 7\right)\text{nJ} = 13.37\text{ nJ}
\end{aligned}
$$

Example 1.6:

Give the cartesian co-ordinates of the vector field $\overline{H} = 20\bar{a}_r - 10\bar{a}_\phi + 3\bar{a}_z$, at point $P(x=5, y=2, z=-1)$.

Solution : The given vector is in cylindrical system.

$\therefore \quad H_x = \overline{H}\cdot\bar{a}_x = 20\bar{a}_r\cdot\bar{a}_x - 10\bar{a}_\phi\cdot\bar{a}_x + 3\bar{a}_z\cdot\bar{a}_x$

$\quad = 20\cos\phi - 10(-\sin\phi) + 0$

At point P, $x = 5$, $y = 2$ and $z = -1$

Now $\quad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{5} = 21.8014°$

$\therefore \quad \cos\phi = 0.9284$ and $\sin\phi = 0.3714$

$\therefore \quad H_x = 20\times(0.9284) + 10\times 0.3714 = 22.282$

Then $\quad H_y = \overline{H}\cdot\bar{a}_y = 20\bar{a}_r\cdot\bar{a}_y - 10\bar{a}_\phi\cdot\bar{a}_y + 3\bar{a}_z\cdot\bar{a}_y$

$\quad = 20\sin\phi - 10\cos\phi + 0$

$\quad = 20\times(0.3714) - 10\times(0.9284) = -1.856$

And $\quad H_z = \overline{H}\cdot\bar{a}_z = 20\bar{a}_r\cdot\bar{a}_z - 10\bar{a}_\phi\cdot\bar{a}_z + 3\bar{a}_z\cdot\bar{a}_z$

$\quad = 20\times 0 - 10\times 0 + 3\times 1 = 3$

$\therefore \quad \overline{H} = \mathbf{22.282}\,\bar{a}_x - \mathbf{1.856}\,\bar{a}_y + \mathbf{3}\,\bar{a}_z$ in cartesian system.

Example1.7:

Transform the vector field $\overline{W} = 10\overline{a}_x - 8\overline{a}_y + 6\overline{a}_z$ to cylindrical co-ordinate system, at point $P(10, -8, 6)$.

Solution : From the given field $\overline{W}$,

$W_x = 10,\ W_y = -8$ and $W_z = 6$

Now $\quad W_r = \overline{W} \cdot \overline{a}_r = [10\overline{a}_x - 8\overline{a}_y + 6\overline{a}_z] \cdot \overline{a}_r$

$$= 10\overline{a}_x \cdot \overline{a}_r - 8\overline{a}_y \cdot \overline{a}_r + 6\overline{a}_z \cdot \overline{a}_r$$

$$= 10(\cos\phi) - 8(\sin\phi) + 6(0)$$

For point P, $x = 10$ and $y = -8$

$\therefore \quad \phi = \tan^{-1}\frac{y}{x}$ ... Relation between cartesian and cylindrical

$$= \tan^{-1}\left[\frac{-8}{10}\right] = -38.6598°$$

As y is negative and x is positive, ϕ is in fourth quadrant. Hence ϕ calculated is correct.

$\therefore \quad \cos\phi = 0.7808$ and $\sin\phi = -0.6246$

$\therefore \quad W_r = 10 \times (0.7808) - 8 \times (-0.6246) = 12.804$

Now $\quad W_\phi = \overline{W} \cdot \overline{a}_\phi = 10\overline{a}_x \cdot \overline{a}_\phi - 8\overline{a}_y \cdot \overline{a}_\phi + 6\overline{a}_z \cdot \overline{a}_\phi$

$$= 10(-\sin\phi) - 8\cos\phi + 0 = 0$$

And $\quad W_z = \overline{W} \cdot \overline{a}_z = 10\overline{a}_x \cdot \overline{a}_z - 8\overline{a}_y \cdot \overline{a}_z + 6\overline{a}_z \cdot \overline{a}_z$

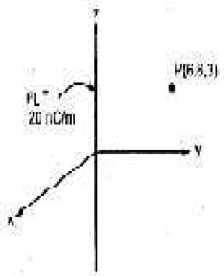
$$= 10 \times 0 - 8 \times 0 + 6 \times 1 = 6$$

$\therefore \quad \overline{W} = 12.804\,\overline{a}_r + 6\overline{a}_z$ in cylindrical system.

Example1.8:

A uniform line charge, infinite in extent with $\rho_L = 20$ nC/m lies along the z axis. Find the $\overline{E}$ at (6,8,3) m

Solution : The line charge is shown in the Fig.



Any point on the line is (0,0,z).

$\therefore \quad \overline{r} = (6-0)\overline{a}_x + (8-0)\overline{a}_y$

$\therefore \quad \overline{a}_r = \frac{\overline{r}}{|\overline{r}|} = \frac{6\,\overline{a}_x + 8\,\overline{a}_y}{\sqrt{6^2+8^2}} = \frac{6\,\overline{a}_x + 8\,\overline{a}_y}{10}$

$$= 0.6\overline{a}_x + 0.8\,\overline{a}_y$$

Key Point: *As line charge is along z axis, $\overline{E}$ can not have any component along z direction. So do not consider z co-ordinate while calculating $\overline{r}$.*

Thus, $\quad \overline{E} = \frac{\rho_L}{2\pi\varepsilon_0 r}\overline{a}_r$

$$= \frac{20 \times 10^{-9}}{2\pi \times 8.854 \times 10^{-12} \times 10}[0.6\overline{a}_x + 0.8\,\overline{a}_y] = 10.7853\,\overline{a}_x + 14.38\,\overline{a}_y \text{ V/m}$$

Example 1.9:

Obtain the spherical coordinates of 10 $\bar{a}_x$ at the point $P(x=-3, y=2, z=4)$.

Solution : Given vector is in cartesian system say $\bar{F} = 10\bar{a}_x$.

Then $$F_r = \bar{F}\bullet\bar{a}_r = 10\bar{a}_x \bullet \bar{a}_r$$

$$= 10 \sin\theta \cos\phi$$

At point P, $x = -3,\ y = 2,\ z = 4$

Using the relationship between cartesian and spherical,

$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$

$$\therefore \quad \phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{-3} = -33.69°$$

But **x is negative** and **y is positive** hence ϕ must be between $+90°$ and $+180°$. So add 180° to the ϕ to get correct ϕ.

$$\therefore \quad \phi = -33.69° + 180° = +146.31°$$

$$\therefore \quad \cos\phi = -0.832 \text{ and } \sin\phi = 0.5547$$

And $$\theta = \cos^{-1}\frac{z}{r} = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$= \cos^{-1}\frac{4}{\sqrt{(-3)^2+(2)^2+(4)^2}} = 42.0311°$$

$$\cos\theta = 0.7428 \text{ and } \sin\theta = 0.6695$$

$$F_r = 10\times 0.6695\times(-0.832) = -5.5702$$

$$F_\theta = \bar{F}\bullet\bar{a}_\theta = 10\bar{a}_x\bullet\bar{a}_\theta = 10\cos\theta\cos\phi$$

$$= 10\times 0.7428\times(-0.832) = -6.18$$

$$F_\phi = \bar{F}\bullet\bar{a}_\phi = 10\bar{a}_x\bullet\bar{a}_\phi = 10(-\sin\phi)$$

$$= 10\times(-0.5547) = -5.547$$

$$\bar{F} = -5.5702\,\bar{a}_r - 6.18\,\bar{a}_\theta - 5.547\,\bar{a}_\phi \text{ in spherical system.}$$

Dot operator •	$\bar{a}_r$	$\bar{a}_\phi$	$\bar{a}_z$
$\bar{a}_x$	$\cos\phi$	$-\sin\phi$	0
$\bar{a}_y$	$\sin\phi$	$\cos\phi$	0
$\bar{a}_z$	0	0	1

Example 1.10:

Given point $P(-2, 6, 3)$ and vector $\mathbf{A} = y\mathbf{a}_x + (x + z)\mathbf{a}_y$, express P and $\mathbf{A}$ in cylindrical and spherical coordinates. Evaluate $\mathbf{A}$ at P in the Cartesian, cylindrical, and spherical systems.

Solution:

At point P: $x = -2, y = 6, z = 3$. Hence,

$$\rho = \sqrt{x^2 + y^2} = \sqrt{4 + 36} = 6.32$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{6}{-2} = 108.43°$$

$$z = 3$$

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{4 + 36 + 9} = 7$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z} = \tan^{-1}\frac{\sqrt{40}}{3} = 64.62°$$

Thus,

$$P(-2, 6, 3) = P(6.32, 108.43°, 3) = P(7, 64.62°, 108.43°)$$

In the Cartesian system, $\mathbf{A}$ at P is

$$\mathbf{A} = 6\mathbf{a}_x + \mathbf{a}_y$$

For vector $\mathbf{A}$, $A_x = y, A_y = x + z, A_z = 0$. Hence, in the cylindrical system

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x + z \\ 0 \end{bmatrix}$$

$$A_\rho = y\cos\phi + (x + z)\sin\phi$$

$$A_\phi = -y\sin\phi + (x + z)\cos\phi$$

$$A_z = 0$$

But $x = \rho\cos\phi$, $y = \rho\sin\phi$, and substituting these yields

$$\mathbf{A} = (A_\rho, A_\phi, A_z) = [\rho\cos\phi\sin\phi + (\rho\cos\phi + z)\sin\phi]\mathbf{a}_\rho + [-\rho\sin^2\phi + (\rho\cos\phi + z)\cos\phi]\mathbf{a}_\phi$$

At P

$$\rho = \sqrt{40}, \qquad \tan\phi = \frac{6}{-2}$$

$$\cos\phi = \frac{-2}{\sqrt{40}}, \qquad \sin\phi = \frac{6}{\sqrt{40}}$$

$$\mathbf{A} = \left[\sqrt{40}\cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + \left(\sqrt{40}\cdot\frac{-2}{\sqrt{40}} + 3\right)\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_\rho + \left[-\sqrt{40}\cdot\frac{36}{40} + \left(\sqrt{40}\cdot\frac{-2}{\sqrt{40}} + 3\right)\cdot\frac{-2}{\sqrt{40}}\right]\mathbf{a}_\phi$$

$$= \frac{-6}{\sqrt{40}}\mathbf{a}_\rho - \frac{38}{\sqrt{40}}\mathbf{a}_\phi = -0.9487\mathbf{a}_\rho - 6.008\mathbf{a}_\phi$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} y \\ x+z \\ 0 \end{bmatrix}$$

$$A_r = y\sin\theta\cos\phi + (x+z)\sin\theta\sin\phi$$

$$A_\theta = y\cos\theta\cos\phi + (x+z)\cos\theta\sin\phi$$

$$A_\phi = -y\sin\phi + (x+z)\cos\phi$$

But $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$. Substituting these yields

$$\begin{aligned}\mathbf{A} &= (A_r, A_\theta, A_\phi) \\ &= r[\sin^2\theta\cos\phi\sin\phi + (\sin\theta\cos\phi + \cos\theta)\sin\theta\sin\phi]\mathbf{a}_r \\ &\quad + r[\sin\theta\cos\theta\sin\phi\cos\phi + (\sin\theta\cos\phi + \cos\theta)\cos\theta\sin\phi]\mathbf{a}_\theta \\ &\quad + r[-\sin\theta\sin^2\phi + (\sin\theta\cos\phi + \cos\theta)\cos\phi]\mathbf{a}_\phi\end{aligned}$$

At P $\quad r = 7, \quad \tan\phi = \dfrac{6}{-2}, \quad \tan\theta = \dfrac{\sqrt{40}}{3}$

$$\cos\phi = \frac{-2}{\sqrt{40}}, \quad \sin\phi = \frac{6}{\sqrt{40}}, \quad \cos\theta = \frac{3}{7}, \quad \sin\theta = \frac{\sqrt{40}}{7}$$

$$\begin{aligned}\mathbf{A} &= 7\cdot\left[\frac{40}{49}\cdot\frac{-2}{\sqrt{40}}\cdot\frac{6}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{\sqrt{40}}{7}\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_r \\ &\quad + 7\cdot\left[\frac{\sqrt{40}}{7}\cdot\frac{3}{7}\cdot\frac{6}{\sqrt{40}}\cdot\frac{-2}{\sqrt{40}} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{3}{7}\cdot\frac{6}{\sqrt{40}}\right]\mathbf{a}_\theta \\ &\quad + 7\cdot\left[\frac{-\sqrt{40}}{7}\cdot\frac{36}{40} + \left(\frac{\sqrt{40}}{7}\cdot\frac{-2}{\sqrt{40}} + \frac{3}{7}\right)\cdot\frac{-2}{\sqrt{40}}\right]\mathbf{a}_\phi \\ &= \frac{-6}{7}\mathbf{a}_r - \frac{18}{7\sqrt{40}}\mathbf{a}_\theta - \frac{38}{\sqrt{40}}\mathbf{a}_\phi \\ &= -0.8571\mathbf{a}_r - 0.4066\mathbf{a}_\theta - 6.008\mathbf{a}_\phi\end{aligned}$$

Example: 1.11:

Determine the divergence of these vector fields:

(a) $\mathbf{P} = x^2yz\,\mathbf{a}_x + xz\,\mathbf{a}_z$

(b) $\mathbf{Q} = \rho \sin\phi\,\mathbf{a}_\rho + \rho^2 z\,\mathbf{a}_\phi + z\cos\phi\,\mathbf{a}_z$

(c) $\mathbf{T} = \dfrac{1}{r^2}\cos\theta\,\mathbf{a}_r + r\sin\theta\cos\phi\,\mathbf{a}_\theta + \cos\theta\,\mathbf{a}_\phi$

Solution:

(a)
$$\begin{aligned}\nabla\cdot\mathbf{P} &= \frac{\partial}{\partial x}P_x + \frac{\partial}{\partial y}P_y + \frac{\partial}{\partial z}P_z\\ &= \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz)\\ &= 2xyz + x\end{aligned}$$

(b)
$$\begin{aligned}\nabla\cdot\mathbf{Q} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho Q_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}Q_\phi + \frac{\partial}{\partial z}Q_z\\ &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi)\\ &= 2\sin\phi + \cos\phi\end{aligned}$$

(c)
$$\begin{aligned}\nabla\cdot\mathbf{T} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(T_\phi)\\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\cos\theta)\\ &= 0 + \frac{1}{r\sin\theta}2r\sin\theta\cos\theta\cos\phi + 0\\ &= 2\cos\theta\cos\phi\end{aligned}$$

Example: 1.12

Given $W = x^2y^2 + xyz$, compute ∇W and the direction derivative dW/dl in the direction $3\mathbf{a}_x + 4\mathbf{a}_y + 12\mathbf{a}_z$ at $(2, -1, 0)$.

Solution:
$$\begin{aligned}\nabla W &= \frac{\partial W}{\partial x}\mathbf{a}_x + \frac{\partial W}{\partial y}\mathbf{a}_y + \frac{\partial W}{\partial z}\mathbf{a}_z\\ &= (2xy^2 + yz)\mathbf{a}_x + (2x^2y + xz)\mathbf{a}_y + (xy)\mathbf{a}_z\end{aligned}$$

At $(2, -1, 0)$: $\nabla W = 4\mathbf{a}_x - 8\mathbf{a}_y - 2\mathbf{a}_z$

Hence,

$$\frac{dW}{dl} = \nabla W\cdot\mathbf{a}_l = (4, -8, -2)\cdot\frac{(3, 4, 12)}{13} = -\frac{44}{13}$$

Example: 1.13

Determine the divergence of these vector fields:

(a) $\mathbf{P} = x^2yz\,\mathbf{a}_x + xz\,\mathbf{a}_z$

(b) $\mathbf{Q} = \rho \sin\phi\,\mathbf{a}_\rho + \rho^2 z\,\mathbf{a}_\phi + z\cos\phi\,\mathbf{a}_z$

(c) $\mathbf{T} = \frac{1}{r^2}\cos\theta\,\mathbf{a}_r + r\sin\theta\cos\phi\,\mathbf{a}_\theta + \cos\theta\,\mathbf{a}_\phi$

Solution:

(a)
$$\begin{aligned}\nabla\cdot\mathbf{P} &= \frac{\partial}{\partial x}P_x + \frac{\partial}{\partial y}P_y + \frac{\partial}{\partial z}P_z \\ &= \frac{\partial}{\partial x}(x^2yz) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(xz) \\ &= 2xyz + x\end{aligned}$$

(b)
$$\begin{aligned}\nabla\cdot\mathbf{Q} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho Q_\rho) + \frac{1}{\rho}\frac{\partial}{\partial\phi}Q_\phi + \frac{\partial}{\partial z}Q_z \\ &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho^2\sin\phi) + \frac{1}{\rho}\frac{\partial}{\partial\phi}(\rho^2 z) + \frac{\partial}{\partial z}(z\cos\phi) \\ &= 2\sin\phi + \cos\phi\end{aligned}$$

(c)
$$\begin{aligned}\nabla\cdot\mathbf{T} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2T_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(T_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(T_\phi) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\cos\theta) \\ &= 0 + \frac{1}{r\sin\theta}2r\sin\theta\cos\theta\cos\phi + 0 \\ &= 2\cos\theta\cos\phi\end{aligned}$$

Example: 1.14

Convert the point P(3, 4, 5) from cartesian to spherical co-ordinates.

Solution : x = 3, y = 4, z = 5

$$r = \sqrt{x^2+y^2+z^2} = \sqrt{3^2+4^2+5^2} = \mathbf{7.071}$$

$$\theta = \cos^{-1}\left[\frac{z}{\sqrt{x^2+y^2+z^2}}\right] = \cos^{-1}\left[\frac{5}{\sqrt{50}}\right] = \mathbf{45^\circ}$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{4}{3} = \mathbf{53.13^\circ}$$

$\therefore$ **(7.071, 45°, 53.13°)**

Example: 1.15

Point charges 1 mC and −2 mC are located at (3, 2, −1) and (−1, −1, 4), respectively. Calculate the electric force on a 10-nC charge located at (0, 3, 1) and the electric field intensity at that point.

Solution:

$$\mathbf{F} = \sum_{k=1,2} \frac{QQ_k}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \sum_{k=1,2} \frac{QQ_k(\mathbf{r}-\mathbf{r}_k)}{4\pi\varepsilon_o|\mathbf{r}-\mathbf{r}_k|^3}$$

$$= \frac{Q}{4\pi\varepsilon_o}\left\{\frac{10^{-3}[(0,3,1)-(3,2,-1)]}{|(0,3,1)-(3,2,-1)|^3} - \frac{2.10^{-3}[(0,3,1)-(-1,-1,4)]}{|(0,3,1)-(-1,-1,4)|^3}\right\}$$

$$= \frac{10^{-3}\cdot 10\cdot 10^{-9}}{4\pi\cdot\dfrac{10^{-9}}{36\pi}}\left[\frac{(-3,1,2)}{(9+1+4)^{3/2}} - \frac{2(1,4,-3)}{(1+16+9)^{3/2}}\right]$$

$$= 9\cdot 10^{-2}\left[\frac{(-3,1,2)}{14\sqrt{14}} + \frac{(-2,-8,6)}{26\sqrt{26}}\right]$$

$$\mathbf{F} = -6.507\mathbf{a}_x - 3.817\mathbf{a}_y + 7.506\mathbf{a}_z \text{ mN}$$

At that point,

$$\mathbf{E} = \frac{\mathbf{F}}{Q}$$

$$= (-6.507, -3.817, 7.506)\cdot\frac{10^{-3}}{10\cdot 10^{-9}}$$

$$\mathbf{E} = -650.7\mathbf{a}_x - 381.7\mathbf{a}_y + 750.6\mathbf{a}_z \text{ kV/m}$$

Example: 1.16

If the point B is at (− 2, 3, 3) in the above example, obtain the potential difference between the points A and B.

Solution : $V_{AB} = V_A - V_B$

where V_A and V_B are the absolute potentials of A and B.

Now $V_A = 3.595$ V ... as calculated earlier.

$$V_B = \frac{Q}{4\pi\varepsilon_0 R_B}$$ where R_B is distance between point B and Q (2, 3, 3)

$$\therefore \quad R_B = \sqrt{(-2-2)^2+(3-3)^2+(3-3)^2} = 4$$

$$\therefore \quad V_B = \frac{0.4\times 10^{-9}}{4\pi\times 8.854\times 10^{-12}\times 4} = 0.8987 \text{ V}$$

$$\therefore \quad V_{AB} = V_A - V_B = 3.595 - 0.8987 = \mathbf{2.6962\ V}$$

Example: 1.17

Two point charges $-4\ \mu C$ and $5\ \mu C$ are located at $(2, -1, 3)$ and $(0, 4, -2)$, respectively. Find the potential at $(1, 0, 1)$ assuming zero potential at infinity.

Solution:

Let

$$Q_1 = -4\ \mu C, \qquad Q_2 = 5\ \mu C$$

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|} + C_o$$

If $V(\infty) = 0$, $C_o = 0$,

$$|\mathbf{r} - \mathbf{r}_1| = |(1, 0, 1) - (2, -1, 3)| = |(-1, 1, -2)| = \sqrt{6}$$

$$|\mathbf{r} - \mathbf{r}_2| = |(1, 0, 1) - (0, 4, -2)| = |(1, -4, 3)| = \sqrt{26}$$

Hence

$$V(1, 0, 1) = \frac{10^{-6}}{4\pi \times \frac{10^{-9}}{36\pi}}\left[\frac{-4}{\sqrt{6}} + \frac{5}{\sqrt{26}}\right]$$
$$= 9 \times 10^3 (-1.633 + 0.9806)$$
$$= -5.872 \text{ kV}$$

Example: 1.18

Two dipoles with dipole moments $-5\mathbf{a}_z$ nC/m and $9\mathbf{a}_z$ nC/m are located at points $(0, 0, -2)$ and $(0, 0, 3)$, respectively. Find the potential at the origin.

Solution:

$$V = \sum_{k=1}^{2} \frac{\mathbf{p}_k \cdot \mathbf{r}_k}{4\pi\varepsilon_o r_k^3}$$
$$= \frac{1}{4\pi\varepsilon_o}\left[\frac{\mathbf{p}_1 \cdot \mathbf{r}_1}{r_1^3} + \frac{\mathbf{p}_2 \cdot \mathbf{r}_2}{r_2^3}\right]$$

where

$$\mathbf{p}_1 = -5\mathbf{a}_z, \qquad \mathbf{r}_1 = (0, 0, 0) - (0, 0, -2) = 2\mathbf{a}_z, \qquad r_1 = |\mathbf{r}_1| = 2$$

$$\mathbf{p}_2 = 9\mathbf{a}_z, \qquad \mathbf{r}_2 = (0, 0, 0) - (0, 0, 3) = -3\mathbf{a}_z, \qquad r_2 = |\mathbf{r}_2| = 3$$

Hence,

$$V = \frac{1}{4\pi \cdot \frac{10^{-9}}{36\pi}}\left[\frac{-10}{2^3} - \frac{27}{3^3}\right] \cdot 10^{-9}$$
$$= -20.25 \text{ V}$$

Example: 1.19

Express vector $\bar{B}$ in cartesian and cylindrical systems. Given, $\bar{B} = \frac{10}{r}\bar{a}_r + r\cos\theta\,\bar{a}_\theta + \bar{a}_\phi$
Then find $\bar{B}$ at (– 3, 4, 0) and (5, π/2, – 2)

Solution :

$$\bar{B} = \frac{10}{r}\bar{a}_r + r\cos\theta\,\bar{a}_\theta + \bar{a}_\phi$$

$\therefore$ $$B_r = \frac{10}{r}, \quad B_\theta = r\cos\theta, \quad B_\phi = 1$$

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \frac{10}{r} \\ r\cos\theta \\ 1 \end{bmatrix}$$

$\therefore$ $$B_x = \frac{10}{r}\sin\theta\cos\phi + r\cos^2\theta\cos\phi - \sin\phi$$

$\therefore$ $$B_y = \frac{10}{r}\sin\theta\sin\phi + r\cos^2\theta\sin\phi + \cos\phi$$

$\therefore$ $$B_z = \frac{10}{r}\cos\theta - r\sin\theta\cos\theta$$

But $$r = \sqrt{x^2+y^2+z^2}, \quad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}, \quad \tan\phi = \frac{y}{x}$$

$\therefore$ $$\sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}, \quad \sin\phi = \frac{y}{\sqrt{x^2+y^2}}, \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}}$$

Using equations (1), (2) and (3), $\bar{B}$ in cartesian system is :

$$\bar{B} = B_x\,\bar{a}_x + B_y\,\bar{a}_y + B_z\,\bar{a}_z \quad \text{where,}$$

$$B_x = \frac{10x}{x^2+y^2+z^2} + \frac{xz^2}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} - \frac{y}{\sqrt{x^2+y^2}}$$

$$B_y = \frac{10y}{x^2+y^2+z^2} + \frac{yz^2}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}} + \frac{x}{\sqrt{x^2+y^2}}$$

$$B_z = \frac{10z}{x^2+y^2+z^2} - \frac{z\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

At (– 3, 4, 0), $x = -3$, $y = 4$, $z = 0$

$\therefore$ $$\bar{B} = -2\,\bar{a}_x + \bar{a}_y$$

For transforming spherical to cylindrical use,

$$\begin{bmatrix} B_\rho \\ B_\phi \\ B_z \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} B_r \\ B_\theta \\ B_\phi \end{bmatrix}$$

$\therefore$ $$B_\rho = \sin\theta\,B_r + \cos\theta\,B_\theta = \frac{10\sin\theta}{r} + r\cos^2\theta$$

$$B_\phi = B_\phi = 1$$

$$B_z = \cos\theta\, B_r - \sin\theta\, B_\theta = \frac{10\cos\theta}{r} - r\sin\theta\cos\theta$$

Now $\rho = r\sin\theta,\ z = r\cos\theta,\ \phi = \phi,\ r = \sqrt{\rho^2+z^2},\ \theta = \tan^{-1}\frac{\rho}{z}$

And $\tan\theta = \frac{\rho}{z}$ hence $\sin\theta = \frac{\rho}{\sqrt{\rho^2+z^2}},\ \cos\theta = \frac{z}{\sqrt{\rho^2+z^2}}$

$\therefore$ $\bar{B} = B_\rho\,\bar{a}_\rho + B_\phi\,\bar{a}_\phi + B_z\,\bar{a}_z$ where,

$$B_\rho = \frac{10\rho}{\rho^2+z^2} + \frac{z^2}{\sqrt{\rho^2+z^2}},\quad B_\phi = 1,\quad B_z = \frac{10z}{\rho^2+z^2} - \frac{\rho z}{\sqrt{\rho^2+z^2}}$$

At given point $\left(5, \frac{\pi}{2}, -2\right)$, $\rho = 5,\ \phi = \frac{\pi}{2}$ and $z = -2$

$$B_\rho = \frac{10\times 5}{5^2+(-2)^2} + \frac{(-2)^2}{\sqrt{5^2+(-2)^2}} = 2.467,\quad B_\phi = 1$$

$$B_z = \frac{10\times(-2)}{5^2+(-2)^2} - \frac{5\times(-2)}{\sqrt{5^2+(-2)^2}} = 1.167$$

$$\bar{B} = 2.467\,\bar{a}_\rho + \bar{a}_\phi + 1.167\,\bar{a}_z$$

SUMMARY

- A field is a function that specifies a quantity in space. For example, $\bar{A}(x, y, z)$ is a vector field whereas $V(x, y, z)$ is a scalar field.
- A vector $\bar{A}$ is uniquely specified by its magnitude and a unit vector along it, that is, A = $A\overline{aA}$.
- The three common coordinate systems we shall use throughout the text are the Cartesian (or rectangular), the circular cylindrical, and the spherical.

	Cartesian coordinate system	***Cylindrical coordinate system***	***Spherical co ordinate system***
The differential length	dl= $\sqrt{dx^2+dy^2+dz^2}$	dl= $\sqrt{(d\rho)^2+(\rho d\varphi)^2+(dz)}$	dl= $\sqrt{(dr)^2+(rd\theta)^2+(rsin\theta}$
The differential area	ds_1=dxdy(x,yplane) ds_2=dydz(y,zplane) ds_3=dzdx(z,xplane)	ds = dρdz (ρ,z plane) ds = ρdρdΦ (ρ,Φ plane) ds = ρdΦdz (Φ,z plane)	ds = r dr dθ (r,θ plane) ds = rsinθ dΦ d (r,Φ plane) ds = r^2sinθ dθ dΦ(θ,Φplane)
The differential volume	dv = dxdydz	dv = ρdρdΦdz	dv = r^2sinθ dr dθ dΦ

- Multiplying two vectors $\bar{A}$ and $\bar{B}$ results in either a scalar $A \cdot B = AB \cos \theta_{AB}$ or a vector $\bar{A} \times \bar{B} = AB \sin \theta_{AB} a_n$. Multiplying three vectors $\bar{A}$, $\bar{B}$, and $\bar{C}$) yields a scalar $\bar{A} \cdot (\bar{B} \times \bar{C})$ or a vector $\bar{A} \times (\bar{B} \times \bar{C})$.
- The scalar projection (or component) of vector $\bar{A}$, onto $\bar{B}$, is $A_B = \bar{A}, \cdot a_B$ whereas vector projection of $\bar{A}$, onto $\bar{B}$ is $A_B = A_B a_B$.
- Vector Transformation:

In matrix form, the transformation of vector A from *(Ax,Ay,Az)* to *(Aρ,* AΦ, Az) as

$$\begin{bmatrix} Ax \\ Ay \\ Az \end{bmatrix} = \begin{bmatrix} \cos \Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos \Phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A\rho \\ A\Phi \\ Az \end{bmatrix}$$

The inverse of the transformation *(Aρ,* AΦ, Az) to *(Ax,Ay,Az)*

$$\begin{bmatrix} A\rho \\ A\Phi \\ Az \end{bmatrix} = \begin{bmatrix} \cos \Phi & -\sin\Phi & 0 \\ \sin\Phi & \cos \Phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Ax \\ Ay \\ Az \end{bmatrix}$$

In matrix form, the *(Ax, Ay, Az)* → (Ar, *Aθ, AΦ)* vector transformation is performed according to

$$\begin{bmatrix} Ar \\ A\theta \\ A\Phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos \Phi & \sin\theta\sin\Phi & \cos\theta \\ \cos\theta\cos\Phi & \cos\theta\sin\Phi & -\sin\theta \\ -\sin\Phi & \cos \Phi & 0 \end{bmatrix} \begin{bmatrix} Ax \\ Ay \\ Az \end{bmatrix}$$

The inverse transformation (Ar, *Aθ, AΦ)* → (Ax, Ay, Az) is similarly obtained,

$$\begin{bmatrix} Ax \\ Ay \\ Az \end{bmatrix} = \begin{bmatrix} \sin\theta\cos \Phi & \cos\theta\cos\Phi & -\sin\Phi \\ \sin\theta\sin\Phi & \cos\theta\sin\Phi & \cos \Phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} Ar \\ A\theta \\ A\Phi \end{bmatrix}$$

In matrix form, the (Ar, *Aθ, AΦ) to (Aρ,AΦ,Az)*vector transformation is performed according to

$$\begin{bmatrix} Ar \\ A\theta \\ A\Phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A\rho \\ A\Phi \\ Az \end{bmatrix}$$

In matrix form, the *(Aρ,AΦ,Az)to* (Ar, *Aθ, AΦ)* vector transformation is performed according to

$$\begin{bmatrix} A\rho \\ A\Phi \\ Az \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} Ar \\ A\theta \\ A\Phi \end{bmatrix}$$

- Fixing one space variable defines a surface; fixing two defines a line; fixing three defines a point.
- The del operator, written ∇, is the vector differential operator.

$$\nabla = \frac{\partial}{\partial x}\overline{ax} + \frac{\partial}{\partial y}\overline{ay} + \frac{\partial}{\partial z}\overline{az}$$

- $(\textbf{curl of } \boldsymbol{H}) = \nabla \times H = \lim_{\Delta S_N \to 0} \left(\frac{\oint H.dl}{\Delta S_N} \right)$
- The volume integral of the divergence of a vector field over a volume is equal to the surface integral of the normal component of this vector over the surface bounding this volume.

$$\iiint_V \nabla.\vec{A}\, dv = \oiint \vec{A}.ds$$

- ***Stokes's theorem*** states that the circulation of a vector field A around a (closed) path is equal to the surface integral of the curl of A over the open surface *S* bounded by L provided that A and ∇ X A are continuous S.

$$\oint H.dl = \iint_s \nabla \times H \, ds$$

- ***Coulomb's law*** states that the force f between two point charges (Q1 and Q2 is):

i. Along the line joining them
ii. Directly proportional to the product Q1Q2of the charges
iii. Inversely proportional to the square of the distance *R* between them.
iv. Point charge is a hypothetical charge located at a single point in space. It is an idealized

$$\boldsymbol{F = \frac{1}{4\pi\varepsilon}\frac{Q_1 Q_2}{R^2}}$$

- The electric field intensity (or electric field strength) K is the force per unit charge when placed in the electric field.

$$E=\frac{F}{Q}$$

- Point, Line, Surface And Volume Charge Distributions

$$dQ = \rho_L dL \rightarrow Q = \int_L \rho_L dL \qquad \text{(line charge)}$$

$$dQ = \rho_S dS \rightarrow Q = \int_S \rho_S dS \text{(surface charge)}$$

$$dQ = \rho_V dS \rightarrow Q = \int_V \rho_V dV \qquad \text{(volume charge)}$$

- The flux due to the electric field E can be calculated using the general definition of flux. For practical reasons, however, this quantity is not usually considered as the most useful flux in electrostatics D = εE

$$D = \frac{Q}{4\pi R_{12}{}^2}$$

- Gauss's law states that the total electric flux *Ψ* through any *closed* surface is equal to the total charge enclosed by that surface.

$$\Psi = Qenc$$

$$\Psi= \int d\Psi = \int D.dS$$

- To move a point charge *Q* from point *A* to point *B* in an electric field E as shown in Figure. From Coulomb's law, the force on *Q* is F = *QE* so that the *work done* in displacing the charge by *dl* is
- An electric dipole is two equal and opposite charges separated by small distance.

define electrostatic energy density W_E (in J/m^3) as

$$W_E = \frac{dW_E}{dV} = \frac{1}{2} D \bullet E = \frac{1}{2}\varepsilon_0 E^2 = \frac{D^2}{2\varepsilon_0}$$

$$W_E = \int W_E \, dV$$