UNIT II
CONDUCTORS AND DIELECTRICS

Conductors and dielectrics in Static Electric Field, Current and current density, Continuity equation, Polarization, Boundary conditions, Method of images, Resistance of a conductor, Capacitance, Parallel plate, Coaxial and Spherical capacitors, Boundary conditions for perfect dielectric materials, Poisson's equation, Laplace's equation, Solution of Laplace equation, Application of Poisson's and Laplace's equations.

2. INTRODUCTION

Just as electric fields can exist in free space, they can exist in material media. Materials are broadly classified in terms of their electrical properties as conductors and nonconductors. Non conducting materials are usually referred to as insulators or dielectrics.

2.1. PROPERTIES OF MATERIALS

Why an electron does not leave a conductor surface, why a current-carrying wire remains un charged, why materials behave differently in an electric field, and why waves travel with less speed in conductors than in dielectrics are easily answered by considering the electrical properties of materials. Here, a brief discussion will suffice to help understand the mechanism by which materials influence an electric field.

Materials may be classified in terms of their conductivity a, in mhos per meter (mho/m) or Siemens per meter (S/m), as conductors and nonconductors, or technically as metals and insulators (or dielectrics). The conductivity of a material usually depends on temperature and frequency. A material with high conductivity ($\sigma >> 1$) is referred to as a metal whereas one with low conductivity ($\sigma << 1$) is referred to as an insulator. A material whose conductivity lies somewhere between those of metals and insulators is called a semiconductor. Copper and aluminum are metals, silicon and germanium are semiconductors, and glass and rubber are insulators. The conductivity of metals generally increases with decrease in temperature. At temperatures near absolute zero (T = 0°K), some conductors exhibit infinite conductivity and are called superconductors. Lead and aluminum are typical examples of such metals. The conductivity of lead at 4°K is of the order of 1020 mhos/m.

The slightly separated charges for these cases form electric dipoles. Dielectric materials have a distribution of such dipoles. Even though these materials are charge neutral because each dipole contains an equal amount of positive and negative charges, a net charge can accumulate in a region if there is a local imbalance of positive or negative dipole ends. The net polarization charge in such a region is also a source of the electric field in addition to any other free charges.

2.2. CONDUCTORS AND DIELECTRICS IN STATIC ELECTRIC FIELD

2.2.1. Conductors

A conductor has abundance of charge that is free to move. Consider an isolated conductor, such as shown in Figure 2.1 (a). When an external electric field Ee is applied, the positive free charges are pushed along the same direction as the applied field, while the negative free charges move in the opposite direction. This charge migration takes place very quickly. The free charges do two things. First, they accumulate on the surface of the conductor and form an induced surface charge. Second, the induced charges set up an internal induced field E, which cancels the externally applied field Ee. The result is illustrated in Figure 2.1(b). This leads to an important property of a conductor:

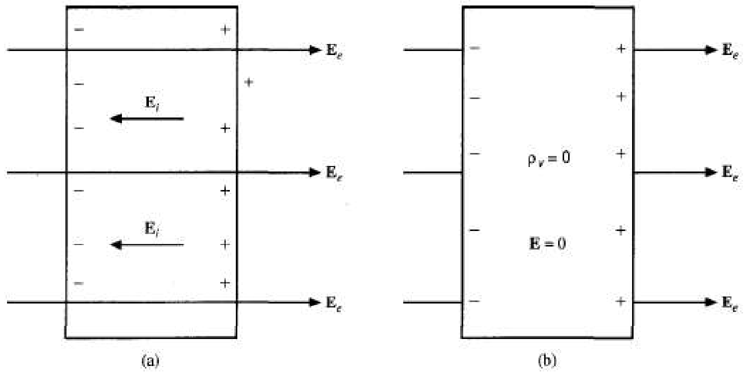


Figure2.1 (a) An isolated conductor under the influence of an applied field; (b) a conductor has zero electric field under static conditions.

A perfect conductor cannot contain an electrostatic field within it. A conductor is called an equipotential body, implying that the potential is the same everywhere in the conductor. This is based on the fact that $E = -\nabla V = 0$. Another way of looking at this is to consider Ohm's law, $J = \sigma E$. To maintain a finite current density J, in a perfect conductor ($\sigma \to \infty$), requires that the electric field inside the conductor must vanish. In other words, $E \to 0$ because $\sigma \to \infty$ in a perfect conductor. If some charges are introduced in the interior of such a conductor, the charges will move to the surface and redistribute themselves quickly in such a manner that the field inside the conductor vanishes.

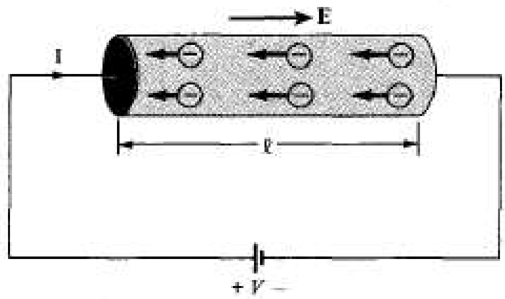


Figure 2.2. A conductor of uniform cross section under an applied E field.

According to Gauss's law, if E = 0, the charge density ρv must be zero. Conclude again that a perfect conductor cannot contain an electrostatic field within it. Under static conditions, consider a conductor whose ends are maintained at a potential difference V, $E \neq 0$ inside the conductor there is no static equilibrium in Figure 2.2 since the conductor is not isolated but wired to a source of electromotive force, which compels the free charges to move and prevents the eventual establishment of electrostatic equilibrium. Thus in the case of an electric field must exist inside the conductor to sustain the flow of current. As the electrons move, they encounter some damping forces called resistance. Based on Ohm's law the resistance of the conducting material. Suppose the conductor has a uniform cross section S and is of length l. The direction of the electric field E produced is the same as the direction of the flow of positive charges or current l. This direction is opposite to the direction of the flow of electrons. The electric field applied is uniform and its magnitude is given by

$$E = \frac{V}{l}$$

Since the conductor has a uniform cross section,

$$J = \frac{I}{S}$$

Substituting

$$\frac{I}{S} = \sigma E = \sigma \frac{V}{l}$$

Hence

$$R = \frac{V}{I} = \frac{I}{\sigma S}$$

$$R = \frac{\rho c\, l}{S}$$

$$where\ \rho c\ = \frac{l}{\sigma}\ is\ the\ resistivity\ of\ the\ material.$$

Equation R is useful in determining the resistance of any conductor of uniform cross section. If the cross section of the conductor is not uniform, equation R is not applicable. However, the basic definition of resistance R as the ratio of the potential difference V between the two ends of the conductor to the current I through the conductor still apply. Therefore, the resistance of a conductor of non uniform cross section is,

$$R = \frac{V}{I} = \frac{\int E.dl}{\int \sigma E.ds}$$

Note that the negative sign before $V = -\int E.dl$ is dropped. Because $\int E.dl < 0$ if $I > 0$. Power P (in watts) is defined as the rate of change of energy W (in joules) or force times velocity. Hence,

$$\int \rho_v\, dv\, E \bullet u = \int E \bullet \rho_v u\, dv$$

or

$$P = \int E.J\, dv$$

which is known as Joule's law. The power density wP (in watts/m^3) is given by the integrand

$$W = \frac{dP}{dV} = E.J = \sigma|E^2|$$

For a conductor with uniform cross section,

$dv = dS\, dl,$

$$P = I^2R$$

which is the more common form of Joule's law in electric circuit theory.

Properties of Conductors

Consider the charge distribution is unbalanced inside the conductors. There are number of electrons trying to reside inside the conductor. All the electrons are negatively charged and they are start repelling each other due to their own electric fields. Such electrons get accelerated away from each other, till all the electron causing interior imbalance, reach at the surface of the conductor. The conductor is surrounded by the insulating medium. And hence electron just driven from the interior of the conductor reside over a surface.

1. Under static condition no charge and no electric field can exists at any point within the conducting material.
2. The charge can exist on the surface of the conductor giving raise to surface charge density.
3. Within a conductor the charge density is always zero.
4. The charge distribution on the surface depends on the shape of the surface.
5. The conductivity of an ideal conductor is infinity.
6. The conductor surface is an equipotential surface

2.2.2. Dielectrics

The theory of dielectrics we have discussed so far assumes ideal dielectrics. Practically speaking, no dielectric is ideal. When the electric field in a dielectric is sufficiently large, it begins to pull electrons completely out of the molecules, and the dielectric becomes conducting. Dielectric breakdown is said to have occurred when a dielectric becomes conducting. Dielectric breakdown occurs in all kinds of dielectric materials (gases, liquids, or solids) and depends on the nature of the material, temperature, humidity, and the amount of time that the field is applied. The minimum value of the electric field at which dielectric breakdown occurs is called the dielectric strength of the dielectric material. The dielectric strength is the maximum electric field that a dielectric can tolerate or withstand without breakdown.

A material is said to be linear if D varies linearly with E and nonlinear otherwise. Materials for which s (or a) does not vary in the region being considered and is therefore the same at all points (i.e., independent of x, y, z) are said to be homogeneous. They are said to be inhomogeneous (or non homogeneous) when e is dependent of the space coordinates. The atmosphere is a typical example of an inhomogeneous medium; its permittivity varies with altitude. Materials for which D and E are in the same direction are said to be isotropic. That is, isotropic dielectrics are those which have the same properties in all directions. For anisotropic (or non isotropic) materials, $\overline{\boldsymbol{D}}, \overline{\boldsymbol{E}}$, and $\overline{\boldsymbol{P}}$ are not parallel.

Properties of dielectric materials:

The various properties of dielectric materials of,

1. The dielectrics does not contain any free charges but contain bounded charges.
2. Bound charges are under the internal molecular and atomic forces and cannot contribute to the conduction.
3. When subjected to the external field the bound charge shifted their relative positions. Due to this small electric dipole get induced inside the dielectric. This is called polarization.

4. Due to the polarization the dielectrics can be stored the energy.
5. Due to the polarization the flux density of the dielectric increases by amount equal to the polarization.
6. The induced dipoles produced their own electric field and align in the direction of the applied electric field.
7. When polarization occurs the volume charge density is formed inside the dielectrics while the surface charge density is formed over the surface of the dielectric.
8. The electric outside and inside the dielectric gets modified due to the induced electric dipoles.

2.3. CURRENT AND CURRENT DENSITY

2.3.1. Electric Current

- The electric current is defined as the state of motion of the electric charge .
- Quantitatively electric current is defined as the time rate of flow of the net charge of the area of cross section of the conductor i.e

Electric current = Total charge flowing / time taken

- If q is the amount of charge flowing through the conductor in t sec, The current through the conductor is given by

$$I = \frac{q}{t}$$

- SI unit of the current is Ampere (A) 1 Ampere= 1 Coulomb/ 1 sec=1 Cs^{-1}
- Thus current through any conductor is said to be 1 ampere. Small amount of currents are accordingly expressed in milli amperes (1mA=10^{-3} A) or in micro ampere (1 mA=10^{-6} A)
- Direction of electric current is in the direction of the flow of positive charged carriers and this current is known as conventional current.
- Direction of the flow of electron in conductor gives the direction of electronic current.

Direction of conventional current is opposite to that of electronic current

- Electric current is a scalar quantity .Although electric current represent the direction of the flow of positive charged carrier in the conductor, still current is treated as scalar quantity as current in wires in a circuit does not follows the laws of vector addition

2.3.2. Current density

The current density at a point in the conductor is defined as the current per unit cross-section area. Thus if the charge is flowing per unit time uniformly over the area of cross-section A of the conductor, then current density J at any point on that area is defined as

$$J = \frac{I}{A}$$

- It is the characteristic property of point inside the conductor nor of the conductor as a whole
- Direction of current density is same as the direction of conventional current
- Note that current density is a vector quantity unlike electric current
- Unit of current density is Ampere/meter^2 (Am^{-2})

2.3.3. Relation between current and current density

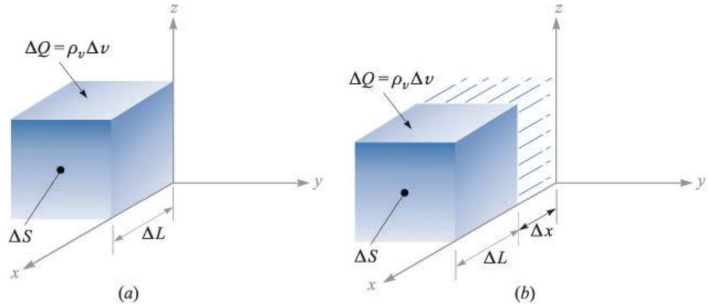


Fig. 2.3. Current and current density

The current is defined as the rate of flow of charge and measured in ampere. One ampere is said to be flowing across surface when a charge of one coulomb is passing across the surface in one second. The current which exists in the conductor due to the drifting of electrons under the influence of the applied voltage is called drift current. In dielectric there can be flow of charge under the influence of the electric field intensity. Such a current is called the displacement current or convection current. The current density is defined as the current passing through the unit surface area, when the surface is held normal to the direction of current unit is A/m^2.

Consider the incremental surface area ds

$$\overline{ds} = ds\,\overline{a_n} \text{ while } \bar{J} = J\overline{a_n}$$

$Current\ dI = \bar{J}.\overline{ds}$

$$dI = J\overline{a_n}.ds\overline{a_n} = J.dS$$

$$I = \int_S J.ds$$

If $\bar{J}$ is not normal to the differential area $\overline{ds}$. The total current is

$$I = \int_S J.ds$$

2.4. CONTINUITY EQUATION

Due to the principle of charge conservation, the time rate of decrease of charge within a given volume must be equal to the net outward current flow through the closed surface of the volume. Thus current Iout coming out of the closed surface is

$I_{out} = \oint J.ds = \frac{-dQ_{in}}{dt}$

where Qin is the total charge enclosed by the closed surface. Invoking divergence theorem

$$\oint J.dS = \int \nabla.J\,dV$$

$$\oint J.ds = \frac{-dQ_{in}}{dt} = -\frac{d}{dt}\int \rho_V.dV$$

Substituting eqn,

$$\int \nabla.J\,dV = -\int \frac{\partial\rho_V}{\partial t}.dV$$

$$\nabla . J = -\frac{\partial \rho_V}{\partial t}$$

which is called the continuity of current equation. It must be kept in mind that the continuity equation is derived from the principle of conservation of charge and essentially states that there can be no accumulation of charge at any point. For steady currents, dpv/dt = 0 and hence ∇• J = 0 showing that the total charge leaving a volume is the same as the total charge entering it. Kirchhoff's current law follows from this. Having discussed the continuity equation and the properties σ and ε of materials, it is appropriate to consider the effect of introducing charge at some interior point of a given material (conductor or dielectric). conjunction with Ohm's law

$$J = \sigma E$$

and Gauss's law

$$\nabla . E = \frac{\rho_V}{\varepsilon}$$

2.5. POLARIZATION

The main difference between a conductor and a dielectric lies in the availability of free electrons in the atomic outermost shells to conduct current. Although the charges in a dielectric are not able to move about freely, they are bound by finite forces and may certainly expect a displacement when an external force is applied. To understand the macroscopic effect of an electric field on a dielectric, consider an atom of the dielectric as consisting of a negative charge - Q (electron cloud) and a positive charge +Q (nucleus) as in Figure (a).

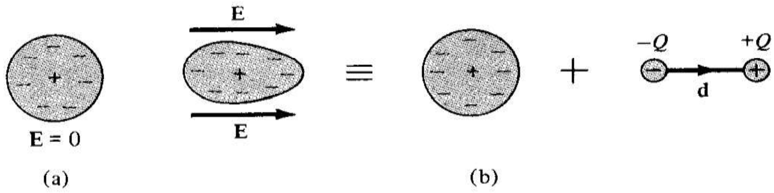


Figure2.4. Polarization of a nonpolar atom or molecule

A similar picture can be adopted for a dielectric molecule; it can treat the nuclei in molecules as point charges and the electronic structure as a single cloud of negative charge. Since have equal amounts of positive and negative charge, the whole atom or molecule is electrically neutral. When an electric field E is applied, the positive charge is displaced from its equilibrium position in the direction of E by the force $\overline{F}_+ = Q\overline{E}$ while the negative charge is displaced in the opposite direction by the force $\overline{F}\_ = Q\overline{E}$. A dipole results from the displacement of the charges and the dielectric is said to be polarized. In the polarized state, the electron cloud is distorted by the applied electric field E. This distorted charge distribution is equivalent, by the principle of superposition, to the original distribution plus a dipole whose moment is

$$\bar{p} = Q\bar{d}$$

where $\overline{d}$ is the distance vector from - Q to +Q of the dipole as in Figure (b). If there are N dipoles in a volume Δv of the dielectric, the total dipole moment due to the electric field is

$$\overline{p} = Q_1\overline{d}_1 + Q_2\overline{d}_2 + \cdots + Q_N\overline{d}_N = \sum_{k=1}^{N} Q_k\overline{d}_k$$

As a measure of intensity of the polarization, define polarization$\overline{P}$ (in colombs/meter square) as the dipole moment per unit volume of the dielectric: that is,

$$\overline{p} = \lim_{\Delta V \to 0} \frac{\sum_{k=1}^{N} Q_k \overline{d}_k}{\Delta V}$$

Thus conclude that the major effect of the electric field $\overline{E}$ on a dielectric is the creation of dipole moments that align themselves in the direction of $\overline{E}$. This type of dielectricis said to be nonpolar. Examples of such dielectrics are hydrogen, oxygen, nitrogen, and the rare gases. Nonpolar dielectric molecules do not possess dipoles until the application of the electric field. Other types of molecules such as water, sulfur dioxide, and hydrochloric acid have built-in permanent dipoles that are randomly oriented as shown in Figure (a) and are said to be polar. When an electric field $\overline{E}$ is applied to a polar molecule, the permanent dipole experiences a torque tending to align its dipole moment parallel with E as in Figure (b). Now calculate the field due to a polarized dielectric.

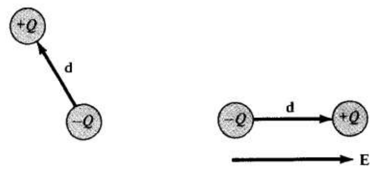


Figure2.5. Polarization of a polar molecule:(a) permanent dipole (E = 0), (b) alignment of permanent dipole (E≠ 0).

Consider the dielectric material shown in Figure. Consisting of dipoles with dipole moment $\overline{p}$ per unit volume. The potential $d\overline{V}$ at an exterior point O due to the dipole moment P dv' is

$$dV = \frac{P a_R \cdot dv'}{4\pi\varepsilon_0 {R_{12}}^2}$$

where $R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2$ and R is the distance between the volume element dv' at (x', y', z') and the field point O (x, y, z). Transform into a form that facilitates physical interpretation the gradient of 1/R with respect to the primed coordinates is

$$\nabla' V = \frac{1}{R} = \frac{a_R}{R^2}$$

$$\frac{P.a_R}{R^2} = P.\nabla'(\frac{1}{R})$$

Applying the vector identity $\nabla'.fA = f.\nabla' \bullet A + A \bullet \nabla' f$,

$$\frac{P.a_R}{R^2} = \nabla'.\frac{P}{R} - \frac{\nabla'.P}{R}$$

Substituting this into eqn. and integrating over the entire volume v' of the dielectric,

$$V = \int \frac{1}{4\pi\varepsilon_0}[\nabla'.\frac{P}{R} - \frac{\nabla'.P}{R}]dv'$$

$$= -\int \frac{1}{4\pi\varepsilon_0}\frac{\nabla'.P}{R}dv' + \int \frac{1}{4\pi\varepsilon_0}\nabla'.\frac{P}{R}dv'$$

Applying divergence theorem to the first term leads finally to

$$V = -\int \frac{1}{4\pi\varepsilon_0}\frac{P.a_{n'}}{R}dS' - \int \frac{1}{4\pi\varepsilon_0}.\frac{\nabla'.P}{R}dv'$$

where a'n is the outward unit normal to surface dS' of the dielectric. Comparing the two terms on the right side of eqn. with potential equation shows that the two terms denote the potential due to surface and volume charge distributions with densities

$$\rho_{PS} = P.\ an$$
$$\rho_{PV} = -\nabla.\ P$$

In other words, polarization occurs, an equivalent volume charge density ρ_{PV} is formed throughout the dielectric while an equivalent surface charge density ρ_{PS} is formed over the surface of the dielectric. ρ_{Ps} and ρ_{PV} as bound (or polarization) surface and volume charge densities, respectively, as distinct from free surface and volume charge densities ρs and ρv. Bound charges are those that are not free to move within the dielectric material; they are caused by the displacement that occurs on a molecular scale during polarization. Free charges are those that are capable of moving over macroscopic distance as electrons in a conductor; they are the stuff control. The total positive bound charge on surface S bounding the dielectric is

$$\oint \bar{P}.\overline{dS} = \int \rho PS\ dS$$

while the charge that remains inside S is

$$-Qb\ = \int \rho_{PV}\ dV = -\int \rho_{PS}\ dS$$

Thus the total charge of the dielectric material remains zero, that is,

$$Total\ charge\ = \int \rho_{PS}\ dS + \int \rho_{PV}\ dV = \ Q_b\ -\ Q_b = 0$$

This is expected because the dielectric was electrically neutral before polarization.
consider the case in which the dielectric region contains free charge. If pv is the free charge volume density, the total volume charge density ρt is given by

$$\rho t\ =\ \rho v\ + \rho_{PV}\ =\ \nabla.\varepsilon 0E$$
$$\rho v\ =\ \nabla.\varepsilon 0E\ -\ \rho_{PV}$$
$$=\ \nabla.(\varepsilon 0E\ +\ P)$$
$$=\ \nabla.D$$
$$D\ =\ \varepsilon 0E\ +\ P$$

The net effect of the dielectric on the electric field $\bar{E}$ is to increase $\bar{D}$ inside it by amount $\bar{P}$. In other words, due to the application of $\bar{E}$ to the dielectric material, the flux density is greater than it would be in free space. It should be noted that the definition of $\bar{D}$ for free space is a special case because $\bar{P} = 0$ in free space.
The polarization $\bar{P}$ would vary directly as the applied electric field $\bar{E}$. For some dielectrics, this is usually the case and

$$\bar{P}\ =\ \chi e\varepsilon 0E$$

where χe, known as the electric susceptibility of the material, is more or less a measure of how susceptible (or sensitive) a given dielectric is to electric fields.

2.6. BOUNDARY CONDITIONS

Considered the existence of the electric field in a homogeneous medium. If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called *boundary conditions.* These conditions are helpful in determining the field on one side of the boundary if the field on the other side is known. Obviously, the conditions will be dictated by the types of material the media are made of. The boundary conditions at an interface separating

- Boundary condition between conductor and dielectric
- Boundary condition between conductor and free space.
- Boundary condition between two dielectrics with different properties (Dielectric ($\varepsilon r1$) and Dielectric ($\varepsilon r2$) *)*

To determine the boundary conditions, Maxwell's equations for elecrostatics are required:

$$\oint \bar{E}.\overline{dl} = 0$$

$$\oint \bar{D}.\overline{dS} = Q_{enc}$$

Also to decompose the electric field intensity$\bar{E}$ into two orthogonal components:

$$\bar{E} = \bar{E}_t + \bar{E}_n$$

where$\overline{E_r}$, and $\bar{E}_n$are, respectively, the tangential and normal components of E to the interface of interest. A similar decomposition can be done for the electric flux density $\bar{D}$.

2.6.1. BOUNDARY CONDITION BETWEEN CONDUCTOR AND FREE SPACE

Consider a boundary between conductor and free space. The conductor is ideal having infinite conductivity. Such conductors are copper, silver etc. having conductivity of the order of 10^6 S/m and can be treated ideal. For ideal conductor

2. The field intensity inside the conductor is zero and the flux density inside the conductor is zero.
3. No charge can exist within conductor. The charge appears on a surface in the form of surface charge density.
4. The charge density within the conductor is zero.

To determine the boundary condition consider the conductor free space boundary.

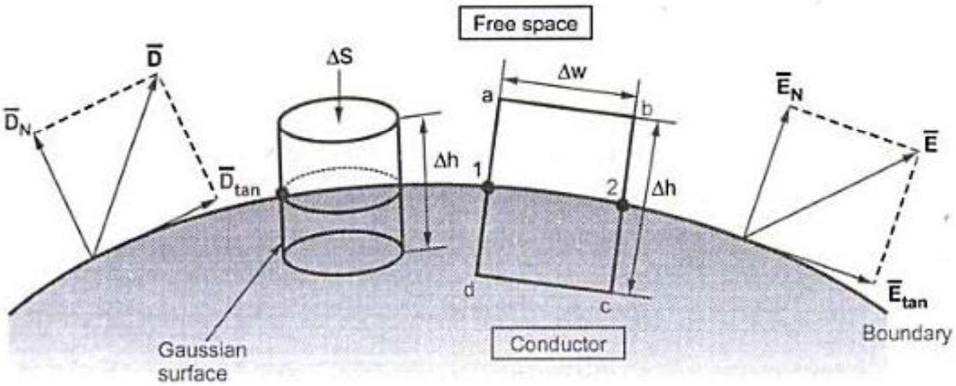


Fig2.6: Boundary between conductor and free space

Ē at the Boundary

Let **Ē** be the electric field intensity, in the direction shown in fig making some angle with the boundary. This $\bar{E}$ can be resolved in to two components:

1. The component tangential to the surface ($\bar{E}_{tan}$).
2. The component normal to the surface. ($\bar{E}_N$)

   It is known that,

$$\oint \bar{E}\,.\,d\bar{L} = 0$$

The integral of $\bar{E}.d\bar{L}$ carried over the closed contour is zero. i.e work done in the carrying unit positive charge along the closed path is zero. Consider the rectangular closed path abcda as shown in fig. It is traced in clockwise direction as a-b-c-d-a and hence $\oint \bar{E}\,.\,d\bar{L}$ can be divide into four parts.

$$\oint \bar{E}\,.\,d\bar{L} = \int_a^b \bar{E}.d\bar{L} + \int_b^c \bar{E}.d\bar{L} + \int_c^d \bar{E}.d\bar{L} + \int_d^a \bar{E}.d\bar{L} = 0 \ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

The closed contour is placed in such a way that its two sides a-b and c-d are parallel to tangential direction to the surface while the other two are normal to the surface, at the boundary. The rectangle is an elementary rectangle with elementary height Δh and elementary width Δw. The rectangle is placed in such a way that half of it conductor and remaining half in the free space. Thus Δh/2 is in the conductor and Δh/2 is in the free space.

Now the portion c-d is in the conductor where $\bar{E}$=0 hence the corresponding integral is zero.

$$\int_a^b \bar{E}.d\bar{L} + \int_b^c \bar{E}.d\bar{L} + \int_d^a \bar{E}.d\bar{L} = 0$$

As the width Δw is very small, $\bar{E}$ over it can be assumed constant and hence can be taken out of integration.

$$\int_a^b \bar{E}.d\bar{L} = \bar{E}.\int_a^b d\bar{L} = \bar{E}(\Delta w)$$

But Δw is along tangential direction to the boundary in which the direction $\bar{E} = \bar{E}_{tan}$

$$\bar{E}.\int_a^b d\bar{L} = E_{tan}\,.\,\Delta w \qquad where\ E_{tan} = |E_{tan}|$$

Now b-c is parallel to the normal component so $\bar{E} = \bar{E}_N$ along in this direction,

$$\bar{E} = |\bar{E}_N|$$

Over the small height Δh, E_N can be assumed constant and can be taken out of integration.

$$\int_b^c \bar{E}.d\bar{L} = \bar{E}.\int_b^c d\bar{L} = \bar{E}_N \int_b^c d\bar{L}$$

But out of b-c is in free space and 2-c is in the conductor where $\bar{E} = 0$.

$$\int_b^c d\bar{L} = \int_b^2 d\bar{L} + \int_2^c d\bar{L} = \frac{\Delta h}{2} + 0 = \frac{\Delta h}{2}$$

$$\int_b^c \bar{E}.d\bar{L} = \bar{E}_N(\frac{\Delta h}{2})$$

Similarly for path d-a the condition is same as for the path b-c only direction is opposite.

$$\int_{d}^{a} \bar{E}.d\bar{L} = -\bar{E}_N(\frac{\Delta h}{2})$$

Substituting in (1)

$$E_{tan}.\Delta w + \bar{E}_N\left(\frac{\Delta h}{2}\right) - \bar{E}_N\left(\frac{\Delta h}{2}\right) = 0$$
$$E_{tan}.\Delta w = 0$$
$$E_{tan} = 0$$

Thus the tangential component of electric field intensity is zero at the boundary between conductor and free space. Now,

$$\bar{D} = \varepsilon_0 \bar{E} \quad for\, free\, space$$
$$D_{tan} = \varepsilon_0 \bar{E}_{tan} = 0$$

Thus the tangential component of electric flux density is zero at the boundary between conductor and free space.

D_N at the Boundary

To find normal component of D select a closed Gaussian surface in the firm of right circular cylinder as shown in fig. Its height is Δh ans is placed in such a way that Δh/2 is in the conductor and remaining Δh/2 is in the free space. Its axis is in the normal direction to the surface.

According to Gauss law,

$$\oint_S \bar{D}.d\bar{S} = Q$$

The surface integral must be evaluated over three surfaces,

i) Top ii) Bottom iii) Lateral

Let the area of top and bottom is same equal to ΔS.

$$\int_{top} \bar{D}.d\bar{S} + \int_{bottom} \bar{D}.d\bar{S} + \int_{lateral} \bar{D}.d\bar{S} = Q$$

The bottom surface in the conductor where $\bar{D}$ =0 hence the corresponding integral is zero. The top surface is in the free space . The lateral surface area is 2πr Δh where r is the radius of the cylinder. But as Δ→0, this area reduced to zero and corresponding integral is zero. While only component of $\bar{D}$ present is the normal component having magnitude D_N can be assumed constant and can be taken out of integration.

$$\int_{top} \bar{D}.d\bar{S} + \int_{lateral} \bar{D}.d\bar{S} = Q$$

$$\int_{top} \bar{D}.d\bar{S} = \overline{D_N}.\int_{top} d\bar{S} = D_N.\Delta S$$

From Gauss law,

$$D_N.\Delta S = Q$$

But at the boundary the charge exists in the form of surface charge density ρ_S C/m^2.

$$Q = \rho s \Delta s$$
$$D_N = \rho s$$

Thus the flux leaves the surface normally and the normal component of flux density is equal to the surface charge density.

$$D_N = \varepsilon_0 E_N = \rho s$$
$$E_N = \frac{\rho s}{\varepsilon_0}$$

2.6.2. BOUNDARY CONDITION BETWEEN CONDUCTOR AND DIELECTRIC

The free space is a dielectric with $\varepsilon = \varepsilon_0$. Thus if the boundary is between conductor and dielectric with $\varepsilon = \varepsilon_r \varepsilon_0$.

$$D_{tan} = \bar{E}_{tan} = 0$$
$$D_N = \rho s$$
$$E_N = \frac{\rho s}{\varepsilon_0} = \frac{\rho s}{\varepsilon_0\, \varepsilon_r}$$

2.6.3. BOUNDARY CONDITION BETWEEN TWO PERFECT DIELECTRICS

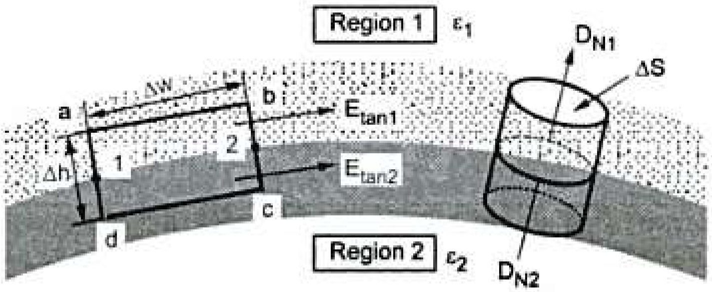


Fig:2.7. **Boundary Condition Between Two Perfect Dielectrics**

Consider the boundary between two different dielectrics. One dielectric has permittivity ε1 and while the other has permittivity ε2. The interface is shown in fig.

The integral of $\bar{E}.d\bar{L}$ carried over the closed contour is zero. i.e work done in the carrying unit positive charge along the closed path is zero. Consider the rectangular closed path abcda as shown in fig. It is traced in clockwise direction as a-b-c-d-a and hence $\oint \bar{E}\,.d\bar{L}$ can be divide into four parts.

$$\oint \bar{E}\,.d\bar{L} = \int_a^b \bar{E}.d\bar{L} + \int_b^c \bar{E}.d\bar{L} + \int_c^d \bar{E}.d\bar{L} + \int_d^a \bar{E}.d\bar{L} = 0 \ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

Consider a closed path abcda rectangular in shape having elementary height Δh and elementary width Δw, as shown in fig. It is placed in such a way that Δh/2 is in dielectric 1 while the remaining is dielectric 2.

Now

$$\overline{E_1} = \overline{E_{t1}} + \bar{E}_{N1}$$
$$\overline{E_2} = \overline{E_{t2}} + \bar{E}_{N2}$$

Now for the rectangle to be reduced at the surface to analyse boundary conditions. Δh→0. As Δh→0 , $\int_c^b and\ \int_a^d$. become zero as these are line integrals along Δh and Δh→0. Hence,

$$\int_a^b \bar{E}.d\bar{L} + \int_c^d \bar{E}.d\bar{L} = 0$$

Now a-b is in dielectric 1 hence the corresponding component of $\bar{E}\ is\ E_{tan\,1}$ as a-b direction is tangential to the surface.

$$\int_a^b \bar{E}.d\bar{L} = E_{\tan 1}\int_a^b d\bar{L} = E_{\tan 1}\,\Delta w$$

While c-d is in dielectric 2 hence the corresponding component of $\bar{E}$ is E $_{tan2}$ as c-d direction is also tangential to the surface. But direction c-d is opposite to a-b hence corresponding integrals is negative of the integral obtained for path a-b.

$$\int_c^d \bar{E}.d\bar{L} = -E_{\tan 2}\,\Delta w$$

Substituting,

$$E_{\tan 1}\,\Delta w - E_{\tan 2}\,\Delta w = 0$$
$$E_{\tan 1} = E_{\tan 2}\,\Delta w$$

Thus the tangential component of field intensity at the boundary in both the dielectrics remain same. i.e. electric fiels intensity is continuous across the boundary.
The relation between D and E is,

$$\bar{D} = \varepsilon\bar{E}$$

Hence if D tan1 and D tan 2 are magnitudes of the tangential components of $\bar{D}$ in dielectric1 and 2 respectively then,

$$D_{tan\,1} = \varepsilon\bar{E}_{tan\,1}\ and\ D_{tan\,2} = \varepsilon\bar{E}_{tan\,2}$$
$$\frac{D_{tan\,1}}{\varepsilon_1} = \frac{D_{tan\,2}}{\varepsilon_2}$$
$$\frac{D_{tan\,1}}{D_{tan\,2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_0\varepsilon_{r1}}{\varepsilon_0\varepsilon_{r2}} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$

Thus the tangential component of $\bar{D}$ undergoes some change across the interface hence tangential $\bar{D}$ is said to be discontinuous across the boundary.

To find the normal component use Gauss law,
According to Gauss law,

$$\oint_S \bar{D}.d\bar{S} = Q$$

The surface integral must be evaluated over three surfaces,

i) Top ii) Bottom iii) Lateral

Let the area of top and bottom is same equal to ∆S.

$$\int_{top} \bar{D}.d\bar{S} + \int_{bottom} \bar{D}.d\bar{S} + \int_{lateral} \bar{D}.d\bar{S} = Q$$

$$But \int_{lateral} \bar{D}.d\bar{S} = 0\ as\ \Delta h \to 0$$

$$\int_{top} \bar{D}.d\bar{S} + \int_{bottom} \bar{D}.d\bar{S} = Q$$

And as top and bottom surfaces are elementary, flux density can be assumed constant and can be taken out of integration.

$$\int_{top} \bar{D}.d\bar{S} = \overline{D_{N1}}.\int_{top} d\bar{S} = D_{N1}.\Delta S$$

For top surface the direction of D_N is entering the boundary while for bottom surface, the direction of D_N is leaving the boundary. Both are opposite in direction at the boundary.

$$\int_{bottom} \bar{D}.d\bar{S} = -D_{N2}.\int_{bottom} d\bar{S} = -D_{N2}.\Delta S$$

$$D_{N1}.\Delta S - D_{N2}.\Delta S = Q$$

But, Q = ρSΔS

$$D_{N1} - D_{N2} = \rho S$$

There is no free charge available in perfect dielectric hence no free charge can exists on the surface. All charges in dielectrics are bound are not free. Hence at the ideal dielectrics media boundary the surface charge density ρs can be assumed zero.

$$\rho s = 0$$

$$D_{N1} - D_{N2} = 0$$

$$D_{N1} = D_{N2}$$

Hence the normal component of flux density is $\bar{D}$ continuous at the boundary between two perfect dielectrics.

$$D_{N1} = \varepsilon 1 E_{N1} and\ D_{N2} = \varepsilon 2 E_{N2}$$

$$\frac{D_{N1}}{D_{N2}} = \frac{\varepsilon 2}{\varepsilon 1} = \frac{\varepsilon r2}{\varepsilon r1}$$

The normal components of electric field intensity $\bar{E}$ are inversely proportional to the relative permittivity of the two media.

Refraction of $\bar{D}$ at the boundary

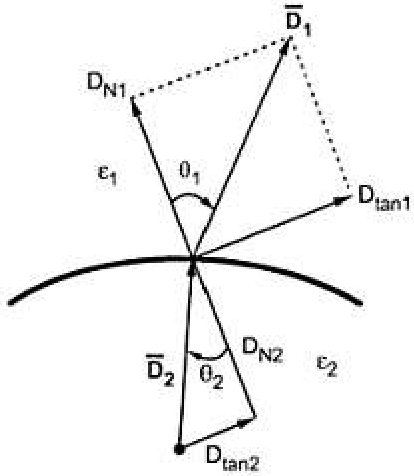


Fig 2.8. Refraction of $\bar{D}$ at the boundary

The direction of $\overline{D}$ and $\overline{E}$ change at the boundary between the two dielectrics. Let $\overline{D_1}$and $\overline{E_1}$ make an angle θ1 with the normal to the surface. $\overline{D_1}$and $\overline{E_1}$ direction is same as $\overline{D_1}$=ε1$\overline{E_1}$,

$$|\overline{D_1}\ | = D_1\ and\ |\overline{D_2}\ | = D_2$$

$$DN_1=D_1 \cos \theta_1$$

$$DN_2=D_2 \cos \theta_2$$

$$\text{But } D_{N1}=D_{N2}$$

$$D_1 \cos \theta_1= D_2 \cos \theta_2$$

$$\frac{D_{tan\,1}}{D_{tan\,2}} = \frac{\varepsilon 1}{\varepsilon 2}$$

$$\frac{D_1\ sin\theta 1}{D_2 sin\theta 2} = \frac{\varepsilon 1}{\varepsilon 2} = \frac{D_{tan\,1}}{D_{tan\,2}}$$

$$\frac{tan\theta 1}{tan\theta 2} = \frac{\varepsilon 1}{\varepsilon 2} = \frac{\varepsilon r2}{\varepsilon r1}$$

This is called law if refraction. Thus the angles θ1 and θ2 are dependent on permitivities of two media and not on $\overline{D}$ or $\overline{E}$. The magnitude of $\overline{D}$ in region 2 can be,

$$D_2^2 = D_{N2}^2 + D_{tan\,2}^2 = (D_1 \cos\theta 1)^2 + D_{tan\,2}^2$$

$$D_{tan\,2} = D_2 sin\theta 2 = \frac{D_1\ sin\theta 1\varepsilon 2}{\varepsilon 1}$$

$$D_2^2 = (D_1 \cos\theta 1)^2 + \left(\frac{D_1\ sin\theta 1\varepsilon 2}{\varepsilon 1}\right)^2$$

$$D_2 = D_1 \sqrt{\cos^2\theta_1 + \left(\frac{\varepsilon 2}{\varepsilon 1}\right)^2 \sin^2\theta_1}$$

Similarly magnitude of E_2 can be obtained as,

$$E_2 = E_1 \sqrt{\sin^2\theta_1 + \left(\frac{\varepsilon 1}{\varepsilon 2}\right)^2 \cos^2\theta_1}$$

The equation shows that

1. $\overline{D}$ is larger in the region of larger permittivity.
2. $\overline{E}$ is larger in the region of smaller permittivity.
3. $|\overline{D}_1| = |\overline{D}_2|$ if θ1=θ2=0.
4. $|\overline{E}_1| = |\overline{E}_2|$ if θ1=θ2=90.

To find the angles θ1 and θ2 with respect to normal use the dot product if normal direction to the boundary is known.

2.7. METHOD OF IMAGES

The method of images, introduced by Lord Kelvin in 1848, is commonly used to determine V, E, D, and ρs due to charges in the presence of conductors. By this method, avoid solving Poisson's or Laplace's equation but rather utilize the fact that a conducting surface is an equipotential. Although the method does not apply to all electrostatic problems, it can reduce a formidable problem to a simple one.

In applying the image method, two conditions must always be satisfied:

- The image charge(s) must be located in the conducting region.
- The image charge(s) must be located such that on the conducting surface(s) the potential is zero or constant.

The first condition is necessary to satisfy Poisson's equation, and the second condition ensures that the boundary conditions are satisfied. Now apply the image theory to two specific problems.

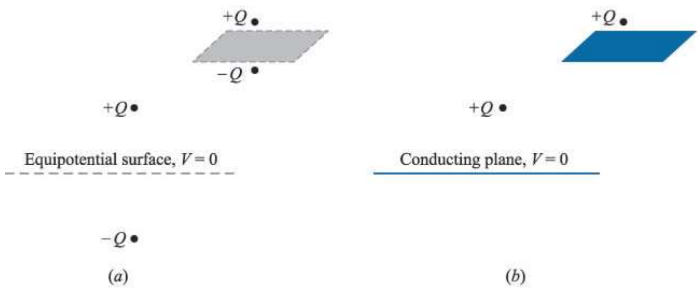


Image system: (a) image configuration with the conducting plane replaced by equipotential surface; (b). charge configurations above a perfectly conducting plane

i. Point Charge Above a Grounded Conducting Plane

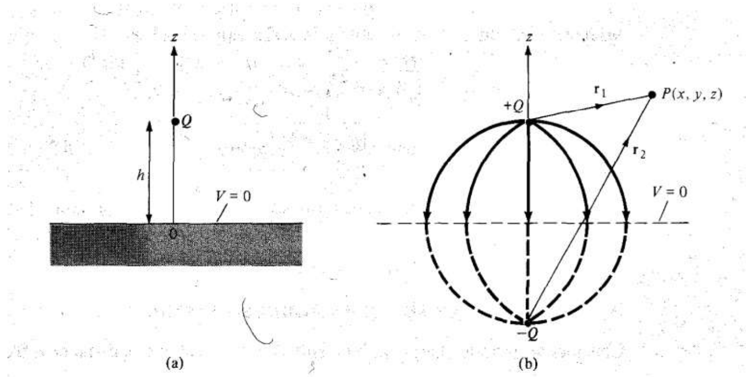


Figure 2.9 (a) Point charge and grounded conducting plane,(b) image configuration and field lines.

Consider a point charge Q placed at a distance h from a perfect conducting plane of infinite extent as in Figure2.9 (a). The image configuration is in Figure 2.9 (b). The electric field at point P(x, y, z) is given by

$$E = E_+ + E_-$$

$$= \frac{Qr_1}{4\pi\varepsilon_0 r_1^3} + \frac{-Qr_2}{4\pi\varepsilon_0 r_2^3}$$

The distance vectors r_1 and r_2 are given by

$$r_1 = (x, y, z) - (0, 0, h) = (x, y, z - h)$$

$$r_2 = (x, y, z) - (0, 0, -h) = (x, y, z + h)$$

$$E = \frac{Q}{4\pi\varepsilon_0}\left[\frac{x\overline{ax} + y\overline{ay} + (z - h)\overline{az}}{(x^2 + y^2 + (z - h)^2)^{\frac{3}{2}}} + \frac{x\overline{ax} + y\overline{ay} + (z + h)\overline{az}}{(x^2 + y^2 + (z + h)^2)^{3/2}}\right]$$

It should be noted that when z = 0, E has only the z-component, confirming that E is normal to the conducting surface. The potential at P is easily obtained using $V = -\int E.dl$.

Thus,

$$V = V_+ + V_-$$

$$= \frac{Q}{4\pi\varepsilon_0 r_1} + \frac{-Q}{4\pi\varepsilon_0 r_2}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{(x^2+y^2+(z-h)^2)^{\frac{3}{2}}} - \frac{1}{(x^2+y^2+(z+h)^2)^{3/2}}\right]$$

for $z \geq 0$ and V = 0 for $z \leq 0$. Note that V(z = 0) = 0. The surface charge density of the induced charge can also be obtained,

$$\rho s = Dn = \varepsilon_0 E_n|_{z=0}$$

$$= \frac{-Qh}{2\pi(x^2+y^2+h^2)^{3/2}}$$

The total induced charge on the conducting plane is

$$Q_i = \int \rho s ds = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{-Qhdxdy}{2\pi(x^2+y^2+h^2)^{3/2}}$$

By changing variables, $\rho^2 = x^2 + y^2$, dx dy = ρdρdΦ

$$Q_i = \frac{-Qh}{2\pi}\int_{-\infty}^{2\pi}\int_{-\infty}^{\infty} \frac{\rho d\rho d\Phi}{(\rho^2+h^2)^{3/2}}$$

Integrating over Φ gives 2π, and letting ρdp = $-\frac{1}{2}d(\rho^2)$ obtain

$$Q_i = \frac{-Qh}{2\pi}2\pi\int_{-\infty}^{\infty} \frac{-\frac{1}{2}d(\rho^2)}{(\rho^2+h^2)^{3/2}}$$

$$= \left|\frac{Qh}{(\rho^2+h^2)^{1/2}}\right|_0^{\infty}$$

$$= -Q$$

as expected, because all flux lines terminating on the conductor would have terminated on the image charge if the conductor were absent.

ii. A Line Charge above a Grounded Conducting Plane

Consider an infinite charge with density ρL C/m located at a distance h from the grounded conducting plane z = 0. The same image system of Figure 2.9(b) applies to the line charge except that Q is replaced by ρL. The infinite line charge ρL may be at x = 0, z = h and the image — ρL at x = 0, z = -h so that the two are parallel to the y-axis.

The electric field at point P is given by,

$$E = E_+ + E_-$$

$$= \frac{\rho_L}{2\pi\varepsilon_0\rho_1}\overline{a\rho}_1 + \frac{-\rho_L}{2\pi\varepsilon_0\rho_2}\overline{a\rho}_2$$

The distance vectors ρ_1 and ρ_2 are given by,

$$\rho_1 = (x, y, z) - (0, 0, h) = (x, y, z - h)$$

$$\rho_2 = (x, y, z) - (0, 0, -h) = (x, y, z + h)$$

$$E = \frac{\rho_L}{2\pi\varepsilon_0}\left[\frac{x\overline{ax} + (z-h)\overline{az}}{x^2+(z-h)^2} - \frac{x\overline{ax} + (z+h)\overline{az}}{x^2+(z-h)^2}\right]$$

Again, notice that when z = 0, E has only the z-component, confirming that E is normal to the conducting surface.

The potential at P is obtained using $V = -\int E.dl$.Thus

$$V = V_+ + V_-$$
$$= -\frac{\rho_L}{2\pi\varepsilon_0} ln\rho_1 - \frac{-\rho_L}{2\pi\varepsilon_0} ln\rho_2$$
$$= -\frac{\rho_L}{2\pi\varepsilon_0} ln\frac{\rho_1}{\rho_2}$$

Substituting $\rho_1 = |\overline{\rho_1}|$ and $\rho_2 = |\overline{\rho_2}|$

$$V = -\frac{\rho_L}{2\pi\varepsilon_0} ln\left[\frac{x^2 + (z - h)^2}{x^2 + (z - h)^2}\right]^{1/2}$$

for z≥0 and V = 0 for z≤0 . Note that V(z = 0) = 0. The surface charge density of the induced charge can also be obtained,

$$\rho s = Dn = \varepsilon_0 E_n|_{z=0}$$
$$= \frac{-\rho_L h}{\pi(x^2+h^2)}$$

The induced charge per length on the conducting plane is

$$\rho_i = \int \rho_s dx = \frac{-\rho_L h}{\pi} \int_{-\infty}^{\infty} \frac{dx}{(x^2+h^2)}$$

By letting x = h tan α,

$$\rho_i = \frac{-\rho_L h}{\pi} \int_{-\pi/2}^{\pi/2} \frac{d\alpha}{h}$$
$$= -\rho_L$$

2.8. RESISTANCE OF A CONDUCTOR

An electric field must exists inside the conductor to sustain the flow of current. As the electrons move, they encounter some damping forces are called resistances.

Based on Ohm's law derive the resistance of the conducting material. Suppose the conductor has a uniform cross section S and is of length l.

$$R = \frac{\rho c l}{S}$$

If the cross section of the conductor is not uniform, eq. R becomes invalid and the resistance is obtained from

$$R = \frac{V}{I} = \frac{\int E.dl}{\int \sigma E.dS}$$

The problem of finding the resistance of a conductor of non uniform cross section can be treated as a boundary-value problem. Using above equation the resistance *R* (or conductance G = 1/R) of a given conducting material can be found by following these steps:

- Choose a suitable coordinate system.
- Assume Vo as the potential difference between conductor terminals.

- Solve Laplace's equation $\nabla^2 V$ to obtain V. Then determine E from E = $-\nabla V$ and I from I=$\int \sigma E.dS$.
- Finally, obtain *R* as $R = \frac{V_0}{I}$

In essence, assume Vo, find I, and determine $R = \frac{V_0}{I}$. Alternatively, it is possible to assume current I_0, find the corresponding potential difference V, and determine *R* from *R = V/Io.* As will be discussed shortly, the capacitance of a capacitor is obtained using a similar technique.

Simply Resistance of A Conductor are,

Consider that the voltage V is applied to a conductor of length L having uniform cross section S, as shown in fig.

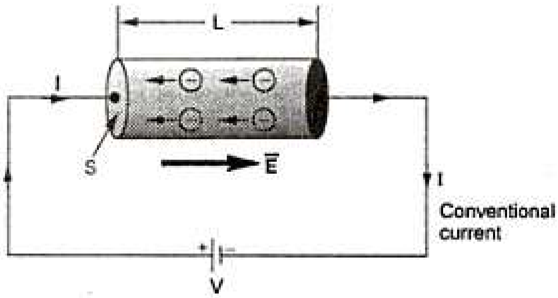


Fig 2.10 Resistance of a Conductor

The direction of $\bar{E}$ is same as the direction of conventional current, which is opposite to the flow of electrons. The electric field applied is uniform and its magnitude is given by,

$$E = \frac{V}{L}$$

The conductor has uniform cross section S and hence,

$$I = \int_S \bar{J}.d\bar{S} = JS$$

The current direction is normal to the surface S.

$$J = \frac{I}{S} = \sigma E$$

And using equation,

$$J = \frac{\sigma V}{L}$$

$$V = \frac{JL}{\sigma} = \frac{IL}{\sigma S} = \left(\frac{L}{\sigma S}\right) I$$

$$R = \frac{V}{I} = \frac{L}{\sigma S}$$

Thus the ratio of potential difference between the two ends of the conductors to the current flowing through it is resistance of the conductor. For the nonuniform field the resistance R is defined as the ratio V to I where V is the potential difference between two specified equipotential surface in the material and I is the current crossing the more positive surface of the two, into the material. Mathematically the resistance for the nonuniform field is given by,

$$R = \frac{V_{ab}}{I} = -\frac{\int_a^b \bar{E}.d\bar{L}}{\int_S \bar{J}.d\bar{S}} = \frac{-\int_a^b \bar{E}.d\bar{L}}{\int_S \sigma\bar{E}.d\bar{S}}$$

The numerator is the line integration giving potential difference across two ends while the denominator is a surface integration giving current flowing through the material. The resistance can also expressed as

$$R = \frac{L}{\sigma S} = \frac{\rho c L}{S} \Omega$$

$$\rho c = \frac{1}{\sigma} = \text{Resistivity of the conductor in } \Omega - \text{m}$$

2.9. CAPACITANCE

A capacitor we must have two (or more) conductors carrying equal but opposite charges. This implies that all the flux lines leaving one conductor must necessarily terminate at the surface of the other conductor. The conductors are sometimes referred to as the plates of the capacitor. The plates may be separated by free space or a dielectric.

Consider two conducting materials M_1 and M_2 which are placed in a dielectric medium having permitivity ε. The material M1 carries a positive charge Q while the material M_2 carries a negative charge, equal in magnitude as Q. There are no other charge present and total charge of the system is zero. In conductor charge cannot reside within the conductor and in resides only on the surface. Thus for M_1 and M_2 charges +Q and –Q reside on the surface of M_1 and M_2 respectively. Shown in fig.

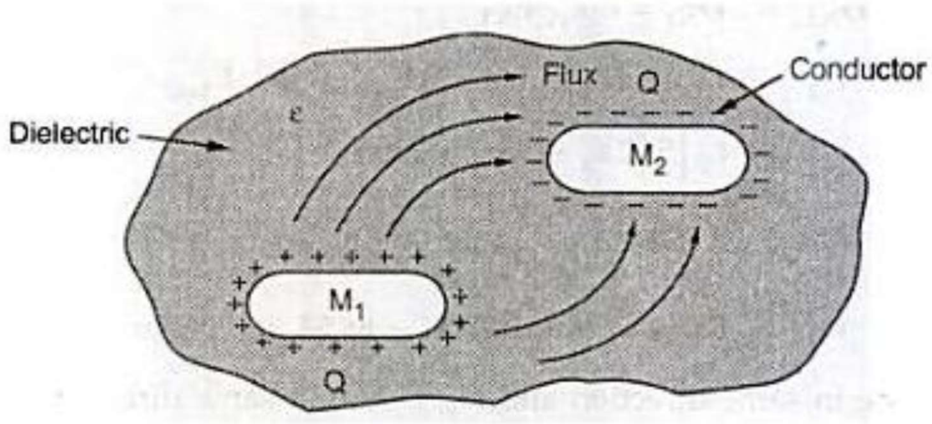


Figure 2.11. A two-conductor capacitor.

Such a system which has two conducting surface carrying equal and opposite charges, separated by a system given raise to a capacitance. The electric field is normal to the conductor surface and the electric flux density is directed from M_1 towards M_2 in such a system. There exists a potential difference between the two surfaces of M_1 and M_2. Let this potential is V_{12}. The ratio of the magnitudes of the total charge on any one of the two conductors and potential difference between the conductor is called Capacitance of two conductor system . denoted as C,

$$C = \frac{Q}{V_{12}}$$

$$In\ general\ C = \frac{Q}{V}$$

Where Q- Charge in coulombs
V- Potential difference in Volts
Capacitance are measured in Farads (F)

$$1\,farad = \frac{1\,Coulomb}{1Volt}$$

As charge Q resides only on the surface of the conductor it can be obtained from the Gauss law as,

$$Q = \oint_S \bar{D}.d\bar{S} = \oint_S \varepsilon_0 \varepsilon_r\,\bar{E}.d\bar{S} = \varepsilon \oint_S \bar{E}.d\bar{S}$$

While V is the work done in moving unit positive charge from negative to the positive surface and can be obtained as,

$$V = -\oint_L \bar{E}.d\bar{L} = -\oint_-^+ \bar{E}.d\bar{L}$$

Hence the capacitance are expressed as,

$$C = \frac{Q}{V} = \frac{\varepsilon \oint_S \bar{E}.d\bar{S}}{-\oint_-^+ \bar{E}.d\bar{L}} F$$

If the charge Q is increased then E and D get increased by the same factor. The voltage V also increased by same factor. Thus the ration Q to V remains constant as C.

The capacitance *C* is a physical property of the capacitor and in measured in farads (F). *C* can be obtained for any given two-conductor capacitance by following either of these methods:

- Assuming Q and determining V in terms of Q (involving Gauss's law)
- Assuming V and determining Q in terms of V(involving solving Laplace's equation)

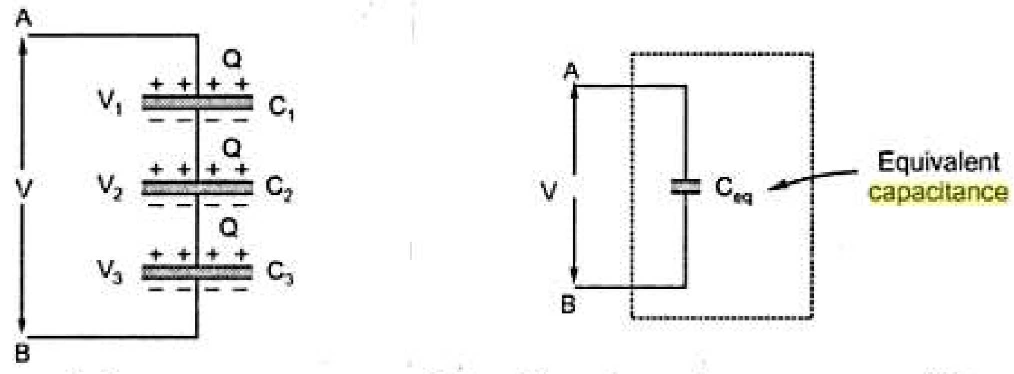
The former method involves taking the following steps:

- Choose a suitable coordinate system.
- Let the two conducting plates carry charges + Q and — Q.
- Determine E using Coulomb's or Gauss's law and find V from V= $-\int E.dl$. The negative sign may be ignored in this case because of interested in the absolute value of V.
- Finally, obtain *C* from $C = \frac{Q}{V}$.

This mathematically attractive procedure to determine the capacitance of some important two-conductor configurations.

2.9.1. CAPACITOR IN SERIES

Consider three capacitors in series connected across the applied voltage V as shown in fig. Suppose this pushes charge Q on C_1 then the opposite plate of C_1 must have the same charge. This charge which is negative must have been obtained from the connecting leads by the charge separation which means that the charge on the upper plate of C_2 is also Q. In short all the three capacitors have the same charge Q.



(a) Fig 2.1 2 Capacitors in series (b)

$$Q = C_1V_1 = C_2V_2 = C_3V_3$$
$$V_1 = \frac{Q}{C_1}; \quad V_2 = \frac{Q}{C_2}; \quad V_3 = \frac{Q}{C_3}$$

If an equivalent capacitance also store the same charge, when applied with the same voltage, then,

$$C_{eq} = \frac{Q}{V} \quad or\, V = \frac{Q}{V_{eq}}$$

But $V=V_1+V_2+V_3$

$$\frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$
$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

It is easy to find V_1,V_2 and V_3 if Q is known,
For n capacitors in series,

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \frac{1}{C_n}$$

2.9.2. CAPACITORS IN PARALLEL

When capacitors are in parallel, the same voltage exists across them, but charges are different.

$$Q_1 = C_1V, \qquad Q_2 = C_2V \quad Q_3 = C_3V$$

The total charge stored by the parallel bank of capacitor Q is given by,

$$Q=Q_1+Q_2+Q_{3]}$$
$$Q = C_1V + C_2V + C_3V$$
$$= (C_1 + C_2 + C_3)V$$

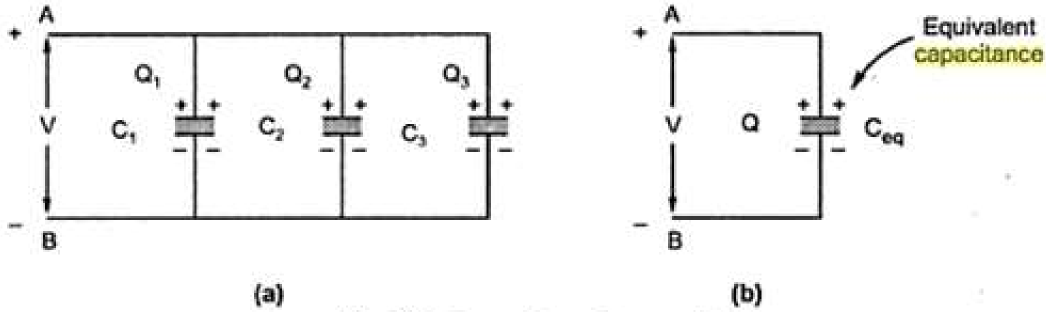


Fig. 2.13 Capacitors in parallel

An equivalent capacitor which is store the same charge Q at the same voltage V, will

$$Q = C_{eq} V$$

Comparing,

$$C_{eq} = C_1 + C_2 + C_3$$
$$Q = C_1 V + C_2 V + C_3 V$$

It is easy to find Q_1,Q_2 and Q_3 if Vis known,
For n capacitors in parallel,

$$C_{eq} = C_1 + C_2 + C_3 + \cdots . + C_n$$

2.9.3. CAPACITANCE OF PARALLEL PLATE

A parallel plate capacitor consists of two parallel metallic plate separated by distance 'd'. The space between the parallel plate is filled with the dielectric of permittivity ε. The lower plate, plate 1 carries the positive charge distributed over it with a density +ρs. The upper plate , plate 2 carries the negative charge distributed over it with a density -ρs. The plate 1 is placed in z=0 while plate 2 is in z=d plane, parallel to xy plane.

A parallel plate capacitor with two dielectrics is shown in the figure.

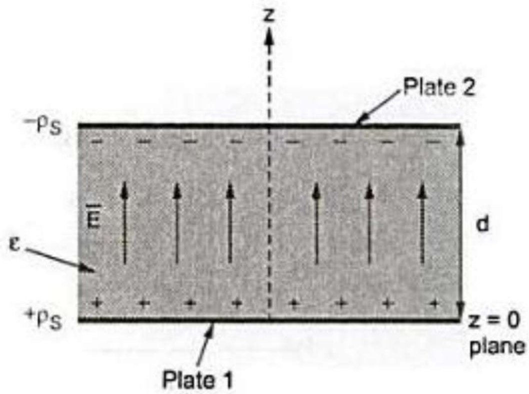


Fig:2.14. Parallel plate capacitor with two dielectrics

The relative permittivity of dielectric medium 1 and medium 2 are ε_{r1} and ε_{r2} respectively. Te potential across the capacitor is V. The potential difference across medium 1 is V_1 and V_2 for medium, 2, E_1 and E_2 are Electric field intensity for medium 1 and medium 2.
Let A- Area of cross section of the plate in m^2.

$$Q = \rho s\ A\ C$$

It is not depends on the charge and potential difference between the plates.

$V_1 = E_1 d_1$ $\qquad\qquad\qquad$ $V_2 = E_2 d_2$

$V = V_1 + V_2 = E_1 d_1 + E_2 d_2$

This is magnitude of charge on any one plate as charge carried by both is equal in magnitude. To find potential difference is,

$$\overline{E_1} = \frac{\rho s}{2\varepsilon}\overline{a_N} = \frac{\rho s}{2\varepsilon}\overline{a_z}\ V/m$$

The E1 is normal at the boundary between conductor and dielectric without any tangential component.
While for plate 2,

$$\overline{E_2} = -\frac{\rho s}{2\varepsilon}\overline{a_N} = -\frac{\rho s}{2\varepsilon}(-\overline{a_z})\ V/m$$

The direction of E2 is downwards in –az direction. In between the plate,

$$\bar{E} = \overline{E_1} + \overline{E_2} = \frac{\rho s}{2\varepsilon}\overline{a_z} + \frac{\rho s}{2\varepsilon}\overline{a_z}$$

$$= \frac{\rho s}{\varepsilon}\overline{a_z}$$

The potential difference is given by,

$$V = -\int_{-}^{+} \bar{E}.d\bar{L} = -\int_{upper}^{lower} \frac{\rho s}{\varepsilon}\overline{a_z}.d\bar{L}$$

$$Now\ d\bar{L} = dx\overline{a_x} + dy\overline{a_y} + dz\overline{a_z}\ in\ cartesial\ system.$$

$$V = -\int_{z=d}^{z=0} \frac{\rho s}{\varepsilon}\overline{a_z}.\left[dx\overline{a_x} + dy\overline{a_y} + dz\overline{a_z}\right]$$

$$V = -\int_{z=d}^{z=0} \frac{\rho s}{\varepsilon}\overline{dz} = \frac{\rho s}{\varepsilon}[z]_d^0$$

$$= -\frac{\rho s}{\varepsilon}[-d]$$

$$V = \frac{\rho s}{\varepsilon}d\ \ V$$

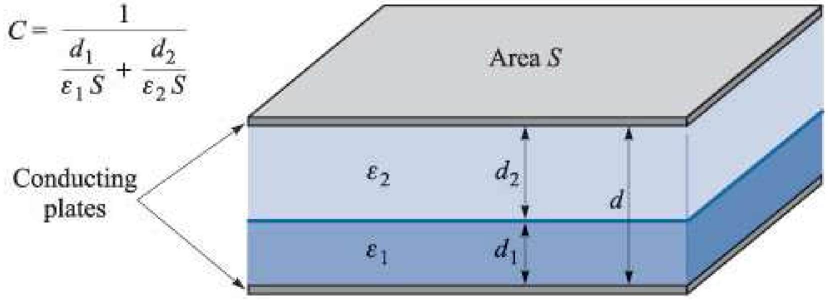
The capacitance is the ratio of charge Q to voltage V,

$$C = \frac{Q}{V} = \frac{\rho s A}{\frac{\rho s}{\varepsilon}d} = \frac{\varepsilon A}{d}\ F$$

$$C = \frac{\varepsilon_0 \varepsilon_r A}{d}\ F$$

Now the value of capacitance depends on,

1. The permittivity of the dielectrics used.
2. The area of cross section of the plates.
3. The distance of separations of the plates.



For three dielectric media,

$$C = \frac{A\varepsilon_0}{\frac{d_1}{\varepsilon_{r1}} + \frac{d_2}{\varepsilon_{r2}} + \frac{d_3}{\varepsilon_{r3}}} \quad \text{(Farads)}$$

2.9.4. CAPACITANCE OF COAXIAL CABLE

Consider a coaxial cable of inner radius ‘a’ and outer radius ‘b’ as shown in figure. The relative permittivity of dielectric filled in between two coaxial cylinders is E. A potential difference V is applied in between two cylinders. The two cylinders are charged at the rate of ρl c/m.

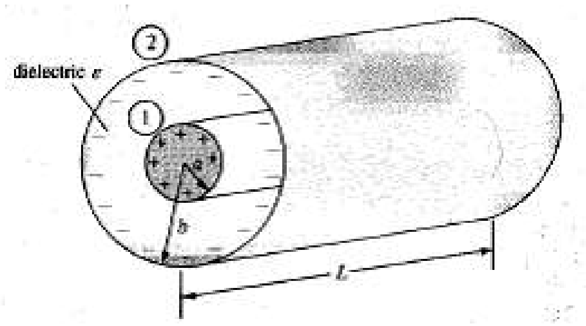


Fig: 2.15 : Coaxial cable

Let a= inner radius
b= outer radius

The two concentric conductors are separated by dielectric of permittivity ε. The length of the cable is L m. The inner conductor carries the charge density +ρL m C/m on its surface then equal and opposite charge density –ρL C/m exists on the outer conductor, By applying Gauss’s law, the electric field E at any distance ‘r’ from the axis of cylinder is given by,

$$\bar{E} = \frac{\rho L}{2\pi\varepsilon r}\overline{a_r}$$

$\bar{E}$ is directed from inner conductor to the outer conductor. The potential difference is work done in moving unit charge against $\bar{E}$ i.e from r=b to r=a. To find the potential difference consider dL in radial direction which is dr$\overline{a_r}$.

$$V = -\int_{-}^{+} \bar{E}.d\bar{L} = -\int_{r=b}^{r=a} \frac{\rho L}{2\pi\varepsilon r}\overline{a_r}.\,\mathrm{dr}\overline{a_r}$$

$$= -\int_{r=b}^{r=a} \frac{\rho L}{2\pi\varepsilon r}\,\mathrm{dr}$$

$$= -\frac{\rho L}{2\pi\varepsilon r}[\ln r]_b^a$$
$$= -\frac{\rho L}{2\pi\varepsilon r}\ln\frac{a}{b}$$
$$= \frac{\rho L}{2\pi\varepsilon r}\ln\frac{b}{a}V$$
$$C = \frac{Q}{V} = \frac{\rho L x L}{\frac{\rho L}{2\pi\varepsilon r}\ln\frac{b}{a}}$$
$$C = \frac{2\pi\varepsilon L}{\ln\frac{b}{a}} F$$

If the length of the cylindrical capacitor is L, capacitance of the capacitor is

$$C = \frac{2\pi\varepsilon}{l_n \frac{b}{a}}.L \text{ Farads,}$$

2.9.5. SPHERICAL CAPACITOR

Consider a spherical capacitor formed of two concentric spherical conducting shells of radius a and b. The capacitor is shown in fig.

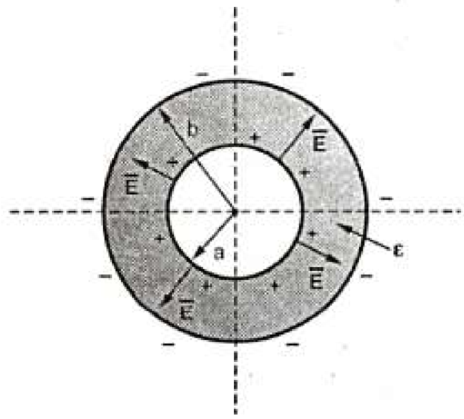


Fig 2.16: Spherical Capacitor

The radius of outer sphere is b while that the inner sphere ia a. Thus b>a. the region between the two spheres is filled with a dielectric if permittivity ε. The inner sphere is given positive charge +Q while for the outer sphere is –Q. Considering Gaussian surface as a sphere of radius r, it can be obtained that $\bar{E}$ is in radial direction as given as,

$$\bar{E} = \frac{Q}{4\pi\varepsilon r^2}\overline{a_r}V/m$$

The potential difference is work done in moving unit positive charge against the direction of $\bar{E}$ i.e. r=b to r=a. . To find the potential difference consider dL in radial direction which is dr$\overline{a_r}$.

$$V = -\int_{-}^{+} \bar{E}.d\bar{L} = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2}\overline{a_r}.\,\mathrm{dr}\overline{a_r}$$

$$= -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2}\mathrm{dr}$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right] V$$

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]}$$

$$C = \frac{4\pi\varepsilon}{\left[\frac{1}{a} - \frac{1}{b}\right]} F$$

2.9.5.1. CAPACITANCE OF SPHERICAL CAPACITORS ISOLATED SPHERE

A sphere of radius r having the charge Q coulombs is shown in figure. The absolute potential is the work done in carrying a positive test charge form infinity to the sphere. It is given by

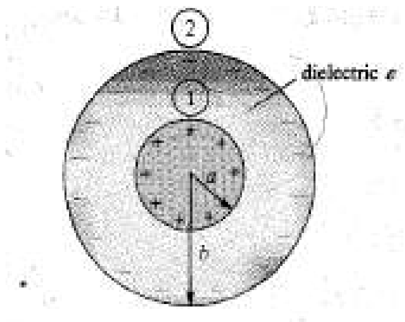


Fig2.17 : Spherical Capacitors

$$V = -\int_{\infty}^{\pi} E.dr$$

$$E = \frac{Q}{4\pi\varepsilon r^2}$$

$$V = -\int_{\infty}^{\pi} \frac{Q}{4\pi\varepsilon r^2} dr$$

$$= \frac{Q}{4\pi\varepsilon}\left[-\int_{\infty}^{\pi} \frac{1}{\pi^2} dr\right]$$

$$= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{\pi} - \frac{1}{\infty}\right]$$

$$V = \frac{Q}{4\pi\varepsilon r}$$

$$V = \frac{Q}{4\pi\varepsilon r}$$

$$C = \frac{Q}{V} = 4\pi\varepsilon r \text{ (Farads)}$$

$$R = \frac{d}{\sigma S} = \frac{1}{4\pi\sigma r} \ (ohms)$$

2.9.5.2. CAPACITANCE OF CONCENTRIC SPHERES

If the charge Q is distributed uniformly over the outer surface of the inner sphere, there will be equal and opposite charge induced on their inner surface of her inner sphere. Te electric field intensity at distance r, in between inner and outer sphere is given as

$$E = \frac{Q}{4\pi\varepsilon r^2} \left(a \le r \le b\right)$$

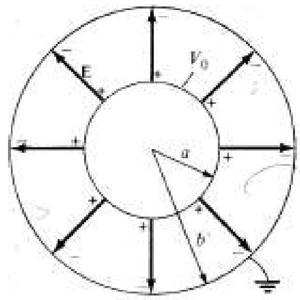


Fig: 2.18 : Concentric Spheres

The potential difference between the spheres is

$$V = -\int_b^a E.dr$$

$$= -\int_b^a \frac{Q}{4\pi\varepsilon r^2} dr$$

$$= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r}\right]_b^a$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{b-a}{ab}\right]$$

Capacitance $C = \frac{Q}{V} = 4\pi\varepsilon\left[\frac{ab}{b-a}\right]$

$$C = 4\pi\varepsilon\left[\frac{ab}{b-a}\right] \text{ Farads.}$$

2.9.6. CAPACITANCE OF PARALLEL CONDUCTORS
(or)
TWO WIRE TRANSMISSION LINE.

Two parallel conductors of radius 'a' separated by a distance 'd' is shown in the figure. If the conductor A has charge of ρ_l c / m along its length, this will induce a charge of + ρ_l c/m on conductor B.

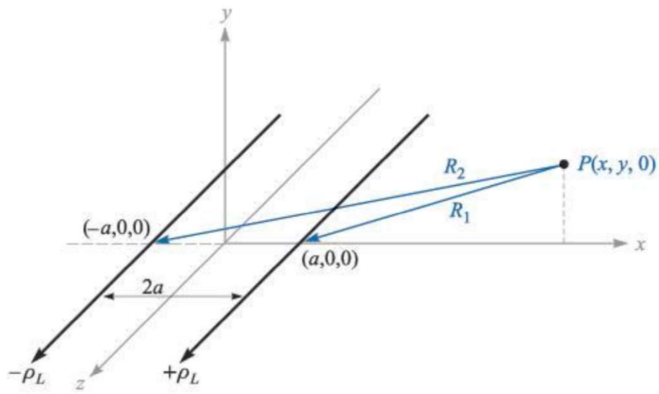


Fig 2.19 : Two Wire Transmission Line

The electric field intensity at any point p with a distance r from the conductor A is the sum of E at P due to conductor A and B.

$$E = \frac{\rho_l}{2\pi\varepsilon_r} + \frac{\rho_l}{2\pi\varepsilon(d-r)}$$

$$E = \frac{\rho_l}{2\pi\varepsilon}\left[\frac{1}{r} + \frac{1}{d-r}\right]$$

Potential difference between the conductors is given by.

$$V = -\int E.dr$$

$$V = \frac{-\rho l}{2\pi\varepsilon}\int_{d-a}^{a}\left(\frac{1}{r} + \frac{1}{d-r}\right)^{dr}$$

$$V = \frac{-\rho l}{2\pi\varepsilon}\left[l_n r - l_n(d-r)\right]_{d-a}^{a}$$

$$V = \frac{-\rho l}{2\pi\varepsilon}\left[l_n a - l_n(d-a) - l_n(d-a) + l_n a\right]$$

$$V = \frac{-\rho l}{2\pi\varepsilon}\left[l_n\left(\frac{a}{d-a}\right) + l_n\left(\frac{a}{d-a}\right)\right]$$

$$= \frac{-\rho l}{2\pi\varepsilon}\left[2l_n\left(\frac{a}{d-a}\right)\right]$$

$$= \frac{-\rho l}{\pi\varepsilon}\left[l_n\left(\frac{a}{d-a}\right)\right]$$

$$C = \frac{\rho l}{2\pi\varepsilon} l_n\left(\frac{d-a}{a}\right)$$

Capacitance per unit length is

$$C = \frac{\rho l}{V} = \frac{\pi\varepsilon}{l_n\left(\frac{d-a}{a}\right)}$$

if d > > a

$$\frac{C}{l} = \frac{\pi\varepsilon}{l_n\left(\frac{d}{a}\right)} \text{ F/m}$$

2.10. POISSON'S &LAPLACE'S EQUATION

Poisson's and Laplace's equations are easily derived from Gauss's law (for a linear material medium). The Maxwell's equation form Gauss Law is

$$\nabla . D = \rho$$

Since $D = \varepsilon E$ and $E = -\nabla V$,

$$D = -\varepsilon \nabla V$$

for an inhomogeneous medium. For a homogeneous medium,

$$\therefore \nabla . \nabla V = \frac{-\rho}{\varepsilon}$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon} \text{ - Poisson's Equation.}$$

This is known as Poisson's equation. A special case of this equation occurs when ρ= 0, (i.e., for a charge-free region).

$$\nabla^2 V = 0 \text{ - Laplace's form.}$$

which is known as Laplace's equation. Assumed that ε is constant throughout the region in which V is defined; for an inhomogeneous region, ε is not constant Poisson's equation for an inhomogeneous medium; it becomes Laplace's equation for an inhomogeneous medium when ρ

= 0. Recall that the Laplacian operator ∇^2 was derived. Thus Laplace's equation in Cartesian, cylindrical, or spherical coordinates respectively is given by

Laplace equation in Cartesian form.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

In cylindrical coordinate system.

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial_r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial V^2}{\partial z^2} = 0$$

In spherical coordinate system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial_r}\right) + \frac{1}{r^2 Sin\theta}\frac{\partial}{\partial\theta}\left[Sin\theta\frac{\partial v}{\partial\theta}\right] + \frac{1}{r^2 Sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

Laplace's equation is of primary importance in solving electrostatic problems involving a set of conductors maintained at different potentials. Examples of such problems include capacitors and vacuum tube diodes.

2.10.1. SOLUTION OF LAPLACE EQUATION

PROCEDURE FOR SOLVING LAPLACE EQUATIONS

Step 1: Solving the Laplace equation using the method of integration. Assume constants of integration as per the requirement.

Step 2: Determine the constants applying the boundary conditions given or known for the region. The solution obtained in step 1 with constants obtained using boundary conditions is an unique solution.

Step 3: Then $\bar{E}$ can be obtained for the potential field V obtained, using gradient operation $-\nabla V$.

Step 4: For homogeneous medium, $\bar{D}$ can be obtained as $\varepsilon\bar{E}$.

Step 5: At the surface ρs= D_N hence once $\bar{D}$ is known the normal component to the surface D_N is known. Hence the charge induced on the conductor surface can be obtained as $Q = -\int \rho_s dS$.

Step 6: Once the charge induced Q is known and potential V is known then the capacitance C of the system can be obtained. If necessary, the capacitance between two conductors can be found using C = Q/V.

If $\rho v \neq 0$ then similar procedure can be adopted to solve the Poisson's equation.

Solving Laplace's (or Poisson's) equation, as in step 1, is not always as complicated as it may seem. In some cases, the solution may be obtained by mere inspection of the problem. Also a solution may be checked by going backward and finding out if it satisfies both Laplace's (or Poisson's) equation and the prescribed boundary conditions.

2.11. SOLUTION OF LAPLACE EQUATION

2.11.1. Calculating the capacitance of Sphere:

Solve the laplace equation for the potential field in the homogenious region between two concentric conducting sphere with radius a and b such that b>a if potential V=0 at r=b and V=V0 at r=a and find the capacitance between the two concentric sphere.

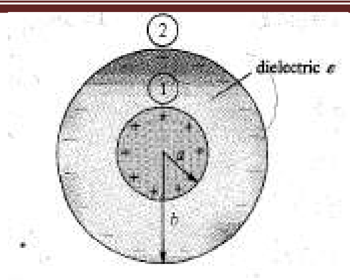


Fig2.20 : Sphere

Consider the sphere with radius a&b such that b>a if potential V = V_0 at r=b and V=V_0 at r=a

According to Laplace equation

$$\nabla^2 V = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$$

$$Integrate\ r^2\frac{\partial V}{\partial r} = \int 0 + C_1 = C_1$$

$$\frac{\partial V}{\partial r} = \frac{C_1}{r^2}$$

$$V = \int \frac{C_1}{r^2} + C_2$$

$$= -\frac{C_1}{r} + C_2$$

Use the boundary condition

$$V=0\ r=b \rightarrow -\frac{C_1}{b} + C_2 = 0$$

$$V=V_0\ r=a \rightarrow -\frac{C_1}{a} + C_2 = V_0$$

Subtract

$$\frac{C_1}{a} - \frac{C_1}{b} = -V_0$$

$$C_1 = \frac{V_0}{\frac{1}{b} - \frac{1}{a}}$$

$$C_2 = -\frac{C_1}{b} = \frac{V_0}{b\left(\frac{1}{b} - \frac{1}{a}\right)}$$

$$V = -\frac{V_0}{r\left(\frac{1}{b} - \frac{1}{a}\right)} + \frac{V_0}{b\left(\frac{1}{b} - \frac{1}{a}\right)}\ \text{Volts}$$

This is the Potential field in the region between the two spheres

$$E = -\nabla V = -\frac{\partial V}{\partial r}\overline{a_r} = -\frac{\partial}{\partial r}\left[-\frac{V_0}{r\left(\frac{1}{b} - \frac{1}{a}\right)}\right]\overline{a_r}$$

$$= \left[-\frac{V_0}{r^2\left(\frac{1}{b} - \frac{1}{a}\right)}\right]\overline{a_r}$$

$$\overline{D} = \overline{\varepsilon E} = -\frac{\varepsilon V_0}{r^2\left(\frac{1}{b}-\frac{1}{a}\right)}\overline{a_r}$$

Surface charge density

$$\rho s = |D_N| = D$$

$$= -\frac{\varepsilon V_0}{r^2\left(\frac{1}{b}-\frac{1}{a}\right)}$$

$$= \frac{\varepsilon V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)} C/m2$$

Q= ρs x Surface area of Sphere

$$= \frac{\varepsilon V_0}{r^2\left(\frac{1}{a}-\frac{1}{b}\right)} 4\pi r^2$$

$$= \frac{4\pi\varepsilon V_0}{\left(\frac{1}{a}-\frac{1}{b}\right)} C$$

$$C = \frac{Q}{V} = \frac{Q}{V_0} = \frac{4\pi\varepsilon V_0}{V_0\left(\frac{1}{a}-\frac{1}{b}\right)}$$

$$C = \frac{4\pi\varepsilon_0}{\left(\frac{1}{a}-\frac{1}{b}\right)} F$$

2.11.2. Calculating the capacitance of Co axial cable:

Use laplace equation to find the capacitance per unit length of co axial cable of inner radius a meter and outer radius b meter. Assume V=V0 at r=a and V=0 at r=b.

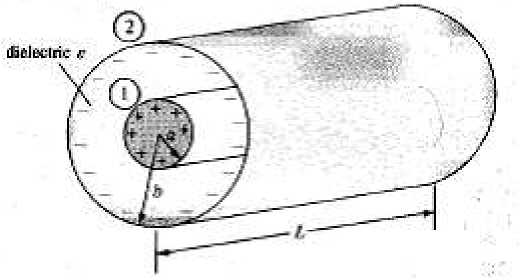


Fig: 2.21 : Coaxial cable

Consider the cylinder with radius a&b such that b>a if potential V = V_0 at r=b and V=V_0 at r=a

According to Laplace equation

$$\nabla^2 V = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0$$

Integrate $r\frac{\partial V}{\partial r} = \int 0 + C_1 = C_1$

$$\frac{\partial V}{\partial r} = \frac{C_1}{r}$$

$$V = \int \frac{C_1}{r} + C_2$$

$$= C_1 \ln r + C_2$$

Use the boundary condition

$$V=0 \quad r=b \rightarrow C_1 \ln b + C_2 = 0$$

$$V=V_0 \quad r=a \rightarrow C_1 \ln a + C_2 = V_0$$

Subtract

$$C_1(\ln b - \ln a) = -V_0$$

$$C_1 = \frac{-V_0}{\ln \frac{b}{a}}$$

$$C_1 = \frac{V_0}{\ln \frac{a}{b}}$$

$$C_2 = -C_1 lnb = -\frac{V_0}{\ln \frac{a}{b}} \ln b$$

$$V = \frac{V_0}{ln \frac{a}{b}} ln\, r - \frac{V_0}{ln \frac{a}{b}} lnb \; Volts$$

This is the Potential field in the region between the two spheres

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r}\overline{a_r} = \frac{\partial}{\partial r}\left[-\frac{V_0 \ln r}{\ln \frac{a}{b}}\right]\overline{a_r}$$

$$= -\frac{V_0}{\ln \frac{a}{b}} \frac{\partial}{\partial r}(\ln r)\overline{a_r}$$

$$= -\frac{V_0}{r \ln \frac{a}{b}}\overline{a_r} \text{ V/m}$$

$$\bar{D} = \varepsilon \bar{E} = -\frac{\varepsilon V_0}{r \ln \frac{a}{b}}\overline{a_r}$$

$$= \frac{\varepsilon V_0}{r \ln \frac{b}{a}}\overline{a_r}$$

Surface charge density

$$\rho s = |D_N| = D = \frac{\varepsilon V_0}{r \ln \frac{b}{a}}$$

$$= \frac{\varepsilon V_0}{r \ln \frac{b}{a}} c/m^2$$

Q= ρs x Surface area of Sphere

$$= \frac{\varepsilon V_0}{r \ln \frac{b}{a}} 2\pi r L$$

$$= \frac{2\pi \varepsilon V_0}{\ln \frac{b}{a}} C$$

$$C = \frac{Q}{V} = \frac{Q}{V_0} = \frac{2\pi \varepsilon V_0}{ln \frac{b}{a}}$$

$$C = \frac{2\pi\varepsilon V_0}{\ln\frac{b}{a}} \, F$$

2.11.3. *Calculating the capacitance of parallel plate*

Two parallel conducting plate are separated distance d apart and filled with dielectrics medium having εr and relative permittivity using Laplace equation. Derive an expression for capacitance per unit length parallel plate capacitor if it is connected to a dc source supplying V volt.

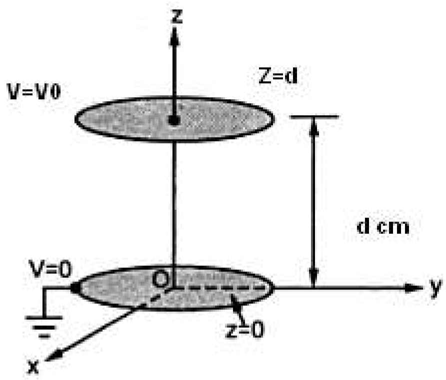


Fig. 2.22 Parallel plate

$$\nabla^2 V = 0$$
$$\frac{\partial^2 V}{\partial Z} = 0$$

On integrating,

$$\frac{\partial V}{\partial Z} = C_1$$

Integrating,

$$V = C_1 + C_2$$

Apply the boundary conditions,

$$Z=0\ ,\ V=0$$
$$Z=d\ ,\ V=V_0$$

When Z=0 , V=0,

$$0 = 0 + C_2$$
$$C_2 = 0$$

When Z=d , V=V_0,

$$V = C_1 d + C_2$$
$$V = C_1 d$$
$$C_1 = \frac{V}{d}$$
$$V = \frac{V}{d} Z + 0$$

The electric field intensity,

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial Z}$$

$$= -\frac{\partial}{\partial Z}\left(\frac{V}{d}Z\right)$$

$$\bar{E} = -\frac{V}{d}$$

The electric flux density is,

$$\bar{D} = \varepsilon \bar{E} = -\frac{V\varepsilon}{d}$$

$$|D_N| = \varepsilon \bar{E} = \frac{V\varepsilon}{d}$$

The Charge,

$$Q = \int \rho s \,.\, ds$$

$$Q = \frac{V\varepsilon}{d} A$$

The Capacitance,

$$\frac{Q}{V} = \frac{\varepsilon A}{d} \;\; F$$

2.12. APPLICATION OF POISSON'S AND LAPLACE'S EQUATIONS.

Laplace's and Poisson's equations are not only useful in solving electrostatic field problem; they are used in various other field problems. For example, V would be interpreted as magnetic potential in magneto statics, as temperature in heat conduction, as stress function in fluid flow, and as pressure head in seepage.

PROBLEMS

Example:2.1

A parallel plate air core capacitor is flowing with 10V across the plates. If the separation between the plate is now increased to double the original value, calculate the new value of the voltage across the capacitor.

Solution: $Q = CV \quad V = \frac{Q}{C}$

$$C = \frac{A\varepsilon_0 A}{d}$$

If d doubles, New capacitance $C' = \frac{\varepsilon_0 A}{2d}$

$$C' = \frac{C}{z}$$

$$V^1 = \frac{Q}{C^1} = \frac{2Q}{C}$$

$V^1 = 2V$

$= 2 \times 10 = 20$ V.

Example:2.2

A parallel plate capacitor with d = 1m and plate area 0.8 m^2 and a dielectric relative permittivity of 2.8. A dc volt of 500 v is applied between the plates. Find the capacitance and energy stored.

Solution:

Capacitance $C = \frac{A\varepsilon}{d}$

d = 1m, A = 0.8 m^2 $\varepsilon = \varepsilon_0 \varepsilon_r = 8.854 \times 10^{-12} \times 2.8$

$C = \frac{8.854 \times 10^{-12} \times 0.8 \times 2.8}{1}$ = 19.833 pF

Energy stored $= \frac{1}{2}CV^2 = \frac{1}{2} \times 19.833 \times 10^{-12} \times (500)^2$

E = 2.48×10^{-6} J.

Example:2.3

Find the resistance of a 1 Km length of is wire which has a conductivity $\sigma = 6.17 \times 10^{-7}\, ohm^{-1}/m$ and r radius of 1×10^{-3} m.

Solution: Resistance $R = \frac{pl}{a} = \frac{l}{\sigma a}$

Area $= \pi r^2 = \pi \times (1 \times 10^{-3})^2 = \pi \times 10^{-6} m^2$

l = 1×10^3 m, $R = \frac{1 \times 10^3}{6.17 \times 10^{-7} \times \pi \times 10^{-6}} = 5.159\Omega$

Example:2.4

Find the frequency at which conduction current density and displacement current density are equal in a medium with $\sigma = 2\times10^{-4}$ ℧/m and $\varepsilon_r = 81$.

Solution : The ratio of amplitudes of the two current densities is given as 1,

$$\frac{|\bar{J}_C|}{|\bar{J}_D|} = \frac{\sigma}{\omega\varepsilon} = 1$$

i.e. $$\omega = \frac{\sigma}{\varepsilon} = \frac{\sigma}{\varepsilon_0 \varepsilon_r}$$

$\therefore$ $$\omega = \frac{2\times10^{-4}}{(8.854\times10^{-12})(81)} = 0.2788\times10^6 \text{ rad/sec}$$

But $$\omega = 2\pi f$$

$\therefore$ $$f = \frac{\omega}{2\pi} = \frac{0.2788\times10^6}{2\pi} = \mathbf{44.372 \ kHz}$$

Hence, the frequency at which the ratio of amplitudes of conduction and displacement current density is unity, is 44.372 kHz.

Example. 2.5

Determine whether or not the following potential fields satisfy the Laplace's equation :

***a)** $V = x^2 - y^2 + z^2$ **b)** $V = r\cos\phi + z$ **c)** $V = r\cos\theta + \phi$*

Solution : a) $V = x^2 - y^2 + z^2$

$$\therefore \ \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{\partial^2}{\partial x^2}[x^2 - y^2 + z^2] + \frac{\partial^2}{\partial y^2}[x^2 - y^2 + z^2] + \frac{\partial^2}{\partial z^2}[x^2 - y^2 + z^2]$$

$$= \frac{\partial}{\partial x}[2x] + \frac{\partial}{\partial y}[-2y] + \frac{\partial}{\partial z}[2z] = 2 - 2 + 2 = 2$$

So $\nabla^2 V \neq 0$

Hence field V does not satisfy Laplace's equation.

b) $V = r\cos\phi + z$

In cylindrical co-ordinate system,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2}$$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}[r\cos\phi + z] = \cos\phi$$

$$\frac{\partial V}{\partial \phi} = \frac{\partial}{\partial \phi}[r\cos\phi + z] = -r\sin\phi$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}[r\cos\phi + z] = 1$$

$$\frac{1}{r^2}\left[\frac{\partial^2 V}{\partial \phi^2}\right] = \frac{1}{r^2}\left[\frac{\partial}{\partial \phi}(-r\sin\phi)\right] = -\frac{r\cos\phi}{r^2} = -\frac{\cos\phi}{r}$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial z}[1] = 0$$

$$\therefore \quad \nabla^2 V = \frac{1}{r}\cos\phi - \frac{\cos\phi}{r} + 0 = 0$$

So this field **satisfies Laplace's equation.**

c) $V = r\cos\theta + \phi$

In spherical system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

$$r^2\frac{\partial V}{\partial r} = r^2\frac{\partial}{\partial r}[r\cos\theta + \phi] = r^2(\cos\theta)$$

$$\sin\theta\frac{\partial V}{\partial \theta} = \sin\theta\frac{\partial}{\partial \theta}[r\cos\theta + \phi] = \sin\theta[-r\sin\theta] = -r\sin^2\theta$$

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}[r\cos\theta + \phi] = \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}[1] = 0$$

$$\therefore \quad \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2\cos\theta] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(-r\sin^2\theta)$$

$$= \frac{1}{r^2}2r\cos\theta + \frac{1}{r^2\sin\theta}[-r\,2\sin\theta\cos\theta] = \frac{2}{r}\cos\theta - \frac{2}{r}\cos\theta$$

$$\therefore \quad \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2\cos\theta] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(-r\sin^2\theta)$$

$$= \frac{1}{r^2}2r\cos\theta + \frac{1}{r^2\sin\theta}[-r\,2\sin\theta\cos\theta] = \frac{2}{r}\cos\theta - \frac{2}{r}\cos\theta$$

$$= 0$$

So this field **satisfies Laplace's equation.**

Example:2.6

Verify that the potential field given below satisfies the Laplace's equ

$V = 2x^2 - 3y^2 + z^2$

Solution : Given field is in cartesian system,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}[2x^2 - 3y^2 + z^2] + \frac{\partial^2}{\partial y^2}[2x^2 - 3y^2 + z^2] + \frac{\partial^2}{\partial z^2}[2x^2 - 3y^2 + z^2]$$

$$\frac{1}{r^2}\left[\frac{\partial^2 V}{\partial \phi^2}\right] = \frac{1}{r^2}\left[\frac{\partial}{\partial \phi}(-r\sin\phi)\right] = -\frac{r\cos\phi}{r^2} = -\frac{\cos\phi}{r}$$

$$\frac{\partial^2 V}{\partial z^2} = \frac{\partial}{\partial z}[1] = 0$$

$$\therefore \quad \nabla^2 V = \frac{1}{r}\cos\phi - \frac{\cos\phi}{r} + 0 = 0$$

So this field **satisfies Laplace's equation.**

c) $V = r\cos\theta + \phi$

In spherical system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

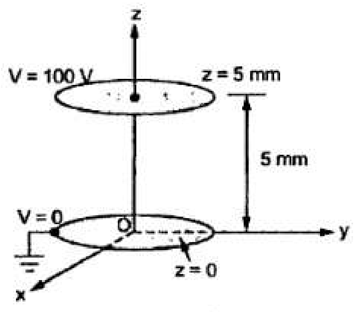
$$r^2\frac{\partial V}{\partial r} = r^2\frac{\partial}{\partial r}[r\cos\theta+\phi] = r^2(\cos\theta)$$

$$\sin\theta\frac{\partial V}{\partial\theta} = \sin\theta\frac{\partial}{\partial\theta}[r\cos\theta+\phi] = \sin\theta[-r\sin\theta] = -r\sin^2\theta$$

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}[r\cos\theta+\phi] = \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}[1] = 0$$

Example:2.7

Two parallel conducting discs are separated by distance 5 mm at z = 0 and z = 5 mm. If V = 0 at z = 0 and V = 100 V at z = 5 mm, find the charge densities on the discs.



Consider cylindrical co-ordinates. The potential V is the function of z alone and is independent of r and ϕ.

$$\therefore \nabla^2 V = \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{... Laplace's equation}$$

Integrating, $\frac{\partial V}{\partial z} = \int 0\,dz + C_1 = C_1$

Integrating, $V = \int C_1 dz + C_2 = C_1 z + C_2$

At z = 0, V = 0 V and at z = 0.005 m, V = 100 V

$\therefore \quad 0 = C_1(0) + C_2 \quad$ thus $C_2 = 0$

and $\quad 100 = C_1 \times 0.005 + C_2 \quad$ thus $C_1 = 20\times 10^3$

$\therefore \quad V = 20\times 10^3 z$ V

Now $\quad \bar{E} = -\nabla V = -\frac{\partial V}{\partial z}\bar{a}_z = -\frac{\partial}{\partial z}[20\times 10^3 z]\bar{a}_z$

$= -20\times 10^3 \bar{a}_z$ V/m

$\therefore \quad \overline{D} = \varepsilon_0 \overline{E} = -8.854\times10^{-12}\times20\times10^{3}\,\bar{a}_z = -1.77\times10^{-7}\,\bar{a}_z \ \text{C/m}^2$

- The $\overline{D}$ acts in the normal direction as per the boundary conditions. Thus $\overline{D} = \overline{D}_N$.

$\therefore \quad \overline{D}_N = -1.7708\times10^{-7}\,a_z$

$\therefore \quad \rho_S = \left|\overline{D}_N\right| = 1.7708\times10^{-7}\ \text{C/m}^2 = \mathbf{177.08\ nC/m^2}$

This is the magnitude of surface charge densities on the discs. So $\rho_S = \pm177.08\ \text{nC/m}^2$, positive on upper plate and negative on lower plate.

Example:2.8

Determine E in spherical co-ordinates from Poisson's equation, assuming a uniform charge density ρ.

Solution : The Poisson's equation for charge density ρ is,

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

In spherical co-ordinates,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = -\frac{\rho_v}{\varepsilon}$$

The charge density is uniform and is a function of r only.

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = -\frac{\rho_v}{\varepsilon} \qquad \text{...Other terms are neglected}$$

$$\text{Integrating} \quad r^2\frac{\partial V}{\partial r} = \int -\frac{\rho_v r^2}{\varepsilon} + A = \frac{-\rho_v r^3}{3\varepsilon} + A$$

$$\text{Integrating} \quad \int\frac{\partial V}{\partial r} = \int\left[\frac{-\rho_v r}{3\varepsilon} + A\,r^{-2}\right]dr + B$$

$$\therefore \quad V = -\frac{\rho_V r^2}{6\varepsilon} - \frac{A}{r} + B$$

$$\overline{E} = -\frac{\partial V}{\partial r}\bar{a}_r = -\frac{\partial}{\partial r}\left[\frac{-\rho_V r^2}{6\varepsilon} - \frac{A}{r} + B\right]\bar{a}_r$$

$$\therefore \quad \boxed{\overline{E} = \left[\frac{\rho_v r}{3\varepsilon} - \frac{A}{r^2}\right]\bar{a}_r} \qquad \text{... where A = Constant}$$

SUMMARY

- Materials may be classified in terms of their conductivity a, in mhos per meter (℧ /m) or Siemens per meter (S/m), as conductors and nonconductors, or technically as metals and insulators (or dielectrics).

- The conductivity of a material usually depends on temperature and frequency. A material with high conductivity (σ>>1) is referred to as a metal whereas one with low conductivity (σ<<1) is referred to as an insulator.
- A **perfect conductor** cannot contain an electrostatic field within it.
- *Dielectric breakdown* is said to have occurred when a dielectric becomes conducting. Dielectric breakdown occurs in all kinds of dielectric materials (gases, liquids, or solids) and depends on the nature of the material, temperature, humidity, and the amount of time that the field is applied.
- The minimum value of the electric field at which dielectric breakdown occurs is called the *dielectric strength* of the dielectric material.
- Relation between current and current density is $I = \int_S J.ds$
- Continuity Equation $\nabla.J = -\frac{\partial \rho_V}{\partial t}$
- As a measure of intensity of the polarization, define polarization$\overline{P}$ (in colombs/meter square) as the dipole moment per unit volume of the dielectric: that is,

$$\bar{p} = \lim_{\Delta V \to 0} \frac{\sum_{k=1}^{N} Q_k \bar{d}_k}{\Delta V}$$

- If the field exists in a region consisting of two different media, the conditions that the field must satisfy at the interface separating the media are called *boundary conditions.*

  The boundary conditions at an interface separating
  1. dielectric ($\varepsilon r1$) and dielectric ($\varepsilon r2$)
  2. conductor and dielectric
  3. conductor and free space
- In applying the image method, two conditions must always be satisfied:
  1. The image charge(s) must be located in the conducting region.
  2. The image charge(s) must be located such that on the conducting surface(s) the potential is zero or constant.
- Based on Ohm's law derive the resistance of the conducting material. Suppose the conductor has a uniform cross section S and is of length l.

$$R = \frac{\rho cl}{S}$$

- The capacitance C of the capacitor as the ratio of the magnitude of the charge on one of the plates to the potential difference between them; that is,

$$C = \frac{Q}{V} = \frac{\varepsilon \oint E.dS}{\int E.dl}$$

- $\nabla^2 V = -\frac{\rho}{\varepsilon}$ - Poisson's Equation.
- $\nabla^2 V = 0$ - Laplace's form.
- Laplace equation in Cartesian form.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

- In cylindrical coordinate system.

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial_r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial V^2}{\partial z^2} = 0$$

- In spherical coordinate system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial_r}\right) + \frac{1}{r^2 Sin\theta}\frac{\partial}{\partial \theta}\left[Sin\theta\frac{\partial v}{\partial \theta}\right] + \frac{1}{r^2 Sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

- Laplace's and Poisson's equations are not only useful in solving electrostatic field problem; they are used in various other field problems.

TWO MARK QUESTIONS

1. State Poisson's equation.

$\nabla^2 V = -\frac{\rho}{\varepsilon}$ - Poisson's Equation.

Laplace equation in Cartesian form.

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon}$$

In cylindrical coordinate system.

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial_r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial V^2}{\partial z^2} = -\frac{\rho}{\varepsilon}$$

In spherical coordinate system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial_r}\right) + \frac{1}{r^2 Sin\theta}\frac{\partial}{\partial \theta}\left[Sin\theta\frac{\partial v}{\partial \theta}\right] + \frac{1}{r^2 Sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = -\frac{\rho}{\varepsilon}$$

Laplace's and Poisson's equations are not only useful in solving electrostatic field problem; they are used in various other field problems.

2. State Uniqueness Theorem.

   The Uniqueness theorem can be stated as, If the solutions of Laplace's equation satisfy the boundary condition then that solution is unique, by whatever method is obtained. The solution of Laplace's equation gives the field which is unique satisfying the same boundary conditions, in a given region.

3. State the applications of Poisson's equation and Laplace's equation.
   - To obtain potential distribution over the region.
   - To obtain E in the region.
   - To check whether given region is free of charge or not.
   - To obtain the charge induced on the surface of the region.

4. Define current density

   The current density is defined as the current passing through the unit surface area, when the surface is held normal to the direction of the current. The current density is measured in A/m^2

5. Define a current and its unit Ampere.