UNIT V
WAVE GUIDES AND CAVITY RESONATORS

General Wave behaviors along uniform Guiding structures, Transverse Electromagnetic waves, Transverse Magnetic waves, Transverse Electric waves, TM and TE waves between parallel plates, TM and TE waves in Rectangular wave guides, Bessel's differential equation and Bessel function, TM and TE waves in Circular wave guides, Rectangular and circular cavity Resonators.

5. INTRODUCTION

Waveguides are basically a device ("a guide") for transporting electromagnetic energy from one region to another. Typically, waveguides are hollow metal tubes (often rectangular or circular in cross section). They are capable of directing power precisely to where it is needed, can handle large amounts of power and function as a high-pass filter. The waveguide acts as a high pass filter in that most of the energy above a certain frequency (the cutoff frequency) will pass through the waveguide, whereas most of the energy that is below the cutoff frequency will be attenuated by the waveguide. Waveguides are often used at microwave frequencies.

It is a guiding structure which is used to transmit EM waves at a certain frequency called microwave frequency. The forms of waveguides

i. Parallel plate waveguide
ii. Rectangular waveguide
iii. Circular wave guide

5.1. GENERAL WAVE BEHAVIOURS ALONG UNIFORM GUIDING STRUCTURES

Maxwell's equation,

$$\nabla x H = (\sigma + j\omega\varepsilon)\vec{E}$$

$$\nabla x E = -j\omega\mu\vec{H}$$

Assume σ = 0, $\therefore \nabla x H = j\omega\varepsilon\vec{E}$

$$\nabla x\vec{H} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = j\omega\varepsilon(Ex\vec{a_x} + Ey\vec{a_y} + Ez\vec{a_z})$$

$$\vec{a_x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) + \vec{a_y}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + \vec{a_z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) = j\omega\varepsilon(Ex\vec{a_x} + Ey\vec{a_y} + Ez\vec{a_z})$$

equating,

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon Ex \ldots\ldots 1$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon Ey \ldots\ldots 2$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon Ez \ldots\ldots 3$$

Similarly, $\nabla x\vec{E}$ = -jωμ$\vec{H}$

$$\nabla x\vec{E} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega\mu(Hx\vec{a_x} + Hy\vec{a_y} + Hz\vec{a_z})$$

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu Hx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 4$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu Hy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 5$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\varepsilon Hz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 6$$

Propagation constant $\gamma = \sqrt{(\sigma + j\omega\varepsilon)j\omega\mu}$

With σ = 0 $\gamma = j\omega\sqrt{\mu\varepsilon}$

The wave equation is $\nabla^2\vec{E} = \gamma^2\vec{E}$

$$\nabla 2\vec{E} = -\omega 2\mu\varepsilon\vec{E}$$

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = -\omega^2\mu\varepsilon\vec{E} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 7$$

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2} = -\omega^2\mu\varepsilon\vec{H} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 8$$

$$Let\ Hy = Hy0e - \gamma z \qquad \frac{\partial H_y}{\partial z} = Hy0e - \gamma z(-\gamma)$$
$$= -\gamma Hy$$

∴ 1,2,3 can be written as,

$$\frac{\partial H_z}{\partial y} + \gamma Hy = j\omega\varepsilon Ex \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 9$$

$$-\gamma Hx - \frac{\partial H_z}{\partial x} = j\omega\varepsilon Ey \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 10$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon Ez \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 11$$

Similarly 4,5,6 can be written as,

$$\frac{\partial E_z}{\partial y} + \gamma Ey = -j\omega\mu Hx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 12$$

$$\gamma Ex + \frac{\partial E_z}{\partial x} = j\omega\mu Hy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 13$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\varepsilon Hz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 14$$

7 and 8 can be written as,

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \gamma^2 E = -\omega^2\mu\varepsilon E$$

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \gamma^2 H = -\omega^2\mu\varepsilon H$$

To obtain the value of the field component Ex,Ey,Hx & Hy equation must be solved simultaneously. From equation 9,

$$Ex = \frac{1}{j\omega\varepsilon}\left[\frac{\partial H_z}{\partial y} + \gamma Hy\right]$$

From equation 13,

$$Hy = \frac{1}{j\omega\mu}\left[\gamma Ex + \frac{\partial E_z}{\partial x}\right]$$

$$\therefore Ex = \frac{1}{j\omega\varepsilon}\left[\frac{\partial H_z}{\partial y} + \gamma\left(\frac{1}{j\omega\mu}\left[\gamma Ex + \frac{\partial E_z}{\partial x}\right]\right)\right]$$

$$= \frac{1}{j\omega\varepsilon}\left[\frac{\partial H_z}{\partial y} + \frac{\gamma^2}{j\omega\mu}Ex + \frac{\gamma}{j\omega\mu}\frac{\partial E_z}{\partial x}\right]$$

$$Ex = \frac{1}{j\omega\varepsilon}\frac{\partial H_z}{\partial y} - \frac{\gamma^2}{\omega^2\mu\varepsilon}Ex - \frac{\gamma}{\omega^2\mu\varepsilon}\frac{\partial E_z}{\partial x}$$

$$Ex\left[1 + \frac{\gamma^2}{\omega^2\mu\varepsilon}\right] = \frac{1}{j\omega\varepsilon}\frac{\partial H_z}{\partial y} - \frac{\gamma}{\omega^2\mu\varepsilon}\frac{\partial E_z}{\partial x} \qquad [h2 = \gamma 2 + \omega 2\mu\varepsilon]$$

$$Ex\left(\frac{h^2}{\omega^2\mu\varepsilon}\right) = \frac{1}{j\omega\varepsilon}\frac{\partial H_z}{\partial y} - \frac{\gamma}{\omega^2\mu\varepsilon}\frac{\partial E_z}{\partial x}$$

$$Ex = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial x}$$

Similarly from equation 10,

$$Ey = \frac{1}{j\omega\varepsilon}\left[-\gamma Hx - \frac{\partial H_z}{\partial x}\right]$$

from equation 12,

$$Hx = \frac{1}{-j\omega\mu}\left[\frac{\partial E_z}{\partial y} + \gamma Ey\right]$$

$$\therefore Ey = \frac{1}{j\omega\varepsilon}\left[\frac{\gamma}{j\omega\mu}\frac{\partial E_z}{\partial y} + \frac{\gamma^2}{j\omega\mu}Ey - \frac{\partial H_z}{\partial x}\right]$$

$$Ey = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

$$\therefore Hx = \frac{1}{-j\omega\mu}\left[\frac{\partial E_z}{\partial y} + \gamma\left(-\frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}\right)\right]$$

$$Hx = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial x} + \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y}$$

from equation 13, $$Hy = \frac{1}{-j\omega\mu}\left[\gamma Ex + \frac{\partial E_z}{\partial x}\right]$$

Sub Ex

$$Hy = \frac{1}{-j\omega\mu}\left[\gamma\frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} - \frac{\gamma^2}{h^2}\frac{\partial E_z}{\partial x} + \frac{\partial E_z}{\partial x}\right]$$

$$Hy = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial y} - \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x}$$

i. Rectangular waveguide

All the Field components are depends on Ez and Hz. So if Ez and Hz is zero all field components within the guide is zero. The two field configuration Ez = 0 for TE waves and Hz = 0 for TM waves in rectangular waveguide.

$$Ex = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial x}$$

$$Ey = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

$$Hx = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial x} + \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y}$$

$$Hy = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial y} - \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x}$$

ii. Parallel plate waveguide

In a parallel plane waveguide the electromagnetic wave propagating through the positive z direction. Hence $e^{\gamma l}$represent that the wave propagate in positive z direction. This factor is also represents the variations of the field components. The term γ in the factor $e^{-\gamma Z}$ is called complex propagation constant. The planes are extending infinitely in y direction. So boundary conditions can be applied along y direction. Hence the derivative with respect to y i.e, $\frac{\partial}{\partial y}$ is zero. The two plane are x = 0 & x = a to obtain the boundary conditions. For the variation of field components $\frac{\partial}{\partial y} \rightarrow 0$

$$Ex = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial x}$$

$$Hy = -\frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x}$$

$$Hx = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial x}$$

$$Ey = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

iii. Circular wave guide

In order to determine the condition for propagation of waves inside a hollow, perfectly conducting cylinder is employed r,Φ,z co ordinates instead of x,y,z.

$$E\Phi = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial r} - \frac{\gamma}{h^2 r}\frac{\partial E_z}{\partial \Phi}$$

$$Hr = \frac{j\omega\varepsilon}{h^2 r}\frac{\partial E_z}{\partial\varphi} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial r}$$

$$H\Phi = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial r} - \frac{\gamma}{rh^2}\frac{\partial H_z}{\partial\Phi}$$

$$Er = \frac{-j\omega\mu}{h^2 r}\frac{\partial H_z}{\partial\varphi} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial r}$$

5.2. EM WAVES

The term wave refers to a phenomenon in which repetition of some activity occurs with respect to space as well as time co ordinates. Formally it can be defined as a physical phenomenon that occurs at one place at the given time and reproduced at the other place at the later times, the time delay being proportional to the space separation from the first location. Then the group of phenomena constitute the wave.

Classifications:

Wherever there exists the time varying field, there exists the wave and the converse is also true. The electromagnetic wave can be classified into the following categories.

5.2.1. Transverse Electromagnetic (TEM) Waves

In this type of wave also known as the principal wave, both the electric vector $\bar{E}$ and magnetic vector $\overline{H}$ are entirely normal to the direction of propagation of the wave. In addition the electric vector $\bar{E}$ and magnetic vector $\overline{H}$ and the direction of propagation, all the three vector form a right handed vector system. The electromagnetic energy travels as TEM waves in free space and other parallel wire transmission line. The coaxial line can also hold this type of wave.

The phase velocity and group velocity are same for TEM wave. Neither one depends upon the frequency. So the TEM wave is non dispersive wave.

The TEM wave can be either a plane wave or a cylindrical or a spherical wave. A plane wave is characterized by a disturbance that at given point in the time has uniform properties across and infinite plane perpendicular to the direction of propagation and similarly for cylindrical spherical waves the disturbance are uniform across the cylindrical and spherical surface. All there are the dimensional waves as they propagates through a volume and its disturbance may be function of all these space variables.

5.2.2. Transverse Magnetic (TM) Waves

In this wave the magnetic vector is entirely normal to the direction of propagation and hence it has no components in the direction of propagation. The electric vector has both the normal and parallel components.

5.2.3. Transverse Electric (TE) Waves

In this wave the electric vector is entirely normal to the direction of propagation and hence it has no components in the direction of propagation. The magnetic vector has both the normal and parallel components. The linear combination of TE and TM wave is hybrid (or) mixed waves. In this wave both the electric & magnetic vectors posses both the components, normal and parallel, to the direction of propagation of the wave.

5.3.TM WAVES BETWEEN PARALLEL PLATES

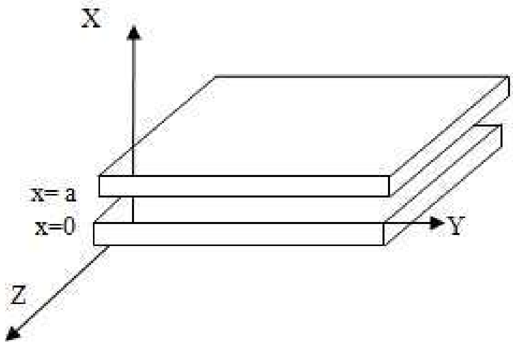

Fig 5.1. Parallel plate waveguide

Consider that the magnetic field is totally along y-axis. As the magnetic field is in y – direction only the field components in x & z directions are absent ie, Hx = Hz = 0 Hy ≠ 0. Hence Ex ≠ 0, Ey = 0; Ex ≠ 0 ie, the components of electric field in the y direction is absent. As the electric field in the direction of magnetic field ie, in y-direction does not exists, the wave is called TM waves. The equation for TM waves are given by,

$$\gamma Hy = j\omega\varepsilon E_x \quad \ldots\ldots\ldots\ldots\ldots\ldots 1\ (a)$$

$$\frac{\partial Hy}{\partial x} = j\omega\varepsilon E_x \quad \ldots\ldots\ldots\ldots\ldots 1(b)$$

$$-\gamma E_x - \frac{\partial Ez}{\partial x} = j\omega\mu Hy \quad \ldots\ldots\ldots\ldots\ldots 1\ (c)$$

differentiating equation 1(b)

$$\frac{\partial^2 \mathrm{Hy}}{\partial \mathrm{x}^2} = \mathrm{j}\omega\varepsilon \frac{\partial \mathrm{Ez}}{\partial \mathrm{x}}$$

But from equation 1(c)

$$\frac{\partial \mathrm{Ez}}{\partial \mathrm{x}} = -\gamma \mathrm{E_x} + \mathrm{j}\omega\mu\mathrm{Hy}$$

But

$$\mathrm{E_x} = \frac{\gamma \mathrm{Hy}}{\mathrm{j}\omega\varepsilon} \text{ [from 1(a)]}$$

$$\therefore \frac{\partial Ez}{\partial x} = \frac{-\gamma^2 Hy}{j\omega\varepsilon} + j\omega\mu Hy$$

$$\therefore \frac{\partial^2 Hy}{\partial x^2} = j\omega\varepsilon\left[\frac{-\gamma^2 Hy}{j\omega\varepsilon} + j\omega\mu Hy\right]$$

$$= -\gamma^2 Hy - \omega^2\mu\varepsilon Hy$$

$$= -Hy\,(\gamma^2 + \omega^2\mu\varepsilon)$$

$$\frac{\partial^2 Hy}{\partial x^2} - h^2 Hy \qquad where \quad h = \sqrt{\gamma^2 + \omega^2\mu\varepsilon}$$

Equation (2) is the second order differential equation representing simple harmonic motion. Then the solution of such equation can be written in the standard form as,

$$Hy_0 = C_3\,Sinhx + C_4 Coshx \qquad \ldots\ldots\ldots(2)$$

WhereC_3 and C_4 are arbitrary constants. Considering the variation of field components in Z direction with respect to time, the complete solution can be written as,

$$Hy = (C_3\,Sinhx + C_4 Coshx)e^{-\gamma z} \qquad \ldots\ldots\ldots(3)$$

The arbitrary constants C_3 & C_4 cannot be found directly by using boundary conditions as in TE Waves. Because the tangential components of H is not zero at the boundary. (or) the surface of the conductor. To overcome this difficulty the boundary conditions are applied for E_z components.

So consider the expression for E_z interms of Hy.

$$1(b) \rightarrow \frac{\partial Hy}{\partial x} = j\omega\varepsilon E_z$$

$$\frac{\partial}{\partial x}[(C_3\,Sinhx + C_4 Coshx)e^{-\gamma z}] = j\omega\varepsilon E_z$$

$$h(C_3\,coshx - C_4 Coshx.h)e^{\gamma z} = j\omega\varepsilon E_z$$

$$E_z = \frac{h}{j\omega\varepsilon}[C_3\,Coshx - C_4 Sinhx)e^{-\gamma z}$$

Now at x = 0 $E_z = 0$

x = a $E_z = 0$

Applying first boundary condition,

$$E_z = \frac{h}{j\omega\varepsilon}[C_3]e^{-\gamma z}$$

$C_3 = 0$

$$\therefore \quad E_z = \frac{h}{j\omega\varepsilon}[-C_4 Sinhx]e^{-\gamma z}$$

Applying second boundary condition,

$$0 = \frac{h}{j\omega\varepsilon} - [C_4 Sinha]e^{-\gamma z}$$

To fulfill the second boundary condition the value of h must be selected as

$$h = \frac{m\pi}{a} \quad where\ m = 1, 2, \ldots..$$

$$\text{Then } E_z = \frac{m\pi/a}{j\omega\varepsilon}(-C_4 Sin\frac{m\pi}{a}x)e^{-\gamma z}$$

$$E_z = \frac{jm\pi}{\omega\varepsilon a}C_4 Sin(\frac{m\pi}{a}x)e^{-\gamma z}$$

The remaining non-zero field components are Hy & E_x To obtain Hy

$$1(b) \rightarrow \frac{\partial Hy}{\partial x} = j\omega\varepsilon E_z$$

$$\frac{\partial Hy}{\partial x} = j\omega\varepsilon\,[\frac{jm\pi}{\omega\varepsilon a}C_4 Sin(\frac{m\pi}{a}x)e^{-\gamma z}]$$

$$= \frac{-m\pi}{a}C_4 Sin\left(\frac{m\pi}{a}x\right) - e^{\gamma z}$$

Integrating both sides with respect to x

$$Hy = \frac{m\pi}{a} C_4 Cos\left(\frac{m\pi}{a}x\right).\frac{m\pi}{a} e^{-\gamma z}$$

$$Hy = C_4 Cos\left(\frac{m\pi}{a}x\right) e^{-\gamma z}$$

$$To\ obtain\ E_x \qquad 1(a) \quad \rightarrow \quad \gamma Hy = j\omega\varepsilon E_x$$

$$E_x = \frac{\gamma}{j\omega\varepsilon}\left[C_4 Cos\left(\frac{m\pi}{a}x\right) e^{-\gamma z}\right]$$

When m = 0, Ex & Hy Components Exit. So the possible Smallest Value of m = 0. Hence the lowest order mode possible with TM Waves, TM_{00} mode.

When the wave propagate without any attenuation $\propto = 0$

$\therefore \qquad \gamma = j\beta$

Thus the field components are

$$E_x = \frac{jm\pi}{\omega\varepsilon a} C_4 Sin\left(\frac{m\pi}{a}x\right) e^{-j\beta z}$$

$$\text{Hy} = C_4 Cos\left(\frac{m\pi}{a}x\right) e^{-j\beta z}$$

$$E_z = \frac{j\beta}{j\omega\varepsilon} C_4 Cos(\frac{m\pi}{a}x) e^{-j\beta z}$$

Characteristics

The properties of the TM waves between parallel conducting planes are altogether different than those of the uniform plane waves in the free space. From the expression of the field components of transverse electric waves or transverse magnetic waves either sinusoidal or co sinusoidal variations or standing wave distribution of each of the component of $\bar{E}$ and $\bar{H}$ in the x direction. In y direction none of the field components vary in magnitude or phase which is according to the assumption made earlier.

Thus x-y plane is an equiphase for each of the field components. The meaning of equiphase plane is that for all the points on the plane, the maximum value of any sinusoidal variation of any field component will reach its maximum value at the same instant.

We know,

$$h = \sqrt{\bar{\gamma}^2 + \omega^2\mu\varepsilon}$$

$$h^2 = \gamma^2 + \omega^2\mu\varepsilon$$

$$h^2 = \bar{\gamma}^2 + \omega^2\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2$$

$$\bar{\gamma}^2 = \omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2$$

$$\bar{\gamma} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}$$

At lower frequency the value of factor $\omega^2\mu\varepsilon$ is found to be less than $\left(\frac{m\pi}{a}\right)^2$. Thus $\bar{\gamma}$ become real with value equal to the attenuation constant α and phase constant β=0. Thus there is only attenuation suffered by the wave without any propagation.

At higher frequency the value of factor $\omega^2\mu\varepsilon$ is becomes greater than that of the factor $\left(\frac{m\pi}{a}\right)^2$. Thus $\bar{\gamma}$ purely imaginary with value equal to the phase constant jβ and attenuation constant α=0.

The cut off frequency can be defined as the frequency at which the propagation constant changes from real to imaginary. F=fc value of the propagation constant is zeso.

Thus,

For f<fc , $\omega^2\mu\varepsilon < \left(\frac{m\pi}{a}\right)^2, \gamma = \alpha \text{ and } \beta = 0$

For f>fc , $\omega^2\mu\varepsilon > \left(\frac{m\pi}{a}\right)^2, \gamma = j\beta \text{ and } \alpha = 0$

For f=fc,

$$\bar{\gamma} = \sqrt{{\omega_c}^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2} = 0$$

$${\omega_c}^2\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2$$

$$4\pi^2{f_c}^2\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2$$

$$2\pi f_c\sqrt{\mu\varepsilon} = \frac{m\pi}{a}$$

$$f_c = \frac{m}{2a\sqrt{\mu\varepsilon}} Hz$$

In general, $\left(\frac{m\pi}{a}\right)^2$ as ${\omega_c}^2\mu\varepsilon$,

$$\bar{\gamma} = \sqrt{{\omega_c}^2\mu\varepsilon - \omega^2\mu\varepsilon}$$

$$\bar{\gamma} = \sqrt{\mu\varepsilon({\omega_c}^2 - \omega^2)}$$

$$\bar{\gamma} = 2\pi\sqrt{\mu\varepsilon}\sqrt{({f_c}^2 - f^2)}$$

$$\bar{\gamma} = j2\pi\sqrt{\mu\varepsilon}\sqrt{(f^2 - {f_c}^2)} = j\beta$$

The phase constant β,

$$\beta = 2\pi\sqrt{\mu\varepsilon}\sqrt{(f^2 - {f_c}^2)}$$

The wavelength is defined as the distance travelled for the phase shift through 2π radius. Thus the wavelength is given by,

$$\lambda = \frac{2\pi}{\beta} = \lambda_g$$

This is the wavelength in the direction of propagation of the guide. Hence this wavelength is called guide wavelength and denoted by λ_g.

The cut off wavelength is given by,

$$\lambda_c = \frac{v}{f_c} \quad where\ v = \frac{1}{\sqrt{\mu\varepsilon}} = velocity\ of\ propagation$$

But the cut off frequency is given by,

$$f_c = \frac{m}{2a\sqrt{\mu\varepsilon}} = \frac{mv}{2a}$$

Hence the cut off wavelength is given by,

$$\lambda_c = \frac{v}{f_c} = \frac{2a}{m}$$

Thus the distance of separation is given by,

$$a = m\frac{\lambda_c}{2}$$

The integer m indicates the number of wavelength variations of either electric and magnetic fields along x direction.

Wave Impedance,

$$Z_0(TM) = \frac{E_x}{H_y} = \frac{\beta}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

$$Z_0(TM) = \eta\sqrt{1-\left(\frac{f_c}{f}\right)^2}$$

5.4. TE WAVES BETWEEN PARALLEL PLATES

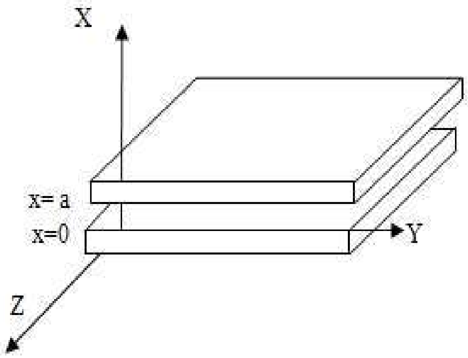

Fig 5.2 Parallel plate waveguide

Consider that the electric field is totally along y – axis. As the electric field is in y – direction only the field components in x & z are absent. Ex= Ez = 0, Ey ≠ 0

∴ Hy = 0 $H_x \neq 0, H_z \neq 0$

ie., the components of magnetic field in the y direction is absent. As the magnetic field in the direction of electric field [ie. in y direction] does not exists, the wave is called TE waves.

The equation for TE waves are given by,

$$-\gamma Hx - \frac{\partial H_z}{\partial x} = j\omega\varepsilon Ey \quad \ldots\ldots\ldots\ldots\ldots 1(a)$$

$$\gamma Ey = -j\omega Hx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1(b)$$

$$\frac{\partial E_y}{\partial x} = -j\omega\varepsilon Hz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 1(c)$$

Differentiating eqn 1(c)➔

$$\frac{\partial^2 E_y}{\partial x^2} = -j\omega\varepsilon \frac{\partial H_z}{\partial x}$$

$$1(a) ➔ \frac{\partial H_z}{\partial x} - j\omega\varepsilon Ey + \gamma Hx$$

$$1(b) ➔ Hx = \frac{\gamma}{-j\omega} Ey$$

$$\therefore \frac{\partial^2 E_y}{\partial x^2} = -j\omega\varepsilon\left[- j\omega\varepsilon Ey + \gamma\left(\frac{\gamma}{-j\omega} Ey\right)\right]$$

$$\frac{\partial^2 E_y}{\partial x^2} = -(\omega^2\mu\epsilon + \gamma^2)E_y$$

$$\frac{\partial^2 E_y}{\partial x^2} + h^2 E_y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots 2$$

The auxillary equation is, $(m^2 + h^2)E_y = 0$

Equation (2) is the second order differential equation representing simple harmonic motion. Then the solution of such equation can be written in standard form as

$$Ey_0 = C_1\, Sinhx + C_2\, Coshx$$

Where C_1 & C_2 are arbitrary constants.

Considering the variations of field components in Z – direction with respect to time the complete solution can be written as,

$$Ey = (C_1 Sinhx + C_2\, Coshx)\, e^{-\gamma z} \quad \ldots\ldots (3)$$

The arbitary constants C_1 & C_2 can be found by using boundary conditions.

The boundary conditions are written for Ey component

ie, at	x = 0	Ey = 0
at	x = a	Ey = 0

The tangential components of Electric field E is Zero at the boundary or the surface of the conductor.

Applying the boundary conditions in (3) x = 0 Ey = 0

$$0 = (C_1\, Sin\, 0 + C_2\, Cos\, 0)\, e^{-\gamma z}$$

$$C_2 = 0$$

$$Ey = (C_1\, Sinhx)\, e^{-\gamma z} \quad \ldots\ldots (4)$$

Applying the second boundary conditions, x = a Ey = 0

$$0 = C_1 \, Sinh \, a \, e^{-\gamma z}$$

$$Sinh \, a = 0$$

To fulfill the second boundry condiions the value h must be selected as,

h= $\frac{m\pi}{a}$ where m= 1,2,3………

$$(4) \rightarrow Ey = C_1 Sin\left(\frac{m\pi}{a}x\right)e^{-\gamma z} \quad \ldots\ldots..(5)$$

The rearranging non-Zero components are H_z and H_x. To obtain H_z

equation (1) C $\rightarrow \frac{\partial Ey}{\partial x}$ = -j$\omega\mu H_z$

$$\frac{\partial}{\partial x}\left[C_1 \, Sin\left(\frac{m\pi}{a}x\right)e^{-\gamma z}\right] = -j\omega\mu H_z$$

$$C_1 e^{-\gamma z} Cos\left(\frac{m\pi}{a}x\right)\left(\frac{m\pi}{a}\right) = -j\omega\mu H_z$$

$$H_z = \frac{-m\pi \, C_1}{j\omega\mu \, 0} \, Cos\left(\frac{m\pi}{a}x\right)e^{-\gamma z}$$

To obtain H_z Equation 1(b) => -j$\omega\mu H_z = \gamma Ey$

$$H_x = \frac{\gamma}{j\omega\mu} C_1 \, Sin\left(\frac{m\pi}{a}x\right)e^{-\gamma z}$$

$$H_x = \frac{\gamma C_1}{j\omega\mu} \, Sin\left(\frac{m\pi}{a}x\right)e^{-\gamma z}$$

In the expression for the field component m is an integer which specifies different field configuration (or) mode for different values of m. Hence the wave associated with m is called TE $_{mo}$ wave (or) TE $_{mo}$ mode. The smallest possible value of m is 0. But when m=0 Ey = H_z = H_x = 0 ie, all the field components are Zero. Hence the smallest possible value of m is 1. Hence the lowest order mode possible with TE waves of TE_{10} mode. We know that $\gamma = \propto + j\beta$

Where $\propto$ is the attenuation constant & β is the phase constant

When the wave propagates without any attenuation α = 0 ∴ $\gamma = j\beta$

The field components are,

$$Ey = C_1 \, Sin\left(\frac{m\pi}{a}x\right)e^{-j\beta z}$$

$$H_z = \frac{-m\pi C_1}{j\omega\mu o} \cos\left(\frac{m\pi}{a}x\right)e^{-j\beta z}$$

$$H_x = \frac{-j\beta C_1}{j\omega\mu} \, Sin\left(\frac{m\pi}{a}x\right)e^{-j\beta z}$$

Characteristics

The properties of the TE waves between parallel conducting planes are altogether different than those of the uniform plane waves in the free space. From the expression of the field components of transverse electric waves or transverse magnetic waves either sinusoidal or co sinusoidal variations or standing wave distribution of each of the component of $\bar{E}$ and $\bar{H}$ in the x direction. In y direction none of the field components vary in magnitude or phase which is according to the assumption made earlier.

i. Propagation constant

Thus x-y plane is an equiphase for each of the field components. The meaning of equiphase plane is that for all the points on the plane, the maximum value of any sinusoidal variation of any field component will reach its maximum value at the same instant.

We know,

$$h = \sqrt{\bar{\gamma}^2 + \omega^2\mu\varepsilon}$$
$$h^2 = \gamma^2 + \omega^2\mu\varepsilon$$
$$h^2 = \bar{\gamma}^2 + \omega^2\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2$$
$$\bar{\gamma}^2 = \omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2$$
$$\bar{\gamma} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}$$

ii. Cut off frequency

At lower frequency the value of factor $\omega^2\mu\varepsilon$ is found to be less than $\left(\frac{m\pi}{a}\right)^2$. Thus $\bar{\gamma}$ become real with value equal to the attenuation constant α and phase constant β=0. Thus there is only attenuation suffered by the wave without any propagation. At higher frequency the value of factor $\omega^2\mu\varepsilon$ is becomes greater than that of the factor $\left(\frac{m\pi}{a}\right)^2$. Thus $\bar{\gamma}$ purely imaginary with value equal to the phase constant jβ and attenuation constant α=0.

The cut off frequency can be defined as the frequency at which the propagation constant changes from real to imaginary. F=fc value of the propagation constant is zeso.

Thus,

For f<fc , $\omega^2\mu\varepsilon < \left(\frac{m\pi}{a}\right)^2, \gamma = \alpha \text{ and } \beta = 0$

For f>fc , $\omega^2\mu\varepsilon > \left(\frac{m\pi}{a}\right)^2, \gamma = j\beta \text{ and } \alpha = 0$

For f=fc,

$$\bar{\gamma} = \sqrt{{\omega_c}^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2} = 0$$
$${\omega_c}^2\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2$$
$$4\pi^2{f_c}^2\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2$$
$$2\pi f_c\sqrt{\mu\varepsilon} = \frac{m\pi}{a}$$
$$f_c = \frac{m}{2a\sqrt{\mu\varepsilon}} Hz$$

In general, $\left(\frac{m\pi}{a}\right)^2$ as ${\omega_c}^2\mu\varepsilon$,

$$\bar{\gamma} = \sqrt{\omega_c{}^2\mu\varepsilon - \omega^2\mu\varepsilon}$$

$$\bar{\gamma} = \sqrt{\mu\varepsilon(\omega_c{}^2 - \omega^2)}$$

$$\bar{\gamma} = 2\pi\sqrt{\mu\varepsilon}\sqrt{(f_c{}^2 - f^2)}$$

$$\bar{\gamma} = j2\pi\sqrt{\mu\varepsilon}\sqrt{(f^2 - f_c{}^2)} = j\beta$$

iii. The phase constant β,

$$\beta = 2\pi\sqrt{\mu\varepsilon}\sqrt{(f^2 - f_c{}^2)}$$

iv. Wavelength

The wavelength is defined as the distance travelled for the phase shift through 2π radius. Thus the wavelength is given by,

$$\lambda = \frac{2\pi}{\beta} = \lambda_g$$

This is the wavelength in the direction of propagation of the guide. Hence this wavelength is called guide wavelength and denoted by λ_g.

v. Cut off wavelength

The cut off wavelength is given by,

$$\lambda_c = \frac{v}{f_c} \quad where\ v = \frac{1}{\sqrt{\mu\varepsilon}} = velocity\ of\ propagation$$

But the cut off frequency is given by,

$$f_c = \frac{m}{2a\sqrt{\mu\varepsilon}} = \frac{mv}{2a}$$

Hence the cut off wavelength is given by,

$$\lambda_c = \frac{v}{f_c} = \frac{2a}{m}$$

Thus the distance of separation is given by,

$$a = m\frac{\lambda_c}{2}$$

The integer m indicates the number of wavelength variations of either electric and magnetic fields along x direction.

vi. Wave Impedance,

$$Z_0(TE) = \frac{E_x}{H_y} = \frac{\beta}{\omega\varepsilon} = \frac{\sqrt{\frac{\mu}{\varepsilon}}}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

$$Z_0(TE) = \frac{\eta}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}}$$

5.5. TRANSMISSION OF TEM WAVES BETWEEN PARALLEL PLANS

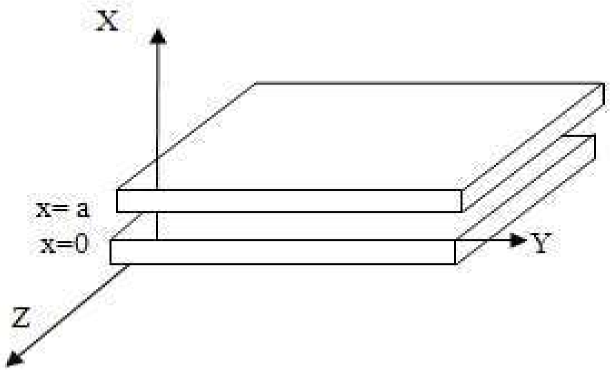

Fig 5.3. Parallel plate waveguide

Consider the electric field is totally along x-axis ie, $E_y = E_x = 0$ and the magnetic field to totally along the y-axis. ie, $H_x = H_z = 0$. As the components of the electric and magnetic fields are transverse to the direction of propagation. ie, Z axis, the wave is called transverse electromagnetic waves.

The equation for the TEM waves are,

$$j\omega\varepsilon E_x = \gamma\, H_y$$

$$j\omega\varepsilon H_y = \gamma\, E_x$$

$$\frac{\partial Hy}{\partial x} = 0$$

***Properties of TEM waves*:**

The TEM wave is a special case of guided wave propagation. Some of the important properties are,

i) The fields are entirely transverse

ii) In the direction perpendicular to the direction of propagation, the amplitude of the field components are constant.

iii) The velocity of propagation of TEM wave is impendent of frequency.

iv) The cut-off frequency of the wave is Zero which indicate all the frequencies below fc can propagate along the guide.

The equation for field components of TE waves are,

$$Ey = C_1 \text{ Sin}\left(\frac{m\pi}{a}x\right) e^{-\gamma z}$$

$$H_z = \frac{-m\pi\, C_1}{j\omega\mu a} \text{ Cos}\left(\frac{m\pi}{a}x\right) e^{-\gamma z}$$

$$H_x = \frac{-j\beta C_1}{j\omega\mu}\, Sin\left(\frac{m\pi}{a}x\right) e^{-\gamma z}$$

If m= 0, all the field components are Zero ie, $E_y = H_z = H_x = 0$

The equation for field components of TM waves are,

$$Ez = C_1 \frac{jm\pi}{\omega\varepsilon a} C_e \text{Sin}\left(\frac{m\pi}{a}x\right) e^{-\gamma z}$$

$$H_y = C_2 \text{ Cos}\left(\frac{m\pi}{a}x\right) e^{-\gamma z}$$

$$E_x = \frac{\gamma}{j\omega\varepsilon} C_2 Cos\left(\frac{m\pi}{a}x\right) e^{-\gamma z}$$

If m = 0, $E_z = 0$ by and E_x components exists.

$$Hy = C2e^{-\gamma z}, \quad Ex = \frac{\gamma}{j\omega\varepsilon} C2e^{-\gamma z}$$

This is the equation for TEM Waves. Comparing the field components of TE & TM Waves for m = 0, $E_z = H_z = 0$. Thus there is not field components along the direction of propagation ie, in the Z- direction.This is called transverse electromagnetic wave. Also for this wave, we get the cut off frequency as Zero. This is called principal wave. The propagation constant $\gamma = j\omega\sqrt{\mu\varepsilon}$

Velocity propagation $V = \frac{1}{\sqrt{\mu\varepsilon}}$

The wavelength $\lambda = \frac{V}{f}$

Wave Impedance,

$$Z_0(TEM) = \frac{E_x}{H_y} = \frac{\beta}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$Z_0(TEM) = \eta$$

5.6. TM WAVES IN RECTANGULAR WAVE GUIDES

For TM wave no components of magnetic field is present in the direction of propagation i.e, Hz = 0. The wave equation in rectangular waveguide is,

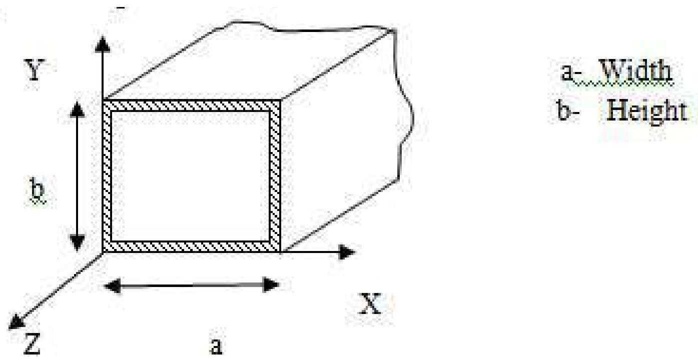

Fig 5.4 Cross section of Rectangular waveguide

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \gamma^2 E = -\omega^2\mu\varepsilon E$$

The wave is propagating in Z direction,

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \gamma^2 E_z = -\omega^2 \mu\varepsilon E_z$$

Let the solution of the equation is, Ez = XY

Where X is the function of x alone and Y is the function of y alone. Substitute the value of Ez in the wave equation , $\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} + \gamma^2 XY = -\omega^2 \mu\varepsilon XY$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + \gamma^2 XY = -\omega^2 \mu\varepsilon XY$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + (\gamma^2 + \omega^2 \mu\varepsilon)XY = 0$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + h^2 XY = 0$$

Divide this equation by XY,

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + h^2 = 0$$

This equation equate the function of x alone to a function of y alone and this can be equated to a constant, $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + h^2 = A^2$ $-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = A^2$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + h^2 - A^2 = 0 \qquad A^2 + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

$$Let\ h^2 - A^2 = B^2 \qquad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + B^2 = 0 \qquad A^2 + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

The solution of this equations are,

$$X = C_1 cosBx + C_2 sinBx \qquad Y = C_3 cosAy + C_4 sinAy$$

The general solution is, Ez = XY

$$Ez = (C_1 cosBx + C_2 sinBx)(C_3 cosAy + C_4 sinAy)$$

$$= C_1C_3 cosBxcosAy + C_1C_4 cosBxsinAy + C_2C_3 sinBxcosAy + C_2C_4 sinBxsinAy$$

The constants C_1,C_2, C_3,C_4 A&B are determined by boundary conditions

Ez =0 when x=0,x=a ,y=0 &y=b

i. When x = 0, Ez = 0

$$C_1C_3 cosAy + C_1C_4 sinAy = 0$$

$$C_1[C_3 cosAy + C_4 sinAy\,] = 0$$

$$C_3 cosAy + C_4 sinAy = 0$$

This is possible only if $C_1 = 0$

The general solution is,

$$Ez = C_2C_3 sinBxcosAy + C_2C_4 sinBxsinAy$$

ii. When y = 0, Ez = 0

$$Ez = C_2C_3 sinBx = 0$$

This is possible only if either $C_2 = 0$ or $C_3 = 0$ if $C_2 = 0$ Ez = 0

$Sub.\, C_3 = 0$ $\therefore Ez = C_2C_4 sinBxsinAy$

iii. If x = a , Ez = 0

$$E_0\ sinBa\ sinAy\ =\ 0 \qquad E0\ =\ C2C4$$

This is possible only if B = $\frac{m\pi}{a}$ for all values of y where m = 1,2,3,….

iv. $If\ \ y\ =\ b, Ez\ =\ 0$

$$E_0 sin\frac{m\pi}{a}xsinAb\ =\ 0$$

This is Possible only if A= $\frac{n\pi}{b}$ for all values of x where n = 1,2,3,...

$$Hence \qquad Ez\ =\ E_0\ sin\frac{m\pi}{a}x\ sin\frac{n\pi}{b}y$$

When the wave is propagate along Z direction with respect to time 'ωt'

$$Ez\ =\ E_0\ sin\frac{m\pi}{a}x\ sin\frac{n\pi}{b}y\ sin\omega t\ e-\gamma z$$

Consider no attenuation, Hence α = 0 γ = jβ

$$Ez =\ E_0\ sin\frac{m\pi}{a}x\ sin\frac{n\pi}{b}y\ e-j\beta z\ sin\omega t$$

$$Ex\ =\ \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y}-\frac{\gamma}{h^2}\frac{\partial E_z}{\partial x}$$

$$For\ TM\ wave\ Hz\ =\ 0,\frac{\partial H_z}{\partial y}=0$$

$$Then \qquad Ex\ =\ -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial x}$$

$$\frac{\partial E_z}{\partial x}=\ E_0\ cos\frac{m\pi}{a}x\ sin\frac{n\pi}{b}y\ \frac{m\pi}{a}\ sin\omega t\ e-j\beta z$$

$$Ex\ =\ -\frac{j\beta}{h^2}E_0\ \frac{m\pi}{a}\ cos\frac{m\pi}{a}x\ sin\frac{n\pi}{b}y\ cos\omega t\ e-j\beta z$$

$Similarly\ know,$

$$Ey\ =\ \frac{\gamma}{h^2}\ \frac{\partial E_z}{\partial y}+\frac{j\omega ?}{h^2}\frac{\partial H_z}{\partial x}\quad Hz\ =\ 0\ Hence,$$

$$Ey\ =\ \frac{\gamma}{h^2}\ \frac{\partial E_z}{\partial y}$$

$$Ey\ =\ \frac{-j\beta}{h^2}E_0\ \frac{n\pi}{b}\ sin\frac{m\pi}{a}x\ cos\frac{n\pi}{b}y\ cos\omega t\ e^{-j\beta z}$$

$Then$

$$Hx\ =\ \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial x}+\frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y} \qquad Hz\ =\ 0$$

$$Hx\ =\ \frac{j\omega\varepsilon}{h^2}E_0\ \frac{n\pi}{b}\ sin\frac{m\pi}{a}x\ cos\frac{n\pi}{b}y\ cos\omega t\ e^{-j\beta z}$$

And

$$Hy\ =\ \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial y}-\frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x} \qquad Now\ Hz\ =\ 0$$

$$Hy\ =\ \frac{j\omega\varepsilon}{h^2}E_0\ cos\frac{m\pi}{a}x\ sin\frac{n\pi}{b}y\ \frac{m\pi}{a}cos\omega t\ e^{-j\beta z}$$

Where a and b are the width and height of the rectangular waveguide, m and n are integer. The above equations are gives the field component eqn for a rectangular waveguide $B^2 = h^2 - A^2$

Now, $A^2+B^2 = h^2$

$h2 = \gamma 2 + \omega 2 \mu\varepsilon \;=>\; \gamma = \sqrt{h^2 - \omega^2\mu\varepsilon} = \sqrt{A^2 + B^2 - \omega^2\mu\varepsilon}$

$$\text{Propagation constant} \quad \gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon}$$

$$\text{At cut off frequency,} \qquad \omega c2\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$\omega c = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$fc = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$= \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

This is the cut off frequency of rectangular waveguide. The corresponding cut off wavelength is,

$$\lambda c = \frac{v}{f} = \frac{c}{f_c} = \frac{1/\sqrt{\mu\varepsilon}}{f_c}$$

$$= \frac{c}{\frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

$$\lambda c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

$\text{The propagation constant } \gamma = j\beta \text{ if } \alpha = 0$

$$\gamma = j\beta = j\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$\therefore \beta = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$\text{The velocity of propagation in waveguide } v = \frac{\omega}{\beta}$

$$V = \frac{\omega}{\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

The corresponding wavelength is rectangular waveguide,

$$\lambda = \frac{v}{f} = \frac{\omega}{f\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

The non zero field components are,

$$Ex = -\frac{j\beta}{\omega c^2\mu\varepsilon} E_0 \frac{m\pi}{a} \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \cos\omega t\, e^{-j\beta z}$$

$$Ey = \frac{-j\beta}{\omega c^2\mu\varepsilon} E_0 \frac{n\pi}{b} \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y \cos\omega t\, e^{-j\beta z}$$

$$Ez = E_0 \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y\, e^{-j\beta z} \sin\omega t$$

$$Hx = \frac{j\omega\varepsilon}{\omega c^2\mu\varepsilon} E_0 \frac{n\pi}{b} \sin\frac{m\pi}{a}x \cos\frac{n\pi}{b}y \cos\omega t\, e^{-j\beta z}$$

$$Hy = \frac{j\omega\varepsilon}{\omega c^2\mu\varepsilon} E_0 \cos\frac{m\pi}{a}x \sin\frac{n\pi}{b}y \frac{m\pi}{a} \cos\omega t\, e^{-j\beta z}$$

$$Hz = 0$$

The general representation for TM wave is TM_{mn} mode. If m=0 or n = 0 the fields of TM wave will be Zero. So the lowest possible value for m and n is 1 for TM wave. This is called TM_{11} wave. This lowest mode is called dominant.

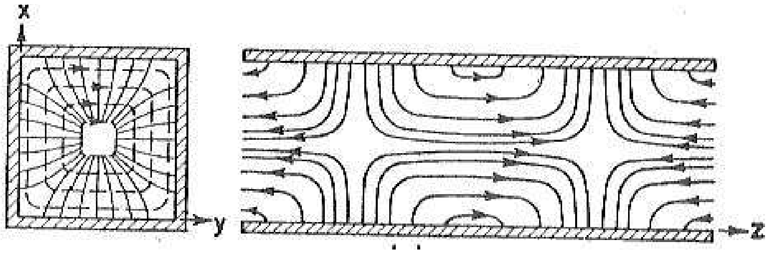

Fig 5.5. Field configuration of Dominant mode(TM_{11})

5.7. TE WAVES IN RECTANGULAR WAVE GUIDES

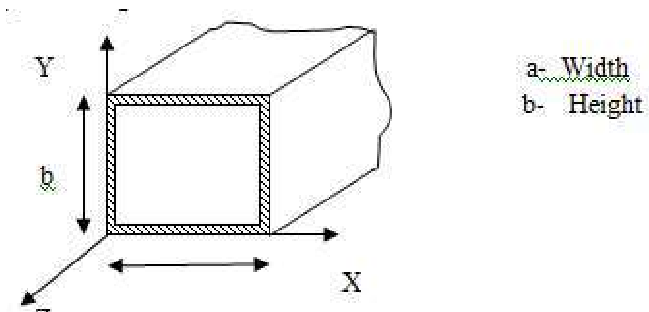

Fig 5.6 Cross section of Rectangular waveguide

For TE wave no components of Electric field is present in the direction of propagation i.e, Ez = 0. The wave equation in rectangular waveguide is,

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \gamma^2 H = -\omega^2\mu\varepsilon H$$

The wave is propagating in Z direction,

$$\frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \gamma^2 H_z = -\omega^2 \mu \varepsilon H_z$$

Let the solution of the equation is, Hz = XY

Where X is the function of x alone

Where Y is the function of y alone

Substitute the value of Ez in the wave equation ,

$$\frac{\partial^2 XY}{\partial x^2} + \frac{\partial^2 XY}{\partial y^2} + \gamma^2 XY = -\omega^2 \mu \varepsilon XY$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + \gamma^2 XY = -\omega^2 \mu \varepsilon XY$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + (\gamma^2 + \omega^2 \mu \varepsilon) XY = 0$$

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} + h^2 XY = 0$$

Divide this equation by XY, $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + h^2 = 0$

This equation equate the function of x alone to a function of y alone and this can be equated to a constant, $\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + h^2 = A^2$ $\quad -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = A^2$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + h^2 - A^2 = 0 \qquad A^2 + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$$

Let $h^2 - A^2 = B^2$ $\quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + B^2 = 0$ $\quad A^2 + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$

The solution of this equations are,

$$X = C_1 \cos Bx + C_2 \sin Bx \qquad Y = C_3 \cos Ay + C_4 \sin Ay$$

The general solution is, $\quad Hz = XY$

$$Hz = (C_1 \cos Bx + C_2 \sin Bx)(C_3 \cos Ay + C_4 \sin Ay)$$

$$= C_1C_3 \cos Bx \cos Ay + C_1C_4 \cos Bx \sin Ay + C_2C_3 \sin Bx \cos Ay + C_2C_4 \sin Bx \sin Ay$$

The constants C_1, C_2, C_3, C_4. *A&B are determined by boundary conditions*

From the Boundary Conditions Hz≠0 and for TE waves Ez = 0

$$\therefore\ Ex = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y}$$

$$Ex = \frac{-j\omega\mu}{h^2}(-AC_1C_3 \cos Bx \sin Ay + AC_1C_4 \cos Bx \cos Ay - AC_2C_3 \sin Bx \sin Ay + AC_2C_4 \sin Bx \cos Ay) \ldots I$$

And $\quad Ey = \frac{j\omega\square}{h^2}\frac{\partial H_z}{\partial x}$

$$Ey = \frac{j\omega\square}{h^2}(-C_1C_3\ B \sin Bx \cos Ay - BC_1C_4 \sin Bx \sin Ay + BC_2C_3 \cos Bx \cos Ay + BC_2C_4 \cos Bx \sin Ay) \ldots II$$

Applying boundry conditions,

i. When $x = 0, Ey = 0$ in I

$$\frac{j\omega?}{h^2}(BC_2C_3\cos Bx\cos Ay + BC_2C_4\cos Bx\sin Ay) = 0$$

$$B\,C_2\,[C_3\cos Bx\cos Ay + C_4\cos Bx\sin Ay] = 0$$

This is possible only if $C2 = 0$

The general solution is, $Ey == \frac{j\omega?}{h^2}(-C_1C_3\,B\sin Bx\cos Ay - BC_1C_4\sin Bx\sin Ay)$

ii. When $x = a, Ey = 0$

$$0 = \frac{j\omega?}{h^2}(-C_1C_3\,B\sin Ba\cos Ay - BC_1C_4\sin Ba\sin Ay)$$

This is possible only if either $A = 0$ or $B = \frac{m\pi}{a}$

If $A = 0$; $Ey = 0$ Hence $B = \frac{m\pi}{a}$

$$\therefore Ey = \frac{j\omega?}{h^2}(-C_1C_3\frac{m\pi}{a}\sin\frac{m\pi}{a}x\cos Ay - \frac{m\pi}{a}C_1C_4\sin\frac{m\pi}{a}x\sin Ay)$$

$$Ey = \frac{j\omega?}{h^2}\frac{m\pi}{a}(-C_1C_3\sin\frac{m\pi}{a}x\cos Ay - C_1C_4\sin\frac{m\pi}{a}x\sin Ay)$$

iii. If $y = 0, Ex = 0$ in II

$$\frac{-j\omega\mu}{h^2}(AC_1C_4\cos Bx\cos Ay + AC_2C_4\sin Bx\cos Ay) = 0$$

This is possible only if $C_4 = 0$

$$Ex = \frac{-j\omega\mu}{h^2}(-AC_1C_3\cos Bx\sin Ay - AC_2C_3\sin Bx\sin Ay)$$

iv. If $y = b, Ex = 0$

$$\frac{-j\omega\mu}{h^2}(-AC_1C_3\cos Bx\sin Ab - AC_2C_3\sin Bx\sin Ab) = 0$$

$$C\sin\frac{m\pi}{a}x\sin Ab = 0$$

This is Possible only if $A = \frac{n\pi}{b}$. If $B = 0$ the whole Ex term vanished Hence $A = \frac{n\pi}{b}$

$$\therefore Ex = \frac{-j\omega\mu}{h^2}(-\frac{n\pi}{b}C_1C_3\cos Bx\sin\frac{n\pi}{b}y - \frac{n\pi}{b}C_2C_3\sin Bx\sin\frac{n\pi}{b}y)$$

We already know $C2 = 0$ Hence $Ex = \frac{j\omega\mu}{h^2}\frac{n\pi}{b}C_1C_3\cos Bx\sin\frac{n\pi}{b}y$

Let $C_1C_3 = H0$ When the wave is propagate along Z direction with respect to time 'ωt'

$$Ex = \frac{j\omega\mu}{h^2}\frac{n\pi}{b}H_0\cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y\cos\omega t\,e^{-\gamma z}$$

Similarly, $C4 = 0$, Hence $Ey = -\frac{j\omega?}{h^2}\frac{m\pi}{a}H_0\sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y\cos\omega t\,e^{-\gamma z}$

In Hz equation substitute $C_2 = C_4 = 0$

$$Hz = C_1C_3\cos Bx\cos Ay$$

$$Hz = H_0 \cos\frac{m\pi}{a}x\cos\frac{n\pi}{b}y \sin \omega t\, e^{-\gamma z}$$

Similarly $Hx = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial x}$ But $Ey = \frac{j\omega?}{h^2}\frac{\partial H_z}{\partial x}$

$$\frac{\partial H_z}{\partial x} = \frac{h^2}{j\omega?}Ey$$

$$Hx = \frac{-\gamma}{h^2}\frac{h^2}{j\omega?}\frac{j\omega?}{h^2}\frac{m\pi}{a} H_0 \sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y \cos \omega t\, e^{-\gamma z}$$

$$\therefore \quad Hx = \frac{-\gamma}{h^2}\frac{m\pi}{a} H_0 \sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y \cos \omega t\, e^{-\gamma z}$$

Similarly $Hy = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial y}$ But $Ex = -\frac{j\omega?}{h^2}\frac{\partial H_z}{\partial y}$

$$\frac{\partial H_z}{\partial y} = -\frac{h^2}{j\omega?}Ex$$

$$\therefore Hy = \frac{\gamma}{h^2}\frac{n\pi}{b}H_0 \cos Bx \sin\frac{n\pi}{b}y \cos \omega t\, e^{-\gamma z}$$

Thus the field equations for TE waves are, if $\alpha = 0$

$$Ex = \frac{j\omega\mu}{h^2}\frac{n\pi}{b}H_0 \cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}b \cos \omega t\, e^{-j\beta z}$$

$$Ey = -\frac{j\omega?}{h^2}\frac{m\pi}{a} H_0 \sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y \cos \omega t\, e^{-j\beta z}$$

$$Ez = 0$$

$$Hx = \frac{-\gamma}{h^2}\frac{m\pi}{a} H_0 \sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y \cos \omega t\, e^{-j\beta z}$$

$$Hy = \frac{\gamma}{h^2}\frac{n\pi}{b} H_0 \cos Bx \sin\frac{n\pi}{b}y \cos \omega t\, e^{-j\beta z}$$

$$Hz = H0 \cos\frac{m\pi}{a}x\cos\frac{n\pi}{b}y \cos \omega t\, e^{-j\beta z}$$

If $m = n = 0$ all the fields of TE wave is vanished. So the lowest possible value is $m = 1$ and $n = 0$. This is called TE10 wave. This lowest mode is called dominant.

$$B^2 = h^2 - A^2$$

$$\text{Now, } A2 + B2 = h2$$

$$h2 = \gamma 2 + \omega 2?\varepsilon \Rightarrow \gamma = \sqrt{h^2 - \omega^2\mu\varepsilon} = \sqrt{A^2 + B^2 - \omega^2\mu\varepsilon}$$

$$\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\varepsilon}$$

This is the propagation constant for a rectangular waveguide for TM waves.

At cut off frequency, $\omega c2?\varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

$$\omega c = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$fc = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$= \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

This is the cut off frequency of rectangular waveguide. The corresponding cot off wavelength is,

$$\lambda c = \frac{v}{f} = \frac{c}{f_c} = \frac{1/\sqrt{\mu\varepsilon}}{f_c}$$

$$= \frac{c}{\frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

$$\lambda c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

The propagation constant $\gamma = j\beta$ if $\alpha = 0$

$$\gamma = j\beta = j\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

$$\therefore \beta = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

The velocity of propagation in waveguide $v = \frac{\omega}{\beta}$

$$V = \frac{\omega}{\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

The corresponding wavelength is rectangular waveguide,

$$\lambda = \frac{v}{f} = \frac{\omega}{f\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}}$$

The lowest propagating mode is termed the *dominant* mode. Wave propagation below cutoff dies out very rapidly, and such a mode is termed *evanescent*. If more than one mode is propagating, the waveguide is termed *overmoded*. In an overmoded waveguide, the multiple modes propagate at different velocities and there will be interference between the waves of the propagating modes which can result in cancellation at frequencies above the cutoff frequency of the mode above the dominant mode. For this reason, waveguide operating bandwidth must be restricted to the dominant mode.

The field Configuration is,

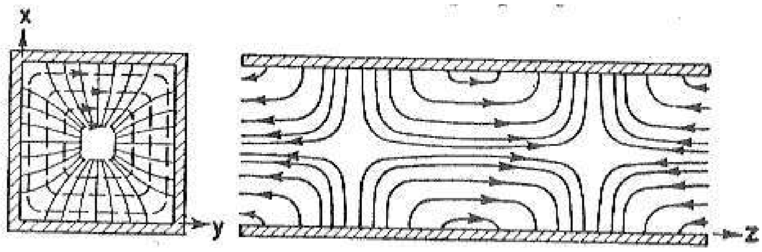

Fig 5.7 Field configuration of Dominant mode

5.7.1. Impossibilities of TEM waves in Rectangular Waveguides

Consider that TEM wave exists within a hollow guide of any shape. By the property of the lines of magnetic field intensity $\overline{H}$ must lie entirely in the transverse plane. For a non magnetic material with condition $\nabla.\overline{H}$=0, the lines of $\overline{H}$ must be in closed loops. So to have existence of the TEM wave inside the guide this $\overline{H}$ lines must be in a plane transverse to the axis of the guide. According to the Maxwell's first equations the magneto motive force around the closed loop must be equal to the axial current. In a guide consisting inner conductor, axial current is nothing but the conductance current of the inner conductor. But in a hollow waveguide like rectangular waveguide there is no inner conductor present. In this case the axial current must be equal to the displacement current. By the property displacement current needs to the components of the electric field in the axial direction. But such axial component of $\overline{E}$ is not present in TEM waves, hence it cannot exist in rectangular waveguide.

5.8. BESSEL'S DIFFERENTIAL EQUATION AND BESSEL FUNCTION

For the circular waveguide the expression for the field components within a hollow circular guide can be obtain conveniently by using the cylindrical co ordinate system (r ,φ, z). The general properties are similar to the rectangular waveguide. Also the basic equations of TE and TM waves are exactly same in both the types of the waveguides. But the solution is totally different. When the cylindrical coordinate system is employed the resulting differential equation has certainly different from and it is known as Bessel's equations. The solution of such a Bessel equation leads to Bessel functions.

The analysis of field component within the hollow perfectly conducting cylinder with uniform circular cross section is carried out using the cylindrical coordinate system. The resulting differential equation is called Bessel equations. The solution of such a Bessel equation is Bessel functions. These Bessels equation are useful in applications such as wave propagation within a cylinder, or circular cross section, the field distribution along the long wire of infinite length, vibration of circular membrane. The surface used to define the cylindrical co ordinates are,

1. Plane of constant z which is parallel to xy plane.
2. A cylinder of radius r with z axis as the axis of cylinder
3. A half plane perpendicular to xy plane and at an angle φ with respect to xz plane where φ is called azimuth angle. The range of variables are,

$$0 \le r \le \infty$$
$$0 \le \varphi \le 2\pi$$
$$-\infty \le z \le \infty$$

Value of Jn(x)

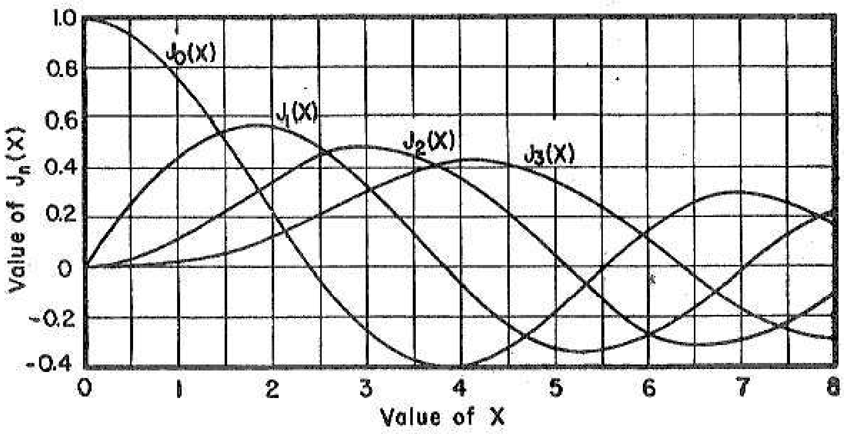

Fig 5.8. Variation of Bessel Function $J_0(x)J_1(x)$ $J_2(x)$ $J_3(x)$

When such a co ordinate system is used for the analysis of the electromagnetic wave propagating with in the circular cylindrical the differential equations obtained is of the form given by,

$$\frac{d^2P}{dr^2} + \frac{1}{r}\frac{dp}{dr} + \left(1 + \frac{n^2}{r^2}\right)P = 0$$

Here n is the integer. Using power series solution one of the solution of equation is,

$$P = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \cdots ..$$

Substitute the value of P, and n=0

$$\frac{d^2P}{dr^2} + \frac{1}{r}\frac{dp}{dr} + P = 0$$

Addition of the coefficient of all powers of r individually equated to zero. Then the resulting series is given by,

$$P = P_1 = C_1\left[1 - \left(\frac{r}{2}\right)^2 + \frac{\left(\frac{1}{2}r\right)^4}{(2!)^2} - \frac{\left(\frac{1}{2}r\right)^6}{(3!)^2} + \cdots\right]$$

$$P = P_1 = C_1 \sum_{r=0}^{\infty} (-1)^r \frac{\left(\frac{1}{2}r\right)^{2r}}{(r!)^2}$$

The series represented in integrable for all values of r. That means it convergent for all real and complex values of r. This is known as Bessel's function of first type of order zero.

Such function is denoted by $J_0(r)$ where suffix zero represent the value of the integer n such as 1,2,3,4…. The Bessel function is indicated by $J_1(r), J_2(r), J_3(r)$ ... respectively.

The second solution is manipulated series. This solution is commonly known as Bessel function of the second type of order zero or Neumann's function. It is denoted by $N_n(r)$. Sometimes a notation $y_n(r)$ is also used. The series obtained for first order Bessel's function of second type for order zero is given by,

$$N_0(r) = \frac{2}{\pi}\left[ln\left(\frac{r}{2}\right) + \gamma\right]J_0(r) - \frac{2}{\pi}\sum_{r=1}^{\infty}(-1)^r \frac{\left(\frac{1}{2}r\right)^{2r}}{(r!)^2}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n})$$

Then the complete solution of Bessel equation of order zero, represented by the equation is the combination of the two individual solution for n = 0. The complete solution is given by,

$$P = AJ_0(r) + BN_0(r)$$

The poles of $J_0(r)$ and $N_0(r)$ as Shown in fig. The important property of the Bessel function of the second type that their values become infinite for all the order, When the arrangement is zero i.e. r=0. Thus while analyzing the wave propagating through circular waveguide; will be considering the region with r=0. So practically it is not possible to have an infinite field, the second type of Bessel function cannot be employed for any physical problem, such as wave through a hollow circular waveguide.

The curve for $J_0(r)$ and $N_0(r)$ are similar to the damped cosine and sine curves respectively. And movever it is observed that for very large valued of r, the functions are represented in the sinusoidal form. The expression of Bessel function for larger is given by,

$$J_0(r) = \sqrt{\frac{2}{\pi r}}\cos\left(r - \frac{\pi}{4}\right)$$

$$N_0(r) = \sqrt{\frac{2}{\pi r}}\sin\left(r - \frac{\pi}{4}\right)$$

5.9. TM WAVES IN CIRCULAR WAVE GUIDES

The field equation in Cylindrical co-ordinate System, Sub $H_z = 0$ in equation

$$h^2H_r = \frac{j\omega\varepsilon}{r}\frac{\partial E_z}{\partial \phi} \quad \text{……..(1)}$$

$$h^2H_\phi = -j\omega\varepsilon\frac{\partial E_z}{\partial_r} \quad \text{…….(2)}$$

$$h^2E_r = -\gamma\frac{\partial E_z}{\partial_r} \quad \text{…….(3)}$$

$$h^2E_\phi = \frac{-\gamma}{r}\frac{\partial E_z}{\partial \phi} \quad \text{…….(4)}$$

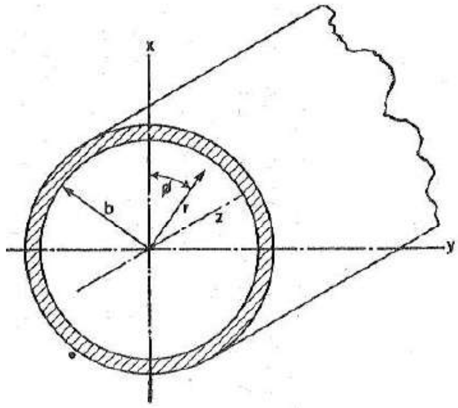

Fig. 5.9. The cylindrical Waveguide

The solution of Maxwell equation in rectangular co ordinates gave fields in terms of trigonometric function of the tube dimensions. Operation on Maxwell's equation in cylindrical co ordinate systems leads to Bessel function J_n (rh) ie. J_n ($r\sqrt{\gamma^2 + \omega^2\mu\varepsilon}$). The manner of J_0,J_1,J_2 and J_3 is,

The expression for Ex of TM wave is, $E_z = E_0 J_n(rh) \cos n\varphi$

$$\frac{\partial E_z}{\partial_r} = E_0 \frac{\partial J_n(rh)}{\partial r} h \cos n\varphi$$

$$\frac{\partial E_z}{\partial_\phi} = -E_0 n J_n(rh) \sin n\varphi$$

$$E_z = E_z \, Sin\, \omega t\, e^{-\gamma z}$$

$$H_z = H_z \, Sin\, \omega t\, e^{-\gamma z}$$

When a wave propagate only in Z direction

Sub these in equation (1) to (4)

$$H_r = \frac{-j\, A_n \varepsilon_1 \varepsilon_0 \omega_n J_n}{(\gamma^2 + \omega^2\mu\varepsilon) r}(r\sqrt{\gamma^2 + \omega^2\mu\varepsilon})\, Sin\, n\varphi\, e^{-\gamma z} \cos \omega t$$

$$H_\phi = \frac{-j\, A_n \varepsilon_1 \varepsilon_0}{\sqrt{\gamma^2 + \omega^2\mu\varepsilon}} \frac{\partial J_n}{\partial p} (\gamma\sqrt{\gamma^2 + \omega^2\mu\varepsilon})\, Cos\, n\varphi\, e^{-\gamma z} \cos \omega t$$

$$E_r = \frac{-j\, \beta_{nA_n}}{\sqrt{(\gamma^2 + \omega^2\mu\varepsilon)}} \frac{\partial J_n}{\partial p} (r\sqrt{\gamma^2 + \omega^2\mu\varepsilon})\, Cos\, n\varphi\, e^{-\gamma z} \cos \omega t \quad = \frac{\beta}{\omega\varepsilon} H_\varphi$$

$$E_\phi = \frac{j\beta}{\gamma\sqrt{(\gamma^2 + \omega^2\mu\varepsilon)}} An_n J_n(r\sqrt{\gamma^2 + \omega^2\mu\varepsilon}))\, Sin\, n\varphi\, e^{-\gamma z} \cos \omega t \quad = \frac{-\beta}{\omega\varepsilon} H_r$$

i. For TM waves Hz = 0

The boundary condition is Jn(ρh) = 0

ii. Let the roots of the equations be Jn(ρh) = 0

The first few roots are,

$(ha)_{01} = 2.405$

$(ha)_{11} = 3.83$

$(ha)_{02} = 5.52$

$(ha)_{12} = 7.02$

The various TM wave will be referred as TMnm waves

The propagation constant $\gamma = \sqrt{h^2 - \omega^2\mu\varepsilon}$

The propagation constant $\gamma = j\beta$ if $\alpha = 0$

$$\therefore \beta = \sqrt{\omega^2\mu\varepsilon - h_{nm}^2}$$

The cut off frequency,

$$\omega_c{}^2 \mu\varepsilon = h_{nm}^2$$

$$f_c = \frac{h_{nm}}{2\pi\sqrt{\mu\varepsilon}}$$

Where $$hnm = \frac{(ha)_{nm}}{a}$$

The phase velocity is,

$$v = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega^2\mu\varepsilon - h_{nm}^2}}$$

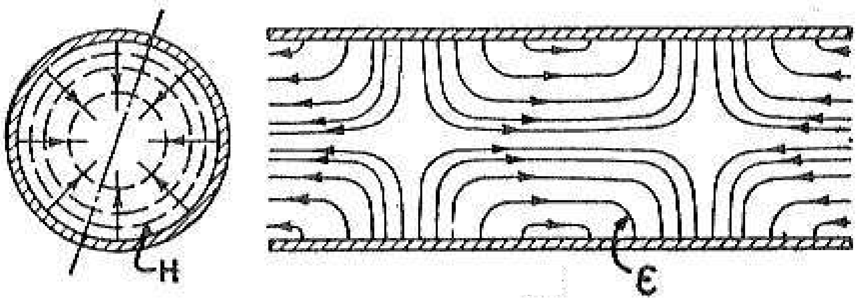

Fig 5.9 Field configuration of Dominant mode

The Roots of Bessel Function (TM_{nm} modes)

x_{nm}	$J_0(x_{0m})$	$J_1(x_{1m})$	$J_2(x_{2m})$	$J_3(x_{3m})$	$J_4(x_{4m})$	$J_5(x_{5m})$
m=1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

5.10. TE WAVES IN CIRCULAR WAVE GUIDES

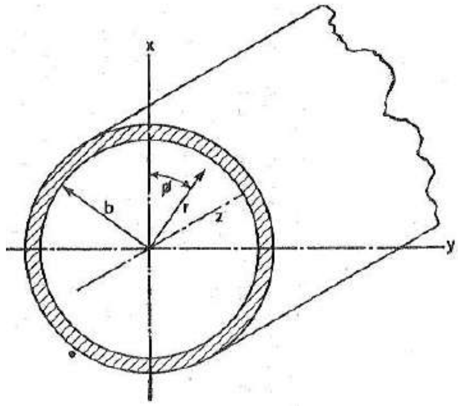

Fig. 5.10. The cylindrical Waveguide

For TE Waves E_Z is Zero. The equations for field components in cylindrical co-ordinate is,

$$h^2H_r = \frac{j\omega\varepsilon}{r}\frac{\partial E_z}{\partial \phi} - \gamma\frac{\partial H_z}{\partial_r} \qquad \text{........(1)}$$

$$h^2H_\phi = -j\omega\varepsilon\frac{\partial E_z}{\partial_r} - \frac{\gamma}{r}\frac{\partial H_z}{\partial \phi} \qquad \text{........(2)}$$

$$h^2E_r = -\gamma\frac{\partial E_z}{\partial_r} - \frac{j\omega\mu}{r}\frac{\partial H_z}{\partial \phi} \qquad \text{.......(3)}$$

$$h^2E_\phi = \frac{-\gamma}{r}\frac{\partial E_z}{\partial \phi} - j\omega\varepsilon\frac{\partial H_z}{\partial_r} \qquad \text{.......(4)}$$

For TE waves, the field equations are obtained by substituting $E^z = 0$ in the above equations.

$$h^2H_r = -\gamma\frac{\partial H_z}{\partial_r} \qquad \text{........(5)}$$

$$h^2H_\phi = \frac{-\gamma}{r}\frac{\partial H_z}{\partial \phi} \qquad \text{.......(6)}$$

$$h^2E_r = \frac{-j\omega\mu}{r}\frac{\partial H_z}{\partial \phi} \qquad \text{.......(7)}$$

$$h^2E_\phi = j\omega\mu\frac{\partial H_z}{\partial_p} \qquad \text{.......(8)}$$

The expressions for Hz for TE wave is, $H_{z0} = H_0J_n^{\ 1}$ (rh) Cos nΦ

If the wave is propagate only in Z direction is,

$$Hz = Hz_0\, e^{-\gamma z}\, Sin\, \omega t$$

$$Hz = H_0Jn1\,(r\sqrt{\gamma^2 + \omega^2\mu\varepsilon}))\, Cos\, n\Phi e^{-\gamma z}\, Sin\, \omega t$$

$$\therefore \quad \frac{\partial H_z}{\partial_r} = H_0 h\frac{\partial J_n(rh)}{\partial r}\, Cos\, n\Phi$$

$$\frac{\partial H_z}{\partial_\phi} = -H_0 Jn\,(rh)\, Sin\, n\Phi$$

Sub these Values I the above equations,

$$h^2H_r = -\gamma H_0 h\frac{\partial}{\partial r}J_n^{\ 1}(r\sqrt{\gamma^2 + \omega^2\mu\varepsilon})\ \text{Cos}\ n\Phi e^{-\gamma z}\ \cos\omega t$$

$$H_r = \frac{-j\beta H_0}{\sqrt{\gamma^2+\omega^2\mu\varepsilon}} \frac{\partial}{\partial r} J_n{}^1(r\sqrt{\gamma^2 + \omega^2\mu\varepsilon})\, e^{-\gamma z} \text{ Cos n}\Phi\text{cos } \omega t$$

$$H_\emptyset = \frac{-j\beta_n H_0}{r\sqrt{\gamma^2+\omega^2\mu\varepsilon}} J_n{}^1(r\sqrt{\gamma^2 + \omega^2\mu\varepsilon}) \text{ Sin n}\Phi\, e^{-\gamma z} \text{ cos } \omega t$$

$$E_r = \frac{\omega\mu}{\beta} H_\emptyset \quad \text{and} \quad E_\emptyset = \frac{-\omega\mu}{\beta} H_r$$

Let the root of the equation be

$J_n{}^1$ (ha) = 0 => $\frac{\partial J_n(ha)}{\partial p} = 0$

i. For TE waves Ez = 0

The boundary condition is $Jn^1(rh) = 0$

ii. Let the roots of the equations be $Jn^1(rh) = 0$

The first few roots are,

$(ha^1)_{01} = 3.832$

$(ha^1)_{11} = 1.841$

$(ha^1)_{02} = 7.02$

$(ha^1)_{12} = 5.331$

The various TE wave will be referred as TEnm waves

The propagation constant $\gamma = \sqrt{h^2 - \omega^2\mu\varepsilon}$ *where* $hnm = \frac{P_{nm}}{a}$

The propagation constant $\gamma = j\beta \text{ if } \alpha = 0$

$$\therefore \beta = \sqrt{\omega^2\mu\varepsilon - h_{nm}^2}$$

The cut off frequency, $\omega_c{}^2\mu\varepsilon = h_{nm}^2$

$$f_c = \frac{h_{nm}}{2\pi\sqrt{\mu\varepsilon}}$$

Where

$$hnm = \frac{(ha)_{nm}}{a}$$

The phase velocity is,

$$v = \frac{\omega}{\beta}$$

$$v = \frac{\omega}{\sqrt{\omega^2\mu\varepsilon - h_{nm}^2}}$$

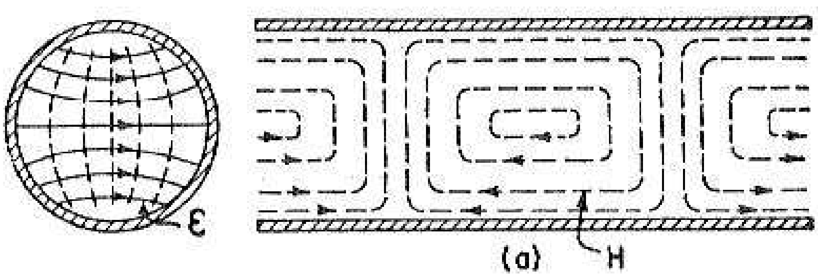

Fig 5.11 Field configuration of Dominant mode

The Roots of Bessel Derivative (TE modes)

x'_{nm}	$J'_0(x'_{0m})$	$J'_1(x'_{1m})$	$J'_2(x'_{2m})$	$J'_3(x'_{3m})$	$J'_4(x'_{4m})$	$J'_5(x'_{5m})$
m=1	3.8317	1.8412	3.0542	4.2012	5.3175	6.4156
2	7.0156	5.3314	6.7061	8.0152	9.2824	10.5199
3	10.1735	8.5363	9.9695	11.3459	12.6819	13.9872
4	13.3237	11.7060	13.1704	14.5858	15.9641	17.3128
5	16.4706	14.8636	16.3475	17.7887	19.1960	20.5755

5.11. THE TEM WAVE IN COAXIAL LINES

We know that, $H = H_0 \sin \omega t\, e^{-\gamma z}$

$E = E_0 \sin \omega t\, e^{-\gamma z}$

For TEM Mode $E_z = H_z = 0$.

∴ The co-ordinate equation are, $E_\emptyset \;\&\; H_r = 0$

$\gamma H_\varphi = j\omega\varepsilon\, E_r$

…….(1)

$\frac{1}{r}\frac{\partial}{\partial r}(\gamma\, H_\emptyset) = 0$ ………(2)

$\frac{\partial}{\partial r}(\gamma\, H_\emptyset) = 0$

$-\gamma E_r = -j\omega\mu\, H_\emptyset$ ……….(3)

$\frac{\partial E_r}{\partial \emptyset} = 0$ ………..(4)

∴ $H\varphi = \frac{j\omega\varepsilon}{\gamma} E_r = \frac{\gamma}{j\omega\mu} E_r$

$\gamma^2 = -\omega^2\mu\varepsilon$

$\gamma = j\omega\sqrt{\mu\varepsilon}$

∴ For a TEM Condition $\propto = 0, \beta -$ obtained for the dissipation less line. For TEM Wave $f_c = 0$ (or) that all frequencies are propagated on the co-axial line by the TEM mode.

$2 \rightarrow \frac{\partial}{\partial r}(r\, H_\emptyset) = 0$ K – constant

$r\, H_\emptyset = K$

$H_\emptyset = \frac{K}{r}$

We know the Ampere octal law, $\int H.dl = I$

$2\pi a\, H_\emptyset = I_0$

$To\ find\ K, r = a$ $k = \frac{I_0}{2\pi}$

$H_\emptyset = \frac{I_0}{2\pi r}$

$3 \rightarrow E_r = \frac{j\omega\varepsilon}{\gamma} H_\emptyset = \frac{\omega\mu}{\sqrt{\omega\mu}} H_\emptyset = \sqrt{\frac{\mu}{\varepsilon}}\, H_\emptyset$

$$\therefore \; E_r = \sqrt{\frac{\mu}{\varepsilon}} \frac{I_0}{2\pi r}$$

The TEM wave in the co-axial line, it is possible for higher order forms of TE & TM wave to exist with components of electric and magnetic field in the direction of the line axis.

5.12. RESONANT CAVITIES

A cavity resonator is a hollow conductor blocked at both ends and along which an electromagnetic wave can be supported. It can be viewed as a waveguide short-circuited at both ends. The cavity's interior surfaces reflect a wave of a specific frequency. When a wave that is resonant with the cavity enters, it bounces back and forth within the cavity, with low loss. As more wave energy enters the cavity, it combines with and reinforces the standing wave, increasing its intensity. Waveguide resonators in its simplest forms are metallic enclosures or cavities. Electric and magnetic energy is stored in this volume thus establishing a resonance condition. The power dissipation is through the surface of the waveguide and the dielectric filling. Coupling energy to the waveguide resonator can be made through a small aperture, a probe or a current loop. Depending on the type of waveguide, two types of resonant cavities

i) Rectangular
ii) Circular.

The cavity resonators are used in tuned circuits and UMF tubes, Klystron amplifier, oscillators. The circular cavity resonators are also used in μ wave frequency meter.

5.12.1. Rectangular Cavity Resonator:

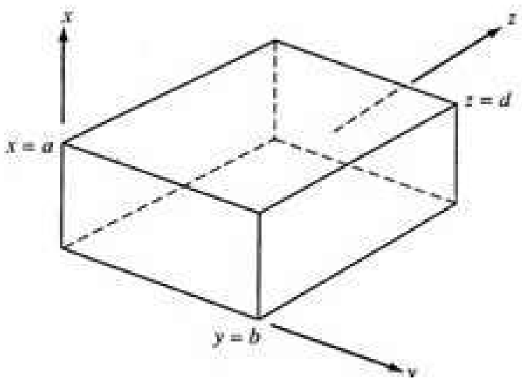

Fig 5.12 Geometry of Rectangular cavity resonator

Consider rectangular cavity shorted at both the ends. The guide wavelength is $\lambda_g = \frac{\lambda_0}{\sqrt{1-\left(\frac{\lambda_0}{\lambda_c}\right)^2}}$. The most dominant mode in rectangular wave is TE_{10} mode. In dominant mode the field frequency the field configuration is lowest. For TE_{10} mode $\lambda_c = 2\,a$

$$\therefore \lambda_g = \frac{\lambda_0}{\sqrt{1-\left(\frac{\lambda_0}{2a}\right)^2}}$$

The dimension 'a' is fixed for the resonator. So λ_g - fixed we know $f = \frac{c}{\lambda_0} = f_0$

In a rectangular wave guide,

$$\gamma^2 + \omega^2\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2$$

$$\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2 - \omega^2\mu\varepsilon}$$

$$\omega^2\mu\varepsilon = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2 - \gamma^2}$$

For $\propto = 0$ $\qquad \omega^2\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2 - (j\beta)^2$

But the condition for the cavity resonator is given as the cavity must be an integer multiple of a half-guide wavelength long at the resonant frequency,

$$\beta = \frac{p\pi}{d} \quad where\ P = 1, 2, 3 \ \ldots\ldots\infty$$

Depending on the Value of P, the general wave mode through the cavity resonators are denoted by TE_{mnp} for TE wave & TM_{mnp} for TM wave.

$$\therefore \quad \omega^2{}_0\mu\varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2 - \left(\frac{p\pi}{d}\right)^2 \quad \Rightarrow \text{At resonant frequency}$$

$$\omega^2{}_0 = \frac{1}{\mu\varepsilon}\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2 - \left(\frac{p\pi}{d}\right)^2\right]$$

$$\omega^2{}_0 = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2 - \left(\frac{p\pi}{d}\right)^2} \text{ rad/s}$$

$$f_0 = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2 - \left(\frac{p\pi}{d}\right)^2}$$

If the resonator cavity is filled in air,

$$\frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\,\varepsilon_0}} = \frac{1}{c}$$

$$f_0 = \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{a}\right)^2 - \left(\frac{p}{d}\right)^2}$$

5.10.2. Circular cavity Resonator:

The circular cavity resonators modes are specified as TM_{nmp} for TM waves and TE_{nmp} for TE waves

For a circular waveguide

$$\gamma^2 + \omega^2\mu\varepsilon = h^2$$

$$\therefore \quad \gamma^2 + \omega^2\mu\varepsilon = \left(\frac{P_{nm}}{a}\right)^2$$

$$-\beta^2 + \omega^2\mu\varepsilon = \left(\frac{P_{nm}}{a}\right)^2$$

$$\omega^2\mu\varepsilon = \left(\frac{P_{nm}}{a}\right)^2 + \beta^2$$

$$\omega^2\mu\varepsilon = \left[\left(\frac{P_{nm}}{a}\right)^2 + \beta^2\right]$$

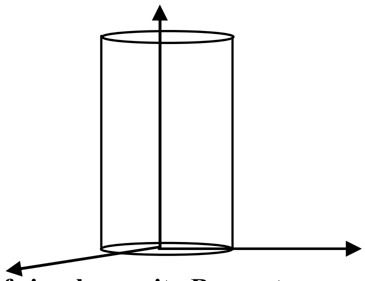

Fig 5.15: Geometry of circular cavity Resonator

But the condition for circular cavity resonator is same as rectangular cavity resonator.

$$ie, \beta = \frac{P\pi}{d} \qquad P = 1, 2, 3 \ldots\ldots\ldots\ldots.. \infty$$

Depending upon the value of P the mode is selected. At resonant frequency,

$$\omega^2{}_0 = \frac{1}{\mu\varepsilon}\left[\left(\frac{P_{nm}}{a}\right)^2 + \left(\frac{P\pi}{d}\right)^2\right]$$

$$\omega_0 = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{P_{nm}}{a}\right)^2 + \left(\frac{P\pi}{d}\right)^2}$$

$$\omega_0 = 2\pi f_0$$

$$f_0 = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{P_{nm}}{a}\right)^2 + \left(\frac{P\pi}{d}\right)^2}\ HZ$$

$$For\ free\ space\ \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = C$$

$$f_0 = \frac{C}{2\pi}\sqrt{\left(\frac{P_{nm}}{a}\right)^2 + \left(\frac{P\pi}{d}\right)^2}\ HZ$$

…. For TM_{nmp} mode

For TE_{nmp} mode,

$$f_0 = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{P_{nm^1}}{a}\right)^2 + \left(\frac{P\pi}{d}\right)^2}\ HZ$$

$$f_0 = \frac{C}{2\pi}\sqrt{\left(\frac{P_{nm^1}}{a}\right)^2 + \left(\frac{P\pi}{d}\right)^2}\ HZ$$

…. For free space

5.13. Q FACTOR OF RESONANT CAVITIES (Q)

The quality factor for the resonant circuit, either series or parallel, is the measure of efficiency with which the energy storing elements can stored maximum energy. It is also a measure of the frequency selectivity of a resonant circuit. Quality factor is also called as figure of merit.

$$Q = 2\pi \times \frac{Maximum\ energy\ stored}{Energy\ dissipated}$$

Let the maximum energy stored in W & energy dissipated be P.

$$\therefore \quad Q = 2\pi \frac{\omega}{P}$$

$$= \frac{2\pi f}{f} \times \frac{\omega}{p}$$

$$= \frac{\omega}{f} \times \frac{\omega}{p} = \omega_0 \frac{W}{P}$$

Where $\frac{\omega}{p} = \omega_0$. For an ideal resonator, the energy dissipation is zero.Therefore Q factor is ∞. The electric energy stored in the cavity resonator is

$$W_e = \oint_v \frac{\varepsilon}{2} E^2 dv$$

The magnetic energy stored is,

$$W_m = \oint_v \frac{\mu}{2} H^2 dv$$

Where E & H one the peak values of electric and magnetic field intensities. Energy loss or energy dissipated in the resonator is,

$$P = \frac{Rs}{2} \oint |H_t|^2 \, ds$$

Where H_t is the peak value of the tangential component of magnetic field intensity. Rs. Surface resistance of the resonator. At resonance, Electric energy stored is equal to Magnetic energy stored. ie, $W_e = W_m$

$$Now\ Q = \omega_0 \frac{w}{p} = \omega_0 \frac{W_m = \oint_v \frac{E}{2} E^2 dv}{\frac{Rs}{2} \oint_s |H_t|^2 \, ds}$$

$$(or)$$

$$Q = \omega_0 \frac{\oint_v {}^{\mu}/_{2} H^2 dv}{\frac{Rs}{2} \oint_s |H_t|^2 \, ds}$$

H_n- Peak value of normal component of a magnetic field intensity.

$H_2 = Ht_2 + Hn_2$

$$|H_t|^2 \cong 2|H|^2$$

$$Q \approx \omega_0 \frac{\oint_v {}^{\varepsilon}/_{2} E^2 dv}{\frac{Rs}{2} \oint_s |H|^2 \, ds} = \omega_0 \frac{\oint_v {}^{\mu}/_{2} H^2 dv}{\frac{Rs}{2} \oint_s |H|^2 \, ds}$$

$$\approx \frac{\omega_0 \varepsilon}{2Rs} \frac{\oint_v E^2 dv}{\frac{Rs}{2} \oint_s |H|^2 ds} \quad (or) \quad Q \simeq \frac{\omega_{0\mu}}{2Rs} \frac{\oint_v H^2 dv}{\frac{Rs}{2} \oint_s |H|^2 ds}$$

The types of Q factor are,

a. Unloaded Q
b. Loaded Q
c. External Q

5.11.1. Unloaded Q:

When the cavity is assumed to be not connected to any load (or) external circuit, the quality factor is denoted as unloaded quality factor Q_o. An unloaded resonator can be represented either by a series resonant circuit or by a parallel resonant circuit.

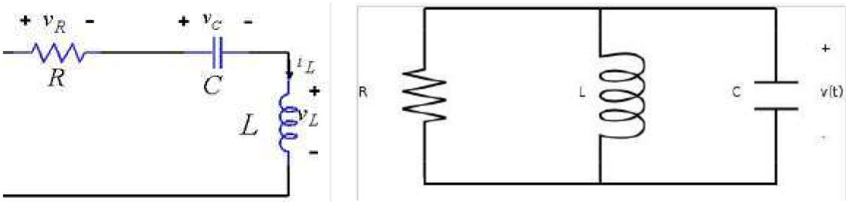

Fig 5.16. equivalent circuit of cavity resonator.

The resonant frequency and the unloaded Q of cavity resonator are,

$$f_r = \frac{1}{2\pi\sqrt{Lc}}$$

$$\& \; Q_0 = \frac{Lw_0}{R}$$

5.11.2. External Q for TE_{101}:

If the cavity is connected to any external load the energy will be dissipated in the external load. This cause the power loss. The quality factor which results due to the presence of external load in the cavity results in an external quality factor Q_0.

$$Q_e = \omega_0 \frac{w}{p} \qquad \omega_0 = \frac{\omega}{f}$$

W – Maximum energy stored

P – Power loss due to the presence of an external load.

5.11.3. Loaded Q for TE_{101}:

When the cavity is connected to any load or external circuit, the quality factor is denoted as loaded Q (Q_L)

$$Q_L = 2\pi \times \frac{Maximum\ energy\ stored}{Energy\ dissipated\ in\ the\ cavity\ per\ cycle\ (Po) + Energy\ dissipated\ due\ to\ the\ external\ load}$$

$$Q_L = \frac{2\pi f}{f} \times \frac{W}{(P_0 + P_e)} = \frac{\omega}{f} \times \frac{W}{(P_e + P_0)}$$

$$Q_L = \omega_0 \frac{W}{P_0 + P_e}$$

$$\frac{1}{Q_L} = \frac{1}{Q_0} + \frac{1}{Q_e}$$

The dominant mode in a rectangular cavity resonator is TE_{101} with dimensions b < a < d.

a – width of the resonator

b – height of the resonator

d – length of the resonator

The energy stored inside the cavity at resonance,

$$w = \frac{\varepsilon}{2}\int_0^a \int_0^b \int_0^d |E_y|^2 \; dx\, dy\, dz$$

$$E_y = E_0{}^2 Sin^2\left(\frac{\pi}{a}x\right) Sin^2\left(\frac{\pi}{d}z\right) \; dx\, dy\, dz$$

$$= \frac{\varepsilon}{2}\,\frac{E_0^2}{4}\; abd$$

Thus the maximum energy stored in TE101 mode is $\frac{\varepsilon_0{}^2}{8}\; abd$

Total power loss,

$$P = \left[\frac{Rs}{2}\; 2\int_0^a \int_0^b (Hx^2 + Hy^2)_{z=0}\right] dxdy \; + \; 2\int_0^b \int_0^d (Hy^2 + Hz^2)_{x=0}$$

$$dydx \; + \; 2\int_0^a \int_0^d (Hx^2 + Hz^2)_{y=0} \; dxdz$$

$$P = \frac{R_s E_0^2 \lambda^2}{8\eta^2}\left[\frac{b(a^3 + d^3) + \frac{1}{2}\, ad\,(a^2 + d^2)}{(ad)^2}\right]$$

The quality factor Q = $\omega_0 = \frac{W}{P}$

$$Q = \omega_0 \frac{\varepsilon/8\; E_0{}^2\; abd}{\frac{R_s E_0^2 \lambda^2}{8\lambda^2}\left[\frac{b(a^3 + d^3) + \frac{1}{2}\, ad\,(a^2 + d^2)}{(ad)^2}\right]}$$

$$= \frac{\pi\eta b(a^2 + d^2)^{3/2}}{2Rs[2b(a^3 + d^3) + ad\,(a^2 + d^2)]}$$

For square base cavity (a=d), Q is maximum. If a = d then,

$$Q\; max = \frac{\pi\eta b(a^2 + d^2)^{3/2}}{2Rs[2b(a^3 + a^3) + a^2\,(a^2 + a^2)]}$$

$$= \frac{\pi\eta b(2a^2)^{3/2}}{2Rs[4a^3 b + 2a^4]} = \frac{\pi\eta b 2\sqrt{2}\, a^3}{8Rsa^3 b\left[1 + \frac{2a^4}{4a^3 b}\right]}$$

$$Q\; max = \frac{\pi\sqrt{2}\eta}{4\; Rs\left[1 + \frac{a}{2b}\right]} = \frac{1.11\eta}{Rs\;\left[1 + \frac{a}{2b}\right]}$$

For Cubic Cavity a = b = d

$$Q = \frac{\pi\eta a(a^2+a^2)^{3/2}}{2Rs[2a(a^3+a^3)+a^2\,(a^2+a^2)]}$$

$$= \frac{\pi\eta b(2a^2)^{3/2}}{2Rs[4a^4+2a^4]} = \frac{\pi\eta a^4 2\sqrt{2}}{2Rs6a^4}$$

$$= \frac{\pi\eta\sqrt{2}}{6Rs}$$

$$Q = 0.74\frac{\eta}{Rs}$$

PROBLEMS

Example 5.1. For a TE_{10} wave travelling through a parallel plans wave guiding system which is air filled with a spacing of 2.286 cm operating at 8 GH_z i. Calculate the cut off frequency ii. Find out whether propagation take place

Solution: Givn m=1, a = 2.286×10^{-2}, f = 8 GH_Z

i. cut off frequency

$$f_c = \frac{m}{2a\sqrt{\mu\varepsilon}}$$

$$= \frac{1}{2 \times 2.86\ \times 10^{-2}\sqrt{4\pi\ \times\ 10^{-7} \times 8.85 \times 10^{-12}}}$$

= 6.6GH_z

ii. If operating frequency f greater than cut off frequency f_c propagation take place.

Here $f > f_c$ 8GH_z > 6.6 GH_z

∴ Propagation take place

Example 5.2. A TEM wave propagates through a parallel plane wave guiding system at 6 GHz, with a spacing of 3 cm. The system is made up of brass whose conductivity is $1.1 \times 10^7 \text{mho}/m$ and is air filled i. Phase constant ii. Attenuation constant iii. Propagation constant iv. Wave velocity v. Wave length vi. Cut off frequency

Solution: f = 6GHz a = 3×10^{-2} $\sigma m = 1.1 \times 10^7 \text{mho}/m$

For free space $\eta = \eta_0 = 377\Omega$

i. Phase constant

$$\beta = \omega\sqrt{\mu_0\varepsilon_0}$$

$$= \frac{\omega}{C} = 125.7\ rad/m$$

ii.

$$\propto = \frac{1}{\eta}\sqrt{\frac{\omega\mu m}{2\sigma m}}$$

$$= \frac{1}{377 \times 0.03}\sqrt{\frac{2\pi \times 6 \times 10^9}{2(1.1 \times 10^7)}}$$

$$= 41.33 \times 10^{-4}\ NP/m$$

iii. Propagation constant $\gamma = \propto \ + j\beta$

$= 41.33\times 10^{-4} + \ j125.7/m$

iv. Wave velocity

$$\vartheta = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$= 3\times 10^{8}$m/s

v. Wave length

$$\lambda \ = \frac{2\pi}{\omega\sqrt{\mu_0 \varepsilon_0}}$$

= 0.05 cm

vi. Cut off frequency

$$f_c = \frac{m}{2a\sqrt{\mu_0 \varepsilon_0}} = 0$$

Example 5.3. An electromagnetic wave travels through a parallel plane wave guiding system with a spacing of 300 and aix-filled at 12 GHz. Calculate the phase construct

i. If the EM wave is TE_{10} ii. If the EM wave is TM_{11}

Solution: Given f = 12 GHz a = 0.03

$$\beta = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}$$

For TE_{10} & TM_{11} mode m = 1

$$\therefore \qquad \beta = \sqrt{(2\pi f)^2 \mu_0 \varepsilon_0 - \left(\frac{\pi}{0.03}\right)^2}$$

$$\beta = 228.66\, rad/m$$

Example 5.4. An air filled parallel plane wave guiding system with spacing of 4 cm and with planes made up of silver whose conductivity is 6.1×10^{7}mho/m, propagates an EM wave at frequency of 10 GHz.i. Calculate the attenuation factor if EM wave is TE_{10} ii. Calculate the attenuation factor if EM wave is TM_{11}

Solution:

$$f = 10\ GHz, \sigma m\ 6.1\ \times\ 10^{7}\quad a = 4\times 10^{-2}$$

$$i. For\ TE\ wave\ \propto = \frac{2(m^2)\pi^2}{a^2\beta\omega\mu} \qquad \beta = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}$$

$$= \ 20.458\ \times\ 10^{-6}\ N/m$$

$$ii)\ For\ TM\ wave, \qquad \propto = \frac{\omega\varepsilon}{\beta a}\frac{\omega\mu m}{2\sigma m} \qquad \beta = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}$$

$$= \ 194.3\, rad/m$$

$$= \ 18.21\ \times\ 10^{-4}\ NP/m$$

Example 5.5 A pair of perfectly conducting planes are separated 4 cm in air, frequency of 5000 MHz with TM1 mode, find following. i) Cut off frequency (ii) Cut off wavelength (iii) Phase constant β

Solution: given A = 4 CM, F = 5000 m MHz

For TM1 mode, m = 1

$$i)\ Cut\ of\ frequency \qquad f_c = \frac{m}{2a\sqrt{\mu\varepsilon}}$$

$$= \frac{1}{2 \times 4 \times 10^{-2} \times \sqrt{4\pi \times 10^{-7} \times \frac{1}{36\pi} \times 10^{-9}}}$$

$$= 3.74\ GHz$$

$$ii)\ Cut\ off\ wavelength \qquad \lambda_c = \frac{2a}{m} = \frac{2 \times 4 \times 10^2}{1} = 8 \times 10^{-2} = m = 8cm$$

$$iii)\ \beta = 2\pi\sqrt{\mu.\varepsilon.}\sqrt{f_2 - f_c^2}$$

$$= 2\pi\sqrt{4\frac{1}{\pi} \times 10^{-7}\left(\frac{1}{36}\pi\right) \times 10^{-9}}\,.\sqrt{(5000 \times 10^6)^2 - (3.75 \times 10^{-9})}$$

$$= 69.26\ degree/m$$

Example 5.6: For a frequency of 6000 MHz and plane separation of 7 cm. following for TE_1 mode.i) Critical frequency ii) Phase constant βiii) Attenuation constant ∝ and phase constant β for f = 0.8 $f_c$iv) Attenuation constant ∝ and phase constant β for f = 1.25 f_cv) Critical wavelength λ_c

f = 600 MHz

d = 7 cm = 7 × $10^{-2}m$

Medium is air,∴ $\mu_1 = \mu_0 = 4 \times \pi \times 10^{-7}\ and\ \varepsilon_1 = \varepsilon_0 = 8.8542 \times 10^{-12}$

i) Critical frequency

$$f_c = \frac{m}{2a} = \sqrt{\frac{1}{\mu_{1\varepsilon_1}}}$$

$$= \frac{1}{2 \times 7 \times 10^{-2}}\sqrt{\frac{1}{4 \times \pi \times 10^{-7} \times 8.8542 \times 10^{-12}}}$$

= 2141.39 MHz

ii) At f = 6000 MHz which is greater than f_c $\propto = 0$

$$\beta = \frac{m\pi}{2a}\left(\frac{f}{f_c}\right)^2\text{-1}$$

$$= \frac{\pi}{7 \times 10^{-2}}\sqrt{\frac{(600 \times 10^6)^2}{(2141.39 \times 10^6)^2} - 1}$$

= 117.46

iii) For f = 0.8 f_c where f < f_c

Propagation constant is purely real ∴ $\beta = 0$

$$\therefore \quad \propto = \frac{m\pi}{2a}\sqrt{1-\left(\frac{f}{f_c}\right)^2} - 1$$

$$\therefore \quad \propto = \frac{\pi}{7\times10^{-2}}\sqrt{1-\left(\frac{0.8f_c}{f_c}\right)^2}$$

$$\therefore \quad \propto = 26.9279$$

iv) For f = 1.25 f_c where f > f_c propagation constant is purely imaginary $\propto = 0$

$$\beta = \frac{m\pi}{2a}\sqrt{\left(\frac{f}{f_c}\right)^2 - 1}$$

$$= \frac{\pi}{7\times10^{-2}}\sqrt{\left(\frac{1.25f_c}{f_c}\right)^2 - 1}$$

$$= 33.66$$

v) Critical wavelength

$$\lambda_c = \frac{2a}{m} = 2\times7\times10^{-2}$$

$$= 14 \text{ cm}$$

Example 5.7. Given a circular waveguide used for a signal at a freq. of 11 GHz Propagated in the TE11 mode and the internal diameter is 4.5 cm. Calculate cut off wavelength guide wavelength group velocity and phase velocity.

Solution: f = 11 GHz, d = 4.5 cm, n = 1, m = 1

a = 2.25 cm

$(ha^1)_{11} = 1.841$

$$\lambda_c = \frac{2\pi a}{(h_a^1)_{11}} = \frac{25\times2.25\,cm}{1.841}$$

$$\lambda_c = 7.68\,cm \qquad \lambda = c/f$$

$$= 0.27$$

$$\lambda_g = \frac{\lambda}{\sqrt{1-\left(\frac{g}{g_c}\right)^2}}$$

$$= 0.0288 \text{ m}$$

$$\vartheta P = \frac{\lambda_g}{\lambda}c$$

$$= 3.176 \times 10^8 \text{ m/sec}$$

$$\vartheta P = \frac{C^2}{\vartheta P} = 2.833\times10^8\,m/s$$

Example 5.8. A circular waveguide has an internal diameter of 5 cm. Calculate the cut off frequency for TE_{01} and TE_{11} mode and also find cut off wavelength.

Solution: d = 5 cm a = 2.5cm

$$f_c = \frac{h_{nm}}{2\pi\sqrt{\mu\varepsilon}}$$

$$f_c = \frac{h_{01}}{2\pi\sqrt{\mu\varepsilon}} = \frac{3.8320}{2\pi\sqrt{\mu\varepsilon}}$$

$$= 1.828 \times 10^8 H_r$$

$$\lambda_c = \frac{2\pi a}{ha_{nm}} = \frac{2\pi a}{h_{01}}$$

$$\frac{TE_{11}}{f_c} = \frac{h_{11}}{2\pi\sqrt{\mu\varepsilon}} = \frac{1.841}{2\pi\sqrt{\mu\varepsilon}} = 8.784 \times 10^7 Hz$$

$$\lambda_c = \frac{2\pi a}{ha_{nm}} = \frac{2\pi a}{h_{11}} = 8.53\ cm$$

Example 5.9. Determine the cut off frequencies of the first two propagating modes of a circular waveguide with a=0.5 cm and $\varepsilon_r = 2.25$ if the guide is operating at f = 13 GHz.

Solution: a = 0.5 cm $\varepsilon_r = 2.25$ f = 13 GHz

$$f_c = \frac{h_{nm}}{2\pi\sqrt{\mu\varepsilon}}$$

$$= \frac{2.405}{2\pi\sqrt{\mu_0\varepsilon_r}} = 7.650 \times 10^7\ Hz$$

$(ha)_{11} = 3.85$

$f_c = 1.224 \times 10^8$ Hz

TE: $(ha)_{01} = 3.832$

$f_c = 1.224 \times 10^8$ Hz

$(ha)_{11} = 1.841$

$f_c = 5.856 \times 10^7$ Hz

Example 5.10. An air filled circular waveguide having an inner radius of 1 cm is excited in dominant mode at 10 GHz. Find a) the cut off freq. of dominant mode at 10 GHz. b) the guide wave length and wave impedance C, the Bw for operation in dominant mode only.

Solution: given: a= 1 cm f = 10 GHz

TE_{11}

$$a,\ \lambda c_{11} = \frac{2\pi a}{h_{11}} = \frac{2\pi \times 1\ cm}{1.841}$$

$$= 0.0341\ m$$

$fc_{11} = \frac{C}{\lambda c_{11}} = 8.8GHz$ $\lambda = c/f = 0.03$

b, $\lambda_g = \dfrac{\lambda}{\sqrt{1-\left(\frac{8}{8_c}\right)^2}}$

= 0.0631 m $\eta = \sqrt{\mu/\varepsilon} = 376.73$

$Z_{TE} = \dfrac{\eta}{\sqrt{1-\left(\frac{8}{8_c}\right)^2}}$

= 792. 44Σ

$TM01 \quad a,\ \lambda c_{01} = \dfrac{2\pi a}{h_{01}} = \dfrac{2\pi \times 1\,cm}{2.405} = 0.0261\,m$

$fc01 = \dfrac{C}{\lambda c_{01}} = 11.49 GHz$

b, $\lambda_g = \dfrac{\lambda}{\sqrt{1-\left(\frac{\lambda}{\lambda_c}\right)^2}}$ $\lambda = c/f = 0.03$

$= \dfrac{0.03}{\sqrt{1-\left(\frac{0.03}{0.0261}\right)^2}}$

$ZTE =$

d, B.W = Cut off frequency of TM_{01} - cut off frequency TM_{01}

= 11.49 GHz – 8.8 Hz

= 2.69 GHz

Example 5.11. A rectangular cavity resonator has diameter of a = 5 cm, b = 2 cm and d=15 cm computer a, the resonant frequency of dominant mode for an air filled cavity b, the resonant frequency of dominant mode for a dielectric filled cavity of ε_r = 2.56

Solution: a= 5 cm b = 2 cm d = 15 cm

$TE_{101_{mnp}}$ =>domininant mode (wave guide)

$a,\ f_r = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{a}\right)^2 + (\frac{p}{d})^2} \qquad (n = 1, \quad m = 0,\ p = 1)$

$= \dfrac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\dfrac{1}{5\,cm}\right)^2 + \left(\dfrac{1}{15\,cm}\right)^2}$

$= 3.1622\,GHz$

$b,\ f_r = \dfrac{1}{2\sqrt{\mu\varepsilon_0\varepsilon_r}}\sqrt{\left(\dfrac{m}{a}\right)^2 + \left(\dfrac{n}{a}\right)^2 + (\dfrac{p}{d})^2}$

$m = 1, \quad n = 0, \qquad p = 1$

$f_r = 1.975\,GHz$

Example 5.12. Calculate resonant frequency of rectangular resonator of dimensions a=3 cm, b= 2 cm, d = 4 cm if the operating mode is TE101 assume free space within cavity.

Solution: a=3 cm, b= 2 cm, d = 4 cm, m = 1, n = 0, p = 1

$$f_r = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{a}\right)^2 + \left(\frac{p}{d}\right)^2}$$

= 6.25 GHz

Example 5.13. Design a rectangular cavity to have a resonant frequency of TE11 mode is 9.8 GHz having dimensions a = d and b = $\frac{a}{d}$

Solution: $f_r = \frac{C}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{a}\right)^2 + \left(\frac{p}{d}\right)^2}$ (m = 1, n = 1, p = 0)

$$9.8 \text{ G} = \frac{3 \times 10^8}{2}\sqrt{\frac{1}{a^2} + \frac{4}{a^2}}$$

$$f_r = \frac{C}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{2/a}\right)^2 - \left(\frac{p}{d}\right)^2}$$

$$9.8\, G = \frac{C}{2}\sqrt{\frac{1}{a^2} + \frac{4}{a^2}} = \frac{C}{2}\sqrt{\frac{5}{a^2}}$$

$$= \frac{C}{2}\frac{\sqrt{5}}{a}$$

$$a = \frac{3 \times 10^8\sqrt{5}}{2 \times 9.8}$$

= 0.0342 m

a = 3.42 cm, d = 3.42 cm

b = $\frac{3.42}{2} = 1.71\, Cm$

Example 5.14: A rectangular waveguide with dimensions a = 2.5 cm and b = 1 cm to operate below 15.1 GHz. How many TE and TM modes can the waveguide transmit if the guide is filled with a medium characterized by $\sigma = 0,\ \varepsilon_1 = 4\ \varepsilon_0\ and\ \mu_{/} = 1$? Calculate Cut off frequencies of the modes.

Solution: Given: a = 2.5 cm = 2.5 × 10^{-2} m

b = 1 cm = 1 × 10^{-2} m

f = 15.1 GHz

The cut-off frequency f_c is given by,

$$f_c = f_{c,m,n} = \frac{V_1}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{a}\right)^2}$$

$$But \qquad V_1 = \frac{1}{\sqrt{\mu_1 \varepsilon_1}} \frac{1}{\sqrt{(\mu_0 \mu_r)(4\varepsilon_0)}} = \frac{\frac{1}{\sqrt{\mu_0 \varepsilon_0}}}{\sqrt{4\mu_r}}$$

$$\therefore \qquad V_1 = \frac{C}{2} = \frac{3 \times 10^8}{2}$$

$$\therefore \qquad f_{c,m,n} = \frac{3 \times 10^8}{2} \sqrt{\left(\frac{m}{2.5 \times 10^{-2}}\right)^2 + \left(\frac{n}{1 \times 10^{-2}}\right)^2} \qquad \ldots \ldots (1)$$

Let us consider different values of m and n.

For $TE_{0,1}$ mode: m = 0, n = 1

$$f_{c,0,1} = \frac{3 \times 10^8}{2} \sqrt{0 + (1 \times 10^2)^2} = 7.5\ GHz$$

For $TE_{0,\,2}$ mode: m = 0, n = 2

For $TE_{1,1}$, $EM_{1,\,1}$, (degenerate modes), $f_{c1,1}$ = 8.078 GHz

$TE_{2,1}$, $TM_{2,1}$, $f_{c\,2,1}$ = 9.6 GHz

$TE_{3,1}$, $TM_{3,1}$, $f_{c3,1}$ = 11.72 GHz

$TE_{4,1}$, $TM_{4,1}$, $f_{c\,4,1}$ = 14.14 GHz

$TE_{1,2}$, $TM_{1,2}$, $f_{c\,1,2}$ = 15.3 GHz

The modes whose cut-off frequencies are less or equal to 15.1 GHz will be transmitted. Thus in all total 11 TE modes and r TM modes are transmitted by the rectangular guide.

Example 5.15: A standard air filled rectangular waveguide with dimensions a = 8.5 cm and b = 4.3 cm is fed by a 4 GHz carrier from co-axial cable. Determine if a TE_{11} mode will be propagated. If so calculate phase velocity and group velocity.

Solution: For $TE_{1,1}$ mode : m = 1, n=1. For air, $V_1 = c = 3 \times 10^8$ m/sec

$$The\ cut-off\ frequency\ is\ given\ by \qquad f_c = \frac{V_1}{2} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\therefore \qquad f_c = \frac{3 \times 10^8}{2} \sqrt{\left(\frac{1}{8.5 \times 10^{-2}}\right)^2 + \left(\frac{n}{4.3 \times 10^{-2}}\right)^2}$$

$$\therefore \qquad f_c = 3.909\ GHz$$

$$As\ f = 4\ GHz > fc, the\ TE1{,}1\ mode\ will\ propagate.$$

$$Hence\ the\ phase\ velocity\ is\ given\ by$$

$$V_p = \frac{V_1}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} = \frac{3 \times 10^8}{\sqrt{1 - \left(\frac{3.909 \times 10^9}{4 \times 10^9}\right)}} = 14.14 \times 10^8\ m/sec$$

$$\therefore \qquad f_{c\ o,2} = \frac{3 \times 10^8}{4} \sqrt{\left(\frac{0}{2.5 \times 10^{-2}}\right)^2 + \left(\frac{2}{1 \times 10^{-2}}\right)^2} = 15\ GHz$$

$For\ TE0,3\ mode: m = 0, \quad n = 2$

$$f_{c\,o,3} = \frac{3 \times 10^8}{4}\sqrt{\left(\frac{0}{2.5 \times 10^{-2}}\right)^2 + \left(\frac{3}{1 \times 10^{-2}}\right)^2} = 22.5\ GHz$$

Thus for $TE_{0,3}$ mode $f_{c\,o,3} > f\ i.e\ f_{c\,o,3} > 15.1\ GHz.$ Hence the mode with n>2 cannot propagate through guide.

$For\ TE10\ mode: m = 1, n = 0 \qquad f_{c\,o,2} = \frac{3 \times 10^8}{4}\sqrt{\left(\frac{(1)}{2.5 \times 10^{-2}}\right)^2 + \left(\frac{0}{1 \times 10^{-2}}\right)^2}$

$= 3\ GHz$

$For\ TE20\ mode: m = 2, n = 0 \qquad f_{c\,2,0} = \frac{3 \times 10^8}{4}\sqrt{\left(\frac{2}{2.5 \times 10^{-2}}\right)^2 + \left(\frac{0}{1 \times 10^{-2}}\right)^2}$

$= 6\ GHz$

$For\ TE30\ mode: m = 3,$

$$n = 0 \quad f_{c\,3,0} = \frac{3 \times 10^8}{4}\sqrt{\left(\frac{3}{2.5 \times 10^{-2}}\right)^2 + \left(\frac{0}{1 \times 10^{-2}}\right)^2} = 9\ GHz$$

$For\ TE40\ mode: m = 4,$

$$n = 0 \quad f_{c\,4,0} = \frac{3 \times 10^8}{4}\sqrt{\left(\frac{4}{2.5 \times 10^{-2}}\right)^2 + \left(\frac{0}{1 \times 10^{-2}}\right)^2} = 12\ GHz$$

$For\ TE50\ mode: m = 5,$

$$n = 0 \quad f_{c\,5,0} = \frac{3 \times 10^8}{4}\sqrt{\left(\frac{5}{2.5 \times 10^{-2}}\right)^2 + \left(\frac{0}{1 \times 10^{-2}}\right)^2} = 15\ GHz$$

$For\ TE60\ mode: m = 6,$

$$n = 0 \quad f_{c\,6,0} = \frac{3 \times 10^8}{4}\sqrt{\left(\frac{6}{2.5 \times 10^{-2}}\right)^2 + \left(\frac{0}{1 \times 10^{-2}}\right)^2} = 18\ GHz$$

$The\ group\ velocity\ is\ given\ by,$

$$V_g = V\sqrt{1 - \left(\frac{f_c}{f}\right)^2} = 3 \times 10^8\sqrt{1 - \left(\frac{3.909 \times 10^9}{4 \times 10^9}\right)}$$

$$= 0.6362 \times 10^8\ m/sec$$

Example 5.16: When the dominant mode is propagated through a waveguide at a frequency of 9 GHz, the wavelength is found to be 4 cm. Find dimension of the breadth a of the guide.

Solution: The dominant mode is TE_{10} mode. $\therefore\ m = 1,\ n = 0$

$Given: guide\ wavelength = \lambda_g = 4\ cm,\ f = 9\ GHz \qquad \lambda = \frac{C}{f} = \frac{3\times10^8}{9\times10^7} =$ 0.033

$The\ guide\ wavelength\ is\ given\ by$

$$\lambda_g = \frac{\lambda_0}{\sqrt{1-\left[\frac{\lambda_0}{\lambda_c}\right]^2}} \qquad \therefore \quad \sqrt{1-\left[\frac{\lambda_0}{\lambda_c}\right]^2} = \left[\frac{\lambda_0}{\lambda_g}\right]^2$$

$$= \sqrt{1-\left[\frac{0.033}{\lambda_c}\right]^2}$$

$$1.0 \times 10^{-3} = \frac{1.10889 \times 10^{-3}}{\lambda_c^2} = 0.69305$$

Dividing all the terms by λ_c^2 *we get,*

$$\frac{1}{\lambda_0^2}\frac{1}{\lambda_c^2} = \frac{1}{\lambda_g^2} \qquad \lambda_c^2 - 0.69305\,\lambda_c^2 = 1.10889 \times 10^{-3}$$

$$\therefore \quad \frac{1}{\lambda_c^2}\frac{1}{\lambda_0^2} = \frac{1}{\lambda_g^2} \qquad \lambda_c^2 = 6.612 \times 10^{-3}$$

$$But \quad \lambda_0 = \frac{c}{f} \times \frac{3\times 10^8}{9 \times 10^9} = 0.0333\,m \qquad \lambda_c = 0.06\,m$$

$$\lambda_g = 4\,cm = 4 \times 10^{-2} \qquad \lambda_c = 6\,cm$$

$$Substituting\ values, we\ can\ write \quad \frac{1}{\lambda_c^2} = \frac{1}{(0.03333)^2} - \frac{1}{(4 \times 10^{-2})^2}$$

$$\lambda_c = 0.06 \ \ m = 6\,cm$$

$$But\ for\ TE10\ mode \quad \lambda_c = 2\,a$$

$$6 = 2\,a$$

$$a = 3\,cm$$

Example 5.17: A TE_{10} mode is propagated through a waveguide with a = 10 cm at frequency 2.5 GHz. Find $\lambda_c, V_p, V_g, \lambda_g, Z_{z(TE)}$ and β.

Solution: Given a = 10 cm = 10×10^{-2} m = 0.1 m

f = 2.5 GHz = 2.5×10^9 Hz

TE_{10} mode i.e. m =1, n = 0

The free space wavelength is given by, $\lambda_0 = \frac{c}{f} \times \frac{3\times10^8}{2.5\times10^9} = 0.12\,m$

i) Cut − off wavelength is given by, $\lambda_c = \lambda_{c10} = 2(a) = 2 \times 0.1$

$= 0.2\,m$

$$ii)\ The\ phase\ velocity\ is\ given\ by,\ V_P = \frac{C}{\sqrt{1-\left[\frac{\lambda_0}{\lambda_c}\right]^2}} = \frac{3\times 10^8}{\sqrt{1-\left(\frac{0.12}{0.2}\right)^2}}$$

$$= 3.75 \times 10^8\,m/s$$

$iii)\ The\ group\ velocity\ is\ given\ by,\ V_g = C\sqrt{1-\left[\frac{\lambda_0}{\lambda_c}\right]^2} = 3\times10^8\sqrt{1-\left(\frac{0.12}{0.2}\right)^2}$

$= 2.4\ \times 10^8\ m/s$

$iv)\ The\ group\ wavelength\ \lambda_g\ is\ given\ by, \lambda_g = \frac{\lambda_0}{\sqrt{1-\left[\frac{\lambda_0}{\lambda_c}\right]^2}} = \frac{0.12}{\sqrt{1-\left(\frac{0.12}{0.2}\right)^2}} = 0.15\ m$

$= 15cm$

$The\ wave\ impedance\ Zz(TE)\ is\ given\ by,\ Z_{z(TE)} = \frac{\eta_0}{\sqrt{1-\left[\frac{\lambda_0}{\lambda_c}\right]^2}} = \frac{377}{\sqrt{1-\left(\frac{0.12}{0.2}\right)^2}}$

$= 471.25\Omega$

$The\ phase\ constant\ \beta\ is\ given\ by,\quad \beta = \frac{2\pi}{\lambda_g} = \frac{2\pi}{15\times10^{-2}} = 41.88\ rad/m$

Example 5.18: The cut-off wavelengths of a rectangular waveguide are measured to be 8 cm and 4.8 cm for TE_{10} and TE_{11} modes respectively. Determine waveguide dimensions.

Solution: Given: For TE_{10} mode : $\lambda_{c10} = 8\ cm$

For TE_{11} mode : $\lambda_{c11} = 4.8\ cm$

For TE_{10} mode which is the dominant mode, cut off wave length is given by,

$$\lambda_{c10} = 2\ a$$

$$\lambda = 2\ a$$

$$a = 4\ cm$$

For TEM_{11} mode, i.e. for dominant mode, the cut-off wavelength is given by,

$$\lambda_{c11} = \frac{2ab}{\sqrt{a^2+b^2}} = 4.8$$

But a = 4 cm

$$\therefore \quad \frac{2(4)(b)}{\sqrt{(4)^2+b^2}} = 4.8$$

$$\therefore \quad \frac{b}{\sqrt{b^2+16}} = 0.6$$

Squaring and cross multiplying $\quad 0.36\ (b^2 + 16) = b^2$

$\therefore$ b = 3 cm

Example 5.19: Waveguide, the guide wavelength measured is 8 cm and when TE11 mode propagation the guide wavelength increases to 12 cm. If operating frequency for both 6 GHz. Calculate a and b for the guide.

Solution: For TE_{10} mode : $\lambda_{g10} = 8\ cm = 8\times10^{-2}\ m$

For TE_{11} mode : $\lambda_{g11} = 12\ cm = 12 \times 10^{-2}\ m$

$f = 6\ GHz = 6 \times 10^9\ Hz$

$$For\ standard\ waveguide, \quad a = 2b$$

$$For\ dominant\ TE10\ mode, \lambda_{c10} = 2\,a$$

$$\lambda_{c11} = \frac{2ab}{\sqrt{m^2b^2 + n^2a^2}} = \frac{2a\left[\frac{a}{2}\right]}{\sqrt{(1)^2\left(\frac{a}{2}\right)^2 + (1)^2(a)^2}} \qquad m = 1, \quad n = 1$$

$$= \frac{a^2}{\sqrt{\frac{5a^2}{4}}} \qquad \therefore \quad \lambda_{c11} = 0.8944\,a$$

$$For\ TE10\ mode,\ \lambda_{g10} = \frac{\lambda_0}{\sqrt{1 - \left[\frac{\lambda_0}{\lambda_{c10}}\right]^2}} = 8$$

$$For\ TE11\ mode, \qquad \lambda_{g11} = \frac{\lambda_0}{\sqrt{1 - \left[\frac{\lambda_0}{\lambda_{c11}}\right]^2}} = 12$$

$$Dividing\ equation\ (1)\ by\ (2) \qquad \frac{\sqrt{1 - \left[\frac{\lambda_0}{\lambda_{c11}}\right]^2}}{\sqrt{1 - \left[\frac{\lambda_0}{\lambda_{c10}}\right]^2}} = \frac{8}{12}$$

$$\lambda_0 = \frac{c}{f} \times \frac{3 \times 10^8}{6 \times 10^9} = 0.05 = 5\ cm$$

$$Hence\ substituting\ all\ values, \frac{\sqrt{1 - \left[\frac{5}{0.8944a}\right]^2}}{\sqrt{1 - \left[\frac{5}{2a}\right]^2}} = \frac{8}{12}$$

$$Solving\ we\ get, a = 7.826\ cm$$

$$Hence \qquad b = \frac{a}{2} = \frac{7.826}{2} = 3.913\ cm$$

SUMMARY

- **Transverse Magnetic (TM) Waves**

In this wave the magnetic vector is entirely normal to the direction of propagation and hence it has no components in the direction of propagation. The electric vector has both the normal and parallel components.

- **Transverse Electric (TE) Waves**

In this wave the electric vector is entirely normal to the direction of propagation and hence it has no components in the direction of propagation. The magnetic vector has both the normal and parallel components.

- *Rectangular Waveguide*

$$Ex = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial y} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial x}$$

$$Ey = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial y} + \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

$$Hx = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial x} + \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial y}$$

$$Hy = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial y} - \frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x}$$

- *Parallel plate waveguide*

$$\frac{\partial}{\partial y} \to 0$$

$$Ex = -\frac{\gamma}{h^2}\frac{\partial E_z}{\partial x}$$

$$Hy = -\frac{j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x}$$

$$Hx = \frac{-\gamma}{h^2}\frac{\partial H_z}{\partial x}$$

$$Ey = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial x}$$

- *Circular wave guide*

$$E\Phi = \frac{-j\omega\mu}{h^2}\frac{\partial H_z}{\partial r} - \frac{\gamma}{h^2 r}\frac{\partial E_z}{\partial \Phi}$$

$$Hr = \frac{j\omega\varepsilon}{h^2 r}\frac{\partial E_z}{\partial \varphi} - \frac{\gamma}{h^2}\frac{\partial H_z}{\partial r}$$

$$H\Phi = \frac{-j\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial r} - \frac{\gamma}{rh^2}\frac{\partial H_z}{\partial \Phi}$$

$$Er = \frac{-j\omega\mu}{h^2 r}\frac{\partial H_z}{\partial \varphi} - \frac{\gamma}{h^2}\frac{\partial E_z}{\partial r}$$

- *Characteristics*

$$h = \sqrt{\bar{\gamma}^2 + \omega^2\mu\varepsilon}$$

propagation constant γ

$$\bar{\gamma} = \sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}$$

- Cut off frequency fc

$$f_c = \frac{m}{2a\sqrt{\mu\varepsilon}} Hz$$

- The phase constant β,

$$\beta = 2\pi\sqrt{\mu\varepsilon}\sqrt{(f^2 - f_c^{\,2})}$$

- wavelength

$$\lambda = \frac{2\pi}{\beta} = \lambda_g$$

- The cut off wavelength $\lambda_c = \frac{v}{f_c}$ $where\ v = \frac{1}{\sqrt{\mu\varepsilon}} = velocity\ of\ propagation$
- Thus the distance of separation is given by, $a = m\frac{\lambda_c}{2}$
- Wave Impedance, $Z_0(TM) = \frac{E_x}{H_y} = \frac{\beta}{\omega\varepsilon} = \sqrt{\frac{\mu}{\varepsilon}}\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$

$$Z_0(TM) = \eta\sqrt{1 - \left(\frac{f_c}{f}\right)^2}$$

- By the property displacement current needs to the components of the electric field in the axial direction. But such axial component of $\bar{E}$ is not present in TEM waves, hence it cannot exist in rectangular waveguide.
- The expression of Bessel function for larger is given by,

$$J_0(r) = \sqrt{\frac{2}{\pi r}}\cos\left(r - \frac{\pi}{4}\right)$$

$$N_0(r) = \sqrt{\frac{2}{\pi r}}\sin\left(r - \frac{\pi}{4}\right)$$

- TM waves The first few roots are,

$(ha)_{01} = 2.405$
$(ha)_{11} = 3.83$
$(ha)_{02} = 5.52$
$(ha)_{12} = 7.02$

- The various TM wave will be referred as TMnm waves

The propagation constant $\gamma = \sqrt{h^2 - \omega^2\mu\varepsilon}$

- The propagation constant $\gamma = j\beta$ if $\alpha = 0$

$\therefore \beta = \sqrt{\omega^2\mu\varepsilon - h_{nm}^2}$

- The cut off frequency,

$$f_c = \frac{h_{nm}}{2\pi\sqrt{\mu\varepsilon}}$$

Where $$hnm = \frac{(ha)_{nm}}{a}$$

The phase velocity is,

$$v = \frac{\omega}{\beta}$$

$$v = \frac{\omega}{\sqrt{\omega^2 \mu \varepsilon - h_{nm}^2}}$$

- For TE wave The first few roots are,

$(ha^1)_{01} = 3.832$
$(ha^1)_{11} = 1.841$
$(ha^1)_{02} = 7.02$
$(ha^1)_{12} = 5.331$

- Properties of TEM waves

The TEM wave is a special case of guided wave propagation. Some of the important properties are,

The fields are entirely transverse

In the direction perpendicular to the direction of propagation, the amplitude of the field components are constant.

The velocity of propagation of TEM wave is impendent of frequency.

The cut-off frequency of the wave is Zero which indicate all the frequencies below fc can propagate along the guide.

- Rectangular wave guides
- *Wavelength* $\lambda = \frac{\lambda_c}{\sqrt{1-(\frac{m\lambda_c}{2a})^2}}$
- *The phase velocity*

$$\vartheta\rho = \frac{C}{\sqrt{1-(\frac{m\lambda}{2a})^2}}$$

- *The group velocity*

$$\vartheta_g = C\sqrt{1-(\frac{m\lambda}{2a})^2}$$

- *Phase shift β,*

$$\beta = \sqrt{\omega^2 \mu \varepsilon - (\frac{m\bar{\pi}}{a})^2}$$

- Relation between phase velocity and group velocity

$$\vartheta_g \,.\, \vartheta_p = \vartheta^2$$

- If $f < f_c$ - The propagation constant is real ie, $\propto \neq 0$, $\beta = 0$
- If $f > f_c$ - The propagation constant is imaginary ie, $\propto = 0, \beta = 0$

TWO MARKS

1. What are the three properties of EM Waves?

(i) EM Waves travel at very high velocity

ii) While traveling, they assume the properties of waves