Unit III

IIR FILTER DESIGN

3.1 ANALOG FILTER DESIGN

The design of IIR filter involves design of a digital filter in the analog domain and transforming the design into the digital domain.

The system function describing an analog filter may be written as

$$H_a(S) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

where $\{a_k\}$ and $\{b_k\}$ are the filter coefficients. The impulse response of these filter coefficients is related to $H_a(s)$ by the Laplace Transform

$$H_a(s) = \int_{-\infty}^{\infty} h(t) e^{-st} \mathrm{d}t$$

The analog filter having the rational function $H(s)$ can also be described by the linear constant-coefficient differential equation

$$\sum_{k=0}^{N} a_k \frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k} = \sum_{k=0}^{M} b_k \frac{\mathrm{d}^k x(t)}{\mathrm{d}t}$$

where $x(t)$ is the input signal and $y(t)$ is the output of the filter

The above three equivalent characterisation of an analog filter leads to three alternative methods for transforming the filter to the digital domain. If the conversion techniques are to be effective, the technique should possess the following properties:

i) The $j\Omega$ axis in the s-plane should map onto the unit circle in the z-plane.

ii) The left half plane of the s-plane should map inside of the unit circle in the z-plane to convert a stable analog filter into a stable digital filter.

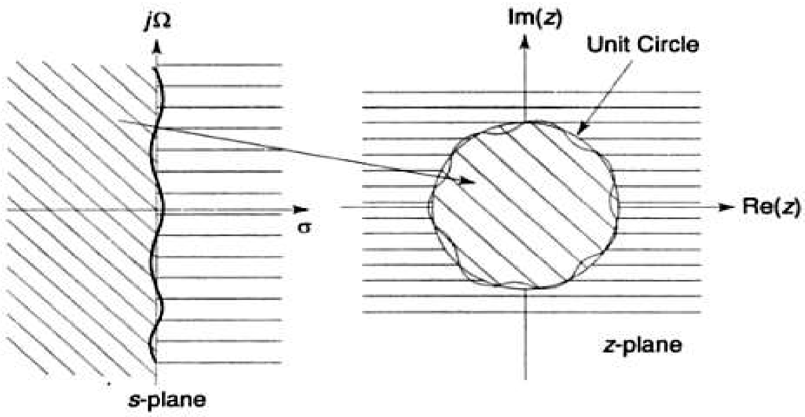


Fig 3.1: Mapping from s-plane to z-plane

3.1.1 IIR FILTER DESIGN BY APPROXIMATION OF DERIVATIVES

In this method the analog filter is converted into a digital filter by approximating the differential equation by an equivalent difference equation. The backward difference formula is substituted for the derivative $dy(t)/dt$ at time $t = nT$. Thus,

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT-T)}{T}$$

$$= \frac{y(n) - y(n-1)}{T}$$

where T is the sampling interval and $y(n) \equiv y(nT)$. The system function of an analog differentiator with the output dy/dt is $H(s) = s$, and the digital system that produces the output $[y(n) - y(n-1)]/T$ has the system function $H(z) = (1 - z^{-1})/T$. These two can be compared to get the frequency-domain equivalent for the relationship as

$$s = \frac{1 - z^{-1}}{T}$$

The second derivative $d^2y(t)/dt^2$is replaced by the second backward difference

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{d}{dT}\left[\frac{dy(t)}{dt}\right]_{t=nT}$$

$$= \frac{[y(nT) - y(nT-T)]/T - [y(nT-T) - y(nT-2T)]/T}{T}$$

$$= \frac{y(n) - 2y(n-1) + y(n-2)}{T^2}$$

The equivalent in the frequency domain is

$$s^2 = \frac{1 - 2z^{-1} + z^{-2}}{T^2} = \left(\frac{1 - z^{-1}}{T}\right)^2$$

The i^{th} derivative of $y(t)$ resulte=s in the equivalent frequency domain relationship

$$s^i = \left(\frac{1 - z^{-1}}{T}\right)^i$$

As a result, the digital filter's system function can be obtained by the method of approximation of the derivatives as,

$$H(z) = H_a(s)\,|_{s=(1-z^{-1})/T}$$

where $H_a(s)$ is the system function of the analog filter characterised by the differential equqtion.

Above equation can be equivalently written as,

$$z = \frac{1}{1 - sT}$$

Substituting $s = j\Omega$ in the above equation

$$z = \frac{1}{1 - j\Omega T}$$
$$= \frac{1}{1+\Omega^2 T^2} + j\frac{\Omega T}{1+\Omega^2 T^2}$$

Varying Ω from -∞ to ∞, the corresponding locus of points in the z-plane is a circle with radius ½ and with the center at z=1/2. It can be seen that the left half of plane of s-domain is mapped into the corresponding points inside the circle of radius 0.5 and center at z=0.5 and the right half of the s-plane is mapped outside the unit circle.. As a result, this mapping results in a stable analog filter transformed to a stable digital filter.

As the location of poles in the z-domain are confined to smaller frequencies, this design method can be used only for transforming analog low-pass and bandpass filters having smaller resonant frequencies. Neither high pass nor band reject filter can be realised using this technique.

The forward difference can be substituted for the derivative instead of the backward difference. This gives,

$$\frac{dy(t)}{dt} = \frac{y(nT+T) - y(nT)}{T}$$
$$= \frac{y(n+1) - y(n)}{T}$$

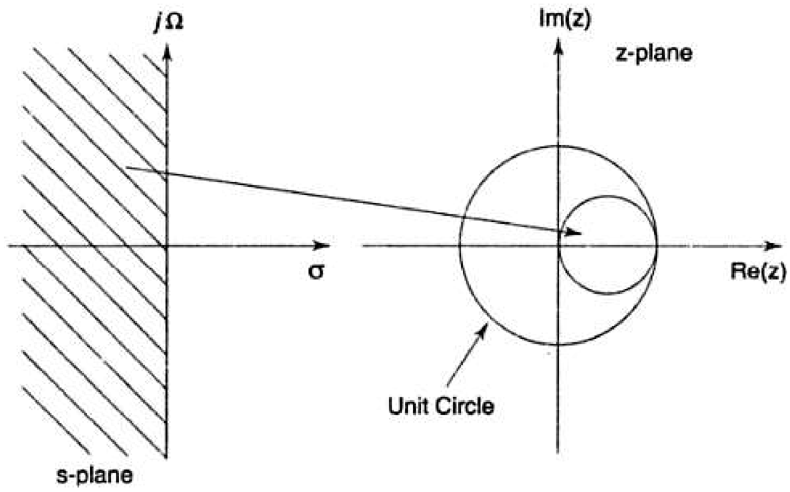


Fig 2.2: z-plane mapping

The transformation formula will be,

$$s = \frac{z-1}{T}$$

or

$$z = 1 + sT$$

The mapping of the above equation is shown in the figure below. This results in a worse situation than the backward difference substitution for the derivative. When $s = j\Omega$, the mapping of these points in the s-domain results in a straight line in the z-domain with coordinates $(Z_{real}, z_{imag}) = (1, \Omega T)$. Consequently, stable analog filters do not always map into stable digital filters.

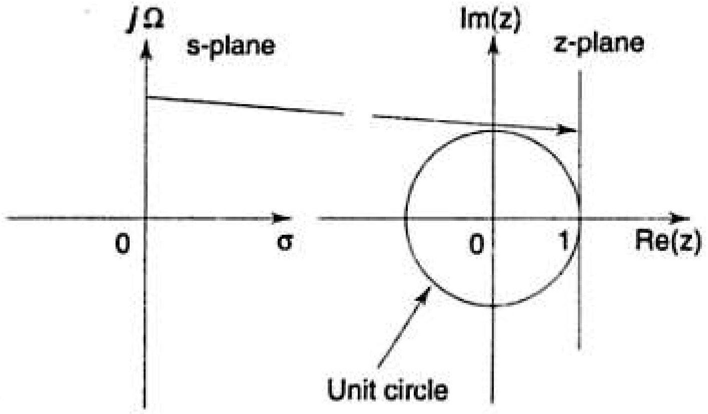


Fig 2.3: z-plane mapping

Example Use the backward difference for the derivative to convert the analog low-pass filter with system function

$$H(s) = \frac{1}{s+2}$$

Solution The mapping formula for the backward difference for the derivative is

$$s = \frac{1-z^{-1}}{T}$$

The system response of the digital filter is

$$H(z) = H(s)|_{s=\frac{1-z^{-1}}{T}} = \frac{1}{\left(\frac{1-z^{-1}}{T}\right)+2}$$

$$= \frac{T}{1-z^{-1}+2T}$$

If $T = 1$s,

$$H(z) = \frac{1}{3-z^{-1}}$$

Example Use the backward difference for the derivative and convert the analog filter with system function

$$H(s) = \frac{1}{s^2 + 16}$$

Solution

$$s = \frac{1 - z^{-1}}{T}$$

The system response of the digital filter is

$$H(z) = H(s)|_{s=\frac{1-z^{-1}}{T}} = \frac{1}{\left(\frac{1-z^{-1}}{T}\right)^2 + 16}$$

$$H(z) = \frac{T^2}{1 - 2z^{-1} + z^{-2} + 16T^2}$$

If $T = 1$s,

$$H(z) = \frac{1}{z^{-2} - 2z^{-1} + 17}$$

Example An analog filter has the following system function. Convert this filter into a digital filter using backward difference for the derivative.

$$H(s) = \frac{1}{(s + 0.1)^2 + 9}$$

Solution The system response of the digital filter is

$$H(z) = H(s)|_{s=\frac{1-z^{-1}}{T}} = \frac{1}{\left(\frac{1-z^{-1}}{T} + 0.1\right)^2 + 9}$$

$$H(z) = \frac{T^2}{z^{-2} - 2(1 + 0.1T)z^{-1} + (1 + 0.2T + 9.01T^2)}$$

$$H(z) = \frac{\dfrac{T^2}{(1 + 0.2T + 9.01T^2)}}{1 - 2\dfrac{(1 + 0.1T)}{(1 + 0.2T + 9.01T^2)}z^{-1} + \dfrac{z^{-2}}{(1 + 0.2T + 9.01T^2)}}$$

If $T = 1$s,

$$H(z) = \frac{0.0979}{1 - 0.2155z^{-1} + 0.09792z^{-2}}$$

3.1.2 IIR FILTER DESIGN BY IMPULSE INVARIANCE

In this technique, the desired impulse response of the digital filter is obtained by uniformly sampling the impulse response of the equivalent analog filter. That is,

$$h(n) = h_a(nT)$$

where T is the sampling interval. The transformation technique can be well understood by first considering the simple distinct pole case for the analog filter's system function, as shown below.

$$H_a(s) = \sum_{i=1}^{M} \frac{A_i}{s - p_i}$$

The impulse response of the system can be obtained by taking the inverse Laplace Transform and it will be of the form

$$h_a(t) = \sum_{i=1}^{M} A_i e^{p_i t} u_a(t)$$

where $u_a(t)$ is the unit step function in continuous time. The impulse response $h(n)$ of the equivalent digital filter is obtained by uniformly sampling $h_a(t)$

$$h(n) = h_a(nT) = \sum_{i=1}^{M} A_i e^{p_i n t} u_a(nT)$$

The system response of the digital system can be obtained by taking the z-transform,

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty} \left[\sum_{i=1}^{M} A_i e^{p_i n T} u_a(nT) \right] z^{-n}$$

Interchanging the order of summation,

$$H(z) = \sum_{i=1}^{M} \left[\sum_{n=0}^{\infty} A_i e^{p_i n T} u_a(nT) \right] z^{-n}$$

$$H(z) = \sum_{i=1}^{M} \frac{A_i}{1 - e^{P_i T} z^{-1}}$$

Now, by comparing, the mapping formula for the impulse invariant transformation is given by

$$\frac{1}{s - P_i} \to \frac{1}{1 - e^{P_i T} z^{-1}}$$

It shows that the analog pole at $s = P_i$ is mapped into a digital pole at $z = e^{P_i T}$. Therefore, the analog poles and the digital poles are related by the relation

$$z = e^{sT}$$

The general characteristic of the mapping $z = e^{sT}$ can be obtained by substituting $s = \sigma + j\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$. With these substitutions,

$$re^{j\omega} = e^{\sigma T} e^{j\Omega T}$$

Clearly,

$$r = e^{\sigma T}$$
$$\omega = \Omega T$$

This shows that, the left half of s-plane is mapped inside the unit circle in the z-plane and the right half of s-plane is mapped into points that fall outside the unit circle in z. This is one of the desirable properties for stability. The $j\Omega$ axis is mapped into the unit circle in z-plane. However, the mapping of the $j\Omega$ axis is not one-to-one. The mapping $\omega = \Omega T$ implies that the interval $-\pi/T \le \Omega \le \pi/T$ maps into the corresponding values of $-\pi \le \omega \le \pi$. Further, the frequency interval $\pi/T \le \Omega \le 3\pi/T$ also maps into the interval $-\pi \le \omega \le \pi$ and in general, any frequency interval $(2k-1)\pi/T \le \Omega \le (2k+1)\pi/T$, where k is an integer, will also map into the interval $-\pi \le \omega \le \pi$ in the z-plane. Thus the mapping from the analog frequency Ω to the frequency variable ω in the digital domain is many-to-one, which simply reflects the effects of aliasing due to sampling of the impulse response.

Some of the properties of the impulse invariant transformation are given below.

$$\frac{1}{(s+s_i)^m} \to \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left[\frac{1}{1-e^{-sT}z^{-1}}\right]; s \to s_i$$

$$\frac{s+a}{(s+a)^2+b^2} \to \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$\frac{b}{(s+a)^2+b^2} \to \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

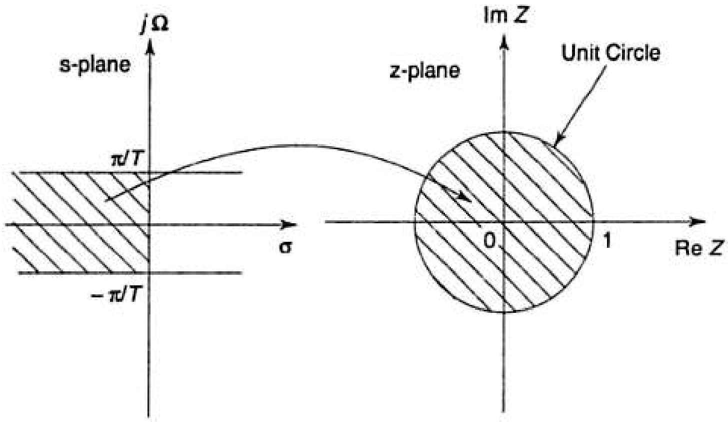


Fig 3.4: Mapping of $z = e^{sT}$

Example Convert the analog filter into a digital filter whose system function is

$$H(s) = \frac{s + 0.2}{(s + 0.2)^2 + 9}$$

Use the impulse invariant technique. Assume $T = 1$s.

Solution The system response of the analog filter is of the standard form

$$H(s) = \frac{s + a}{(s + a)^2 + b^2}$$

where $a = 0.2$ and $b = 3$. The system response of the digital filter can be obtained using

$$H(z) = \frac{1 - e^{-aT}(\cos bT)\, z^{-1}}{1 - 2e^{-aT}(\cos bT)\, z^{-1} + e^{-2aT}\, z^{-2}}$$

$$= \frac{1 - e^{-0.2T}(\cos 3T)\, z^{-1}}{1 - 2e^{-0.2T}(\cos 3T)\, z^{-1} + e^{-0.4T}\, z^{-2}}$$

Taking $T = 1$s,

$$H(z) = \frac{1 - (0.8187)(-0.99)\, z^{-1}}{1 - 2(0.8187)(-0.99)\, z^{-1} + 0.6703\, z^{-2}}$$

That is,

$$H(z) = \frac{1 + (0.8105)\, z^{-1}}{1 + 1.6210\, z^{-1} + 0.6703\, z^{-2}}$$

Example *For the analog transfer function*

$$H(s) = \frac{1}{(s+1)(s+2)}$$

determine H(z) using impulse invariant technique. Assume $T = 1$ s.

Solution Using partial fractions, $H(s)$ can be written as

$$H(s)=\frac{1}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}=\frac{A(s+2)+B(s+1)}{(s+1)(s+2)}$$
$$1=A(s+2)+B(s+1)$$

Letting $s=-2$, we get $B=-1$ and letting, $s=-1$, we get $A=1$. Therefore,

$$H(s)=\frac{1}{s+1}-\frac{1}{s+2}$$

The system function of the digital filter is obtained by using Eq.

$$H(z)=\frac{1}{1-e^{-T}z^{-1}}-\frac{1}{1-e^{-2T}z^{-1}}$$
$$=\frac{z^{-1}[e^{-T}-e^{-2T}]}{1-(e^{-T}+e^{-2T})z^{-1}+e^{-3T}z^{-2}}$$

Since $T=1$ s,

$$H(z)=\frac{0.2326z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}}$$

Example *Determine $H(z)$ using the impulse invariant technique for the analog system function*

$$H(s)=\frac{1}{(s+0.5)(s^2+0.5s+2)}$$

Solution Using partial fractions, $H(s)$ can be written as

$$H(s)=\frac{1}{(s+0.5)(s^2+0.5s+2)}=\frac{A}{s+0.5}+\frac{Bs+C}{s^2+0.5s+2}$$

Therefore,

$$A(s^2+0.5s+2)+(Bs+C)(s+0.5)=1$$

Comparing the coefficients of s^2, s and the constants on either side of the above expression, we get

$$A+B=0$$
$$0.5A+0.5B+C=0$$
$$2A+0.5C=1$$

Solving the above simultaneous equations, we get $A = 0.5$, $B = -0.5$ and $C = 0$. The system response can be written as,

$$H(s) = \frac{0.5}{s+0.5} - \frac{0.5s}{s^2+0.5s+2}$$

$$= \frac{0.5}{s+0.5} - 0.5\left(\frac{s}{(s+0.25)^2+(1.3919)^2}\right)$$

$$= \frac{0.5}{s+0.5} - 0.5\left(\frac{s+0.25}{(s+0.25)^2+(1.3919)^2} - \frac{0.25}{(s+0.25)^2+(1.3919)^2}\right)$$

$$= \frac{0.5}{s+0.5} - 0.5\left(\frac{s+0.25}{(s+0.25)^2+(1.3919)^2}\right) + 0.0898\left(\frac{1.3919}{(s+0.25)^2+(1.3919)^2}\right)$$

Using Eqs.

$$H(z) = \frac{0.5}{1-e^{-0.5T}z^{-1}} - 0.5\left[\frac{1-e^{-0.25T}(\cos 1.3919T)z^{-1}}{1-2e^{-0.25T}(\cos 1.3919T)z^{-1}+e^{-0.5T}z^{-2}}\right] + 0.0898\left[\frac{e^{-0.25T}(\sin 1.3919T)z^{-1}}{1-2e^{-0.25T}(\cos 1.3919T)z^{-1}+e^{-0.5T}z^{-2}}\right]$$

Letting $T = 1$ s,

$$H(z) = \frac{0.5}{1-0.6065\,z^{-1}} - 0.5\left(\frac{1-0.1385\,z^{-1}}{1+0.277\,z^{-1}+0.606\,z^{-2}}\right) + 0.0898\left[\frac{0.7663\,z^{-1}}{1-0.277\,z^{-1}+0.606\,z^{-2}}\right]$$

3.1.3 IIR FILTER DESIGN BY THE BILINEAR TRANSFORMATION

The above two techniques are not suitable for high-pass and band-reject filters. This limitation is overcome in the mapping technique called bilinear transformation. This is a one-to-one mapping from the s-domain to z-domain. That is, the bilinear transformation is a conformal mapping that transforms the $j\Omega$ axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components. Also, the transformation of a stable analog filter results in a stable digital filter as all the poles in the left half of the s-plane are mapped onto the points inside the unit circle of the z-domain. The bilinear transformation is obtained by using the trapezoidal formula for numerical integration. Let the system function of the analog filter be

$$H(s) = \frac{b}{s+a}$$

The differential equation describing the analog filter can be obtained as,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$sY(s) + aY(s) = bX(s)$$

Taking inverse Laplace transform,

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

By integrating between the limits $(nT - T)$ and nT

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt} dt + a \int_{nT-T}^{nT} y(t)\, dt = b \int_{nT-T}^{nT} x(t)\, dt$$

The trapezoidal rule for numeric integration is given by

$$\int_{nT-T}^{nT} a(t)dt = \frac{T}{2}[a(nT) + a(nT-T)]$$

Applying the above equation we get,

$$y(nT) - y(nT-T) + \frac{aT}{2}y(nT) + \frac{aT}{2}y(nT-T) = \frac{bT}{2}x(nT) + \frac{bT}{2}x(nT-T)$$

Taking z-transform, the system function of the digital filter is,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

By comparing we get,

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

The general characteristic of the mapping $z = e^{sT}$ can be obtained by substituting $s = \sigma + j\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{re^{j\omega} - 1}{re^{j\omega} + 1}\right)$$

Substituting $e^{j\omega} = \cos\omega - j\, jsin\,\omega$ and simplifying, we get

$$s = \frac{2}{T}\left(\frac{r^2 - 1}{1 - r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right)$$

Therefore,

$$\sigma = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos\omega}\right)$$

$$\Omega = \frac{2}{T}\left(\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right)$$

This shows that, the left half of s-plane maps onto the points inside the unit circle in the z-plane and the transformation results in a stable digital system. Consider r=1 and σ is zero. In this case,

$$\Omega = \frac{2}{T}\left(\frac{\sin\omega}{1+\cos\omega}\right)$$
$$= \frac{2}{T}\left(\frac{2\sin(\omega/2)\cos(\omega/2)}{\cos^2(\omega/2)+\sin^2(\omega/2)+\cos^2(\omega/2)-\sin^2(\omega/2)}\right)$$
$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

or equivalently,

$$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$

This can be noted that the entire region in Ω is mapped only once into the range $-\pi \le \omega \le \pi$.

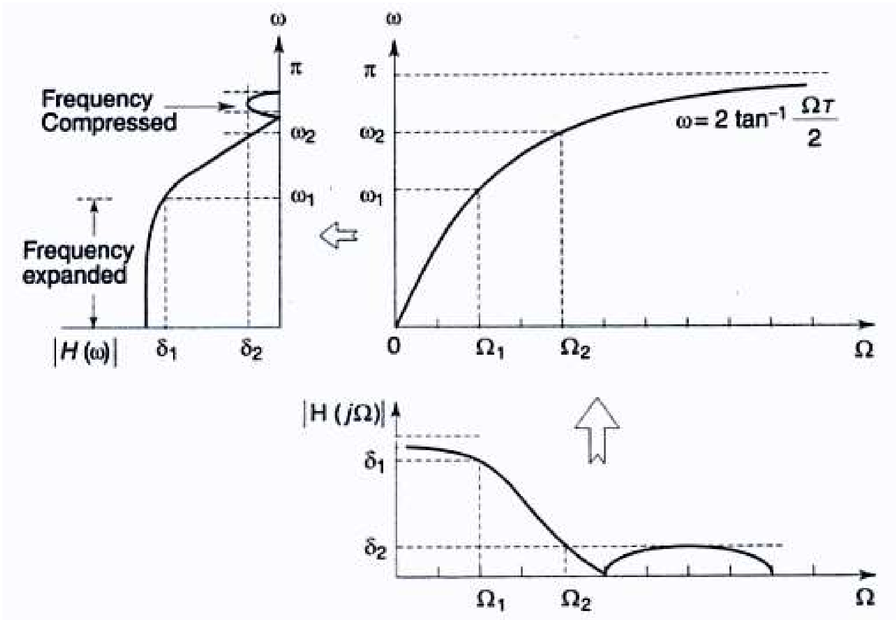


Fig 3.6: Relationship between ω and Ω

As seen in figure, this mapping is non-linear and the lower frequencies in analog domain are expanded in the digital domain, whereas the higher frequencies are compressed. This is due to the non-linearity of the arc tangent function and is usually called as frequency warping.

Example *Convert the analog filter with system function*

$$H(s) = \frac{s+0.1}{(s+0.1)^2+9}$$

into a digital IIR filter using bilinear transformation. The digital filter should have a resonant frequency of $\omega_r = \frac{\pi}{4}$.

Solution From the system function, we note that $\Omega_c = 3$. The sampling period T can be determined using

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_r}{2}$$

The sampling peiiod is obtained from the above equation using

$$T = \frac{2}{\Omega_c}\tan\frac{\omega_r}{2} = \frac{2}{3}\tan\frac{\pi}{8} = 0.276\,\text{s}$$

Using bilinear transformation,

$$H(z) = H(s)\big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$H(z) = \frac{\frac{2}{T}\frac{(z-1)}{(z+1)}+0.1}{\left[\frac{2}{T}\frac{(z-1)}{(z+1)}+0.1\right]^2+9}$$

$$= \frac{(2/T)(z-1)(z+1)+0.1(z+1)^2}{[(2/T)(z-1)+0.1(z+1)]^2+9(z+1)^2}$$

Substituting $T = 0.276$ s,

$$H(z) = \frac{1+0.027z^{-1}-0.973z^{-2}}{8.572-11.84z^{-1}+8.177z^{-2}}$$

Example *Apply bilinear transformation to*

$$H(s) = \frac{2}{(s+1)(s+3)}$$

with $T = 0.1$ s.

Solution For bilinear transformation,

$$H(z) = H(s)|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$= \frac{2}{\left(\frac{2}{T}\frac{(z-1)}{(z+1)}+1\right)\left(\frac{2}{T}\frac{(z-1)}{(z+1)}+3\right)}$$

Using $T = 0.1\,\text{s}$,

$$H(z) = \frac{2}{\left(20\frac{(z-1)}{(z+1)}+1\right)\left(20\frac{(z-1)}{(z+1)}+3\right)}$$

$$= \frac{2(z+1)^2}{(21z-19)(23z-17)}$$

Simplifying further,

$$H(z) = \frac{0.0041(1+z^{-1})^2}{1-1.644z^{-1}+0.668z^{-2}}$$

Example *A digital filter with a 3 dB bandwidth of* 0.25π *is to be designed from the analog filter whose system response is*

$$H(s) = \frac{\Omega_c}{s+\Omega_c}$$

Use bilinear transformation and obtain H(z).

Solution Using

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_r}{2} = \frac{2}{T}\tan 0.125\pi = 0.828/T$$

The system response of the digital filter is given by

$$H(z) = H(s)|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$= \frac{\Omega_c}{\frac{2}{T}\frac{(z-1)}{(z+1)}+\Omega_c} = \frac{\frac{0.828}{T}}{\frac{2}{T}\frac{(z-1)}{(z+1)}+\frac{0.828}{T}}$$

$$= \frac{0.828(z+1)}{2(z-1)+0.828(z+1)}$$

Simplifying we get further,

$$H(z) = \frac{1+z^{-1}}{3.414-1.414z^{-1}}$$

Example *Using bilinear transformation obtain H(z) if*

$$H(s) = \frac{1}{(s+1)^2}$$

and $T = 0.1$ s.

Solution For the bilinear transformation,

$$H(z) = H(s)\big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$= \frac{1}{\left(\frac{2}{T}\frac{(z-1)}{(z+1)}+1\right)^2}$$

Substituting $T = 0.1$ s,

$$H(z) = \frac{1}{\left(20\frac{(z-1)}{(z+1)}+1\right)^2} = \frac{(z+1)^2}{(21z-19)^2}$$

Further simplifying,

$$H(z) = \frac{0.0476(1+z^{-1})^2}{(1-0.9048z^{-1})^2}$$

3.2 DISCRETE TIME IIR FILTER FROM ANALOG FILTER

3.2.1 BUTTERWORTH FILTERS

The Butterworth low-pass filter has a magnitude response given by,

$$|H(j\Omega)| = \frac{A}{\left[1+(\Omega/\Omega_c)^{2N}\right]^{0.5}}$$

where A is the filter gain and Ω_c is the 3 dB cut-off frequency and N is the order of the filter. The magnitude response is shown in the figure below.. the magnitude response has a maximally flat passband and stopband. By increasing the filter order N, the Butterworth response approximates the ideal response. However, the phase response of the Butterworth filter becomes more non-linear with increasing.

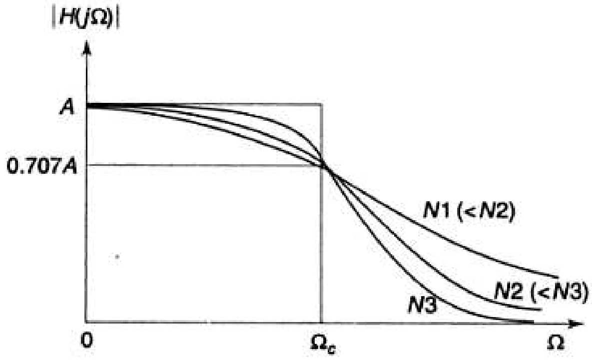


Fig 3.7: Magnitude response of a Butterworth Low-pass filter

The design parameters of the butterworth filter are obtained by considering the low-pass filter with the desired specifications as given below:

$$\delta_1 \le |H(j\Omega)| \le 1, 0 \le \Omega \le \Omega_1$$
$$|H(j\Omega)| \le \delta_2, \Omega_2 \le \Omega \le \pi$$

The order of the filter N is given by,

$$N = \frac{1}{2}\frac{log\{((1/\delta_2^2 - 1)/((1/\delta_1^2) - 1)\}}{\log(\Omega_2/\Omega_1)}$$

Cut-off frequency,

$$\Omega_c = \frac{\Omega_1}{\left[(1/\delta_1^2) - 1\right]^{1/2N}}$$

The values of Ω_1 and Ω_2 are obtained using the bilinear transformation or impulse invariant transformation techniques,

$$\Omega = \frac{2}{T}\tan(\omega/2) \text{ for bilinear transformation}$$

or

$$\Omega = \frac{\omega}{T} \text{ for impulse inavriant tarnsformation}$$

The transfer function of the Butterworth filter is given as:

$$H(s)=\prod_{k=1}^{N/2}\frac{B_k\Omega_c^2}{s^2+b_k\Omega_c s+c_k\Omega_c^2}\quad N=2,4,6,\ldots$$

$$H(s)=\frac{B_0\Omega_c}{s+c_0\Omega_c}\prod_{k=1}^{(N-1)/2}\frac{B_k\Omega_c^2}{s^2+b_k\Omega_c s+c_k\Omega_c^2}\quad N=3,5,7,\ldots.$$

The coefficients b_k and c_k are given by

$$b_k = 2\sin[(2k-1)\pi/2N] \text{ and } c_k = 1$$

The parameter B_k can be obtained from

$$A=\prod_{k=1}^{N/2}B_k,\quad \text{for even } N$$

$$A=\prod_{k=1}^{(N-1)/2}B_k,\quad \text{for odd } N$$

The system function of the equivalent digital filter is obtained from H(s) using the specified transformation technique, viz. Impulse invariant or bilinear transformation.

Example Design a digital Butterworth filter that satisfies the following constrait using bilinear transformation. Assume T=1 s.

$$0.9 \le |H(e^{j\omega})| \le 1\,;\quad 0 \le \omega \le \frac{\pi}{2}$$

$$|H(e^{j\omega})| \le 0.2\ ;3\pi/4 \le \omega \le \pi$$

Solution

Given $\delta_1 = 0.9$; δ_2=0.2; ω_1=π/2; ω_2=3π/4

Step (i) Determination of analog filter's edge frequencies

$$\Omega_1 = \frac{2}{T}\tan\left(\frac{\omega_1}{2}\right) = 2\tan\frac{\pi}{4} = 2$$

$$\Omega_2 = \frac{2}{T}\tan(\omega_2/2) = 2\tan\frac{3\pi}{8} = 4.828$$

Therefore, $\Omega_2/\Omega_1 = 2.414$

Step (ii) Determination of the order of the filter.

From Eq.

$$N \ge \frac{1}{2}\frac{\log\{((1/\delta_2^2)-1)/((1/\delta_1^2)-1)\}}{\log(\Omega_2/\Omega_1)}$$

$$=\frac{1}{2}\frac{\log\{24/0.2346\}}{\log(2.414)} = 2.626$$

Let $N = 3$.

Step (iii) Determination of −3 dB cut-off frequency.

From Eq.

$$\Omega_c = \frac{\Omega_1}{[(1/\delta_1^2)-1]^{1/2N}} = \frac{2}{[(1/0.9^2)-1]^{1/6}} = 2.5467$$

Step (iv) Determination of $H_a(s)$:

From Eq.

$$H(s) = \frac{B_0\Omega_c}{s+c_0\Omega_c}\prod_{k=1}^{(N-1)/2}\frac{B_k\Omega_c^2}{s^2+b_k\Omega_c s+c_k\Omega_c^2}$$

$$= \left(\frac{B_0\Omega_c}{s+c_0\Omega_c}\right)\left(\frac{B_1\Omega_c^2}{s^2+b_1\Omega_c s+c_1\Omega_c^2}\right)$$

From Eq.

$$b_1 = 2\sin\frac{\pi}{6} = 1,\ c_0 = 1 \text{ and } c_1 = 1$$

$$B_0 B_1 = 1. \text{ Therefore, } B_0 = B_1 = 1.$$

Therefore,

$$H(s) = \left(\frac{2.5467}{s+2.5467}\right)\left(\frac{6.4857}{s^2+2.5467s+6.4857}\right)$$

Step (v) Determination of $H(z)$.

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

That is,

$$H(z) = \left(\frac{2.5467}{2\frac{(z-1)}{(z+1)}+2.5467}\right)\left(\frac{6.4857}{\left[2\frac{(z-1)}{(z+1)}\right]^2+2.5467s+6.4857}\right)$$

Simplifying, we get

$$H(z) = \frac{16.5171(z+1)^3}{70.83z^3+31.1205z^2+27.2351z+2.948}$$

or

$$H(z) = \frac{0.2332(1+z^{-1})^3}{1+0.4394z^{-1}+0.3845z^{-2}+0.0416z^{-3}}$$

3.2.2 CHEBYSHEV FILTERS

The Chebyshev filter has the magnitude response given by

$$|H(j\Omega)| = \frac{A}{\left[1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_c}\right)\right]^{0.5}}$$

where A is the filter gain, ε is a constant and Ω_c is the 3 dB cut-off frequency. The Chebyshev polynomial of the I kind of Nth order, $C_N(x)$ is given by

$$C_N(x) = \begin{cases} \cos(N\cos^{-1}x), & for\ |x| \le 1 \\ \cosh(N\cos H^{-1}x), & for\ |x| \ge 1 \end{cases}$$

The magnitude response has equiripple passband and maximally flat stopband. By increasing the order N, the Chebyshev response approximates the ideal response. The phase response of the Chebyshev filter is more non-linear than the Butterworth filter for a given filter length N.

The design parameters of the Chebyshev filter are obtained by considering the low-pass filter with the desired specifications as below.

$$\delta_1 \le \left|H(e^{j\omega})\right| \le 1\,;\ \ 0 \le \omega \le \omega_1$$

$$\left|H(e^{j\omega})\right| \le \delta_2; \omega_2 \le \omega \le \pi$$

The expression for ε is

$$\varepsilon = \left[\frac{1}{\delta_1^2} - 1\right]^{0.5}$$

Cut-off frequency,

$$\Omega_c = \Omega_1$$

Order of the filter is given by,

$$N \ge \frac{\cosh^{-1}\left\{\frac{1}{\varepsilon}\left[\frac{1}{\delta_2^2} - 1\right]^{0.5}\right\}}{\cosh^{-1}(\Omega_2/\Omega_1)}$$

The values of Ω_1 and Ω_2 are obtained using the bilinear transformation or impulse invariant transformation techniques,

$$\Omega = \frac{2}{T}\tan(\omega/2)\ \text{ for bilinear transformation}$$

or

$$\Omega = \frac{\omega}{T}\ \text{ for impulse inavriant tarnsformation}$$

The transfer function of the Butterworth filter is given as:

$$H(s) = \prod_{k=1}^{N/2} \frac{B_k \Omega_c^2}{s^2 + b_k \Omega_c s + c_k \Omega_c^2}\ N = 2, 4, 6, \ldots$$

$$H(s)=\frac{B_0\Omega_c}{s+c_0\Omega_c}\prod_{k=1}^{(N-1)/2}\frac{B_k\Omega_c^2}{s^2+b_k\Omega_c s+c_k\Omega_c^2}\quad N=3,5,7,\dots$$

The coefficients b_k and c_k are given by

$$b_k = 2y_N \sin[(2k-1)\pi/2N]$$

$$c_k = y_N^2 + \cos^2\frac{(2k-1)\pi}{2N}$$

$$c_0 = y_N$$

The parameter y_N is given by

$$y_N=\frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2}+1\right)^{0.5}+\frac{1}{\varepsilon}\right]^{\frac{1}{N}}-\left[\left(\frac{1}{\varepsilon^2}+1\right)^{0.5}+\frac{1}{\varepsilon}\right]^{-\frac{1}{N}}\right\}$$

The parameter B_k can be obtained from

$$\frac{A}{(1+\varepsilon^2)^{0.5}}=\prod_{k=1}^{N/2}\frac{B_k}{c_k},\text{ for } N \text{ even}$$

$$A=\prod_{k=0}^{\frac{N-1}{2}}\frac{B_k}{c_k}\text{ for } N \text{ odd}$$

The system function of the equivalent digital filter is obtained from H(s) using the specified transformation technique, viz. Impulse invariant or bilinear transformation.

Example Design a digital Chebyshev filter to satisfy the constraints

$$0.707 \le |H(e^{j\omega})| \le 1,\quad 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.1,\quad 0.5\pi \le \omega \le \pi$$

Using bilinear transformation and assuming T=1s.

Solution

Given $\delta_1 = 0.707,\ \delta_2 = 0.1,\ \omega_1 = 0.2\pi,\ \omega_2 = 0.5\pi$

Step (i) Determination of the analog filter's edge frequencies

$$\Omega_c = \Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2} = 2\tan 0.1\pi = 0.6498$$

$$\Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} = 2\tan 0.25\pi = 2$$

Therefore, Ω_2/Ω_1=3.0779

Step (ii) Determination of the order of the filter

$$\varepsilon = \left[\frac{1}{\delta_1^2}-1\right]^{0.5} = \left[\frac{1}{0.707^2}-1\right]^{0.5} = 1$$

Order,

$$N \geq \frac{\cosh^{-1}\left\{\frac{1}{\varepsilon}\left[\frac{1}{\delta_2^2}-1\right]^{0.5}\right\}}{\cosh^{-1}(\Omega_2/\Omega_1)} = \frac{\cosh^{-1}\left\{1\left[\frac{1}{0.1^2}-1\right]^{0.5}\right\}}{\cosh^{-1}(3.0779)} = 1.669$$

Let N=2.

Step (iii) Determination of H(s)

$$H(s) = \prod_{k=1}^{N/2} \frac{B_k\Omega_c^2}{s^2 + b_k\Omega_c s + c_k\Omega_c^2} = \frac{B_1\Omega_c^2}{s^2 + b_1\Omega_c s + c_1\Omega_c^2}$$

and

$$y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\varepsilon^2}+1\right)^{0.5} + \frac{1}{\varepsilon}\right]^{\frac{1}{N}} - \left[\left(\frac{1}{\varepsilon^2}+1\right)^{0.5} + \frac{1}{\varepsilon}\right]^{-\frac{1}{N}}\right\}$$

$$= \frac{1}{2}\left\{[2.414]^{\frac{1}{2}} - [2.414]^{-\frac{1}{2}}\right\} = 0.455$$

$$b_1 = 2y_2 \sin\left[\frac{(2k-1)\pi}{2N}\right] = 0.6435$$

$$c_1 = y_2^2 + \cos^2\frac{(2k-1)\pi}{2N} = 0.707$$

For even N,

$$\prod_{k=1}^{\frac{N}{2}} \frac{B_k}{c_k} = \frac{A}{(1+\varepsilon^2)^{0.5}} = 0.707$$

That is,

$$\frac{B_1}{c_1} = 0.707$$

And hence $B_1 = 0.5$

The system function is

$$H(s) = \frac{0.5(0.6498)^2}{s^2 + (0.6435)(0.6498)s + (0.707)(0.6498)^2}$$

On simplifying we get,

$$H(s) = \frac{0.2111}{s^2 + 0.4181s + 0.2985}$$

Step (iv) determination of H(z). Using bilinear transformation,

$$H(z) = H(s)|_{s=\frac{2(z-1)}{T(z+1)}}$$

That is,

$$H(z) = \frac{0.2111}{\left[2\frac{(z-1)}{(z+1)}\right]^2 + 0.4181\left[2\frac{(z-1)}{(z+1)}\right] + 0.2985}$$

$$= \frac{0.2111(z+1)^2}{5.1347z^2 - 7.403z - 3.4623}$$

Rearranging,

$$H(z) = \frac{0.041(1+z^{-1})^2}{1 - 1.4418z^{-1} + 0.6743z^{-2}}$$

3.3 FREQUENCY TRANSLATION

There are basically four types of frequency selective filters, viz. Low-pass, highpass, band pass and bandstop. This low-pass filter can be considered as aprototype filter and its system function can be obtained. Then, if a highpass or bandpass or bandstop filter is to be designed, it can be easily obtained by using frequency transformation. Frequency transformation can be accomplished in two ways. In the analog frequency transformation, the analog system function $H_p(s)$ of the prototype filter is converted into another analog system function H(s) of the desired filter. Then by using any of the mapping techniques, it is converted into the digital filter having a system function H(z). In the digital frequency transformation, the analog prototype filter is first transformed to the digital domain, to have a system function $H_p(z)$. then by using frequency transformation, it can be converted into the desired digital filter.

3.3.1 ANALOG FREQUENCY TRANSFORMATION

Table *Analog frequency transformation*

Type	*Transformation*
Low-pass	$s \rightarrow \frac{\Omega_c}{\Omega_c^*} s$
High-pass	$s \rightarrow \frac{\Omega_c \Omega_c^*}{s}$
Bandpass	$s \rightarrow \Omega_c \frac{s^2 + \Omega_1 \Omega_2}{s(\Omega_2 - \Omega_1)}$
Bandstop	$s \rightarrow \Omega_c \frac{s(\Omega_2 - \Omega_1)}{s^2 + \Omega_1 \Omega_2}$

Example *A prototype low-pass filter has the system response* $H(s) = \frac{1}{s^2 + 2s + 1}$. *Obtain a bandpass filter with* $\Omega'' = 2$ *rad/s and* $Q = 10$.

$\Omega_0^2 = \Omega_1 \cdot \Omega_2$ and $Q = \frac{\Omega_0}{\Omega_2 - \Omega_1}$.

Solution

$$s \rightarrow \Omega_c \frac{s^2 + \Omega_1 \Omega_2}{s(\Omega_2 - \Omega_1)}, \text{ i.e.}$$

$$s = \Omega_c \frac{s^2 + \Omega_0^2}{s(\Omega_0 / Q)} = \Omega_c \frac{s^2 + 2^2}{s(2/10)} = 5\Omega_c \left(\frac{s^2 + 4}{s} \right)$$

Therefore,

$$H(s) = H(s)|_{s = 5\Omega_c \left(\frac{s^2+4}{s} \right)}$$

$$H(s) = \frac{0.04 s^2}{\Omega_c^2 s^4 + 0.4 \Omega_c s^3 + (8\Omega_c^2 + 0.01) s^2 + 1.6 \Omega_c s + 16 \Omega_c^2}$$

Example *Transform the prototype low-pass filter with system function*

$$H(s) = \frac{\Omega_c}{s + \Omega_c}$$

into a high-pass filter with cut-off frequency Ω_c^*.

Solution

$$s \rightarrow \frac{\Omega_c \Omega_c^*}{s}$$

Thus we have,

$$H_{hpf}(s) = \frac{\Omega_c}{\left(\frac{\Omega_c \Omega_c^*}{s} \right) + \Omega_c}$$

$$= \frac{s}{s + \Omega_c^*}$$

3.3.2 DIGITAL FREQUENCY TRANSFORMATION

Table *Digital frequency transformation*

Type	*Transformation*	*Design Parameter*
Low–pass	$z^{-1} \rightarrow \dfrac{z^{-1}-a}{1-az^{-1}}$	$a = \dfrac{\sin[(\omega_c - \omega_c^*)/2]}{\sin[(\omega_c + \omega_c^*)/2]}$
High-pass	$z^{-1} \rightarrow -\dfrac{z^{-1}+a}{1+az^{-1}}$	$a = \dfrac{\cos[(\omega_c - \omega_c^*)/2]}{\cos[(\omega_c + \omega_c^*)/2]}$
Bandpass	$z^{-1} \rightarrow -\dfrac{z^{-2}-a_1 z^{-1}+a^2}{a_2 z^{-2}-a_1 z^{-1}+1}$	$a_1 = -2\alpha K/(K+1)$ $a_2 = (K-1)/(K+1)$ $\alpha = \dfrac{\cos[(\omega_2+\omega_1)/2]}{\cos[(\omega_2-\omega_1)/2]}$ $K = \cot\left(\dfrac{\omega_2-\omega_1}{2}\right)\tan\left(\dfrac{\omega_c}{2}\right)$
Bandstop	$z^{-1} \rightarrow -\dfrac{z^{-2}-a_1 z^{-1}+a_2}{a_2 z^{-2}-a_1 z^{-1}+1}$	$a_1 = -2\alpha/(K+1)$ $a_2 = (1-K)/(1+K)$ $\alpha = \dfrac{\cos[(\omega_2+\omega_1)/2]}{\cos[(\omega_2-\omega_1)/2]}$ $K = \tan\left(\dfrac{\omega_2-\omega_1}{2}\right)\tan\left(\dfrac{\omega_c}{2}\right)$

The frequency transformation may be accomplished in ay of the available two techniques, however, caution must be taken to which technique to use.. For example, the impulse invariant transformation is not suitable for high-pass or bandpass filters whose resonant frequencies are higher. In such a case, suppose a low-pass prototype filter is converted into a high-pass filter using analog frequency transformation and transformed later to a digital filter using impulse invariant technique. This will result in aliasing problems. However, if the same prototype low-pass filter is first transformed into a digital filter using impulse invariant technique and later converted into a high pass filter using digital frequency transformation, it will not have any aliasing problem. Whenever the bilinear transformation is used, it is of no significance whether analog frequency transformation is used or digital frequency transformation. In this case, both analog and digital frequency transformation techniques will give the same result.

3.4 STRUCTURES OF IIR FILTER

In this section, the most important filter structures namely direct forms I and II, cascade and parallel realisations for IIR systems are discussed.

3.4.1 DIRECT FORM I

The digital system structure determined directly from the equation,

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

is called direct form I.

$$H(z) = H_1(z) \cdot H_2(z)$$

Here

$$H_1(z) = \sum_{k=0}^{M} b_k z^{-k}$$

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

In this case, the system function is divided into two parts connected in cascade, the first part $H_1(z)$containing the zeros followed by the part only containing the poles $H_2(z)$.

We can write $H_1(z)$ as

$$H_1(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}$$

Since $H_1(z) = \frac{Y_1(z)}{X_1(z)}$ above equation can be written as,

$$Y_1(z) = b_0 X_1(z) + b_1 z^{-1} X_1(z) + b_2 z^{-2} X_1(z) + \ldots + b_M z^{-M} X_1(z)$$

Taking inverse z-transform of above equation,

$$y_1(n) = b_0 x_1(n) + b_1 x_1(n-1) + b_2 x_1(n-1) + b_2 x_1(n-2) + \ldots + b_M x_1(n-M)$$

The following figure shows the direct form realisation of the above equation.

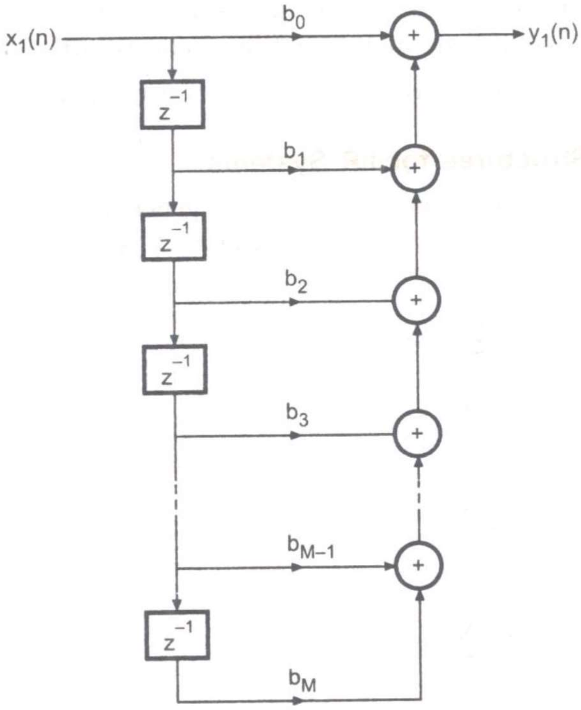


Fig 3.8: Direct form realisation of system function $H_1(z)$

Next consider realisation of $H_2(z)$. This is all pole system. Consider $H_2(z)$ given by the equation,

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

We know that $H_2(z) = \frac{Y_2(z)}{X_2(z)}$, hence above equation becomes,

$$\frac{Y_2(z)}{X_2(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$\therefore \; Y_2(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = X_2(z)$$

$$\therefore \; Y_2(z) = -\sum_{k=1}^{N} a_k z^{-k} Y_2(z) + X_2(z)$$

Expanding summation of this equation we get,

$$Y_2(z) = -a_1 z^{-1} Y_2(z) - a_2 z^{-2} Y_2(z) - a_3 z^{-3} Y_2(z) - \ldots - a_N z^{-N} Y_2(z) + X_2(z)$$

Taking inverse z-transform of above equation,

$$y_2(n) = -a_1\, y_2(n-1) - a_2\, y_2(n-2) - a_3\, y_2(n-3) - \ldots - a_N\, y_2(n-N) + x_2(n)$$

The following figure shows the direct form realisation of the above equation. Here, observe that the feedback terms are also present.

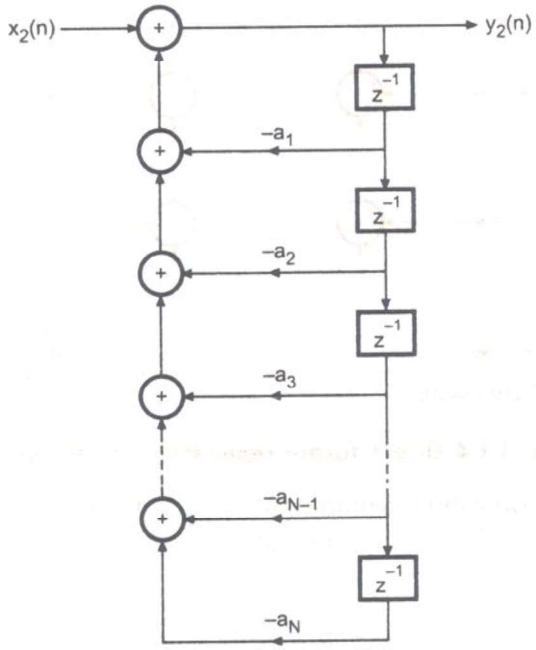


Fig 3.9: Direct form realisation of system function $H_2(z)$

We know that,

$$H(z) = H_1(z) \cdot H_2(z)$$

This represents cascading of two systems $H_1(z)$ and $H_2(z)$. This cascading is shown in figure below.

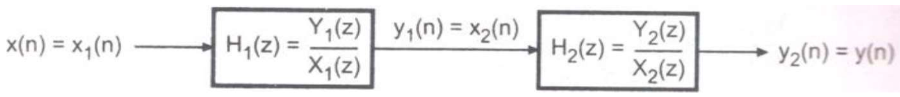


Fig 3.10: $H(z) = H_1(z).H_2(z)$ represents cascading of two systems

By connecting $H_1(z)$ and $H_2(z)$ in cascade we will get direct form realisation for H(z). This connection is shown in figure below.

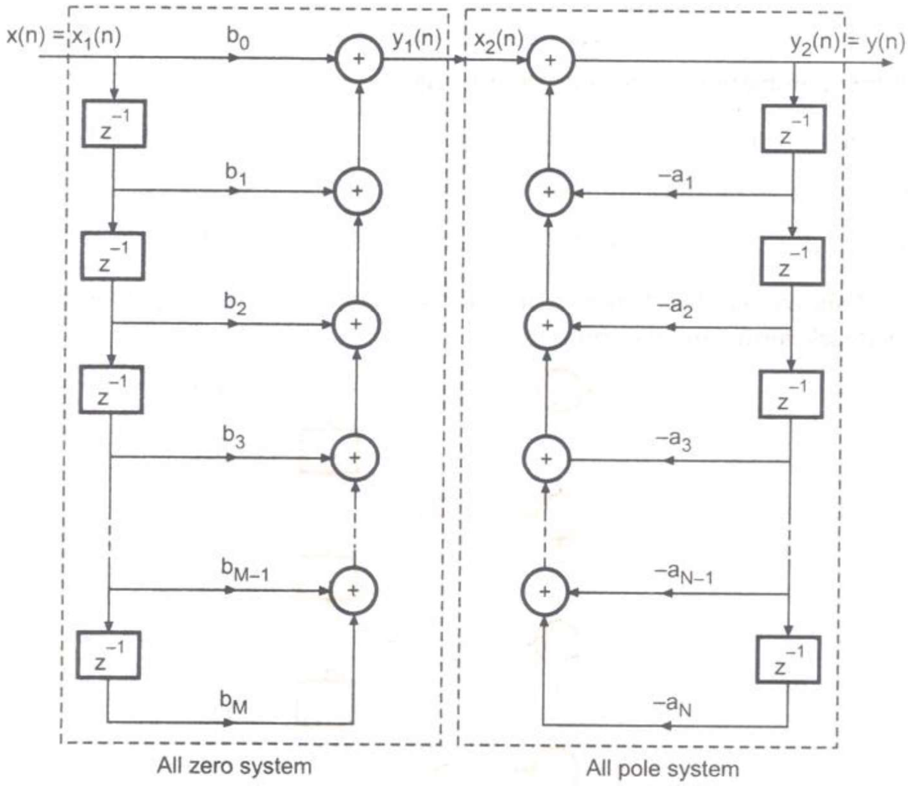


Fig 3.12: Direct form-I realisation of IIR system

3.4.2 DIRECT FORM II

In direct form II, the poles of H(z) are realised first and the zeros second. Here, the transfer function H(z) is broken into a product of two transfer functions $H_1(z)$ and $H_2(z)$, where $H_1(z)$ has only poles and $H_2(z)$ contains only the zeros as given below:

$$H(z) = H_1(z) \cdot H_2(z)$$

$$H(z) = \frac{\sum_{k=0}^{m} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Let $H(z)$ be written as,

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} \quad \text{By rearranging terms}$$

$$= \frac{W(z)}{X(z)} \cdot \frac{Y(z)}{W(z)} = H_1(z) \cdot H_2(z)$$

Now let $H_1(z)$ be given as,

$$H_1(z) = \frac{W(z)}{X(z)}$$

$$= \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

and

$$H_2(z) = \frac{Y(z)}{W(z)}$$

$$= \sum_{k=0}^{M} b_k z^{-k}$$

By cross multiplication,

$$W(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = X(z)$$

$$\therefore \quad W(z) = X(z) - \sum_{k=1}^{N} a_k z^{-k} W(z)$$

$$= X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) - a_3 z^{-3} W(z) - \dots\dots\dots - a_N z^{-N} W(z)$$

Taking inverse z-transform of this equation,

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) - \dots - a_N w(n-N)$$

Figure below shows the direct form implementation of above equation.

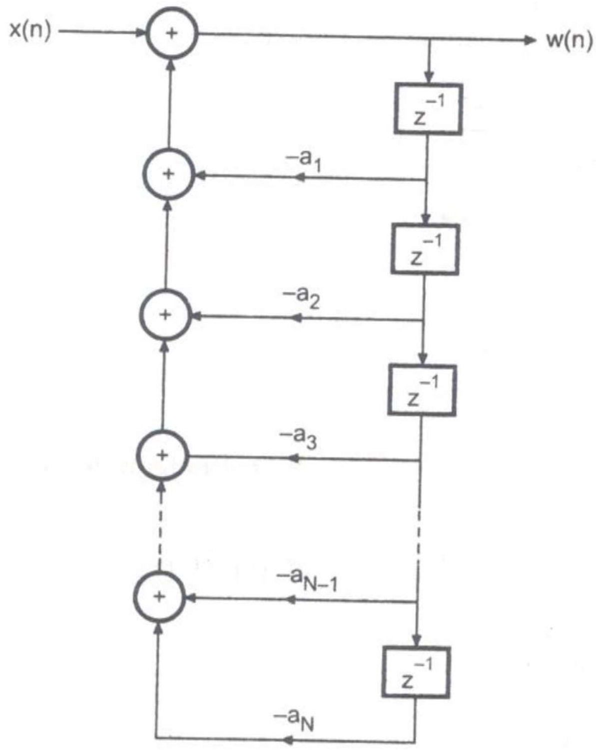


Fig 3.13: Direct form implementation. All pole system

Now let us obtain the realisation of $H_2(z)$. From this equation we obtain,

$$Y(z) = \sum_{k=0}^{M} b_k z^{-k} W(z)$$

$$= b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + \ldots\ldots + b_M z^{-M} W(z)$$

Taking inverse z-transform of this equation we get,

$$y(n) = b_0 w(n) + b_1 w(n-1) + b_2 w(n-2) + \ldots\ldots + b_M w(n-M)$$

The following figure shows the direct form implementation of above equation.

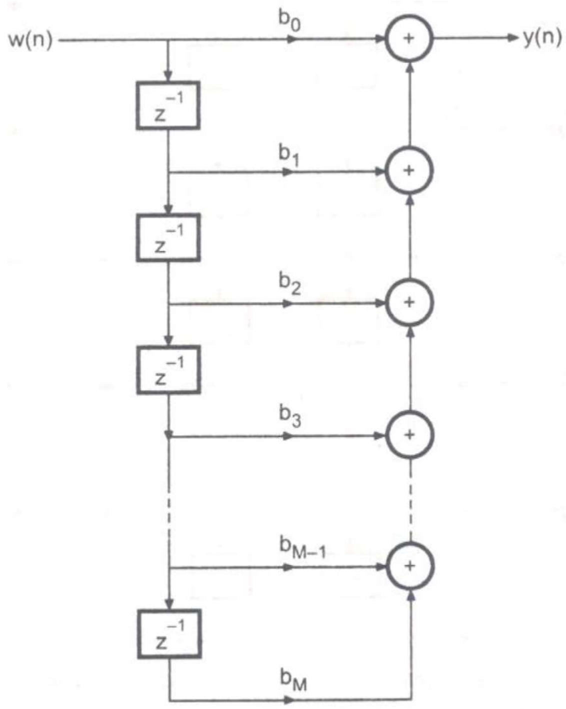


Fig 3.14: Direct form implementation. All zero system

By connecting $H_1(z)$ and $H_2(z)$ in cascade we will get direct form realisation for H(z). This connection is shown in figure below.

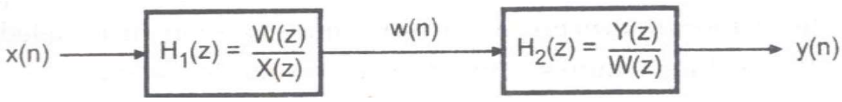


Fig: Cascade connection of $H_1(z)$ and $H_2(z)$

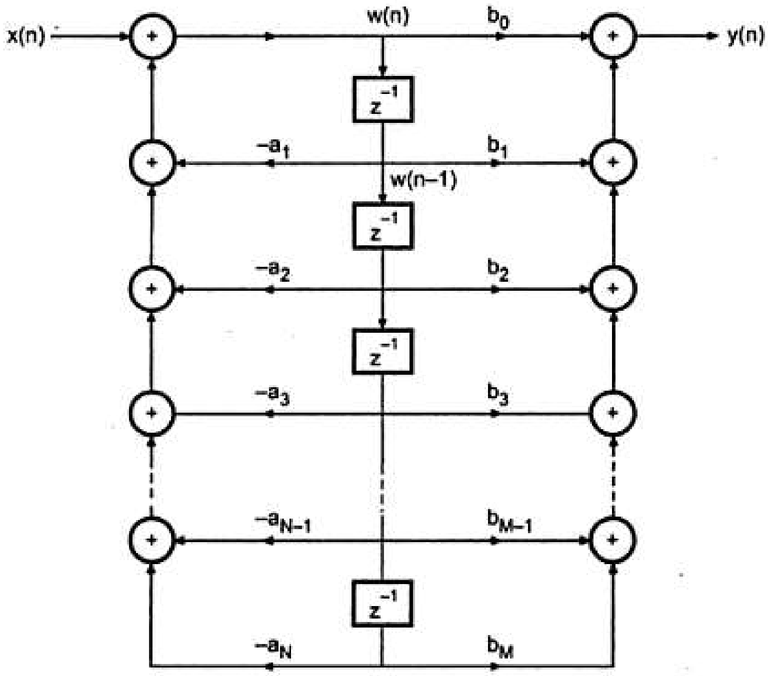


Fig 3.15: Direct form-II realisation of IIR system. Here N=M is considered

3.4.3 CASCADE FORM STRUCTURE

Consider the rational system function of equation

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_z z^{-2} + \dots + a_N z^{-N}}$$

The numerator and denominator polynomials of above equation can be expressed as multiplication of second order polynomials. i.e.,

$$H(z) = H_1(z) \times H_2(z) \times H_3(z) \times \ldots \times H_k(z)$$

where

$$H_k(z) = \frac{b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}, \quad k = 1, 2, \ldots, K$$

We know that the system function $H_1(z), H_2(z)$ ... etc can be connected in cascade to obtain realisation of H(z). This is shown in following figure.

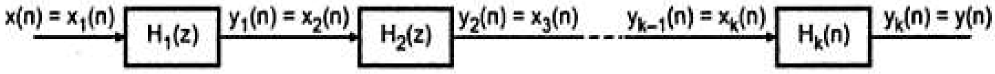


Fig 3.16: Cascade form realisation

Now each $H_1(z)$, $H_1(z)$,.... etc can be realized by direct form I and II structures.

This direct form structure is shown below:

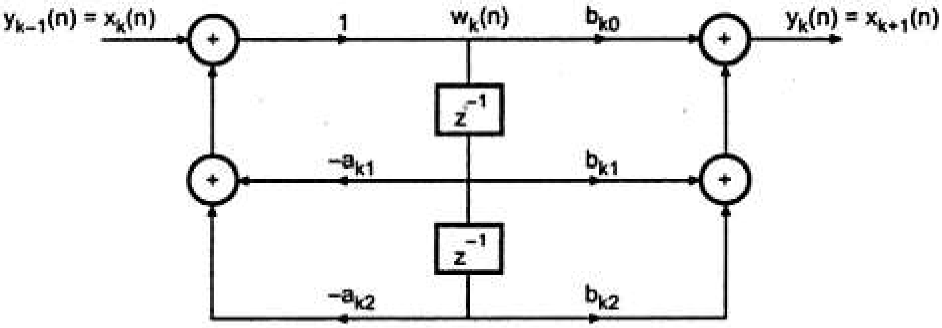


Fig 3.17: Direct form II realisation of second order subsystems used in cascade connection of IIR systems

Ex. *Realize the following system function in cascade form.*

$$H(z) = \frac{1+\frac{1}{5}z^{-1}}{\left(1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}\right)\left(1+\frac{1}{4}z^{-2}\right)}$$

Sol. : The given transfer function can be written as the product of two functions. i.e.,

$$H(z) = H_1(z)\cdot H_2(z) = \frac{1}{1+\frac{1}{4}z^{-1}}\cdot\frac{1+\frac{1}{5}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}$$

Here $$H_1(z) = \frac{1}{1+\frac{1}{4}z^{-1}}$$

and $$H_2(z) = \frac{1+\frac{1}{5}z^{-1}}{1-\frac{1}{2}z^{-1}+\frac{1}{3}z^{-2}}$$

$$H_1(z) = \frac{b_{10}}{1 + a_{11} z^{-1}} = \frac{1}{1 + \frac{1}{4} z^{-1}}$$

$$H_2(z) = \frac{b_{20} + b_{21} z^{-1}}{1 + a_{21} z^{-1} + a_{22} z^{-2}} = \frac{1 + \frac{1}{5} z^{-1}}{1 - \frac{1}{2} z^{-1} + \frac{1}{3} z^{-2}}$$

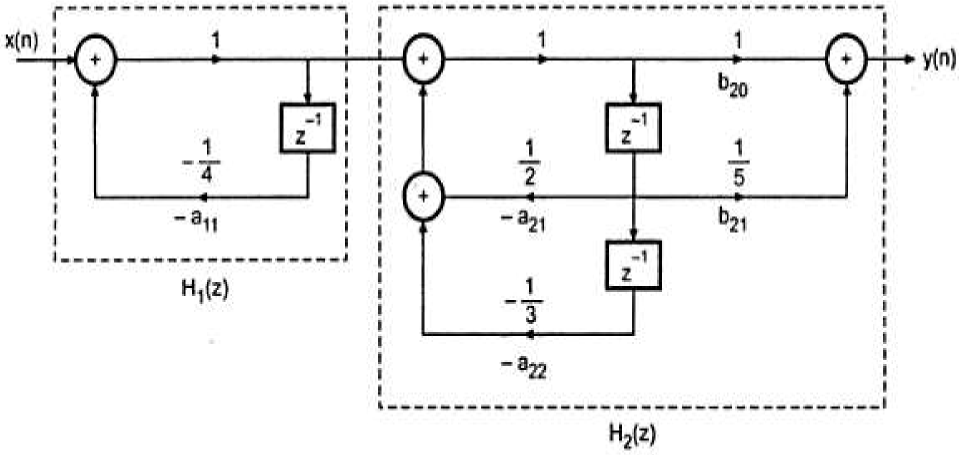


Fig 3.18: Cascade realisation of H(z)

3.4.4 PARALLEL FORM STRUCTURE

The rational system function of IIR filter is given by,

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}}$$

The above system function can be expanded in partial fractions as follows:

$$H(z) = C + H_1(z) + H_2(z) + ... + H_k(z)$$

Here C is a constant and $H_1(z)$, $H_1(z)$,.... etc is the second order subsystem which is given by,

$$H_k(z) = \frac{b_{k0} + b_{k1} z^{-1}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}$$

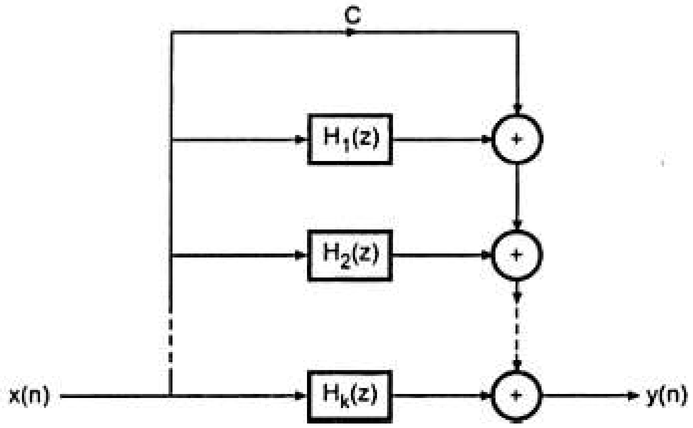


Fig 3.19: Parallel form realisation structure for IIR systems

Here each $H_1(z)$, $H_1(z)$,.... etc can be realized by direct form I and II structures.

Ex. *The transfer function of a discrete causal system is given as follows :*

$$H(z) = \frac{1 - z^{-1}}{1 - 0.2z^{-1} - 0.15z^{-2}}$$

i) Find the difference equation
ii) Draw cascade and parallel realization
iii) Calculate the impulse response of the system.

Sol. : i) To determine the difference equation

The given system function can be expressed as,

$$\frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - 0.2z^{-1} - 0.15z^{-2}}$$

$\therefore Y(z)\,[1 - 0.2\,z^{-1} - 0.15\,z^{-2}] = [1 - z^{-1}]\,X(z)$

$\therefore \quad Y(z) = 0.2\,z^{-1}\,Y(z) + 0.15\,z^{-2}\,2\,Y(z) + X(z) - z^{-1}\,X(z)$

Taking inverse z-transform of above equation

$$y(n) = 0.2\,y(n-1) + 0.15\,y(n-2) + x(n) - x(n-1)$$

This is the required difference equation.

ii) Cascade and parallel realization

Cascade realization

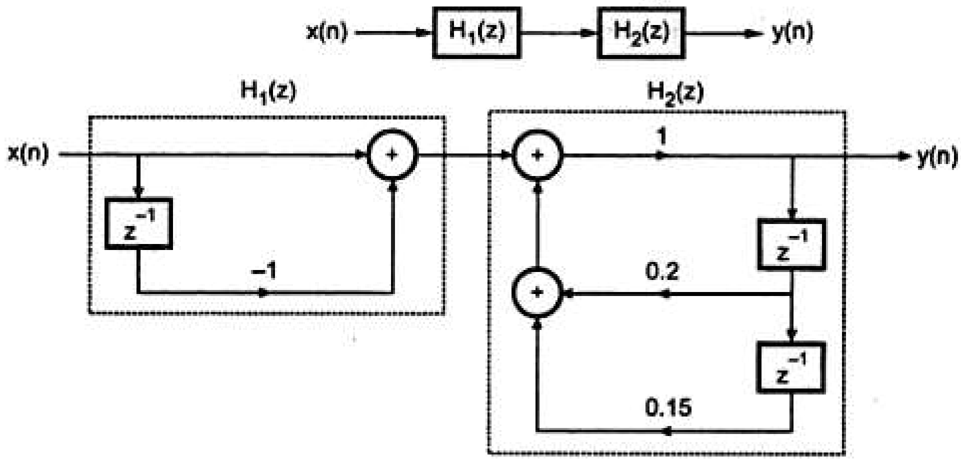
The given system is second order. Hence let us write it as,

$$H(z) = H_1(z) \cdot H_2(z) = (1 - z^{-1}) \cdot \frac{1}{1 - 0.2z^{-1} - 0.15z^{-2}}$$

Here $H_1(z) = 1 - z^{-1}$ is all zero section

and $H_2(z) = \frac{1}{1 - 0.2z^{-1} - 0.15z^{-2}}$ is all pole section.

The cascade realization will be as shown in Fig.



Parallel Realization

Consider the system function,

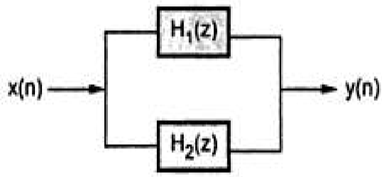
$$H(z) = \frac{1 - z^{-1}}{1 - 0.2z^{-1} - 0.15z^{-2}}$$

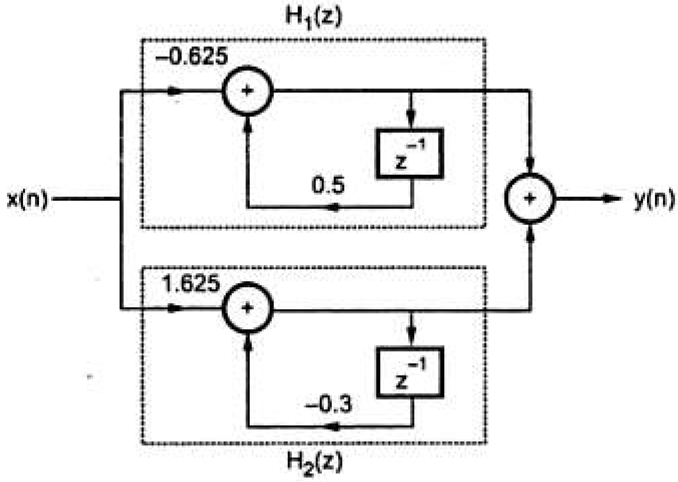
$$\therefore \quad \frac{H(z)}{z} = \frac{-0.625}{z - 0.5} + \frac{1.625}{z + 0.3}$$

$$\therefore \quad H(z) = \frac{-0.625}{1 - 0.5z^{-1}} + \frac{1.625}{1 + 0.3z^{-1}}$$

$$= H_1(z) + H_2(z)$$

These two system functions can be connected in parallel. This is shown in Fig.





iii) To obtain unit sample response

H(z) is expressed earlier as,

$$H(z) = \frac{-0.625}{1-0.5z^{-1}} + \frac{1.625}{1+0.3z^{-1}}$$

Taking inverse z-transform,

$$h(n) = -0.625\,(05)^n\;u(n) + 1.625\,(-0.3)^n\;u(n)$$

$$\text{ROC}: |z| > 0.5$$

This is the unit sample response of the system.

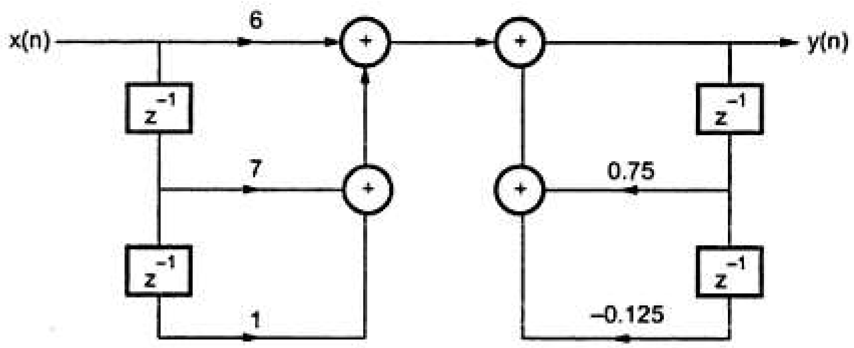
Ex. *Obtain the direct form-I, direct form-II cascade and parallel form realization for the following system.*

$$y(n) = 0.75\,y(n-1) - 0.125\,y(n-2) + 6x(n) + 7x\,(n-1) + x\,(n-2)$$

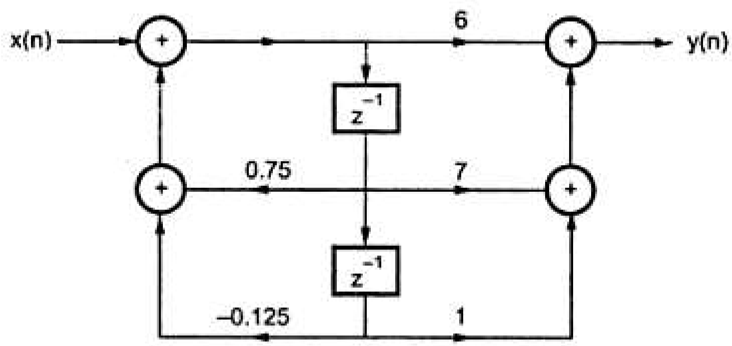
Sol. : i) Direct form-I

Here $b_0 = 6,\ b_1 = 7,\ b_2 = 1$

and $-a_1 = 0.75,\ -a_2 = -0.125$



ii) Direct form - II



iii) Cascade realization

Consider the given difference equation,

$$y(n) = 0.75\, y(n-1) - 0.125\, y(n-2) + 6\, x(n) + 7x(n-1) + x(n-2)$$

Taking z-transform of above equation,

$$Y(z) = 0.75\, z^{-1}\, Y(z) - 0.125\, z^{-2}\, Y(z) + 6\, X(z) + 7\, z^{-1}\, X(z) + z^{-2}\, X(z)$$

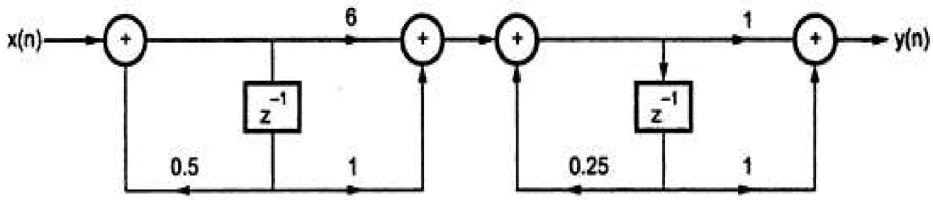
$$\therefore \quad H(z) = \frac{Y(z)}{X(z)} = \frac{6 + 7z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

$$= \frac{6z^2 + 7z + 1}{z^2 - 0.75z + 0.125}$$

$$= \frac{(6z+1)(z+1)}{(z-0.5)(z-0.25)}$$

$$= \frac{(6+z^{-1})(1+z^{-1})}{(1-0.5z^{-1})(1-0.25z^{-1})} = H_1(z) \cdot H_2(z)$$

Here $H_1(z) = \frac{6+z^{-1}}{1-0.5z^{-1}}$ and $H_2(z) = \frac{1+z^{-1}}{1-0.25z^{-1}}$



iv) Parallel realization

We know that,

$$H(z) = \frac{6z^2+7z+1}{z^2+0.75z+0.125}$$

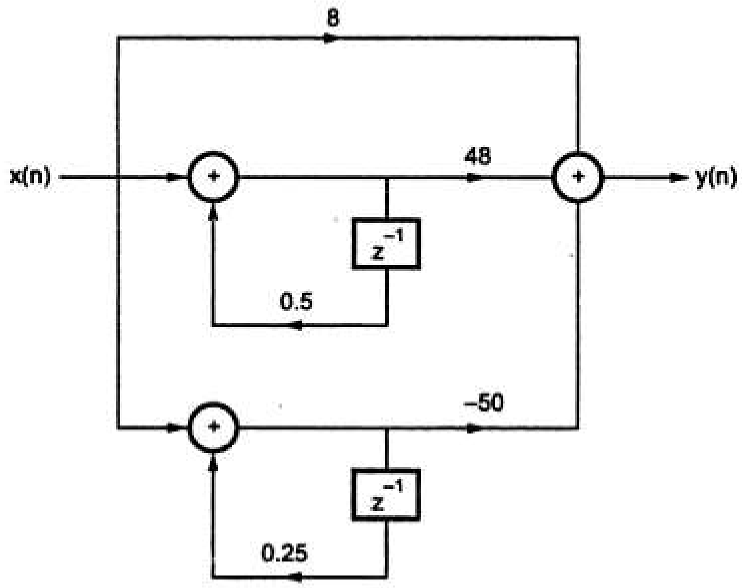
Let us divide both sides by z i.e.,

$$\frac{H(z)}{z} = \frac{6z^2+7z+1}{z(z^2+0.75z+0.125)}$$

$$= \frac{8}{z} + \frac{48}{z-0.5} + \frac{-50}{z-0.25}$$

$$\therefore \quad H(z) = 8 + \frac{48}{1-0.5z^{-1}} + \frac{-50}{1-0.25z^{-1}}$$

$$= H_1(z) + H_2(z) + H_3(z)$$



Ex. *A system function is specified by its transfer function H(z) given by,*

$$H(z) = \frac{(z-1)(z-2)(z+1)(z)}{\left[z-\left(\frac{1}{2}+j\frac{1}{2}\right)\right]\left[z-\left(\frac{1}{2}-j\frac{1}{2}\right)\right]\left(z-\frac{j}{4}\right)\left(z+\frac{j}{4}\right)}$$

Realize the system function in following forms :
i) Cascade of two biquadratic sections ii) Parallel realization in constant, linear and biquadratic sections.

Sol. :

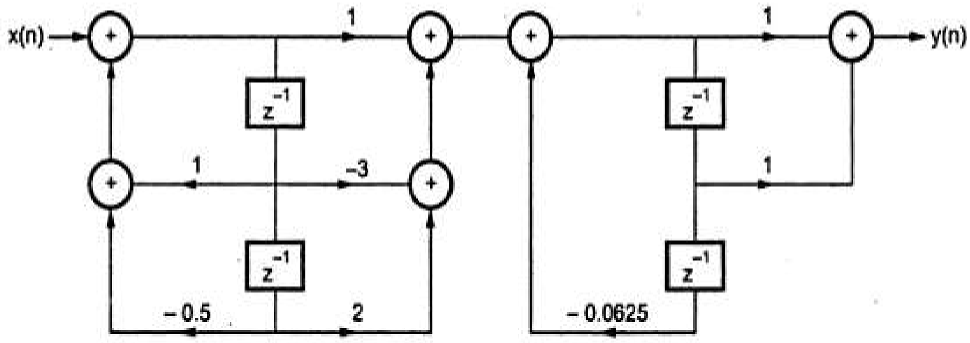
i) Cascade realization

The given H(z) can be represented as,

$$H(z) = \frac{(z-1)(z-2)}{[z-(0.5+j0.5)][z-(0.5-j0.5)]}\cdot\frac{(z+1)z}{(z-j0.25)(z+j0.25)}$$

$$= \frac{z^2-3z+2}{z^2-z+0.5}\cdot\frac{z^2+z}{z^2+j0.0625}$$

$$= H_1(z)\cdot H_2(z)$$



ii) Parallel form realization

Expanding H(z) in partial fractions we get

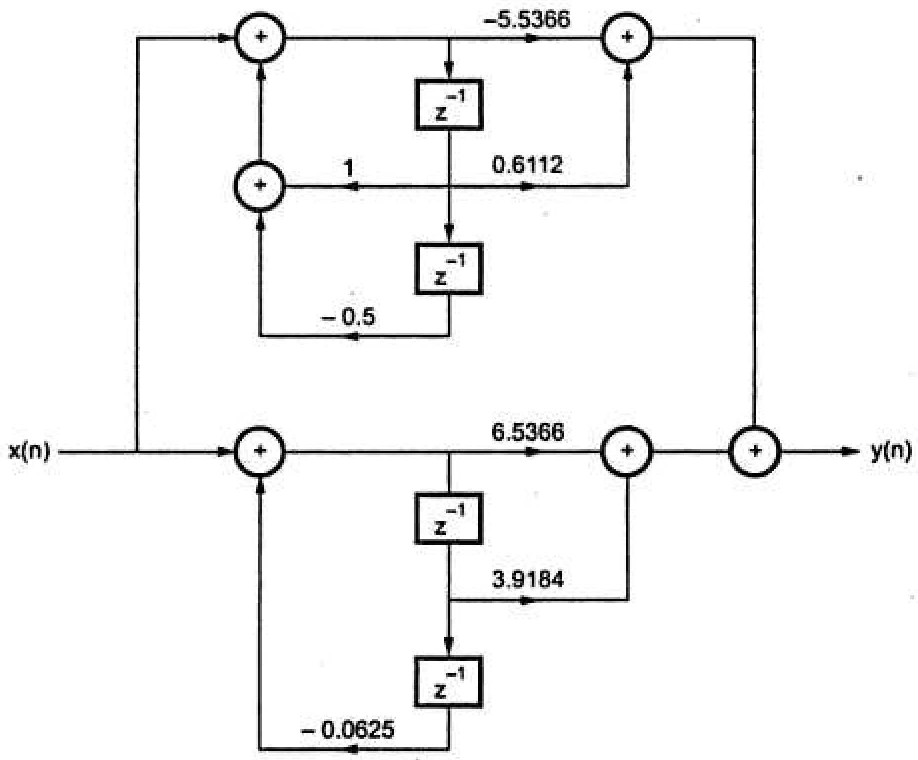
$$\frac{H(z)}{z}=\frac{-2.7683+j2.1517}{z-(0.5+j0.5)}+\frac{-2.7683-j2.1517}{z-(0.5-j0.5)}+\frac{3.2683-j7.8371}{z-j0.25}+\frac{3.2683+j7.8371}{z+j0.25}$$

The realization using second order sections is asked. Combining complex conjugate poles in above equation,

$$\frac{H(z)}{z}=\frac{-5.5366z+0.6112}{z^2-z+0.5}+\frac{6.5366z+3.9184}{z^2+0.0625}$$

$$=\frac{-5.5366+0.6112z^{-1}}{1-z^{-1}+0.5z^{-2}}+\frac{6.5366+3.9184z^{-1}}{1+0.0625z^{-2}}$$

$$= H_1(z) + H_2(z)$$



Ex. *Obtain the direct form-II (canonic) and cascade realization of*

$$H(z) = \frac{(z-1)(z^2+5z+6)(z-3)}{(z^2+6z+5)(z^2-6z+8)}$$

The cascade section should consist of two biquadratic sections.

Sol. : i) Cascade form realization

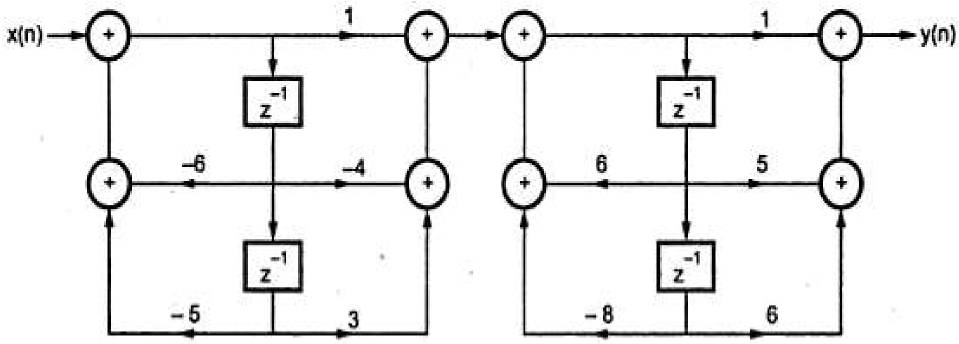
The given system function is,

$$H(z) = \frac{(z-1)(z^2+5z+6)(z-3)}{(z^2+6z+5)(z^2-6z+8)}$$

$$= \frac{z^2-4z+3}{z^2+6z+5} \cdot \frac{z^2+5z+6}{z^2-6z+8}$$

$$= \frac{1-4z^{-1}+3z^{-2}}{1+6z^{-1}+5z^{-2}} \cdot \frac{1+5z^{-1}+6z^{-2}}{1-6z^{-1}+8z^{-2}}$$

$$= H_1(z) \cdot H_2(z)$$



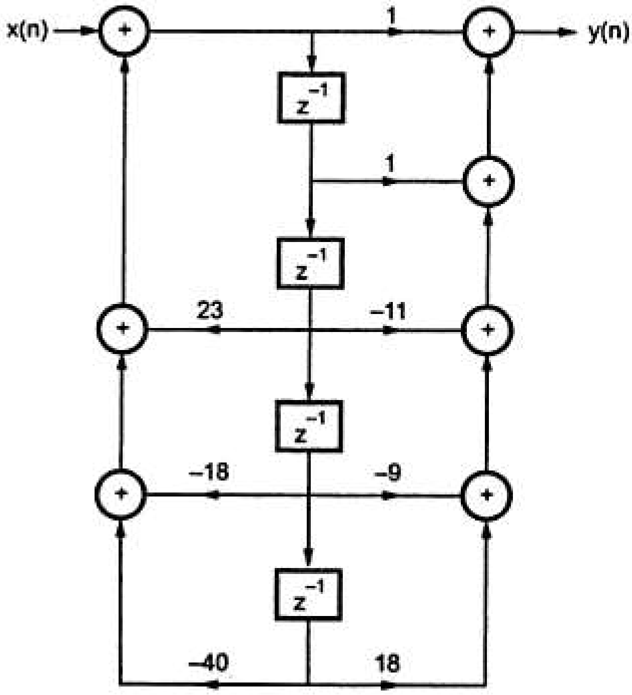
ii) Direct form-II realization

The given system function is,

$$H(z) = \frac{(z-1)(z^2+5z+6)(z-3)}{(z^2+6z+5)(z^2-6z+8)}$$

$$= \frac{z^4+z^3-11z^2-9z+18}{z^4+0z^3-23z^2+18z+40}$$

$$= \frac{1+z^{-1}-11z^{-2}-9z^{-3}+18z^{-4}}{1+0z^{-1}-23z^{-2}+18z^{-3}+40z^{-4}}$$



Ex. *A discrete time system H(z) is expressed as,*

$$H(z) = \frac{10\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)(1+2z^{-1})}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)\left[1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right]\left[1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right]}$$

i) Find the difference equation of the system
ii) Realize the system in direct form I and II
iii) Realize parallel form using second order sections

Sol. : i) To obtain difference equation

The given system can be expressed in polynomials ratio as,

$$H(z) = \frac{10+8.33z^{-1}-20z^{-2}+6.667z^{-3}}{1-1.875z^{-1}+1.14688z^{-2}-0.5313z^{-3}+0.0469z^{-4}}$$

$$\therefore \quad \frac{Y(z)}{X(z)} = \frac{10+8.33z^{-1}-20z^{-2}+6.667z^{-3}}{1-1.875z^{-1}+1.14688z^{-2}-0.5313z^{-3}+0.0469z^{-4}}$$

$$\therefore \quad Y(z) = 1.875\, z^{-1}\, Y(z) - 1.14688\, z^{-2} Y(z) + 0.5313\, z^{-3}\, Y(z)$$
$$- 0.0469\, z^{-4}\, Y(z) + 10\, X(z) + 8.33\, z^{-1}\, X(z)$$
$$- 20\, z^{-2}\, X(z) + 6.667\, z^{-3}\, X(z)$$

Taking inverse z-transform,

$$y(n) = 1.875\, y(n-1) - 1.14688\, y(n-2) + 0.5313\, y(n-3)$$
$$- 0.0469\, y(n-4) + 10\, x(n) + 8.33\, x(n-1)$$
$$- 20\, x(n-2) + 6.667\, x(n-3)$$

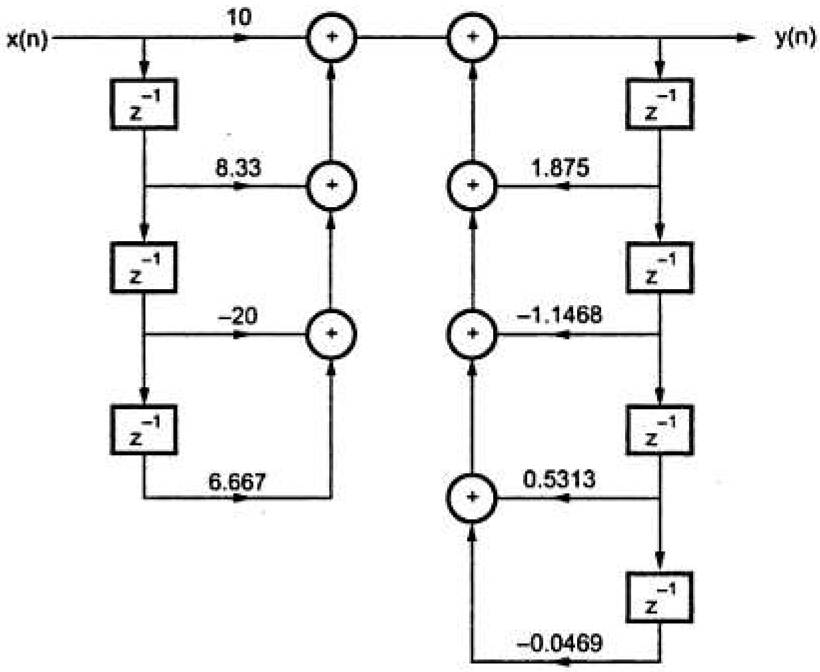
This is the required difference equation.

ii) To realize in direct form-I and II

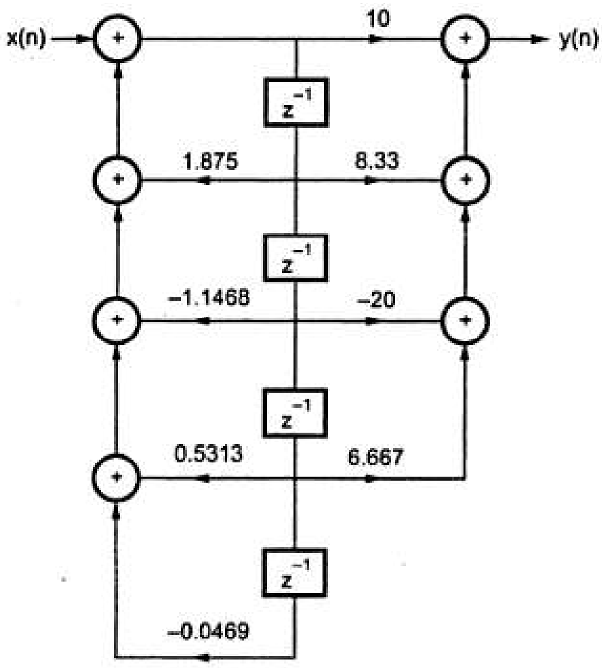
We have,

$$b_0 = 10,\ b_1 = 8.33,\ b_2 = -20,\ b_3 = 6.667$$

$$a_1 = -1.875,\ a_2 = 1.1468,\ a_3 = -0.5313,\ a_4 = 0.0469$$



(a) Direct form - I realization



(b) Direct form - II realization

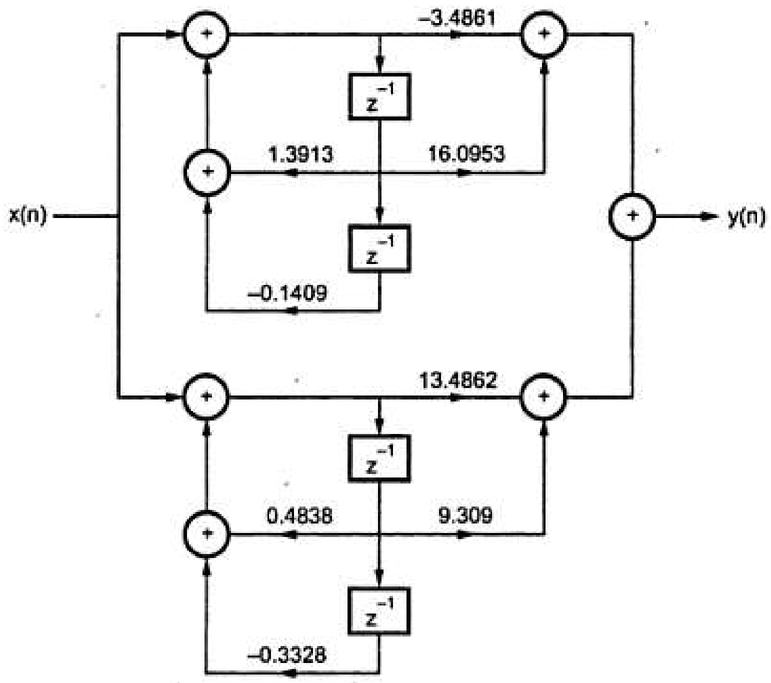
iii) Parallel form realization

Let us combine the complex conjugate poles and convert to second order sections :

$$\frac{H(z)}{z} = \frac{-3.4861z + 16.0953}{z^2 - 1.3913z + 0.1409} + \frac{13.4862z + 9.309}{z^2 - 0.4838z + 0.3328}$$

$$\therefore \quad H(z) = \frac{-3.4861 + 16.0953z^{-1}}{1 - 1.3913z^{-1} + 0.1409z^{-2}} + \frac{13.48622 + 9.304z^{-1}}{1 - 0.48382z^{-1} + 0.3328z^{-2}}$$

$$= H_1(z) + H_2(z)$$



Two Mark Question and Answers:

1. Why impulse invariant method is not preferred in the design of IIR filters other than low pass filter?

In this method the mapping from s plane to z plane is many to one. Thus there are an infinite number of poles that map to the same location in the z plane, producing an aliasing effect. It is inappropriate in designing high pass filters. Therefore this method is not much preferred.

2.What do you understand by backward difference?

One of the simplest methods of converting analog to digital filter is to approximate the differential equation by an equivalent difference equation.

d/dt(y(t)/t=nT=(y(nT)-y(nT-T))/T

3. What are the properties of chebyshev filter?