FIR FILTER DESIGN

Structures of FIR – Linear phase FIR filter – Fourier Series - Filter design using windowing techniques (Rectangular Window, Hamming Window, Hanning Window), Frequency sampling techniques – Finite word length effects in digital Filters: Errors, Limit Cycle, Noise Power Spectrum.

4.1 INTRODUCTION:

A digital filter is a mathematical algorithm implemented in hardware/software that operates on a digital input to produce a digital output. Digital filters often operate on digitized analog signals stored in a computer memory. Digital filters play very important roles in DSP. Compared with analog filters, they are preferred in a number of applications like data compression, speech processing, image processing, etc., because of the following advantages.

(i)Digital filters can have characteristics which are not possible with analog filters such as linear phase response.
(ii).The performance of digital filters does not vary with environmental changes, for example, thermal variations.
(iii).The frequency response of a digital filter can be adjusted if it is implemented using a programmable processor.
(iv).Several input signals can be filtered by one digital filter without the need to replicate the hardware.
(v).Digital filters can be used at very low frequencies.

The following are the main advantages of digital filters compared with analog filters:
(i).Speed limitation
(ii).Finite word length effects
(iii).Long design and development times.

A discrete-time filter produces a discrete time output sequence y(n) for the discrete-time sequence x(n).A filter may be required to have a given frequency response, or a specific response to an impulse, step, or ramp, or simulate an analog system. Digital filters are classified either as finite duration unit pulse response(FIR) filters or infinite duration unit pulse response (IIR) filters, depending on the form of the unit pulse response of the systems..In the FIR system, the impulse response sequence is of finite duration, i.e., it has a finite number of non-zero terms. The IIR system has an infinite number of non-zero terms, i.e., its impulse response sequence is of infinite duration. The system with the impulse response

$$h(n)=\begin{cases}2, & |n|\leq 4\\ 0, & \text{otherwise}\end{cases}$$

has only a finite number of non-zero terms. Thus, the system is an FIR system. The system with the impulse response $h(n)=a^n h(n)$ is non-zero for $n\geq 0$.It has an infinite number of non-zero

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terms and is an IIR system.IIR filters are usually implemented using structures having feedback (recursive structures-poles and zeros) and FIR filters are usually implemented using structures with no feedback(non-recursive structures-all zeros).

The unit sample response of FIR filter is symmetric if it satisfies following conditions

h(n)=h(M-1-n),n=0,1,2….M-1

The unit sample response of FIR filter is Anti symmetric if it satisfies following conditions

h(n)=-h(M-1-n),n=0,1,2….M-1

FIR filters have the following advantages over IIF filters

(i).They can have an exact linear phase
(ii).They are always stable
(iii).The design methods are generally linear.
(iv).They can be realized efficiently in hardware
(v).The filter start-up transients have finite duration

FIR filters are employed in filtering problems where linear phase characteristics within the pass band of the filter are required. If this is not required, either an IIR or an FIR filter may be employed. An IIR filter has lesser number of side lobes in the stop band than an FIR filter with the same number of parameters. For this reason if some phase distortion is tolerable, an IIR filter is preferable. Also, the implementation of an IIR filter involves fewer parameters, less memory requirements and lower computational complexity.

4.2 MAGNITUDE AND PHASE RESPONSE OF FIR FILTERS AND LINEAR PHASE PROPERTY:

The symmetry /Antisymmetry of the unit sample response of FIR filter is related to their linearity of phase. The unit sample response of FIR filter is symmetric if it satisfies following conditions.

h(n)=h(M-1-n),n=0,1,2…M-1……………………………(1)

The unit sample response of FIR filters is antisymmetric if it satisfies the following condition.

h(n)=-h(n-M-1),n=0,1,2….M-1……………………………(2)

The phase of FIR filters is piecewise linear if its unit sample response symmetric or antisymmetric. This can be proved separately for odd and even length of FIR filters. Consider the Fourier transform of unit sample response.

$$H(w)=\sum_{n=0}^{M-1} h(n)e^{-jwn}$$

Let us assume that M is odd

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H(w)= $\sum_{n=0}^{\frac{M-3}{2}} h(n)e^{-jwn} + h(\frac{M-1}{2})e^{-jw(\frac{M-1}{2})} + \sum_{n=\frac{M+1}{2}}^{M-1} h(n)e^{-jwn}$ ……………………….(3)

We know that h(n)=h(M-1-n),n=0,1,2…M-1 for symmetric FIR filter with this condition the last summation term in above equation becomes,

$$= \sum_{n=\frac{M+1}{2}}^{M-1} h(M-1-n)e^{-jwn}$$

Put M-1-n=k

N= $\frac{M+1}{2}, k = M-1-\frac{M+1}{2} = \frac{M-3}{2}$

n=M-1,K=m-1-m+1=0

$$\sum_{n=\frac{M+1}{2}}^{M-1} h(n)e^{-jwn} = \sum_{k=0}^{\frac{M-3}{2}} h(k)e^{-jw(M-1-k)} \quad \text{…………………….(4)}$$

Putting (4) in (3)

H(w)= $\sum_{n=0}^{\frac{M-3}{2}} h(n)e^{-jwn} + h(\frac{M-1}{2})e^{-jw(\frac{M-1}{2})} + \sum_{n=0}^{\frac{M-3}{n=0}} h(n)e^{-jw(M-1-n)}$

H(w)=h($\frac{M-1}{2})e^{-jw(\frac{M-1}{2})} + \sum_{n=0}^{\frac{M-3}{2}} h(n)[e^{-jwn} + e^{-jw(M-1-n)}]$ …………………..(5)

$$e^{-jwn} = e^{-jwn}.e^{jw(\frac{M-1}{2})}e^{-jw(\frac{M-1}{2})}$$

$$e^{-jwn} = e^{-jw(\frac{M-1}{2})}e^{-jw(n-\frac{M-1}{2})}$$

Similarly, $e^{-jw(M-1-n)} = e^{-jw(M-1)}e^{-jwn}$

$$= .e^{-jw(\frac{M-1}{2})}e^{-jw(\frac{M-1}{2})}e^{jwn}$$

$$e^{-jw(M-1-n)} = e^{-jw(\frac{M-1}{2})}e^{jw(n-\frac{M-1}{2})}$$

$$[e^{-jwn} + e^{-jw(M-1-n)}] = e^{-jw(\frac{M-1}{2})}e^{-jw(n-\frac{M-1}{2})} + e^{-jw(\frac{M-1}{2})}e^{jw(n-\frac{M-1}{2})}$$

$$= e^{-jw(\frac{M-1}{2})}[e^{-jw(n-\frac{M-1}{2})} + e^{jw(n-\frac{M-1}{2})}]$$

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$$= e^{-jw(\frac{M-1}{2})} 2\cos(n - \frac{M-1}{2}) \text{..............................(6)}$$

Using (6) in (5)

$$H(w)=h(\frac{M-1}{2})e^{-jw(\frac{M-1}{2})} + \sum_{n=0}^{\frac{M-3}{2}} h(n)e^{-jw(\frac{M-1}{2})} 2\cos w(n - \frac{M-1}{2})\,]$$

$$H(w)= e^{-jw(\frac{M-1}{2})} \{h(\frac{M-1}{2}) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos w(n - \frac{M-1}{2})\}$$

The polar form of H(w) can be expressed as,

$H(w)=|H(w)|e^{j\angle H(w)}$

$|H(w)|$ is magnitude of H(w)

$\angle H(w)$ is angle or phase of H(w)

$$\text{Magnitude } |H(w)| = h(\frac{M-1}{2}) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n) 2\cos w(n - \frac{M-1}{2})$$

$$\text{Phase } \angle H(w) = \begin{cases} -w(\frac{M-1}{2}), & |H(w)| > 0 \\ -w(\frac{M-1}{2}) + \pi, & |H(w)| < 0 \end{cases}$$

In the above equation observe that M-1/2 is constant. Hence phase $\angle H(w)$ is linear function of 'w'. This phase is linearly proportional to frequency. When |H(w)| changes sign phase changes by π. Hence the phase is said to be piecewise linear. Thus FIR filter are linear phase filters.

For even value of M:

$$H(w)= e^{-jw(\frac{M-1}{2})} \{2\sum_{n=0}^{\frac{M}{2}-1} h(n)\cos w(n - \frac{M-1}{2})\}$$

Comparing above equation with the polar form of H(w) we get

$$\text{Magnitude } \{H(w)| = 2\sum_{n=0}^{\frac{M}{2}-1} h(n)\cos w(n - \frac{M-1}{2})$$

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Phase $\angle H(w) = \begin{cases} -w(\frac{M-1}{2}), |H(w)| > 0 \\ -w(\frac{M-1}{2}) + \pi, |H(w)| < 0 \end{cases}$

The above equation shows that phase is piecewise linear. Foe linear phase FIR filter
h(n)=±h(M-1-n)……………………..(8)

Linear phase is the most important feature of FIR filters. IIR filter cannot be designed with linear phase. The linear phase in FIR filter can be obtained if unit sample response satisfies equation (8) many applications need linear phase filtering for example the speech related applications require linear phase.

4.2. Structures of FIR:

4.2.1 Direct form realization of FIR systems:

The convolution sum relationship gives the system response as

$$y(n) = \sum_{K=0}^{M-1} h(k)\,x(n-k)$$

Where y(n) and x(n) are the output and input sequence, respectively. This equation gives the input-output relation of the FIR filter in the time-domain.

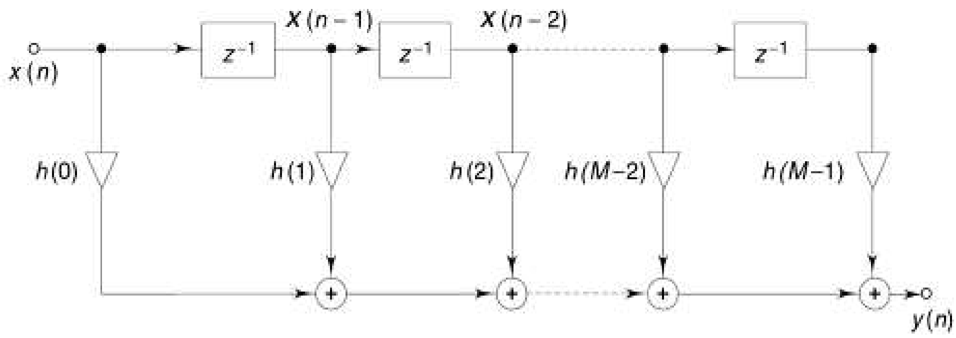

Fig 4.1: direct form structure

The transposed structure shown in given figure is the second direct form structure. Both of these direct form structures are canonic with respect to delays.

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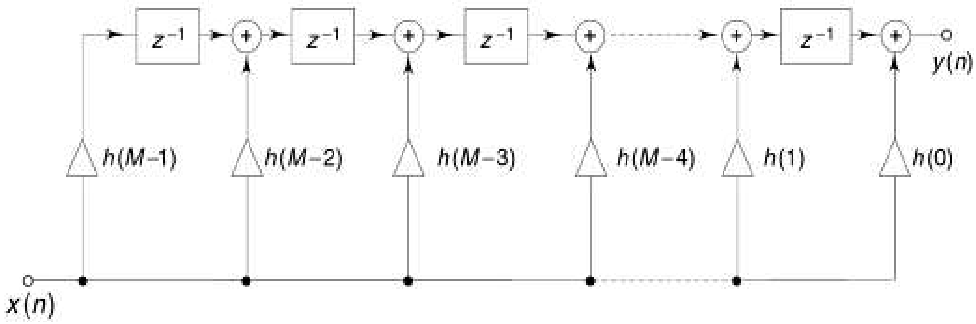

Fig 4.2: The transposed structure

4.2.2.Cascade Form Realization of FIR System:

As an alternative to the Direct-Form ,we factor the FIR transfer function H(z) as a product of second order factors given by,

$$H(z) = \prod_{k=1}^{M/2} (\beta_{0k} + \beta_{1k} z^{-1} + \beta_{2k} z^{-2})$$

A realization of above equation is shown in given figure for a cascade of second order sections. This cascade form is canonic with respect to delay, and also employs (M-1) two input adders and M multipliers for an $(M-1)^{th}$ order FIR transfer function.

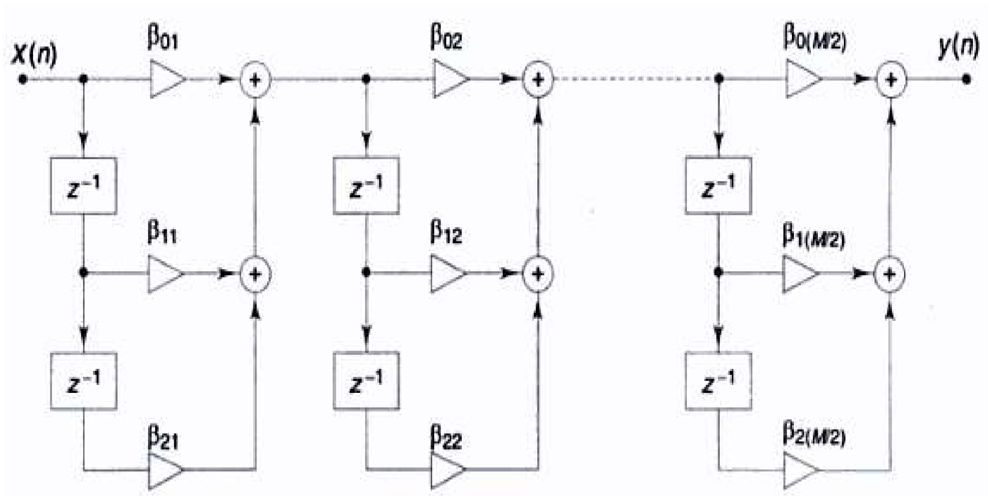

Fig 4.3:Cascade form realization of FIR filter

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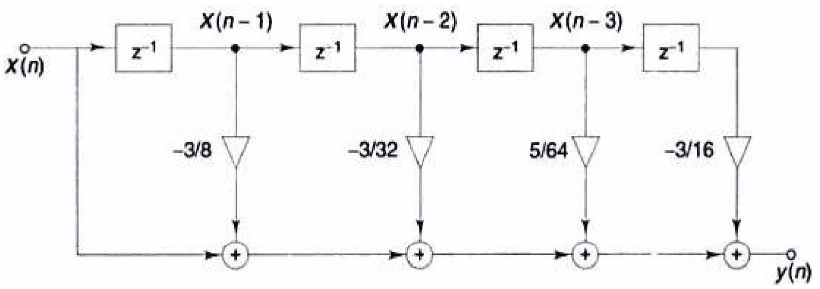
Example *Obtain direct form and cascade form realisation for the transfer function of an FIR system given by*

$$H(z)=\left(1-\frac{1}{4}z^{-1}+\frac{3}{8}z^{-2}\right)\left(1-\frac{1}{8}z^{-1}-\frac{1}{2}z^{-2}\right)$$

Solution:

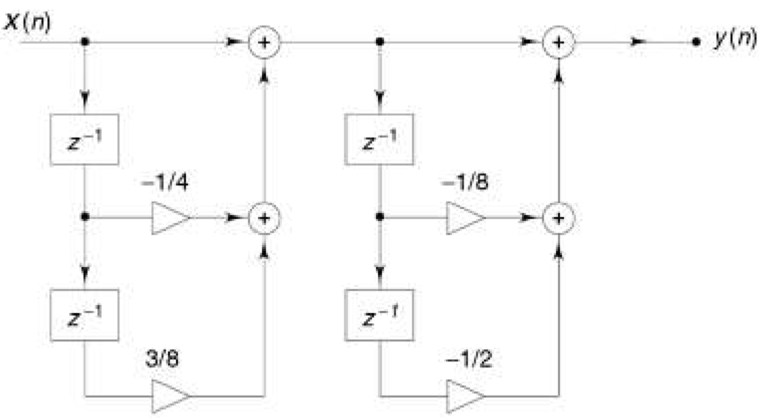
$$H(z)=1-\frac{3}{8}z^{-1}-\frac{3}{32}z^{-2}+\frac{5}{64}z^{-3}-\frac{3}{16}z^{-4}$$



Cascade Form Realization:

$$H(z)=\left(1-\frac{1}{4}z^{-1}+\frac{3}{8}z^{-2}\right)\left(1-\frac{1}{8}z^{-1}-\frac{1}{2}z^{-2}\right)$$

$$= H_1(z)\cdot H_2(z)$$



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4.2.3 Realization of linear Phase FIR System:

If the impulse response is symmetric about its origin, linear phase in results. If all zero filter has a linear phase response, a special non-recursive structure that reduces the number of multiplications by approximately one-half can be implemented. The impulse response for a causal filter begins at zero and ends at M-1.A linear phase filter of length M is characterized by,

$$h(n) = h(M-1-n)$$

The symmetry property of a linear phase FIR filter is used to reduce the multipliers required in these realizations. Using this condition, the z-transform of the impulse response can be expressed as,

$$H(z) = Z\left[h(n)\right] = \sum_{n=0}^{M-1} h(n) z^{-n}$$

For even value of M,

$$H(z) = \sum_{n=0}^{M/2-1} h(n)\left[z^{-n} + z^{-(M-1-n)}\right]$$

For odd value of M,

$$H(z) = h\left(\frac{M-1}{2}\right) z^{-(M-1)/2} + \sum_{n=0}^{(M-3)/2} h(n)\left[z^{-n} + z^{-(M-1-n)}\right]$$

We know that the output transform $Y(z) = H(z)\,X(z)$. When M is even,

$$\begin{aligned} Y(z) &= \sum_{n=0}^{\frac{M}{2}-1} h(n)\left[z^{-n} + z^{-(M-1-n)}\right] X(z) \\ &= h(0)\left[1 + z^{-(M-1)}\right] X(z) + h(1)\left[z^{-1} + z^{-(M-2)}\right] X(z) \\ &\quad + \ldots + h\left(\frac{M}{2} - 1\right)\left[z^{-(M/2-1)} + z^{-M/2}\right] X(z) \end{aligned}$$

The inverse z-transform or $Y(z)$ is

$$\begin{aligned} y(n) &= h(0)[x(n) + x\{n - (M-1)\}] + h(1)\,[x(n-1) \\ &\quad + x\{n - (M-2)\}] \\ &\quad + \ldots + h\left(\frac{M}{2} - 1\right)\left[x\left\{n - \left(\frac{M}{2} - 1\right)\right\}\right] + x\left(n - \frac{M}{2}\right) \end{aligned}$$

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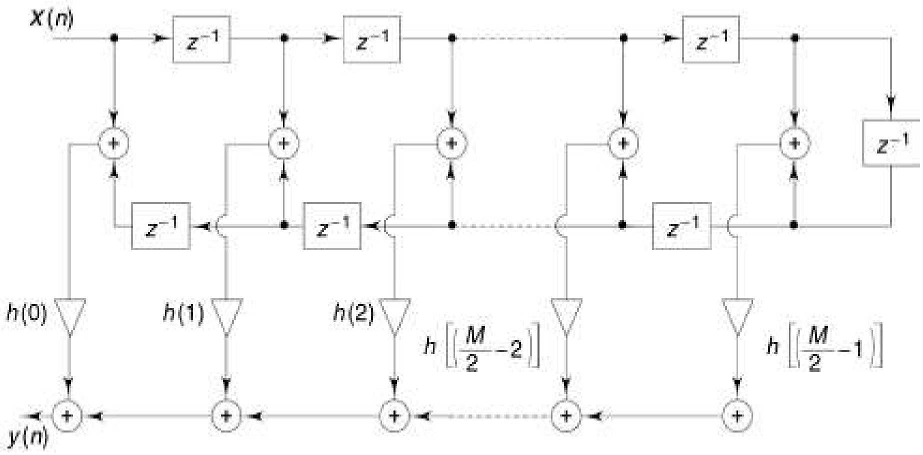

Fig 4.4: Direct form realization structure of a linear phase FIR system when M is even

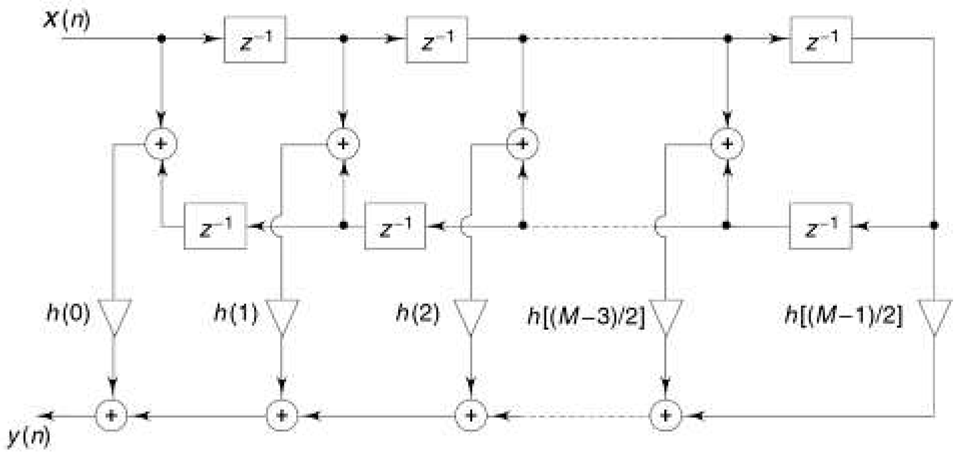

Fig 4.5: Direct form realization structure of a linear phase FIR system when M is odd

The equal –valued coefficients are combined and the system function is expressed as,

$$H(z) = z^{-1} + \frac{2}{3}\left[1 + z^{-2}\right]$$

4.3.Design techniques for FIR Filter:

4.3.1.Window techniques:

The infinite duration impulse response can be converted to a finite duration impulse response by truncating the infinite series at n=±N. But, this results in undesirable oscillations in the pass band and stop band of the digital filter. This is due to the slow convergence of the Fourier series near the points of discontinuity. These undesirable oscillations can be reduced by

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using a set of time-limited weighting functions, w(n),referred to as window functions, to modify the Fourier coefficients.

The desirable characteristics can be listed as follows.
(i).The Fourier transform of the window function $W(e^{jw})$ should have a small width of main lobe containing as much of the total energy as possible.
(ii).The Fourier transform of the window function $W(e^{jw})$ should have side lobes that decrease in energy rapidly as ω tends to π.

4.3.2 Different Types of windows:

1.Hamming Window:

The causal Hamming window function is expressed by

$$w_H(n) = \begin{cases} 0.54 - 0.46\cos\frac{2\pi n}{M-1}, & 0 \le n < M-1 \\ 0, & \text{otherwise} \end{cases}$$

The non-causal Hamming window function is given by

$$w_H(n) = \begin{cases} 0.54 + 0.46\cos\frac{2\pi n}{M-1}, & \text{for } |n| \le \frac{M-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

2.Hanning Window:

The window function of a causal Hanning window is given by

$$w_{Hann}(n) = \begin{cases} 0.5 - 0.5\cos\frac{2\pi n}{M-1}, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

The window function of a non-causal Hanning window is expressed by

$$w_{Hann}(n) = \begin{cases} 0.5 + 0.5\cos\frac{2\pi n}{M-1}, & 0 < |n| < \frac{M-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

3.Blackmann Window:

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The window function of a causal Blackman window is expressed by

$$w_B(n) = \begin{cases} 0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

The window function of a non-causal Blackman window is given by

$$w_B(n) = \begin{cases} 0.42 + 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi n}{M-1}, & \text{for } |n| < \dfrac{M-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

4.Rectangular Window:

The weighting function for the rectangular window is given by

$$\omega_R(n) = \begin{cases} 1, & \text{for } |n| \le \dfrac{M-1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Example *A filter is to be designed with the following desired frequency response*

$$H_d(e^{j\omega}) = \begin{cases} 0, & -\pi/4 \le \omega \le \pi/4 \\ e^{-j2\omega}, & \pi/4 |\omega| \le \pi \end{cases}$$

Determine the filter coefficients $h_d(n)$ if the window function is defined as

$$w(n) = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Also, determine the frequency response $H(e^{j\omega})$ of the designed filter.

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Solution Given

$$H_d(e^{j\omega}) = \begin{cases} 0, & -\pi/4 \le \omega \le \pi/4 \\ e^{-j2\omega}, & \pi/4 < |\omega| \le \pi \end{cases}$$

Therefore,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\pi/4} e^{-j2\omega}e^{j\omega n}\,d\omega + \frac{1}{2\pi}\int_{\pi/4}^{\pi} e^{-j2\omega}e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\pi/4} e^{j\omega(n-2)}\,d\omega + \frac{1}{2\pi}\int_{\pi/4}^{\pi} e^{j\omega(n-2)}\,d\omega$$

$$= \frac{1}{\pi(n-2)}\left\{\left[\frac{e^{j(n-2)\pi} - e^{-j(n-2)\pi}}{2j}\right] - \left[\frac{e^{j(n-2)\pi/4} - e^{-j(n-2)\pi/4}}{2j}\right]\right\}$$

$$= \frac{1}{\pi(n-2)}[\sin\pi(n-2) - \sin(n-2)\pi/4], \quad n \ne 2$$

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Example *The desired response of a low-pass filter is*

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & -3\pi/4 \le \omega \le 3\pi/4 \\ 0, & 3\pi/4 < |\omega| \le \pi \end{cases}$$

Determine $H(e^{j\omega})$ for $M = 7$ using a Hamming window.

Solution The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-3\pi/4}^{3\pi/4} e^{-j3\omega}e^{j\omega n}\,d\omega$$

$$h_d(n) = \frac{\sin 3\pi(n-3)/4}{\pi(n-3)}, n \ne 3 \text{ and } h_d(3) = \frac{3}{4}$$

The filter coefficients are,

$$h_d(0) = 0.0750, h_d(1) = -0.1592, h_d(2) = 0.2251, h_d(3) = 0.75$$
$$h_d(4) = 0.2251, h_d(5) = -0.1592, h_d(6) = 0.0750$$

The Hamming window function is,

$$w(n) = \begin{cases} 0.54 - 0.46\cos\frac{2\pi n}{M-1}, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, with $M = 7$,

$$w(0) = 0.08, w(1) = 0.31, w(2) = 0.77, w(3) = 1, w(4) = 0.77,$$
$$w(5) = 0.31, w(6) = 0.08.$$

The filter coefficients of the resultant filter are then,

$$h(n) = h_d(n).w(n)\ n = 0, 1, 2, 3, 4, 5, 6.$$

Therefore,

$$h(0) = 0.006, h(1) = -0.0494, h(2) = 0.1733, h(3) = 0.75,$$
$$h(4) = 0.1733, h(5) = -0.0494 \text{ and } h(6) = 0.006.$$

The frequency response is given by

$$H(e^{j\omega}) = \sum_{n=0}^{6} h(n)e^{-j\omega n}$$
$$= e^{-j3\omega}[h(3) + 2h(0)\cos 3\omega + 2h(1)\cos 2\omega + 2h(2)\cos\omega]$$
$$= e^{-j3\omega}[0.75 + 0.3466\cos\omega - 0.0988\cos 2\omega + 0.012\cos\omega]$$

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Example *Design a high pass filter using Hamming window with a cut-off frequency of 1.2 rad/sec and N = 9.*

Solution $\omega_c = 1.2\,\text{rad/sec}$ and $N = 9$.

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\alpha}, & -\pi \le \omega \le -\omega_c \text{ and } \omega_c \le \omega \le \pi \\ 0, & \text{otherwise} \end{cases}$$

To find $h_d(n)$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega\alpha}e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha}e^{j\omega n}d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha}e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{j\omega(n-\alpha)}d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{j\omega(n-\alpha)}d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_c}^{\pi}$$

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$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_c(n-\alpha)} - e^{-\pi(n-\alpha)} + e^{j\omega(n-\alpha)} - e^{j\omega_c(n-\alpha)}}{j(n-\alpha)}\right]$$

$$= \frac{\sin(n-\alpha)\pi - \sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$$

$$h_d(n) = \frac{\sin(n-\alpha)\pi - \sin\omega_c(n-\alpha)}{\pi(n-\alpha)} \text{ for } n \neq \alpha$$

$$h_d(n) = \frac{1}{\pi}\left(\underset{n\to\infty}{Lt}\frac{\sin(n-\alpha)\pi}{(n-\alpha)} - \underset{n\to\infty}{Lt}\frac{\sin(n-\alpha)\pi}{(n-\alpha)}\right)$$

$$= \frac{1}{\pi}(\pi - \omega_c) = \left[1 - \frac{\omega_c}{\pi}\right], \quad \text{for} \quad n \neq \alpha$$

$$h(n) = h_d(n) w_H(n)$$

$w_H(n)$=Window sequence for Hamming window.

$$= 0.54 - 0.46\cos\left[\frac{2\pi n}{N-1}\right] \quad \text{for} \quad n = 0 \text{ to } N-1$$

$$h(n) = \frac{1}{\pi(n-\alpha)}\left[\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_c\right] \left[0.54 - 0.46\left[\frac{2\pi n}{N-1}\right]\right]; \text{ for } n \neq \alpha$$

$$= \left(1 - \frac{\omega_c}{\pi}\right)\left[0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\right]; \text{ for } n = \alpha$$

Here

$$\alpha = \frac{N-1}{2} = \frac{5-1}{2} = 4$$

$$h(n) = \frac{-\sin(n-4)\omega_c}{\pi(n-4)}\left[0.54 - 0.46\cos\frac{n\pi}{4}\right]; \text{ for } n \neq 4$$

$$= \left(1 - \frac{\omega_c}{\pi}\right) \times \left[0.54 - 0.46\cos\frac{n\pi}{4}\right]; \text{ for } n = 4$$

$$h(0) = \frac{-\sin(-4)\times(1.2)}{\pi\times(0-4)}\left[0.54 - 0.46\cos 0\right] = 0.0063$$

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Similarly,

$h(1) = 0.0101$
$h(2) = 0.0581$
$h(3) = 0.2567$

$$h(4) = \left[\left(1 - \frac{1.2}{\pi}\right) \times (0.54 - 0.46\cos\pi)\right] = 0.6180$$

$h(5) = -0.2567$
$h(6) = -0.0581$
$h(7) = -0.0101$
$h(8) = -0.0063$

Since impulse response is symmetrical with centre of symmetry at $n = 4$.

$h(0) = h(8)$
$h(1) = h(7)$
$h(7) = h(6)$
$h(8) = h(5)$

To find magnitude response

$$|H(\omega)| = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{N-1/2} 2h\left(\frac{N-1}{2} - n\right)\cos\omega n$$

$$= h(4) + 2h(3)\cos\omega + 2h(2)\cos 2\omega + 2h(1)\cos 3\omega + 2h(0)\cos 4\omega$$

$$= 0.618 - 0.5134\cos\omega - 0.1162\cos 2\omega - 0.0202\cos 3\omega + 0.0126\cos 4\omega$$

To find the transfer function

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

$$= h(0)\left[z^{0} + z^{-8}\right] + h(1)\left[z^{-1} + z^{-7}\right] + h(2)\left[z^{-2} + z^{-6}\right] + h(3)\left[z^{-3} + z^{-5}\right] + h(4)z^{-}$$

$$H(z) = \frac{Y(z)}{X(z)}$$

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Example *Design an FIR digital filter to approximate an ideal low-pass filter with passband gain of unity, cut-off frequency of 850 Hz and working at a sampling frequency of f_s = 5000 Hz. The length of the impulse response should be 5. Use a rectangular window.*

Solution The desired response of the ideal low-pass filter is given by

$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le f \le 850\,\text{Hz} \\ 0, & f > 850\,\text{Hz} \end{cases}$$

The above response can be equivalently specified in terms of the normalised ω_c. The normalised $\omega_c = 2\pi f_c / f_s = 2\pi\ (850)/(5000) = 1.068$ rad/sec. Hence, the desired response is

$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le 1.068 \\ 0, & 1.068 < |\omega| \le \pi \end{cases}$$

The filter coefficients are given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-1.068}^{1.068} e^{j\omega n}\,d\omega$$

$$h_d(n) = \frac{\sin 1.068n}{\pi n}, \quad n \ne 0 \text{ and } h_d(0) = \frac{1.068}{\pi} = 0.3400$$

Using the rectangular window function and for $M = 5$,

$$h(n) = h_d(n)\,.\,w(n)\ n = 0, 1, 2, 3, 4.$$

Therefore,

$$h(0) = 0.34,\ h(1) = 0.2789,\ h(2) = 0.1344,\ h(3) = -0.0066,$$
$$h(4) = -0.0720.$$

4.3.2.Frequency sampling Method:

In this method, a set of samples is determined from the desired frequency response and are identified as discrete Fourier Transform(DFT) coefficients. The inverse discrete Fourier Transform (IDFT) of this set of samples then gives the filter coefficients. The set of sample points used in this procedure can be determined by sampling a desired frequency response Hd(e^{jw}) at M points ω_k=0,1,.M-1,uniformly spaced around the unit circle. Two design techniques are available, viz., Type-I design and design-II design. In the type-I design, the set of frequency samples includes the sample at frequency ω=0.In some cases. it may be desirable to omit the sample at ω=0 and use some other set of samples. Such a design procedure is referred to as the type-II design.

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Type-I Design:

The samples are taken at the frequency

$$\omega_k = \frac{2\pi k}{M}, k = 0, 1, \ldots, M-1$$

The samples of the desired frequency response at these frequencies are given by

$$\begin{aligned}\tilde{H}(k) &= H_d(e^{j\omega})|_{\omega=\omega_k}, \quad k = 0, 1, \ldots, M-1 \\ &= H_d(e^{j2\pi k/M}), \quad k = 0, 1, \ldots, M-1\end{aligned}$$

This set of points can be considered as DFT samples, then the filter coefficients h(n) can be computed using the IDFT,

$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} \tilde{H}(k) e^{j2\pi nk/M}, \quad n = 0, 1, \ldots, M-1$$

If $\overline{H}$ (0) is real and

(i) For M odd $\tilde{H}(M-k) = \tilde{H}^*(k), \quad k = 1, 2, \ldots, (M-1)/2$

(ii) For M even $\tilde{H}(M-k) = \tilde{H}^*(k), \quad k = 1, 2, \ldots, M/2-1$

$$h(n) = \frac{1}{M}\left[\tilde{H}(0) + 2\sum_{k=1}^{(M-1)/2} \text{Re}\left[\tilde{H}(k) e^{j2\pi nk/M}\right]\right], M \text{ odd}$$

$$h(n) = \frac{1}{M}\left[\tilde{H}(0) + 2\sum_{k=1}^{M/2-1} \text{Re}\left[\tilde{H}(k) e^{j2\pi nk/M}\right]\right], M \text{ even}$$

The system function of the filter can be determined from the filter coefficients h(n),

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$

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$$H(e^{j2\pi k/M}) = \sum_{n=0}^{M-1} h(n) e^{-j2\pi kn/M}$$

Replacing z by e^{jw} we get,

$$H(e^{j2\pi k/M}) = \tilde{H}(k) = H_d(e^{-j2\pi k/M})$$

Type –II Design:

$$\omega_k = \frac{2\pi(2k+1)}{2M}, \quad k = 0, 1, \ldots, M-1$$
$$= \frac{2\pi(k+1/2)}{M}, \quad k = 0, 1, \ldots, M-1$$

$$\tilde{H}(k) = H_d(e^{j\pi(2k+1)M}), \quad k = 0,1,\ldots,M-1$$

(i) *For M odd*

$$\tilde{H}(M-k-1) = \tilde{H}^*(k), \quad k = 0,1,\ldots,\left(\frac{M-1}{2}\right)$$
$$\tilde{H}\left(\frac{M-1}{2}\right) = 0$$

(ii) *For M even*

$$\tilde{H}(M-k-1) = \tilde{H}^*(k), \quad k = 0,1,\ldots,\left[\frac{M}{2}-1\right]$$

The filter coefficients are then written as,

$$h(n) = \frac{2}{M} \sum_{k=0}^{(M-3)/2} \mathrm{Re}\left[\tilde{H}(k) e^{j\pi n(2k+1)/M}\right], M \text{ odd}$$

$$h(n) = \frac{2}{M} \sum_{k=0}^{\frac{M}{2}-1} \mathrm{Re}\left[\tilde{H}(k) e^{j\pi n(2k+1)/M}\right], M \text{ even}$$

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Problem:

1.Design a LPF using frequency sampling technique having cutoff frequency of π/2 rad/sample M=17.
Solution:

$$H_d(w) = \begin{cases} e^{-jw\tau}, 0 \le w \le wc \\ 0, otherwise \end{cases}$$

$$H_d(w) = \begin{cases} e^{-jw8}, 0 \le w \le wc \\ 0, otherwise \end{cases}$$

H(k)=H_d(w)/w= $\frac{2\pi k}{17}$,k=0,1,2,…………16

$$H_d(w) = \begin{cases} e^{-jw8}, 0 \le w \le wc \\ 0, otherwise \end{cases}$$

$$H(k) = \begin{cases} e^{\frac{-j2\pi k8}{17}}, 0 \le \frac{2\pi k}{17} \le \frac{\pi}{2} \\ 0, otherwise \end{cases}$$

$$H(k) = \begin{cases} e^{\frac{-j\pi k16}{17}}, 0 \le k \le \frac{\pi}{2}\frac{17}{2\pi} \\ 0, otherwise \end{cases}$$

$$H(k) = \begin{cases} e^{\frac{-j\pi k16}{17}}, 0 \le k \le 4 \\ 0, otherwise \end{cases}$$

h(n)= $\frac{1}{M}[H(0) + 2\sum_{k=0}^{P} \mathrm{Re}[H(k)e^{\frac{j2\pi kn}{17}}]]$

h(n)= $\frac{1}{17}[H(0) + 2\sum_{k=0}^{4} \mathrm{Re}[e^{\frac{-j16\pi k}{17}} e^{\frac{j2\pi kn}{17}}]]$

h(n)= $\frac{1}{17}[1 + 2\sum_{k=1}^{4} \mathrm{Re}[e^{j2\pi(n-8)/17}]]$

$$h(n) = \frac{1}{17}[1 + 2(\cos(2\pi(n-8)/17) + \cos(2\pi 2(n-8)/17) + \cos(2\pi 3(n-8)/17) + \cos(2\pi 4(n-8)/17)]$$

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n	h(n)
0	0.0397
1	-0.048
2	-0.0345
3	0.0659
4	0.0315
5	-0.1074
6	-0.0299
7	0.31876
8	0.5294
9	0.31876
10	-0.0299
11	-0.1074
12	0.0315
13	0.0659
14	-0.0345
15	-0.0488
16	0.0397

Two Marks Questions and Answers:

1) Compare the digital and analog filter.

Digital filter	Analog filter
i) Operates on digital samples of the signal.	i) Operates on analog signals.
ii) It is governed by linear difference equation.	ii) It is governed by linear difference equation.
iii) It consists of adders, multipliers and delays implemented in digital logic.	iii) It consists of electrical components like resistors, capacitors and inductors.
iv) In digital filters the filter coefficients are designed to satisfy the desired frequency response.	iv) In digital filters the approximation problem is solved to satisfy the desired frequency response.

2.What is filter?

Filter is a frequency selective device , which amplify particular range of frequencies and attenuate particular range of frequencies.

3. What are the different types of filters based on impulse response?

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