UNIT II
ANALYSIS OF CONTINUOUS TIME SIGNALS

INTRODUCTION

The French mathematician Jean Baptiste Joseph Fourier (J.B.J. Fourier) has shown that any periodic non-sinusoidal signal can be expressed as a linear weighted sum of harmonically related sinusoidal signals. This leads to a method called ***Fourier series*** in which a periodic signal is represented as a function of frequency.

The Fourier representation of periodic signals has been extended to non-periodic signals by letting the fundamental period T tend to infinity, and this Fourier method of representing non-periodic signals as a function of frequency is called ***Fourier transform***. The Fourier represention of signals is also known as frequency domain representation. In general, the Fourier series representation can be obtained only for periodic signals, but the Fourier transform technique can be applied to both periodic and non-periodic signals to obtain the frequency domain representation of the signals.

The Fourier representation of signals can be used to perform frequency domain analysis of signals, in which we can study the various frequency components present in the signal, magnitude and phase of various frequency components. The graphical plots of magnitude and phase as a function of frequency are also drawn. The plot of magnitude versus frequency is called ***magnitude spectrum*** and the plot of phase versus frequency is called ***phase spectrum***. In general, these plots are called ***frequency spectrum***.

2.1 FOURIER SERIES

Definition of Trigonometric Form of Fourier Series

The ***trigonometric form of Fourier series*** of a periodic signal, x(t), with period T is defined as,

$$x(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

$$\therefore x(t) = \frac{1}{2}a_0 + a_1 \cos\Omega_0 t + a_2 \cos 2\Omega_0 t + a_3 \cos 3\Omega_0 t + \ldots..$$
$$+ b_1 \sin\Omega_0 t + b_2 \sin 2\Omega_0 t + b_3 \sin 3\Omega_0 t + \ldots\ldots\ldots..$$

where, $\Omega_0 = 2\pi F_0 = \frac{2\pi}{T}$ = Fundamental frequency in rad/sec

F_0 = Fundamental frequency in cycles/sec or Hz

$n\Omega_0$ = Harmonic frequencies

a_0, a_n, b_n = Fourier coefficients of trigonometric form of Fourier series

Note : 1. Here $a_0/2$ is the value of constant component of the signal x(t).
2. The Fourier coefficient a_n and b_n are maximum amplitudes of n^{th} harmonic component

The ***Fourier coefficients*** can be evaluated using the following formulae.

$$a_0 = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\,dt \quad \text{(or)} \quad a_0 = \frac{2}{T}\int_0^T x(t)\,dt \qquad \ldots..(4.2)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\cos n\Omega_0 t\,dt \quad \text{(or)} \quad a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt \qquad \ldots..(4.3)$$

$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\sin n\Omega_0 t\,dt \quad \text{(or)} \quad b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\,dt \qquad \ldots..(4.4)$$

In the above formulae, the limits of integration are either **–T/2 to +T/2** or **0 to T**. In general, the limit of integration is one period of the signal and so the limits can be from t_0 to $t_0 + T$, where t_0 is any time instant.

Conditions for Existence of Fourier Series

The Fourier series exists only if the following Dirichlet's conditions are satisfied.

1. The signal x(t) is well defined and single valued, except possibly at a finite number of points.
2. The signal x(t) must possess only a finite number of discontinuities in the period T.
3. The signal must have a finite number of positive and negative maxima in the period T.

Note : 1. The value of signal x(t) at $t = t_0$ is $x(t_0)$ if $t = t_0$ is a point of continuity.

2. The value of signal x(t) at $t = t_0$ is $\frac{x(t_0^+) + x(t_0^-)}{2}$ if $t = t_0$ is a point of discontinuity.

2.1.1 Exponential Form of Fourier Series

Definition of Exponential Form of Fourier Series

The ***exponential form of Fourier series*** of a periodic signal x(t) with period T is defined as,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}$$

where, $\Omega_0 = 2\pi F_0 = \frac{2\pi}{T}$ = Fundamental frequency in rad/sec

F_0 = Fundamental frequency in cycles/sec or Hz

$\pm n\Omega_0$ = Harmonic frequencies

c_n = Fourier coefficients of exponential form of Fourier series.

The ***Fourier coefficient*** c_n can be evaluated using the following equation.

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, e^{-jn\Omega_0 t}\, dt \quad \text{(or)} \quad c_n = \frac{1}{T}\int_{0}^{T} x(t)\, e^{-jn\Omega_0 t}\, dt$$

In equation (4.10), the limits of integration are either **–T/2 to +T/2** or **0 to T**. In general, the limit of integration is one period of the signal and so the limits can be from t_0 to $t_0 + T$, where t_0 is any time instant.

2.1.2 Relation Between Fourier Coefficients of Trigonometric and Exponential Form

The relation between Fourier coefficients of trigonometric form and exponential form are given below.

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n) \quad \text{for } n = 1, 2, 3, 4, \ldots\ldots$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n) \quad \text{for } -n = -1, -2, -3, -4, \ldots\ldots$$

$$\therefore |c_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} \quad \text{for all values of n, except when } n = 0.$$

2.1.3 Frequency Spectrum(or Line Spectrum) of Periodic Continuous Time Signals

Let x(t) be a periodic continuous time signal. Now, exponential form of Fourier series of x(t) is,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}$$

where, c_n is the Fourier coefficient of n^{th} harmonic component.

The Fourier coefficient, c_n is a complex quantity and so it can be expressed in the polar form as shown below.

$$c_n = |c_n| \angle c_n$$

where, $|c_n|$ = Magnitude of c_n ; $\angle c_n$ = Phase of c_n

The term, $|c_n|$ represents the magnitude of n^{th} harmonic component and the term $\angle c_n$ represents the phase of the n^{th} harmonic component.

The plot of harmonic magnitude / phase of a signal versus "n" (or harmonic frequency $n\Omega_0$) is called ***Frequency spectrum (or Line spectrum)***. The plot of harmonic magnitude versus "n" (or $n\Omega_0$) is called ***magnitude (line) spectrum*** and the plot of harmonic phase versus "n" (or $n\Omega_0$) is called ***phase (line) spectrum***.

Consider the ramp waveform shown in fig 4.1. The Fourier coefficient c_n for this ramp waveform is given by,

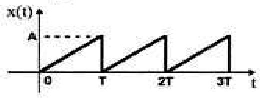


Fig 4.1 : Ramp waveform.

$$c_0 = \frac{A}{2}, \quad c_n = \frac{jA}{2n\pi}$$

(Please refer example 4.12 for derivation of c_n)

$$\text{Let, } A = 20, \qquad \therefore c_n = \frac{j20}{2n\pi} = \frac{j10}{n\pi}$$

$$\vdots$$

$$\text{When } n = -3, \; c_{-3} = -j\frac{10}{3\pi} = -j1.061 = 1.061 \angle -90^\circ = 1.061 \angle -\pi/2$$

$$\text{When } n = -2, \; c_{-2} = -j\frac{10}{2\pi} = -j1.592 = 1.592 \angle -90^\circ = 1.592 \angle -\pi/2$$

$$\text{When } n = -1, \; c_{-1} = -j\frac{10}{\pi} = -j3.183 = 3.183 \angle -90^\circ = 3.183 \angle -\pi/2$$

$$\text{When } n = 0, \; c_0 = \frac{20}{2} = 10 = 10 \angle 0$$

$$\text{When } n = 1, \; c_1 = j\frac{10}{\pi} = j3.183 = 3.183 \angle +90^\circ = 3.183 \angle \pi/2$$

$$\text{When } n = 2, \; c_2 = j\frac{10}{2\pi} = j1.592 = 1.592 \angle +90^\circ = 1.592 \angle \pi/2$$

$$\text{When } n = 3, \; c_3 = j\frac{10}{3\pi} = j1.061 = 1.061 \angle +90^\circ = 1.061 \angle \pi/2$$

$$\vdots$$

Using the above calculated values the magnitude and phase spectrums are sketched as shown in fig

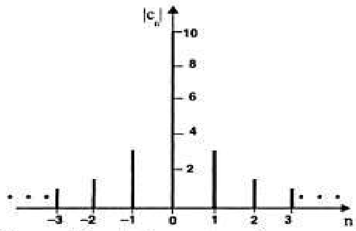


Fig : Magnitude spectrum of ramp waveform.

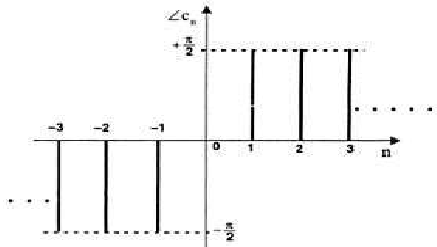


Fig : Phase spectrum of ramp waveform.

2.1.4 Fourier Coefficients of Signals With Symmetry

Even Symmetry

A signal, x(t) is called ***even signal***, if the signal satisfies the condition $x(-t) = x(t)$.

The waveform of an even periodic signal exhibits symmetry with respect to t = 0 (i.e., with respect to vertical axis) and so the symmetry of a waveform with respect to t = 0 or vertical axis is called ***even symmetry***.

Examples of even signals are, $x(t) = 1 + t^2 + t^4 + t^6$ $x(t) = A \cos \Omega_0 t$

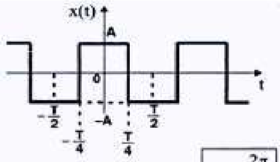
In order to determine the even symmetry of a waveform, fold the waveform with respect to vertical axis. After folding, if the waveshape remains same then it is said to have even symmetry.

For even signals the Fourier coefficient a_0 is optional, a_n exists and b_n are zero. The Fourier coefficient a_0 is zero if the average value of one period is equal to zero. For an even signal the Fourier coefficients are given by,

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt \quad \text{(or)} \quad a_0 = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\,dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt \quad \text{(or)} \quad a_n = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\cos n\Omega_0 t\,dt\ ; \quad b_n = 0$$

The waveform shown in fig has even symmetry, half wave symmetry and quarter wave symmetry.Hence for this waveform, $a_0 = 0$, $b_n = 0$ and a_n exists only for odd values of n. Therefore the Fourier series consists of odd harmonics of cosine terms. The trigonometric Fourier series representation of the waveform of fig 4.4 is given by equation (4.17). [Please refer example 4.1 for the derivation of Fourier series]

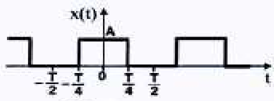


Fig

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{4A}{\pi}\left[\frac{\cos\Omega_0 t}{1} - \frac{\cos 3\Omega_0 t}{3} + \frac{\cos 5\Omega_0 t}{5} - \frac{\cos 7\Omega_0 t}{7} + \frac{\cos 9\Omega_0 t}{9} - \dots\right]$$

The waveform shown in fig has even symmetry and so $b_n = 0$. If the dc component $(a_0/2)$ is substracted from this waveform then it will have half wave and quarter wave symmetry, and so the Fourier series has odd harmonics of cosine terms. The trigonometric Fourier series representation of the waveform of fig 4.5 is given by equation (4.18). [Please refer example 4.3 for the derivation of Fourier series]

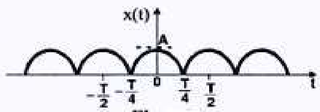


Fig

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{A}{2} + \frac{2A}{\pi}\left[\frac{\cos\Omega_0 t}{1} - \frac{\cos 3\Omega_0 t}{3} + \frac{\cos 5\Omega_0 t}{5} - \frac{\cos 7\Omega_0 t}{7} + \frac{\cos 9\Omega_0 t}{9} - \dots\right]$$

The waveform shown in fig has even symmetry and so $b_n = 0$. The trigonometric Fourier series representation of the waveform of fig is given by equation

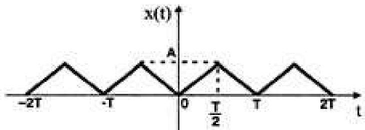


Fig

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{2A}{\pi} + \frac{4A}{\pi}\left[\frac{\cos 2\Omega_0 t}{(2^2 - 1)} - \frac{\cos 4\Omega_0 t}{(4^2 - 1)} + \frac{\cos 6\Omega_0 t}{(6^2 - 1)} - \frac{\cos 8\Omega_0 t}{(8^2 - 1)} + \dots\right]$$

The waveform shown in fig. has even symmetry and so $b_n = 0$. If the dc component $(a_0/2)$ is subtracted from this waveform then it will have half wave and quarter wave symmetry, and so the Fourier series has odd harmonics of cosine terms. The Fourier series representation of the waveform of fig 4.8 is given by equation . [Please refer example 4.2 for the derivation of Fourier series].



Fig

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{A}{2} - \frac{4A}{\pi^2}\left[\frac{\cos\Omega_0 t}{1^2} + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + \frac{\cos 7\Omega_0 t}{7^2} + \dots\right]$$

The waveform shown in fig. has even symmetry and so $b_n = 0$. If the dc component $(a_0/2)$ is subtracted from this waveform then it will have half wave and quarter wave symmetry, and so the Fourier series has odd harmonics of cosine terms. The Fourier series representation of the waveform of fig is given by equation [Please refer example 4.11 for the derivation of Fourier series].

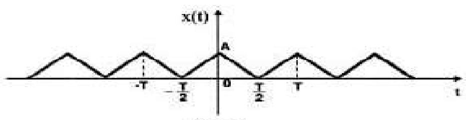


Fig 4.9.

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{A}{2} + \frac{4A}{\pi^2}\left[\frac{\cos\Omega_0 t}{1^2} + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + \frac{\cos 7\Omega_0 t}{7^2} + \dots\right]$$

Odd Symmetry

A signal, x(t) is called ***odd signal*** if it satisfies the condition ***x(–t) = –x(t).***

The waveform of odd periodic signal will exhibit anti-symmetry with respect to t = 0 (i.e., with respect to vertical axis) and so the anti-symmetry of a waveform with respect to t = 0 or vertical axis is called ***odd symmetry.***

> Examples of odd signals are,
> $x(t) = t + t^3 + t^5 + t^7$
> $x(t) = A \sin \Omega_0 t$

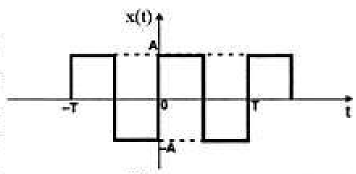
In order to determine the odd symmetry of a waveform, invert either the right side (or the left side) of the waveform with respect to horizontal axis and then fold the waveform with respect to vertical axis. After inverting one half and folding, if the waveshape remains same then it is said to have odd symmetry.

For odd signals a_0 and a_n are zero and b_n exists. For odd signal the Fourier coefficients are given by,

$a_0 = 0$; $a_n = 0$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt \quad \text{or} \quad b_n = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\sin n\Omega_0 t\,dt$$

The waveform shown in fig has odd symmetry, half wave symmetry and quarter wave symmetry. Hence for this waveform, $a_0 = 0$, $a_n = 0$ and b_n exists only for odd values of n. Therefore the Fourier series consists of odd harmonics of sine terms. The trigonometric Fourier series representation of the waveform of fig is given by equation [Please refer example 4.5 for derivation of Fourier series].



Fig

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{4A}{\pi}\left[\sin\Omega_0 t + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 5\Omega_0 t}{5} + \frac{\sin 7\Omega_0 t}{7} + \ldots\right]$$

The waveform shown in fig has odd symmetry, half wave symmetry and quarter wave symmetry. Hence for this waveform, $a_0 = 0$, $a_n = 0$ and b_n exists only for odd values of n. Therefore the Fourier series consists of odd harmonics of sine terms. The trigonometric Fourier series representation of the waveform of fig is given by equation . [Please refer example 4.6 for derivation of Fourier series].

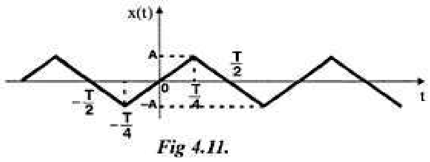


Fig 4.11.

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{8A}{\pi^2}\left[\frac{\sin\Omega_0 t}{1} - \frac{\sin 3\Omega_0 t}{3^2} + \frac{\sin 5\Omega_0 t}{5^2} - \frac{\sin 7\Omega_0 t}{7^2} + \ldots\right]$$

The waveform shown in fig has odd symmetry and so $a_0 = 0$, $a_n = 0$, and b_n exists for all values of n. Hence the Fourier series has both even and odd harmonics of sine terms. The trigonometric Fourier series representation of the waveform of fig is given by equation . [Please refer example for derivation of Fourier series].

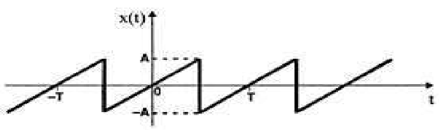
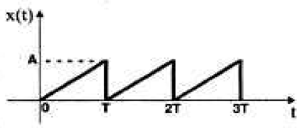


Fig 4.12.

$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{2A}{\pi}\left[\frac{\sin\Omega_0 t}{1} - \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3} - \frac{\sin 4\Omega_0 t}{4} + \frac{\sin 5\Omega_0 t}{5} - \ldots\right]$$

The waveform shown in fig is neither even nor odd. But it can be shown that if the dc component ($a_0/2$) is subtracted from this waveform it becomes odd signal. Hence the Fourier coefficients $a_n = 0$ and b_n exists for all values of n. Therefore the Fourier series has a dc component and all harmonics (both even and odd harmonics) of sine terms. The trigonometric Fourier series representation of the waveform of fig 4.13 is given by equation . [Please refer example 4.8 for derivation of Fourier series].



$\Omega_0 = \frac{2\pi}{T}$

$$x(t) = \frac{A}{2} - \frac{A}{\pi}\left[\frac{\sin\Omega_0 t}{1} + \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 4\Omega_0 t}{4} + \frac{\sin 5\Omega_0 t}{5} + ...\right] \quad .....(4.26)$$

2.1.5 Properties of Exponential Form of Fourier Series Coefficients

Note : c_n and d_n are exponential form of Fourier series coefficients of x(t) and y(t) respectively.

Property	Continuous time periodic signal	Fourier series coefficients
Linearity	$A\,x(t) + B\,y(t)$	$A\,c_n + B\,d_n$
Time shifting	$x(t - t_0)$	$c_n\, e^{-jn\Omega_0 t_0}$
Frequency shifting	$e^{-jm\Omega_0 t}\, x(t)$	c_{n-m}
Conjugation	$x^*(t)$	c^*_{-n}
Time reversal	$x(-t)$	c_{-n}
Time scaling	$x(\alpha t)$; $\alpha > 0$ (x(t) is period with period T/α)	c_n (No change in Fourier coefficient)
Multiplication	$x(t)\, y(t)$	$\sum_{m=-\infty}^{+\infty} c_m\, d_{n-m}$
Differentiation	$\frac{d}{dt}x(t)$	$jn\Omega_0\, c_n$
Integration	$\int_{-\infty}^{t} x(t)\, dt$ (Finite valued and periodic only if $a_0 = 0$)	$\frac{1}{jn\Omega_0}\, c_n$
Periodic convolution	$\int_T x(\tau)\, y(t-\tau)\, d\tau$	$T\, c_n\, d_n$
Symmetry of real signals	x(t) is real	$c_n = c^*_{-n}$ $\lvert c_n\rvert = \lvert c_{-n}\rvert$; $\angle c_n = -\angle c_{-n}$ $Re\{c_n\} = Re\{c_{-n}\}$ $Im\{c_n\} = -Im\{c_{-n}\}$
Real and even	x(t) is real and even	c_n are real and even
Real and odd	x(t) is real and odd	c_n are imaginary and odd
Parseval's relation	Average power, P of x(t) is defined as, $P = \frac{1}{T}\int_T \lvert x(t)\rvert^2\, dt$	The average power, P in terms of Fourier series coefficients is, $P = \sum_{n=-\infty}^{+\infty} \lvert c_n\rvert^2$
	$\frac{1}{T}\int_T \lvert x(t)\rvert^2\, dt = \sum_{n=-\infty}^{+\infty} \lvert c_n\rvert^2$	

Example 4.1

Determine the trigonometric form of Fourier series of the waveform shown in fig 4.1.1.

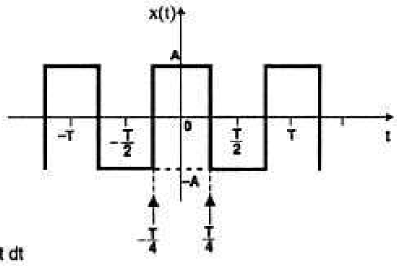


Fig 4.1.1.

Solution

The waveform shown in fig 4.1.1 has even symmetry, half wave symmetry and quarter wave symmetry.

$$\therefore a_0 = 0, \; b_n = 0 \text{ and } a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\, dt$$

The mathematical equation of the square wave is,

$$x(t) = A \;;\; \text{for } t = 0 \text{ to } \frac{T}{4}$$
$$= -A \;;\; \text{for } t = \frac{T}{4} \text{ to } \frac{T}{2}$$

Evaluation of a_n

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\, dt = \frac{4}{T}\int_0^{T/4} A\cos n\Omega_0 t\, dt + \frac{4}{T}\int_{T/4}^{T/2} (-A)\cos n\Omega_0 t\, dt$$

$$= \frac{4A}{T}\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_0^{T/4} - \frac{4A}{T}\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_{T/4}^{T/2} = \frac{4A}{T}\left[\frac{\sin n\frac{2\pi}{T}t}{n\frac{2\pi}{T}}\right]_0^{T/4} - \frac{4A}{T}\left[\frac{\sin n\frac{2\pi}{T}t}{n\frac{2\pi}{T}}\right]_{T/4}^{T/2} \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{4A}{T}\left[\frac{\sin\left(n\frac{2\pi}{T}\frac{T}{4}\right)}{n\frac{2\pi}{T}} - \frac{\sin 0}{n\frac{2\pi}{T}}\right] - \frac{4A}{T}\left[\frac{\sin\left(n\frac{2\pi}{T}\frac{T}{2}\right)}{n\frac{2\pi}{T}} - \frac{\sin\left(n\frac{2\pi}{T}\frac{T}{4}\right)}{n\frac{2\pi}{T}}\right]$$

$$= \frac{4A}{T}\left[\frac{T}{2n\pi}\sin\frac{n\pi}{2} - 0\right] - \frac{4A}{T}\left[\frac{T}{2n\pi}\sin n\pi - \frac{T}{2n\pi}\sin\frac{n\pi}{2}\right] \qquad \boxed{\sin 0 = 0}$$

$$= \frac{2A}{n\pi}\sin\frac{n\pi}{2} + \frac{2A}{n\pi}\sin\frac{n\pi}{2} = \frac{4A}{n\pi}\sin\frac{n\pi}{2} \qquad \boxed{\sin n\pi = 0 \text{ for integer } n}$$

For even values of n, $\sin\frac{n\pi}{2} = 0$

For odd values of n, $\sin\frac{n\pi}{2} = \pm 1$

$$\therefore a_n = 0 \quad ; \text{ for even values of n}$$
$$a_n = \frac{4A}{n\pi}\sin\frac{n\pi}{2} \quad ; \text{ for odd values of n}$$

$$\therefore a_1 = \frac{4A}{1\times\pi}\sin\frac{\pi}{2} = +\frac{4A}{\pi}$$
$$a_3 = \frac{4A}{3\times\pi}\sin\frac{3\pi}{2} = -\frac{4A}{3\pi}$$
$$a_5 = \frac{4A}{5\times\pi}\sin\frac{5\pi}{2} = +\frac{4A}{5\pi}$$
$$a_7 = \frac{4A}{7\times\pi}\sin\frac{7\pi}{2} = -\frac{4A}{7\pi} \text{ and so on.}$$

The trigonometric form of Fourier series of x(t) is,

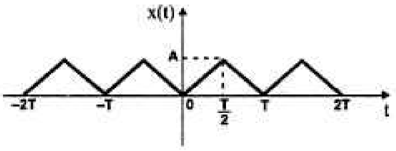
$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $a_0 = 0$, $b_n = 0$ and a_n exists only for odd values of n.

$$\therefore x(t) = \sum_{n=odd} a_n \cos n\Omega_0 t$$
$$= a_1\cos\Omega_0 t + a_3\cos 3\Omega_0 t + a_5\cos 5\Omega_0 t + a_7\cos 7\Omega_0 t + \ldots$$
$$= \frac{4A}{\pi}\cos\Omega_0 t - \frac{4A}{3\pi}\cos 3\Omega_0 t + \frac{4A}{5\pi}\cos 5\Omega_0 t - \frac{4A}{7\pi}\cos 7\Omega_0 t + \ldots$$
$$= \frac{4A}{\pi}\left[\cos\Omega_0 t - \frac{\cos 3\Omega_0 t}{3} + \frac{\cos 5\Omega_0 t}{5} - \frac{\cos 7\Omega_0 t}{7} + \ldots\right]$$

Example

Find the Fourier series of the waveform shown in fig



Solution

The given waveform has even symmetry and so $b_n = 0$

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt \;;\quad a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt \;;\quad b_n = 0$$

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

∴ The equation of straight line can be written as, $\frac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \frac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point-Q = $[t_2, x(t_2)] = \left[\frac{T}{2}, A\right]$

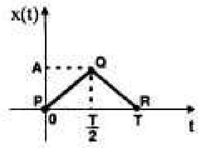


Fig 1.

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - 0}{0 - A} = \frac{t - 0}{0 - \frac{T}{2}} \Rightarrow \frac{x(t)}{-A} = \frac{-2t}{T} \Rightarrow x(t) = \frac{2A}{T}t$$

$$\therefore x(t) = \frac{2A}{T}t \quad ; \quad \text{for } t = 0 \text{ to } \frac{T}{2}$$

Evaluation of a_0

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt = \frac{4}{T}\int_0^{T/2} \frac{2A}{T}t\,dt = \frac{8A}{T^2}\int_0^{T/2} t\,dt$$

$$= \frac{8A}{T^2}\left[\frac{t^2}{2}\right]_0^{T/2} = \frac{8A}{T^2}\left[\frac{T^2}{8} - 0\right] = A$$

Evaluation a_n

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/2} \frac{2A}{T}t\cos n\Omega_0 t\,dt = \frac{8A}{T^2}\int_0^{T/2} t\cos n\Omega_0 t\,dt$$

$$= \frac{8A}{T^2}\left[t\frac{\sin n\Omega_0 t}{n\Omega_0} - \int 1 \times \left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right)dt\right]_0^{T/2}$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = t$	$v = \cos\Omega_0 t$

$$= \frac{8A}{T^2}\left[\frac{t\sin n\Omega_0 t}{n\Omega_0} - \left(\frac{-\cos n\Omega_0 t}{n^2\Omega_0^2}\right)\right]_0^{T/2} = \frac{8A}{T^2}\left[\frac{t\sin n\frac{2\pi}{T}t}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}t}{n^2\frac{4\pi^2}{T^2}}\right]_0^{T/2}$$

$\Omega_0 = \frac{2\pi}{T}$

$$= \frac{8A}{T^2}\left[\frac{\frac{T}{2}\sin n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}\frac{T}{2}}{n^2\frac{4\pi^2}{T^2}} - \frac{0 \times \sin 0}{n\frac{2\pi}{T}} - \frac{\cos 0}{n^2\frac{4\pi^2}{T^2}}\right]$$

$\sin 0 = 0$
$\cos 0 = 1$

$$= \frac{8A}{T^2}\left[\frac{T^2}{4n\pi}\sin n\pi + \frac{T^2}{4n^2\pi^2}\cos n\pi - \frac{T^2}{4n^2\pi^2}\right] = \frac{2A}{n^2\pi^2}[\cos n\pi - 1]$$

$\sin n\pi = 0$ for integer values of n

For even integer values of n, $\cos n\pi = +1$

For odd integer values of n, $\cos n\pi = -1$

∴ $a_n = 0$; for even values of n, and

$$a_n = \frac{2A}{n^2\pi^2}[\cos n\pi - 1] = -\frac{4A}{n^2\pi^2} \quad ; \quad \text{for odd values of n.}$$

$$\therefore a_1 = -\frac{4A}{1^2\pi^2} \quad ; \quad a_3 = -\frac{4A}{3^2\pi^2} \quad ; \quad a_5 = -\frac{4A}{5^2\pi^2} \quad ; \quad \text{and so on.}$$

The trigonometric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $b_n = 0$, and a_n exists only for odd values of n.

$$\therefore x(t) = \frac{a_0}{2} + \sum_{n=\text{odd}} a_n \cos n\Omega_0 t$$

$$= \frac{a_0}{2} + a_1 \cos\Omega_0 t + a_3 \cos 3\Omega_0 t + a_5 \cos 5\Omega_0 t + \ldots\ldots$$

$$= \frac{A}{2} - \frac{4A}{1^2\pi^2}\cos\Omega_0 t - \frac{4A}{3^2\pi^2}\cos 3\Omega_0 t - \frac{4A}{5^2\pi^2}\cos 5\Omega_0 t - \ldots\ldots$$

$$= \frac{A}{2} - \frac{4A}{\pi^2}\left[\cos\Omega_0 t + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + \ldots\ldots\right]$$

Example 4.3

Determine the trigonometric form of Fourier series of the waveform shown in fig 4.3.1.

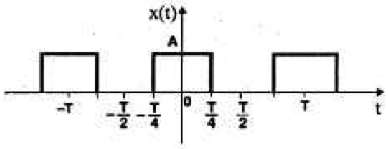


Fig 4.3.1.

Solution

The waveform of fig 4.3.1 has even symmetry.

$$\therefore b_n = 0, \quad a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt \;;\quad a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

The mathematical equation of the given periodic rectangular pulse is,

$$x(t) = A \;;\quad \text{for } t = 0 \text{ to } \frac{T}{4}$$

$$= 0 \;;\quad \text{for } t = \frac{T}{4} \text{ to } \frac{T}{2}$$

Evaluation of a_0

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt = \frac{4}{T}\int_0^{T/4} A\,dt = \frac{4}{T}\left[At\right]_0^{T/4}$$

$$= \frac{4}{T}\left[A\frac{T}{4} - 0\right] = A$$

Evaluation of a_n

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/4} A\cos n\Omega_0 t\,dt = \frac{4A}{T}\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_0^{T/4}$$

$$= \frac{4A}{T}\left[\frac{\sin n\frac{2\pi}{T}t}{n\frac{2\pi}{T}}\right]_0^{T/4} = \frac{4A}{T}\left[\frac{\sin n\frac{2\pi}{T}\frac{T}{4}}{n\frac{2\pi}{T}} - \frac{\sin 0}{n\frac{2\pi}{T}}\right]$$

$$\Omega_0 = \frac{2\pi}{T}, \quad \sin 0 = 0$$

$$= \frac{4A}{T} \times \frac{T}{2n\pi}\sin\frac{n\pi}{2} = \frac{2A}{n\pi}\sin\frac{n\pi}{2}$$

For even values of n, $\sin\frac{n\pi}{2} = 0$

For odd values of n, $\sin\frac{n\pi}{2} = \pm 1$

$\therefore a_n = 0$; for even values of n, and

$$a_n = \frac{2A}{n\pi}\sin\frac{n\pi}{2} \;;\quad \text{for odd values of n.}$$

$$\therefore a_1 = \frac{2A}{1\times\pi}\sin\frac{\pi}{2} = +\frac{2A}{\pi}$$

$$a_3 = \frac{2A}{3\times\pi}\sin\frac{3\pi}{2} = -\frac{2A}{3\pi}$$

$$a_5 = \frac{2A}{5\times\pi}\sin\frac{5\pi}{2} = +\frac{2A}{5\pi}$$

$$a_7 = \frac{2A}{7\times\pi}\sin\frac{7\pi}{2} = -\frac{2A}{7\pi} \quad \text{and so on.}$$

Fourier Series

The trigonomteric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $b_n = 0$ and a_n exists only for odd values of n.

$$\therefore x(t) = \frac{a_0}{2} + \sum_{n=\text{odd}} a_n \cos n\Omega_0 t$$

$$= \frac{a_0}{2} + a_1\cos\Omega_0 t + a_3\cos 3\Omega_0 t + a_5 \cos 5\Omega_0 t + a_7\cos 7\Omega_0 t + ...$$

$$= \frac{A}{2} + \frac{2A}{\pi}\cos\Omega_0 t - \frac{2A}{3\pi}\cos 3\Omega_0 t + \frac{2A}{5\pi}\cos 5\Omega_0 t - \frac{2A}{7\pi}\cos 7\Omega_0 t + ...$$

$$= \frac{A}{2} + \frac{2A}{\pi}\left[\cos\Omega_0 t - \frac{\cos 3\Omega_0 t}{3} + \frac{\cos 5\Omega_0 t}{5} - \frac{\cos 7\Omega_0 t}{7} + ...\right]$$

Example 4.4

Determine the trigonometric form of Fourier series of the full wave rectified sine wave shown in fig 4.4.1.

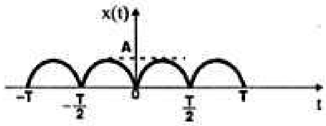


Fig 4.4.1.

Solution

The waveform shown in fig 4.4.1 is the output of full wave rectifier and it has even symmetry.

$$\therefore b_n = 0,\quad a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt \;;\quad a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

The mathematical equation of full wave rectified output is,

$$x(t) = A\sin\Omega_0 t \;;\quad \text{for } t = 0 \text{ to } \frac{T}{2} \quad \text{and} \quad \Omega_0 = \frac{2\pi}{T}$$

Evaluation of a_0

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt = \frac{4}{T}\int_0^{T/2} A\sin\Omega_0 t\,dt = \frac{4A}{T}\left[-\frac{\cos\Omega_0 t}{\Omega_0}\right]_0^{T/2}$$

$$= \frac{4A}{T}\left[-\frac{\cos\frac{2\pi}{T}t}{\frac{2\pi}{T}}\right]_0^{T/2} = \frac{4A}{T}\left[-\frac{\cos\frac{2\pi}{T}\frac{T}{2}}{\frac{2\pi}{T}} + \frac{\cos 0}{\frac{2\pi}{T}}\right] \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{2A}{\pi}\left[-\cos\pi + \cos 0\right] = \frac{2A}{\pi}[1+1] = \frac{4A}{\pi} \qquad \boxed{\cos\pi = -1} \; \boxed{\cos 0 = 1}$$

Evaluation of a_n

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/2} A\sin\Omega_0 t\cos n\Omega_0 t\,dt$$

$$= \frac{4A}{T}\int_0^{T/2}\frac{\sin(\Omega_0 t + n\Omega_0 t) + \sin(\Omega_0 t - n\Omega_0 t)}{2}\,dt \qquad \boxed{2\sin A\cos B = \sin(A+B) + \sin(A-B)}$$

$$= \frac{2A}{T}\int_0^{T/2} \sin(1+n)\Omega_0 t\,dt + \frac{2A}{T}\int_0^{T/2} \sin(1-n)\Omega_0 t\,dt$$

$$= \frac{2A}{T}\left[\frac{-\cos(1+n)\Omega_0 t}{(1+n)\Omega_0}\right]_0^{T/2} + \frac{2A}{T}\left[\frac{-\cos(1-n)\Omega_0 t}{(1-n)\Omega_0}\right]_0^{T/2}$$

$$= \frac{2A}{T}\left[\frac{-\cos(1+n)\frac{2\pi}{T}t}{(1+n)\frac{2\pi}{T}}\right]_0^{T/2} + \frac{2A}{T}\left[\frac{-\cos(1-n)\frac{2\pi}{T}t}{(1-n)\frac{2\pi}{T}}\right]_0^{T/2} \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{2A}{T}\left[\frac{-\cos(1+n)\frac{2\pi}{T}\frac{T}{2}}{(1+n)\frac{2\pi}{T}} + \frac{\cos 0}{(1+n)\frac{2\pi}{T}}\right] + \frac{2A}{T}\left[\frac{-\cos(1-n)\frac{2\pi}{T}\frac{T}{2}}{(1-n)\frac{2\pi}{T}} + \frac{\cos 0}{(1-n)\frac{2\pi}{T}}\right] \qquad \boxed{\cos 0 = 1}$$

$$= -\frac{A\cos(1+n)\pi}{(1+n)\pi} + \frac{A}{(1+n)\pi} - \frac{A\cos(1-n)\pi}{(1-n)\pi} + \frac{A}{(1-n)\pi} \qquad .....(1)$$

The equation (1) for a_n can be evaluated for all values of n except n = 1. For n = 1, a_n has to be estimated seperately as shown below.

$$a_1 = \frac{4}{T}\int_0^{T/2} x(t)\cos\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/2} A\sin\Omega_0 t\cos\Omega_0 t\,dt$$

$$= \frac{4}{T}\int_0^{T/2} A\frac{\sin 2\Omega_0 t}{2}\,dt = \frac{2A}{T}\int_0^{T/2}\sin 2\Omega_0 t\,dt \qquad \boxed{\sin 2\theta = 2\sin\theta\cos\theta}$$

$$= \frac{2A}{T}\left[\frac{-\cos 2\Omega_0 t}{2\Omega_0}\right]_0^{T/2} = \frac{2A}{T}\left[\frac{-\cos\left(2\times\frac{2\pi}{T}\times\frac{T}{2}\right)}{2\times\frac{2\pi}{T}} + \frac{\cos 0}{2\times\frac{2\pi}{T}}\right] \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{2A}{T}\left[-\frac{T}{4\pi}\cos 2\pi + \frac{T}{4\pi}\right] = \frac{2A}{T}\left[-\frac{T}{4\pi} + \frac{T}{4\pi}\right] = 0 \qquad \boxed{\cos 2\pi = \cos 0 = 1}$$

For values of n > 1, the a_n are calculated using equation (1) as shown below.

$$\therefore a_n = -\frac{A\cos(1+n)\pi}{(1+n)\pi} + \frac{A}{(1+n)\pi} - \frac{A\cos(1-n)\pi}{(1-n)\pi} + \frac{A}{(1-n)\pi}$$

When n is even integer, (1 + n) and (1 − n) will be odd, $\therefore \cos(1+n)\pi = -1 \;;\; \cos(1-n)\pi = -1$

When n is odd integer, (1 + n) and (1 − n) will be even, $\therefore \cos(1+n)\pi = 1 \;;\; \cos(1-n)\pi = 1$

$\therefore a_n = 0$; for odd values of n

$$a_n = \frac{A}{(1+n)\pi} + \frac{A}{(1+n)\pi} + \frac{A}{(1-n)\pi} + \frac{A}{(1-n)\pi} \;;\; \text{for even values of n}$$

$$= \frac{2A}{(1+n)\pi} + \frac{2A}{(1-n)\pi} = \frac{2A(1-n) + 2A(1+n)}{(1+n)(1-n)\pi} = \frac{4A}{(1-n^2)\pi}$$

$$\therefore a_2 = \frac{4A}{(1-2^2)\pi} = -\frac{4A}{3\pi}$$

$$a_4 = \frac{4A}{(1-4^2)\pi} = -\frac{4A}{15\pi}$$

$$a_6 = \frac{4A}{(1-6^2)\pi} = -\frac{4A}{35\pi}$$

$$a_8 = \frac{4A}{(1-8^2)\pi} = -\frac{4A}{63\pi} \quad \text{and so on}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $b_n = 0$, and a_n exists only for even values of n.

$$\therefore x(t) = \frac{a_0}{2} + \sum_{n=\text{even}} a_n \cos n\Omega_0 t$$

$$= \frac{a_0}{2} + a_2 \cos 2\Omega_0 t + a_4 \cos 4\Omega_0 t + a_6 \cos 6\Omega_0 t + a_8 \cos 8\Omega_0 t + ...$$

$$= \frac{2A}{\pi} - \frac{4A}{3\pi}\cos 2\Omega_0 t - \frac{4A}{15\pi}\cos 4\Omega_0 t - \frac{4A}{35\pi}\cos 6\Omega_0 t - \frac{4A}{63\pi}\cos 8\Omega_0 t - .....$$

$$= \frac{2A}{\pi} - \frac{4A}{\pi}\left[\frac{\cos 2\Omega_0 t}{3} + \frac{\cos 4\Omega_0 t}{15} + \frac{\cos 6\Omega_0 t}{35} + \frac{\cos 8\Omega_0 t}{63} + ...\right]$$

Example 4.5

Determine the Fourier series of the square wave shown in fig 4.5.1.

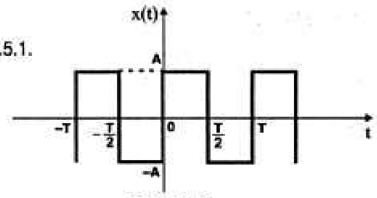


Fig 4.5.1.

Solution

The given waveform has odd symmetry, half-wave symmetry and quarter wave symmetry.

$$\therefore a_0 = 0, \quad a_n = 0, \quad b_n = \frac{4}{T}\int_0^{T/2} x(t) \sin n\Omega_0 t \, dt$$

The mathematical equation of the given waveform is,

$$x(t) = A \quad ; \quad \text{for } t = 0 \text{ to } \frac{T}{2}$$

$$= -A \quad ; \quad \text{for } t = \frac{T}{2} \text{ to } T$$

Evaluation of b_n

$$b_n = \frac{4}{T}\int_0^{T/2} x(t) \sin n\Omega_0 t \, dt = \frac{4}{T}\int_0^{T/2} A \sin n\Omega_0 t \, dt = \frac{4A}{T}\left[\frac{-\cos n\Omega_0 t}{n\Omega_0}\right]_0^{T/2}$$

$$\boxed{\Omega_0 = \frac{2\pi}{T}, \quad \cos 0 = 1}$$

$$= \frac{4A}{T}\left[\frac{-\cos n\frac{2\pi}{T}t}{n\frac{2\pi}{T}}\right]_0^{T/2} = \frac{4A}{T}\left[\frac{-\cos n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} + \frac{\cos 0}{n\frac{2\pi}{T}}\right] = \frac{4A}{T}\left[-\frac{T}{2n\pi}\cos n\pi + \frac{T}{2n\pi}\right]$$

$\cos n\pi = -1$, for n = odd

$\cos n\pi = +1$, for n = even

$$\therefore b_n = 0 \quad ; \text{ for even values of n}$$

$$= \frac{4A}{T}\left[\frac{T}{2n\pi} + \frac{T}{2n\pi}\right] = \frac{4A}{n\pi}; \text{ for odd values of n}$$

$$\therefore b_1 = \frac{4A}{\pi} \quad ; \quad b_3 = \frac{4A}{3\pi} \quad ; \quad b_5 = \frac{4A}{5\pi} \quad \text{and so on.}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $a_0 = 0$, $a_n = 0$ and b_n exists only for odd values of n.

$$\therefore x(t) = \sum_{n=\text{odd}} b_n \sin n\Omega_0 t$$

$$= b_1 \sin \Omega_0 t + b_3 \sin 3\Omega_0 t + b_5 \sin 5\Omega_0 t + ...$$

$$= \frac{4A}{\pi}\sin \Omega_0 t + \frac{4A}{3\pi}\sin 3\Omega_0 t + \frac{4A}{5\pi}\sin 5\Omega_0 t + ...$$

$$= \frac{4A}{\pi}\left[\sin \Omega_0 t + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 5\Omega_0 t}{5} + ...\right]$$

Example 4.6

Determine the trigonometric form of Fourier series of the signal shown in fig 4.6.1.

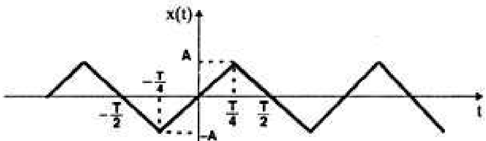


Fig 4.6.1.

Solution

The given signal has odd symmetry, half wave symmetry and quarter wave symmetry, and so $a_0 = 0$, $a_n = 0$,

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\, dt \quad \text{(or)} \quad b_n = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\sin n\Omega_0 t\, dt$$

> ***Note :** Here x(t) is governed by single mathematical equation in the range $-\frac{T}{4}$ to $+\frac{T}{4}$. And so the calculations will be simple, if the integral limit is $-\frac{T}{4}$ to $+\frac{T}{4}$*

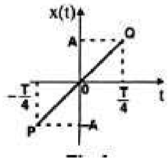
To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\frac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \frac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.



Coordinates of point-P = $[t_1, x(t_1)] = \left[-\frac{T}{4}, -A\right]$

Coordinates of point-Q = $[t_2, x(t_2)] = \left[\frac{T}{4}, A\right]$

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - (-A)}{-A - A} = \frac{t - \left(-\frac{T}{4}\right)}{-\frac{T}{4} - \frac{T}{4}} \Rightarrow \frac{x(t) + A}{-2A} = \frac{t + \frac{T}{4}}{-\frac{T}{2}}$$

$$\Downarrow$$

$$-\frac{x(t)}{2A} - \frac{1}{2} = -\frac{2t}{T} - \frac{1}{2} \Rightarrow -\frac{x(t)}{2A} = -\frac{2t}{T} \Rightarrow x(t) = \frac{4A}{T}t$$

$$\therefore x(t) = \frac{4A}{T}t \; ; \quad \text{for } t = -\frac{T}{4} \text{ to } +\frac{T}{4}$$

Evaluation of b_n

$$b_n = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\sin n\Omega_0 t\, dt = \frac{4}{T}\int_{-T/4}^{+T/4} \frac{4A}{T} t\sin n\Omega_0 t\, dt = \frac{16A}{T^2}\int_{-T/4}^{+T/4} t\sin n\Omega_0 t\, dt$$

$$= \frac{16A}{T^2}\left[t\left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) - \int 1 \times \left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) dt\right]_{-T/4}^{+T/4}$$

$\int uv = u\int v - \int\left[du \int v\right]$	
$u = t$	$v = \sin n\Omega_0 t$

$$= \frac{16A}{T^2}\left[-t\frac{\cos n\Omega_0 t}{n\Omega_0} + \frac{\sin n\Omega_0 t}{n^2\Omega_0^2}\right]_{-T/4}^{T/4} = \frac{16A}{T^2}\left[-t\frac{\cos n\frac{2\pi}{T}t}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}t}{n^2\frac{4\pi^2}{T^2}}\right]_{-T/4}^{+T/4}$$

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{16A}{T^2}\left[-\frac{T}{4}\frac{\cos n\frac{2\pi}{T}\frac{T}{4}}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}\frac{T}{4}}{n^2\frac{4\pi^2}{T^2}} + \frac{T}{4}\frac{\cos n\frac{2\pi}{T}\left(-\frac{T}{4}\right)}{n\frac{2\pi}{T}} - \frac{\sin n\frac{2\pi}{T}\left(-\frac{T}{4}\right)}{n^2\frac{4\pi^2}{T^2}}\right]$$

$$b_n = \frac{16A}{T^2}\left[-\frac{T^2}{8n\pi}\cos\frac{n\pi}{2} + \frac{T^2}{4n^2\pi^2}\sin\frac{n\pi}{2} + \frac{T^2}{8n\pi}\cos\frac{n\pi}{2} + \frac{T^2}{4n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= -\frac{2A}{n\pi}\cos\frac{n\pi}{2} + \frac{4A}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2A}{n\pi}\cos\frac{n\pi}{2} + \frac{4A}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \frac{8A}{n^2\pi^2}\sin\frac{n\pi}{2}$$

For odd integer values of n, $\sin\frac{n\pi}{2} = \pm 1$

For even integer values of n, $\sin\frac{n\pi}{2} = 0$

$\therefore b_n = 0$; for even values of n

$= \frac{8A}{n^2\pi^2} \sin\frac{n\pi}{2}$; for odd values of n

$$\therefore b_1 = \frac{8A}{1^2\pi^2} \sin\frac{\pi}{2} = +\frac{8A}{\pi^2}$$

$$b_3 = \frac{8A}{3^2\pi^2} \sin\frac{3\pi}{2} = -\frac{8A}{3^2\pi^2}$$

$$b_5 = \frac{8A}{5^2\pi^2} \sin\frac{5\pi}{2} = +\frac{8A}{5^2\pi^2}$$

$$b_7 = \frac{8A}{7^2\pi^2} \sin\frac{7\pi}{2} = -\frac{8A}{7^2\pi^2} \quad \text{and so on.}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is given by,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $a_0 = 0$, $a_n = 0$ and b_n exists only for odd values of n.

$$\therefore x(t) = \sum_{n=\text{odd}} b_n \sin n\Omega_0 t$$

$$= b_1 \sin\Omega_0 t + b_3 \sin 3\Omega_0 t + b_5 \sin 5\Omega_0 t + b_7 \sin 7\Omega_0 t + ...$$

$$= \frac{8A}{\pi^2} \sin\Omega_0 t - \frac{8A}{3^2\pi^2} \sin 3\Omega_0 t + \frac{8A}{5^2\pi^2} \sin 5\Omega_0 t - \frac{8A}{7^2\pi^2} \sin 7\Omega_0 t + ...$$

$$= \frac{8A}{\pi^2}\left[\sin\Omega_0 t - \frac{\sin 3\Omega_0 t}{3^2} + \frac{\sin 5\Omega_0 t}{5^2} - \frac{\sin 7\Omega_0 t}{7^2} + ...\right]$$

Example 4.11

Determine the exponential form of the Fourier series representation of the signal shown in fig 4.11.1. Hence determine the trigonometric form of Fourier series.

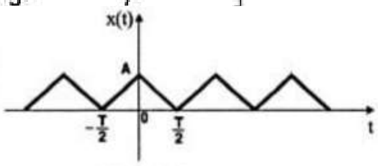


Fig 4.11.1.

Solution

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\frac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \frac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P, Q and R as shown in fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = \left[\frac{-T}{2}, 0\right]$

Coordinates of point-Q = $[t_2, x(t_2)] = [0, A]$

Coordinates of point-R = $[t_3, x(t_3)] = \left[\frac{T}{2}, 0\right]$

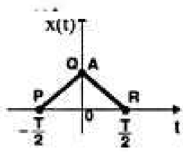


Fig 1.

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - 0}{0 - A} = \frac{t + \frac{T}{2}}{\frac{-T}{2} - 0} \Rightarrow \frac{x(t)}{-A} = \frac{-2t}{T} - 1 \Rightarrow x(t) = A + \frac{2At}{T}$$

On substituting the coordinates of points Q and R in equation (1) we get,

$$\frac{x(t) - A}{A - 0} = \frac{t - 0}{0 - \frac{T}{2}} \Rightarrow \frac{x(t)}{A} - 1 = \frac{-2t}{T} \Rightarrow x(t) = A - \frac{2At}{T}$$

$$\therefore x(t) = A + \frac{2At}{T} \quad ; \quad \text{for } t = -\frac{T}{2} \text{ to } 0$$
$$= A - \frac{2At}{T} \quad ; \quad \text{for } t = 0 \text{ to } \frac{T}{2}$$

Evaluation of c_0

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, e^{-jn\Omega_0 t}\, dt$$

$$\text{When } n = 0,\ c_0 = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, e^{0}\, dt = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, dt$$
$$= \frac{1}{T}\int_{-T/2}^{0}\left(A + \frac{2At}{T}\right) dt + \frac{1}{T}\int_{0}^{+T/2}\left(A - \frac{2At}{T}\right) dt$$

$$\therefore c_0 = \frac{A}{T}\int_{-T/2}^{0} dt + \frac{2A}{T^2}\int_{-T/2}^{0} t\, dt + \frac{A}{T}\int_{0}^{T/2} dt - \frac{2A}{T^2}\int_{0}^{T/2} t\, dt$$
$$= \frac{A}{T}[t]_{-T/2}^{0} + \frac{2A}{T^2}\left[\frac{t^2}{2}\right]_{-T/2}^{0} + \frac{A}{T}[t]_0^{T/2} - \frac{2A}{T^2}\left[\frac{t^2}{2}\right]_0^{T/2}$$
$$= \frac{A}{T}\left[0 + \frac{T}{2}\right] + \frac{2A}{T^2}\left[0 - \frac{T^2}{8}\right] + \frac{A}{T}\left[\frac{T}{2} - 0\right] - \frac{2A}{T^2}\left[\frac{T^2}{8} - 0\right]$$
$$= \frac{A}{2} - \frac{A}{4} + \frac{A}{2} - \frac{A}{4} = \frac{2A}{2} - \frac{2A}{4} = A - \frac{A}{2} = \frac{A}{2}$$

Evaluation of c_n

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, e^{-jn\Omega_0 t}\, dt$$
$$= \frac{1}{T}\int_{-T/2}^{0}\left(A + \frac{2At}{T}\right) e^{-jn\Omega_0 t}\, dt + \frac{1}{T}\int_{0}^{T/2}\left(A - \frac{2At}{T}\right) e^{-jn\Omega_0 t}\, dt$$
$$= \frac{A}{T}\int_{-T/2}^{0} e^{-jn\Omega_0 t}\, dt + \frac{2A}{T^2}\int_{-T/2}^{0} t\, e^{-jn\Omega_0 t}\, dt + \frac{A}{T}\int_{0}^{T/2} e^{-jn\Omega_0 t}\, dt - \frac{2A}{T^2}\int_{0}^{T/2} t\, e^{-jn\Omega_0 t}\, dt$$
$$= \frac{A}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{-T/2}^{0} + \frac{2A}{T^2}\left[t\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0} - \int 1 \times \frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\, dt\right]_{-T/2}^{0}$$

$\int uv = u\int v - \int\left[du \int v\right]$	
$u = t$	$v = e^{-jn\Omega_0 t}$

$$+ \frac{A}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{0}^{T/2} - \frac{2A}{T^2}\left[t\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0} - \int 1 \times \frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\, dt\right]_{0}^{T/2}$$

$$= \frac{A}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{-T/2}^{0} + \frac{2A}{T^2}\left[t\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0} - \frac{e^{-jn\Omega_0 t}}{(-jn\Omega_0)^2}\right]_{-T/2}^{0} + \frac{A}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{0}^{T/2}$$
$$- \frac{2A}{T^2}\left[\frac{t\, e^{-jn\Omega_0 t}}{-jn\Omega_0} - \frac{e^{-jn\Omega_0 t}}{(-jn\Omega_0)^2}\right]_{0}^{T/2}$$

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{A}{T}\left[\frac{e^{0}}{-jn\frac{2\pi}{T}} - \frac{e^{-jn\frac{2\pi}{T}\left(-\frac{T}{2}\right)}}{-jn\frac{2\pi}{T}}\right] + \frac{2A}{T^2}\left[\frac{0 \times e^{0}}{-jn\frac{2\pi}{T}} - \frac{e^{0}}{-n^2\frac{4\pi^2}{T^2}} + \frac{T}{2}\frac{e^{-jn\frac{2\pi}{T}\left(-\frac{T}{2}\right)}}{-jn\frac{2\pi}{T}} + \frac{e^{-jn\frac{2\pi}{T}\left(-\frac{T}{2}\right)}}{-n^2\frac{4\pi^2}{T^2}}\right]$$
$$+ \frac{A}{T}\left[\frac{e^{-jn\frac{2\pi}{T}\frac{T}{2}}}{-jn\frac{2\pi}{T}} - \frac{e^{0}}{-jn\frac{2\pi}{T}}\right] - \frac{2A}{T^2}\left[\frac{T}{2}\frac{e^{-jn\frac{2\pi}{T}\frac{T}{2}}}{-jn\frac{2\pi}{T}} - \frac{e^{-jn\frac{2\pi}{T}\frac{T}{2}}}{-n^2\frac{4\pi^2}{T^2}} - \frac{0 \times e^{0}}{-jn\frac{2\pi}{T}} + \frac{e^{0}}{-n^2\frac{4\pi^2}{T^2}}\right]$$

$$= -\frac{A}{j2n\pi} + \frac{A\,e^{jn\pi}}{j2n\pi} - 0 + \frac{A}{2n^2\pi^2} - \frac{A\,e^{jn\pi}}{j2n\pi} - \frac{A\,e^{jn\pi}}{2n^2\pi^2} - \frac{A\,e^{-jn\pi}}{j2n\pi}$$

$$+ \frac{A}{j2n\pi} + \frac{A\,e^{-jn\pi}}{j2n\pi} - \frac{A\,e^{-jn\pi}}{2n^2\pi^2} - 0 + \frac{A}{2n^2\pi^2}$$

$$= \frac{A}{n^2\pi^2} - \frac{A\,e^{jn\pi}}{2n^2\pi^2} - \frac{A\,e^{-jn\pi}}{2n^2\pi^2}$$

We know that,

$$e^{\pm jn\pi} = \cos n\pi \pm j\sin n\pi$$
$$= +1 \pm j0 = 1 \quad ; \quad \text{for even } n$$
$$= -1 \pm j0 = -1 \quad ; \quad \text{for odd } n.$$

$\therefore$ When n is even,

$$c_n = \frac{A}{n^2\pi^2} - \frac{A}{2n^2\pi^2} - \frac{A}{2n^2\pi^2} = \frac{A}{n^2\pi^2} - \frac{A}{n^2\pi^2} = 0$$

$\therefore$ When n is odd,

$$c_n = \frac{A}{n^2\pi^2} + \frac{A}{2n^2\pi^2} + \frac{A}{2n^2\pi^2} = \frac{A}{n^2\pi^2} + \frac{A}{n^2\pi^2} = \frac{2A}{n^2\pi^2}$$

$$\therefore c_{-1} = \frac{2A}{(-1)^2\pi^2} = \frac{2A}{1^2\pi^2} \qquad c_1 = \frac{2A}{1^2\pi^2}$$

$$c_{-3} = \frac{2A}{(-3)^2\pi^2} = \frac{2A}{3^2\pi^2} \qquad c_3 = \frac{2A}{3^2\pi^2}$$

$$c_{-5} = \frac{2A}{(-5)^2\pi^2} = \frac{2A}{5^2\pi^2} \qquad c_5 = \frac{2A}{5^2\pi^2}$$

and so on | and so on

Exponential Form of Fourier Series

The exponential form of Fourier series is,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n\,e^{jn\Omega_0 t} = \sum_{n=-\infty}^{-1} c_n\,e^{jn\Omega_0 t} + c_0 + \sum_{n=1}^{\infty} c_n\,e^{jn\Omega_0 t}$$

Here c_n exist only for odd values of n.

$$\therefore x(t) = \sum_{\substack{n = \text{negative}\\ \text{odd integer}}} c_n\,e^{jn\Omega_0 t} + c_0 + \sum_{\substack{n = \text{positive}\\ \text{odd integer}}} c_n\,e^{jn\Omega_0 t}$$

$$= \ldots + c_{-5}\,e^{-j5\Omega_0 t} + c_{-3}\,e^{-j3\Omega_0 t} + c_{-1}\,e^{-j\Omega_0 t} + c_0 + c_1\,e^{j\Omega_0 t} + c_3\,e^{j3\Omega_0 t} + c_5\,e^{j5\Omega_0 t} + \ldots$$

$$x(t) = \ldots + \frac{2A}{5^2\pi^2}e^{-j5\Omega_0 t} + \frac{2A}{3^2\pi^2}e^{-j3\Omega_0 t} + \frac{2A}{1^2\pi^2}e^{-j\Omega_0 t} + \frac{A}{2} + \frac{2A}{1^2\pi^2}e^{j\Omega_0 t}$$

$$+ \frac{2A}{3^2\pi^2}e^{j3\Omega_0 t} + \frac{2A}{5^2\pi^2}e^{j5\Omega_0 t} + \ldots$$

$$= \frac{2A}{\pi^2}\left(\ldots + \frac{1}{5^2}e^{-j5\Omega_0 t} + \frac{1}{3^2}e^{-j3\Omega_0 t} + \frac{1}{1^2}e^{-j\Omega_0 t}\right) + \frac{A}{2}$$

$$+ \frac{2A}{\pi^2}\left(\frac{1}{1^2}e^{j\Omega_0 t} + \frac{1}{3^2}e^{j3\Omega_0 t} + \frac{1}{5^2}e^{j5\Omega_0 t} + \ldots\right)$$

Trigonometric Form of Fourier Series

The trigonometric form of Fourier series can be obtained as shown below.

$$x(t) = \frac{A}{2} + \frac{2A}{\pi^2}\left[\frac{1}{1^2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right) + \frac{1}{3^2}\left(e^{j3\Omega_0 t} + e^{-j3\Omega_0 t}\right) + \frac{1}{5^2}\left(e^{j5\Omega_0 t} + e^{-j5\Omega_0 t}\right) + ...\right]$$

$$= \frac{A}{2} + \frac{2A}{\pi^2}\left[\frac{1}{1^2}\, 2\cos\Omega_0 t + \frac{1}{3^2}\, 2\cos 3\Omega_0 t + \frac{1}{5^2}\, 2\cos 5\Omega_0 t + ...\right]$$

$$= \frac{A}{2} + \frac{4A}{\pi^2}\left[\frac{\cos\Omega_0 t}{1^2} + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + ...\right] \qquad \boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

Example 4.12

Determine the exponential form of the Fourier series representation of the signal shown in fig 4.12.1. Hence determine the trigonometric form of Fourier series.

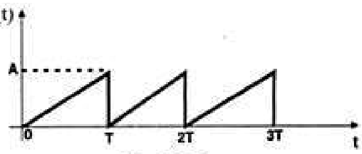


Fig 4.12.1.

Solution

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\frac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \frac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point-Q = $[t_2, x(t_2)] = [T, A]$

On substituting the coordinates of points P and Q in equation (1) we get,

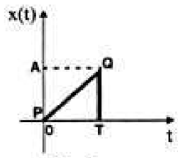


Fig 1.

$$\frac{x(t) - 0}{0 - A} = \frac{t - 0}{0 - T} \Rightarrow \frac{x(t)}{-A} = \frac{t}{-T} \Rightarrow x(t) = \frac{At}{T}$$

$$\therefore x(t) = \frac{At}{T} \;;\; \text{for } t = 0 \text{ to } T$$

Evaluation of c_0

$$c_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\Omega_0 t}\, dt$$

When $n = 0$, $c_0 = \frac{1}{T}\int_0^T x(t)\, e^0\, dt = \frac{1}{T}\int_0^T x(t)\, dt$

$$= \frac{1}{T}\int_0^T \frac{At}{T}\, dt = \frac{A}{T^2}\int_0^T t\, dt = \frac{A}{T^2}\left[\frac{t^2}{2}\right]_0^T$$

$$= \frac{A}{T^2}\left[\frac{T^2}{2} - 0\right] = \frac{A}{2}$$

Evaluation of c_n

$$c_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\Omega_0 t}\, dt = \frac{1}{T}\int_0^T \frac{At}{T}\, e^{-jn\Omega_0 t}\, dt = \frac{A}{T^2}\int_0^T t\, e^{-jn\Omega_0 t}\, dt \qquad \boxed{\int uv = u\int v - \int\left[du\int v\right]} \quad u = t,\; v = e^{-jn\Omega_0 t}$$

$$= \frac{A}{T^2}\left[t\,\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0} - \int 1 \times \frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\, dt\right]_0^T = \frac{A}{T^2}\left[\frac{t\, e^{-jn\Omega_0 t}}{-jn\Omega_0} - \frac{e^{-jn\Omega_0 t}}{(-jn\Omega_0)^2}\right]_0^T$$

$$= \frac{A}{T^2}\left[\frac{t\,e^{-jn\frac{2\pi}{T}t}}{-jn\frac{2\pi}{T}} + \frac{e^{-jn\frac{2\pi}{T}t}}{n^2\frac{4\pi^2}{T^2}}\right]_0^T = \frac{A}{T^2}\left[\frac{T\,e^{-jn\frac{2\pi}{T}T}}{-jn\frac{2\pi}{T}} + \frac{e^{-jn\frac{2\pi}{T}T}}{n^2\frac{4\pi^2}{T^2}} - 0 - \frac{e^0}{n^2\frac{4\pi^2}{T^2}}\right] \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= -\frac{A}{jn2\pi}e^{-jn2\pi} + \frac{A}{n^2 4\pi^2}e^{-jn2\pi} - \frac{A}{n^2 4\pi^2}$$

$$= -\frac{A}{jn2\pi} + \frac{A}{n^2 4\pi^2} - \frac{A}{n^2 4\pi^2} = -\frac{A}{jn2\pi} \qquad \boxed{\begin{aligned} e^{-jn2\pi} &= \cos n2\pi - j\sin n2\pi \\ &= 1 - j0 = 1;\ \text{for integer } n \end{aligned}}$$

$\therefore c_{-1} = \frac{A}{j2\pi}$	$c_1 = -\frac{A}{j2\pi}$
$c_{-2} = \frac{A}{j4\pi}$	$c_2 = -\frac{A}{j4\pi}$
$c_{-3} = \frac{A}{j6\pi}$	$c_3 = -\frac{A}{j6\pi}$
and so on.	and so on.

Exponential Form of Fourier Series

The exponential form of Fourier series is,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t} = \sum_{n=-\infty}^{-1} c_n e^{jn\Omega_0 t} + c_0 + \sum_{n=1}^{\infty} c_n e^{jn\Omega_0 t}$$

$$= \ldots\ c_{-3}e^{-j3\Omega_0 t} + c_{-2}e^{-j2\Omega_0 t} + c_{-1}e^{-j\Omega_0 t} + c_0 + c_1 e^{j\Omega_0 t} + c_2 e^{j2\Omega_0 t} + c_3 e^{j3\Omega_0 t} + \ldots$$

$$= \ldots + \frac{A}{j6\pi}e^{-j3\Omega_0 t} + \frac{A}{j4\pi}e^{-j2\Omega_0 t} + \frac{A}{j2\pi}e^{-j\Omega_0 t} + \frac{A}{2} - \frac{A}{j2\pi}e^{j\Omega_0 t} - \frac{A}{j4\pi}e^{j2\Omega_0 t} - \frac{A}{j6\pi}e^{j3\Omega_0 t} \ldots$$

$$= \frac{A}{j2\pi}\left[\ldots + \frac{e^{-j3\Omega_0 t}}{3} + \frac{e^{-j2\Omega_0 t}}{2} + \frac{e^{-j\Omega_0 t}}{1}\right] + \frac{A}{2} - \frac{A}{j2\pi}\left[\frac{e^{j\Omega_0 t}}{1} + \frac{e^{j2\Omega_0 t}}{2} + \frac{e^{j3\Omega_0 t}}{3} + \ldots\right]$$

Trigonometric Form of Fourier Series

The trigonometric form of Fourier series can be obtained as shown below.

$$x(t) = \frac{A}{2} - \frac{A}{\pi}\left[\frac{1}{1}\left(\frac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{2j}\right) + \frac{1}{2}\left(\frac{e^{j2\Omega_0 t} - e^{-j2\Omega_0 t}}{2j}\right) + \frac{1}{3}\left(\frac{e^{j3\Omega_0 t} - e^{-j3\Omega_0 t}}{2j}\right) + \ldots\right]$$

$$= \frac{A}{2} - \frac{A}{\pi}\left[\frac{\sin\Omega_0 t}{1} + \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3} + \ldots\right] \qquad \boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

2.2 FOURIER TRANSFORM

Definition of Fourier Transform

Let, $x(t)$ = Continuous time signal

$X(j\Omega)$ = Fourier transform of $x(t)$

The Fourier transform of continuous time signal, $x(t)$ is defined as,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

Also, $X(j\Omega)$ is denoted as $\mathcal{F}\{x(t)\}$ where "$\mathcal{F}$" is the symbol used to denote the Fourier transform operation.

$$\boxed{\therefore\ \mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt} \qquad \ldots(4.35)$$

> **Note :** *Sometimes the Fourier transform is expressed as a function of cyclic frequency F, rather than radian frequency Ω. The Fourier transform as a function of cyclic frequency F, is defined as,*
>
> $$X(jF) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi Ft}\, dt$$

2.2.1 Condition for Existence of Fourier Transform

The Fourier transform of x(t) exists if it satisfies the following Dirichlet condition.

1. The x(t) be absolutely integrable.

   i.e., $\int_{-\infty}^{+\infty} x(t)\, dt < \infty$

2. The x(t) should have a finite number of maxima and minima within any finite interval.
3. The x(t) can have a finite number of discontinuities within any interval.

2.2.2 Definition of Inverse Fourier Transform

The ***inverse Fourier transform*** of X(jΩ) is defined as,

$$x(t) = \mathcal{F}^{-1}\{X(j\Omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega \qquad .....(4.36)$$

The signals x(t) and X(jΩ) are called ***Fourier transform pair*** and can be expressed as shown below,

$$x(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\longleftrightarrow}} X(j\Omega)$$

> **Note :** *When Fourier transform is expressed as a function of cyclic frequency F, the inverse Fourier transform is defined as,*
>
> $$x(t) = F^{-1}\{X(jF)\} = \int_{-\infty}^{+\infty} X(jF)\, e^{j2\pi Ft}\, dF$$

2.2.3 Fourier Transform of Some Important Signals

Fourier Transform of Unit Impulse Signal

The impulse signal is defined as,

$$x(t) = \delta(t) = \infty \;;\; t = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$
$$= 0 \;;\; t \neq 0$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\Omega t}\, dt$$
$$= 1 \times e^{-j\Omega t}\Big|_{t=0} = 1 \times e^{0} = 1 \qquad \boxed{\delta(t) \text{ exists only for } t = 0}$$

$$\therefore \boxed{\mathcal{F}\{x(t)\} = 1}$$

The plot of impulse signal and its magnitude spectrum are shown in fig 4.18 and fig 4.19 respectively.

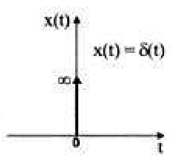


Fig 4.18 : Impulse signal.

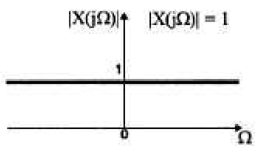


Fig 4.19 : Magnitude spectrum of impulse signal.

Fourier Transform of Single Sided Exponential Signal

The single sided exponential signal is defined as,

$$x(t) = A\,e^{-at} \quad ; \quad \text{for } t \geq 0$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\,e^{-j\Omega t}\,dt = \int_{0}^{+\infty} A\,e^{-at}\,e^{-j\Omega t}\,dt$$

$$= \int_{0}^{+\infty} A\,e^{-(a+j\Omega)t}\,dt = \left[\frac{A\,e^{-(a+j\Omega)t}}{-(a+j\Omega)}\right]_0^{+\infty}$$

$$= \left[\frac{A\,e^{-\infty}}{-(a+j\Omega)} - \frac{A\,e^{0}}{-(a+j\Omega)}\right] = \frac{A}{a+j\Omega} \qquad \boxed{e^{-\infty}=0}$$

$$\therefore \boxed{\mathcal{F}\{A\,e^{-at}\,u(t)\} = \frac{A}{a+j\Omega}} \qquad .....(4.56)$$

The plot of exponential signal and its magnitude spectrum are shown in fig 4.20 and fig 4.21 respectively.

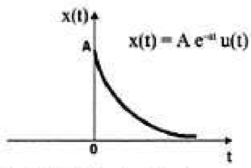


Fig 4.20: Single sided exponential signal.

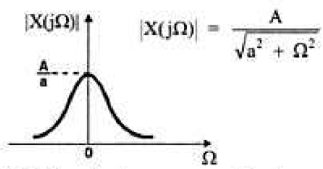


Fig 4.21 : Magnitude spectrum of single sided exponential signal.

Fourier Transform of Double Sided Exponential Signal

The double sided exponential signal is defined as,

$$x(t) = A\,e^{-a|t|} \quad ; \text{ for all } t$$

$$\therefore\ x(t) = A\,e^{+at} \quad ; \text{ for } t = -\infty \text{ to } 0$$

$$= A\,e^{-at} \quad ; \text{ for } t = 0 \text{ to } +\infty$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\,e^{-j\Omega t}\,dt = \int_{-\infty}^{0} A\,e^{at}\,e^{-j\Omega t}\,dt + \int_{0}^{+\infty} A\,e^{-at}\,e^{-j\Omega t}\,dt$$

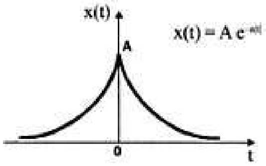
$$= \int_{-\infty}^{0} A\,e^{(a-j\Omega)t}\,dt + \int_{0}^{\infty} A\,e^{-(a+j\Omega)t}\,dt = \left[\frac{A\,e^{(a-j\Omega)t}}{a-j\Omega}\right]_{-\infty}^{0} + \left[\frac{A\,e^{-(a+j\Omega)t}}{-(a+j\Omega)}\right]_0^{\infty}$$

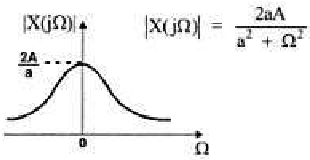
$$= \frac{A\,e^{0}}{a-j\Omega} - \frac{A\,e^{-\infty}}{a-j\Omega} + \frac{A\,e^{-\infty}}{-(a+j\Omega)} - \frac{A\,e^{0}}{-(a+j\Omega)} = \frac{A}{a-j\Omega} + \frac{A}{a+j\Omega} \qquad \boxed{e^{-\infty}=0}$$

$$= \frac{A(a+j\Omega) + A(a-j\Omega)}{(a-j\Omega)\,(a+j\Omega)} = \frac{2aA}{a^2+\Omega^2} \qquad \boxed{(a+b)(a-b) = a^2-b^2}\ \boxed{j^2=-1}$$

$$\therefore \boxed{\mathcal{F}\{A\,e^{-a|t|}\} = \frac{2aA}{a^2 + \Omega^2}} \qquad .....(4.57)$$

The plot of double sided exponential signal and its magnitude spectrum are shown in fig 4.22 and fig 4.23 respectively.





Fourier Transform of Signum Function

The signum function is defined as,

$$x(t) = \text{sgn}(t) = 1 \quad ; \quad t > 0$$
$$= -1 \quad ; \quad t < 0$$

The signum function can be expressed as a sum of two one sided exponential signal and taking limit "a" tends to 0 as shown below.

$$\therefore \ \text{sgn}(t) = \underset{a \to 0}{\text{Lt}} \left[e^{-at}\,u(t) - e^{at}\,u(-t)\right]$$

$$\therefore \quad x(t) = \text{sgn}(t) = \underset{a \to 0}{\text{Lt}} \left[e^{-at}\,u(t) - e^{at}\,u(-t)\right]$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\,e^{-j\Omega t}\,dt = \int_{-\infty}^{+\infty} \underset{a \to 0}{\text{Lt}} \left[e^{-at}\,u(t) - e^{at}\,u(-t)\right] e^{-j\Omega t}\,dt$$

$$= \underset{a \to 0}{\text{Lt}} \left[\int_{0}^{+\infty} e^{-at}\,e^{-j\Omega t}\,dt - \int_{-\infty}^{0} e^{at}\,e^{-j\Omega t}\,dt\right]$$

$$= \underset{a \to 0}{\text{Lt}} \left[\int_{0}^{+\infty} e^{-(a + j\Omega)t}\,dt - \int_{-\infty}^{0} e^{+(a - j\Omega)t}\,dt\right]$$

$$= \underset{a \to 0}{\text{Lt}} \left[\left[\frac{e^{-(a + j\Omega)t}}{-(a + j\Omega)}\right]_{0}^{\infty} - \left[\frac{e^{(a - j\Omega)t}}{(a - j\Omega)}\right]_{-\infty}^{0}\right] \qquad \boxed{e^{0} = 1\ ;\ e^{-\infty} = 0}$$

$$= \underset{a \to 0}{\text{Lt}} \left[\frac{e^{-\infty}}{-(a + j\Omega)} - \frac{e^{0}}{-(a + j\Omega)} - \frac{e^{0}}{a - j\Omega} + \frac{e^{-\infty}}{a - j\Omega}\right]$$

$$= \underset{a \to 0}{\text{Lt}} \left[\frac{1}{a + j\Omega} - \frac{1}{a - j\Omega}\right] = \frac{1}{j\Omega} + \frac{1}{j\Omega} = \frac{2}{j\Omega}$$

$$\therefore \boxed{\mathcal{F}\{\text{sgn}(t)\} = \frac{2}{j\Omega}}$$

The plot of signum function and its magnitude spectrum are shown in fig 4.26 and fig 4.27 respectively.

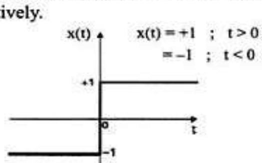


Fig 4.26 : Signum function.

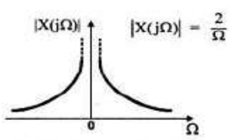


Fig 4.27 : Magnitude spectrum of signum function.

Fourier Transform of Unit Step Signal

The unit step signal is defined as,

$$u(t) = 1 \;;\; t \geq 0$$
$$= 0 \;;\; t < 0$$

If can be proved that, $\mathrm{sgn}(t) = 2u(t) - 1 \;\Rightarrow\; u(t) = \frac{1}{2}[1 + \mathrm{sgn}(t)]$

$$\therefore\; x(t) = u(t) = \frac{1}{2}[1 + \mathrm{sgn}(t)]$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\frac{1}{2}[1 + \mathrm{sgn}(t)]\right\}$$

$$\therefore\; X(j\Omega) = \mathcal{F}\left\{\frac{1}{2}\right\} + \mathcal{F}\left\{\frac{1}{2}\,\mathrm{sgn}(t)\right\} = \frac{1}{2}\mathcal{F}\{1\} + \frac{1}{2}\mathcal{F}\{\mathrm{sgn}(t)\}$$

$$= \frac{1}{2}[2\pi\,\delta(\Omega)] + \frac{1}{2}\left[\frac{2}{j\Omega}\right] = \pi\,\delta(\Omega) + \frac{1}{j\Omega}$$

Using equations (4.58) and (4.59)

$$\therefore\; \boxed{\mathcal{F}\{u(t)\} = \pi\,\delta(\Omega) + \frac{1}{j\Omega}} \quad .....(4.60)$$

The plot of unit step signal and its magnitude spectrum are shown in fig 4.28 and fig 4.29 respectively.

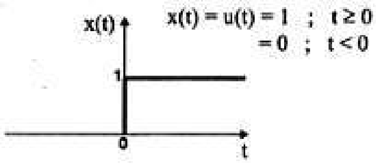


Fig 4.28 : Unit step signal.

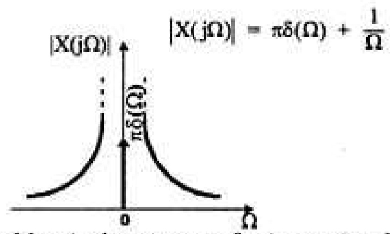


Fig 4.29 : Magnitude spectrum of unit step signal.

Fourier Transform of a Periodic Signal

Let, $x(t)$ = Continuous time periodic signal

$X(j\Omega) = \mathcal{F}\{x(t)\}$ = Fourier transform of $x(t)$

The exponential form of Fourier series representation of $x(t)$ is given by,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n\, e^{jn\Omega_0 t}$$

From equation (4.9)

On taking Fourier transform of the above equation we get,

$$X(j\Omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{+\infty} c_n\, e^{jn\Omega_0 t}\right\} = \sum_{n=-\infty}^{+\infty} c_n\, \mathcal{F}\left\{e^{jn\Omega_0 t}\right\}$$

$$= \sum_{n=-\infty}^{+\infty} c_n\, 2\pi\,\delta(\Omega - n\Omega_0) = 2\pi \sum_{n=-\infty}^{+\infty} c_n\, 2\pi\,\delta(\Omega - n\Omega_0)$$

Using equation (4.61)

$$= \ldots + 2\pi c_{-3}\,\delta(\Omega + 3\Omega_0) + 2\pi c_{-2}\,\delta(\Omega + 2\Omega_0) + 2\pi c_{-1}\,\delta(\Omega + \Omega_0) + 2\pi c_0\,\delta(\Omega) + 2\pi c_1\,\delta(\Omega - \Omega_0) + 2\pi c_2\,\delta(\Omega - 2\Omega_0) + 2\pi c_3\,\delta(\Omega - 3\Omega_0) + \ldots \quad \ldots(4.65)$$

The magnitude of each term of equation (4.65) represents an impulse, located at its harmonic frequency in the magnitude spectrum. Hence we can say that the Fourier transform of a periodic continuous time signal consists of impulses located at the harmonic frequencies of the signal. The magnitude of each impulse is 2π times the magnitude of Fourier coefficient, i.e., the magnitude of n^{th} impulse is $2\pi\,|c_n|$.

2.2.4 Summary of Laplace and Fourier Tranform for Causal Signals

x(t) for t = 0 to ∞	X(s)	X(jΩ) $\left[X(j\Omega) = X(s)\vert_{s=j\Omega}\right]$
$\delta(t)$	1	1
$u(t)$	$\frac{1}{s}$	$\frac{1}{j\Omega}$
$t\,u(t)$	$\frac{1}{s^2}$	$\frac{1}{(j\Omega)^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)$ where, n = 1, 2, 3, ...	$\frac{1}{s^n}$	$\frac{1}{(j\Omega)^n}$
$t^n\,u(t)$ where, n = 1, 2, 3, ...	$\frac{n!}{s^{n+1}}$	$\frac{n!}{(j\Omega)^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\frac{1}{j\Omega + a}$
$t\,e^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\frac{1}{(j\Omega + a)^2}$
$\sin\Omega_0 t\,u(t)$	$\frac{\Omega_0}{s^2 + \Omega_0^2}$	$\frac{\Omega_0}{(j\Omega)^2 + \Omega_0^2} = \frac{\Omega_0}{\Omega_0^2 - \Omega^2}$
$\cos\Omega_0 t\,u(t)$	$\frac{s}{s^2 + \Omega_0^2}$	$\frac{j\Omega}{(j\Omega)^2 + \Omega_0^2} = \frac{j\Omega}{\Omega_0^2 - \Omega^2}$
$\sinh\Omega_0 t\,u(t)$	$\frac{\Omega_0}{s^2 - \Omega_0^2}$	$\frac{\Omega_0}{(j\Omega)^2 - \Omega_0^2} = \frac{-\Omega_0}{\Omega^2 + \Omega_0^2}$
$\cosh\Omega_0 t\,u(t))$	$\frac{s}{s^2 - \Omega_0^2}$	$\frac{j\Omega}{(j\Omega)^2 - \Omega_0^2} = \frac{-j\Omega}{\Omega^2 + \Omega_0^2}$
$e^{-at}\sin\Omega_0 t\,u(t)$	$\frac{\Omega_0}{(s+a)^2 + \Omega_0^2}$	$\frac{\Omega_0}{(j\Omega + a)^2 + \Omega_0^2}$
$e^{-at}\cos\Omega_0 t\,u(t)$	$\frac{s+a}{(s+a)^2 + \Omega_0^2}$	$\frac{j\Omega + a}{(j\Omega + a)^2 + \Omega_0^2}$

2.2.5

Find fourier transform of following signals

(b) Given that, $x(t) = e^{-at}\cos\Omega_0 t\, u(t)$

Since $u(t) = 1$, for $t \ge 0$, we can write,

$$x(t) = e^{-at}\cos\Omega_0 t \quad ; \quad \text{for } t \ge 0$$

By definition of Fourier transform,

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{0}^{\infty} e^{-at}\cos\Omega_0 t\, e^{-j\Omega t}\, dt = \int_{0}^{\infty} e^{-at}\left(\frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}\right) e^{-j\Omega t}\, dt$$

$$= \frac{1}{2}\int_{0}^{\infty} e^{-at}\, e^{j\Omega_0 t}\, e^{-j\Omega t}\, dt + \frac{1}{2}\int_{0}^{\infty} e^{-at}\, e^{-j\Omega_0 t}\, e^{-j\Omega t}\, dt$$

$$= \frac{1}{2}\int_{0}^{\infty} e^{-(a - j\Omega_0 + j\Omega)t}\, dt + \frac{1}{2}\int_{0}^{\infty} e^{-(a + j\Omega_0 + j\Omega)t}\, dt$$

$$= \frac{1}{2}\left[\frac{e^{-(a - j\Omega_0 + j\Omega)t}}{-(a - j\Omega_0 + j\Omega)}\right]_0^{\infty} + \frac{1}{2}\left[\frac{e^{-(a + j\Omega_0 + j\Omega)t}}{-(a + j\Omega_0 + j\Omega)}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(a - j\Omega_0 + j\Omega)} - \frac{e^{0}}{-(a - j\Omega_0 + j\Omega)}\right] + \frac{1}{2}\left[\frac{e^{-\infty}}{-(a + j\Omega_0 + j\Omega)} - \frac{e^{0}}{-(a + j\Omega_0 + j\Omega)}\right]$$

$$= \frac{1}{2}\left[0 + \frac{1}{a - j\Omega_0 + j\Omega}\right] + \frac{1}{2}\left[0 + \frac{1}{a + j\Omega_0 + j\Omega}\right] \qquad e^{-\infty} = 0\ ;\ e^{0} = 1$$

$$= \frac{1}{2}\left[\frac{1}{(a + j\Omega) - j\Omega_0} + \frac{1}{(a + j\Omega) + j\Omega_0}\right] \qquad (a+b)(a-b) = a^2 - b^2 \quad j^2 = -1$$

$$= \frac{1}{2}\left[\frac{(a + j\Omega) + j\Omega_0 + (a + j\Omega) - j\Omega_0}{(a + j\Omega)^2 + \Omega_0^2}\right]$$

$$= \frac{1}{2}\,\frac{2(a + j\Omega)}{(a + j\Omega)^2 + \Omega_0^2} = \frac{a + j\Omega}{(a + j\Omega)^2 + \Omega_0^2}$$

Example 4.14

Determine the Fourier transform of the rectangular pulse shown in fig 4.14.1.

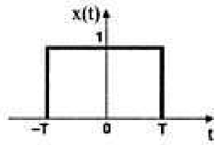


Fig 4.14.1.

Solution

The mathematical equation of the rectangular pulse is,

$$x(t) = 1 \quad ; \quad \text{for } t = -T \text{ to } +T$$

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-T}^{+T} 1 \times e^{-j\Omega t}\, dt = \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-T}^{+T}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{e^{-j\Omega T}}{-j\Omega} - \frac{e^{j\Omega T}}{-j\Omega} = \frac{1}{j\Omega}\left(e^{j\Omega T} - e^{-j\Omega T}\right) = \frac{1}{j\Omega}\, 2j\sin\Omega T$$

$$= 2\,\frac{\sin\Omega T}{\Omega} = 2T\,\frac{\sin\Omega T}{\Omega T} \qquad \frac{\sin\theta}{\theta} = \text{sinc}\,\theta$$

$$= 2T\,\text{sinc}\,\Omega T$$

Example 4.15

Determine the Fourier transform of the triangular pulse shown in fig 4.15.1.

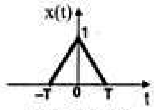


Fig 4.15.1.

Solution

The mathematical equation of triangular pulse is,

$$x(t) = 1 + \frac{t}{T} \quad ; \quad \text{for } t = -T \text{ to } 0$$
$$= 1 - \frac{t}{T} \quad ; \quad \text{for } t = 0 \text{ to } T$$

(Please refer example 4.11 for the mathematical equation of triangular pulse).

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-T}^{0} \left(1 + \frac{t}{T}\right) e^{-j\Omega t}\, dt + \int_{0}^{T} \left(1 - \frac{t}{T}\right) e^{-j\Omega t}\, dt$$

$$= \int_{-T}^{0} e^{-j\Omega t}\, dt + \frac{1}{T}\int_{-T}^{0} t\, e^{-j\Omega t}\, dt + \int_{0}^{T} e^{-j\Omega t}\, dt - \frac{1}{T}\int_{0}^{T} t\, e^{-j\Omega t}\, dt \qquad \boxed{\int uv = u\int v - \int \left[du \int v\right]}$$

$$= \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-T}^{0} + \frac{1}{T}\left[t\,\frac{e^{-j\Omega t}}{-j\Omega} - \int 1 \times \frac{e^{-j\Omega t}}{-j\Omega}\, dt\right]_{-T}^{0} + \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{0}^{T} - \frac{1}{T}\left[t\,\frac{e^{-j\Omega t}}{-j\Omega} - \int 1 \times \frac{e^{-j\Omega t}}{-j\Omega}\, dt\right]_{0}^{T}$$

$$= -\frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{-T}^{0} - \frac{1}{j\Omega T}\left[t\, e^{-j\Omega t} - \int e^{-j\Omega t}\, dt\right]_{-T}^{0} - \frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{0}^{T} + \frac{1}{j\Omega T}\left[t\, e^{-j\Omega t} - \int e^{-j\Omega t}\, dt\right]_{0}^{T}$$

$$= -\frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{-T}^{0} - \frac{1}{j\Omega T}\left[t\, e^{-j\Omega t} - \frac{e^{-j\Omega t}}{-j\Omega}\right]_{-T}^{0} - \frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{0}^{T} + \frac{1}{j\Omega T}\left[t\, e^{-j\Omega t} - \frac{e^{-j\Omega t}}{-j\Omega}\right]_{0}^{T}$$

$$= -\frac{1}{j\Omega}\left[e^{0} - e^{j\Omega T}\right] - \frac{1}{j\Omega T}\left[0 - \frac{e^{0}}{-j\Omega} + T e^{j\Omega T} + \frac{e^{j\Omega T}}{-j\Omega}\right] - \frac{1}{j\Omega}\left[e^{-j\Omega T} - e^{0}\right]$$
$$+ \frac{1}{j\Omega T}\left[T e^{-j\Omega T} - \frac{e^{-j\Omega T}}{-j\Omega} - 0 + \frac{e^{0}}{-j\Omega}\right]$$

$$= -\frac{1}{j\Omega} + \frac{e^{j\Omega T}}{j\Omega} - 0 + \frac{1}{T\Omega^2} - \frac{e^{j\Omega T}}{j\Omega} - \frac{e^{j\Omega T}}{T\Omega^2} - \frac{e^{-j\Omega T}}{j\Omega} + \frac{1}{j\Omega} + \frac{e^{-j\Omega T}}{j\Omega} - \frac{e^{-j\Omega T}}{T\Omega^2} - 0 + \frac{1}{T\Omega^2}$$

$$= \frac{2}{T\Omega^2} - \frac{1}{T\Omega^2}\left(e^{j\Omega T} + e^{-j\Omega T}\right) = \frac{2}{T\Omega^2} - \frac{1}{T\Omega^2}\, 2\cos\Omega T \qquad \boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \frac{2}{T\Omega^2}\left(1 - \cos\Omega T\right)$$

Alternatively the above result can be expressed as shown below.

$$\mathcal{F}\{x(t)\} = \frac{2}{T\Omega^2}\left(1 - \cos\Omega T\right) = \frac{2}{T\Omega^2}\left(1 - \cos 2\left(\frac{\Omega T}{2}\right)\right)$$

$$= \frac{2}{T\Omega^2}\left(2\sin^2\frac{\Omega T}{2}\right) = T\,\frac{4}{T^2\Omega^2}\sin^2\frac{\Omega T}{2} = T\,\frac{\sin^2\left(\frac{\Omega T}{2}\right)}{\left(\frac{\Omega T}{2}\right)^2} \qquad \boxed{\sin^2\theta = \frac{1 - \cos 2\theta}{2}}$$

$$= T\left(\frac{\sin\frac{\Omega T}{2}}{\frac{\Omega T}{2}}\right)^2 = T\left(\text{sinc}\,\frac{\Omega T}{2}\right)^2 \qquad \boxed{\frac{\sin\theta}{\theta} = \text{sinc}\,\theta}$$

2.3 Definition of Laplace Transform

In order to transform a time domain signal x(t) to s-domain, multiply the signal by e^{-st} and then integrate from $-\infty$ to ∞. The transformed signal is represented as X(s) and the transformation is denoted by the script letter $\mathcal{L}$.

Symbolically the ***Laplace transform*** of x(t) is denoted as,

$$X(s) = \mathcal{L}\{x(t)\}$$

Let x(t) be a continuous time signal defined for all values of t. Let X(s) be Laplace transform of x(t). Now the ***Laplace transform*** of x(t) is defined as,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt \qquad .....(3.6)$$

If x(t) is defined for $t \geq 0$, (i.e., if x(t) is causal) then,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{0}^{+\infty} x(t)\, e^{-st}\, dt \qquad .....(3.7)$$

The definition of Laplace transform as given by equation (3.6) is called ***Two sided Laplace transform*** or ***Bilateral Laplace Transform*** and the definition of Laplace transform as given by equation (3.7) is called ***One sided Laplace transform*** or ***Unilateral Laplace transform***.

Definition of Inverse Laplace Transform

The s-domain signal X(s) can be transformed to time domain signal x(t) by using inverse Laplace transform.

The ***Inverse Laplace transform*** of X(s) is defined as,

$$\mathcal{L}^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j} \int_{s=\sigma-j\Omega}^{s=\sigma+j\Omega} X(s)\, e^{st}\, ds$$

2.3.1 Existence of Laplace Transform

The computation of Laplace transform involves integral of x(t) from $t = -\infty$ to $+\infty$. Therefore Laplace transform of a signal exists if the integral, $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$ converges (i.e., finite). The integral will converge if the signal x(t) is sectionally continuous in every finite interval of t and if it is of exponential order as t approaches infinity.

A causal signal x(t) is said to be ***exponential order*** if a real, positive constant σ (where σ is real part of s) exists such that the function, $e^{-\sigma t}|x(t)|$ approaches zero as t approaches infinity.

i.e., if, $\underset{t \to \infty}{Lt}\ e^{-\sigma t}\,|x(t)| = 0$, then x(t) is of exponential order.

For a causal signal, if $\underset{t \to \infty}{Lt}\ e^{-\sigma t}\,|x(t)| = 0$ for $\sigma > \sigma_c$, and if $\underset{t \to \infty}{Lt}\ e^{-\sigma t}|x(t)| = \infty$ for $\sigma < \sigma_c$, then σ_c is called ***abscissa of convergence***, (where σ_c is a point on real axis in s-plane).

The integral $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$ converges only if the real part of s is greater than the abscissa of convergence σ_c. The values of s for which the integral $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$ converges is called ***Region Of Convergence (ROC)***. Therefore for a causal signal the ROC includes all points on the s-plane to the right of abscissa of convergence.

Example

Determine the Laplace transform of the following continuous time signals and their ROC.

a) $x(t) = A\,u(t)$ b) $x(t) = t\,u(t)$ c) $x(t) = e^{-3t}\,u(t)$ d) $x(t) = e^{-3t}\,u(-t)$ e) $x(t) = e^{-4|t|}$

Solution

a) Given that, $x(t) = A\,u(t) = A\ ;\ t \geq 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt = \int_0^{\infty} A\,e^{-st}\,dt = A\int_0^{\infty} e^{-st}\,dt$$

Put, $s = \sigma + j\Omega$

$$= A\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = A\left[\frac{e^{-(\sigma+j\Omega)t}}{-s}\right]_0^{\infty} = A\left[\frac{e^{-(\sigma+j\Omega)\infty}}{-s} - \frac{e^{0}}{-s}\right] = A\left[\frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{-s} + \frac{1}{s}\right]$$

When, $\sigma > 0$, (i.e., when σ is positive), $e^{-\sigma\infty} = e^{-\infty} = 0$
When, $\sigma < 0$, (i.e., when σ is negative), $e^{-\sigma\infty} = e^{\infty} = \infty$

Therefore we can say that, X(s) converges when $\sigma > 0$.

When $\sigma > 0$, the X(s) is given by,

$$X(s) = A\left[\frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{-s} + \frac{1}{s}\right] = A\left[\frac{0 \times e^{-j\Omega\times\infty}}{-s} + \frac{1}{s}\right] = \frac{A}{s}$$

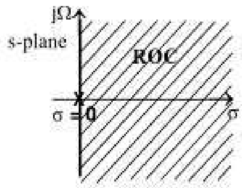


Fig 1 : ROC of x(t) = A u(t).

$\therefore\ \mathcal{L}\{A\,u(t)\} = \frac{A}{s}$; with ROC as all points is s-plane to the right of line passing through $\sigma = 0$.
(or ROC is right half of s-plane).

b) Given that, $x(t) = t\,u(t) = t\ ;\ t \geq 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt = \int_0^{\infty} t\,e^{-st}\,dt$$

$\int uv = u\int v - \int\left[du \int v\right]$

$$= \left[t\,\frac{e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} 1 \times \frac{e^{-st}}{-s}\,dt = \left[t\,\frac{e^{-st}}{-s}\right]_0^{\infty} - \left[\frac{e^{-st}}{s^2}\right]_0^{\infty} = \left[t\,\frac{e^{-(\sigma+j\Omega)t}}{-s}\right]_0^{\infty} - \left[\frac{e^{-(\sigma+j\Omega)t}}{s^2}\right]_0^{\infty}$$

Put, $s = \sigma + j\Omega$

$$= \left[\infty \times \frac{e^{-(\sigma+j\Omega)\infty}}{-s} - 0 \times \frac{e^{0}}{-s} - \frac{e^{-(\sigma+j\Omega)\infty}}{s^2} + \frac{e^{0}}{s^2}\right]$$

$$= \left[\infty \times \frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{-s} - 0 - \frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{s^2} + \frac{1}{s^2}\right]$$

When, $\sigma > 0$, (i.e., when σ is positive), $e^{-\sigma\infty} = e^{-\infty} = 0$
When, $\sigma < 0$, (i.e., when σ is negative), $e^{-\sigma\infty} = e^{\infty} = \infty$

Therefore we can say that, X(s) converges when $\sigma > 0$.

When $\sigma > 0$, the X(s) is given by,

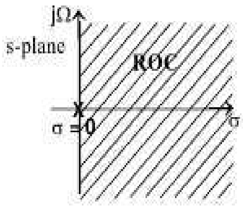


Fig 2 : ROC of x(t) = t u(t).

$$X(s) = \left[\infty \times \frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{-s} - \frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{s^2} + \frac{1}{s^2}\right]$$

$$= \left[\infty \times \frac{0 \times e^{-j\Omega\times\infty}}{-s} - \frac{0 \times e^{-j\Omega\times\infty}}{s^2} + \frac{1}{s^2}\right] = \frac{1}{s^2}$$

c) Given that, $x(t) = e^{-3t}\,u(t) = e^{-3t}$; $t \geq 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt = \int_0^{\infty} e^{-3t}\,e^{-st}\,dt = \int_0^{\infty} e^{-(s+3)t}\,dt$$

$$= \left[\frac{e^{-(s+3)t}}{-(s+3)}\right]_0^{\infty} = \frac{e^{-(s+3)\infty}}{-(s+3)} - \frac{e^0}{-(s+3)} = -\frac{e^{-(\sigma+j\Omega+3)\infty}}{s+3} + \frac{1}{s+3}$$

Put, $s = \sigma + j\Omega$

$$= -\frac{e^{-(\sigma+3)\times\infty}\,e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3} = -\frac{e^{-k\times\infty}\,e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3}$$

where, $k = \sigma + 3 = \sigma - (-3)$

When, $\sigma > -3$, $k = \sigma - (-3)$ = Positive. $\therefore e^{-k\infty} = e^{-\infty} = 0$

When, $\sigma < -3$, $k = \sigma - (-3)$ = Negative. $\therefore e^{-k\infty} = e^{\infty} = \infty$

Therefore we can say that, X(s) converges when $\sigma > -3$.

When $\sigma > -3$, the X(s) is given by,

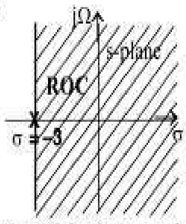


Fig 3 : ROC of $x(t) = e^{-3t}u(t)$.

$$X(s) = -\frac{e^{-k\times\infty}\,e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3} = -\frac{0\times e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3} = \frac{1}{s+3}$$

$$\therefore \mathcal{L}\{e^{-3t}\,u(t)\} = \frac{1}{s+3}$$; with ROC as all points in s-plane to the right of line passing through $\sigma = -3$.

d) Given that, $x(t) = e^{-3t}\,u(-t) = e^{-3t}$; $t \leq 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt = \int_{-\infty}^{0} e^{-3t}\,e^{-st}\,dt = \int_{-\infty}^{0} e^{-(s+3)t}\,dt$$

$$= \left[\frac{e^{-(s+3)t}}{-(s+3)}\right]_{-\infty}^{0} = \frac{e^0}{-(s+3)} - \frac{e^{(s+3)\infty}}{-(s+3)} = -\frac{1}{s+3} + \frac{e^{(\sigma+j\Omega+3)\infty}}{s+3}$$

Put, $s = \sigma + j\Omega$

$$= -\frac{1}{s+3} + \frac{e^{(\sigma+3)\times\infty}\,e^{j\Omega\times\infty}}{s+3} = -\frac{1}{s+3} + \frac{e^{k\times\infty}\,e^{j\Omega\times\infty}}{s+3}$$

where, $k = \sigma + 3 = \sigma - (-3)$

When, $\sigma > -3$, $k = \sigma - (-3)$ = Positive. $\therefore e^{k\infty} = e^{\infty} = \infty$

When, $\sigma < -3$, $k = \sigma - (-3)$ = Negative. $\therefore e^{k\infty} = e^{-\infty} = 0$

Therefore we can say that, X(s) converges when $\sigma < -3$.

When $\sigma < -3$, the X(s) is given by,

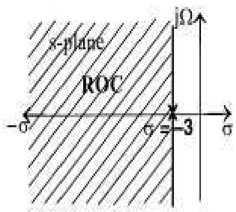


Fig 4 : ROC of $x(t) = e^{-3t}u(-t)$.

$$X(s) = -\frac{1}{s+3} + \frac{e^{k\times\infty}\,e^{j\Omega\times\infty}}{s+3} = -\frac{1}{s+3} + \frac{0\times e^{j\Omega\times\infty}}{s+3} = -\frac{1}{s+3}$$

$$\therefore \mathcal{L}\{e^{-3t}\,u(-t)\} = -\frac{1}{s+3}$$; with ROC as all points in s-plane to the left of line passing through $\sigma = -3$.

e) Given that, $x(t) = e^{-4|t|} = e^{4t}\ ;\ t \le 0$

$= e^{-4t}\ ;\ t \ge 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\,dt = \int_{-\infty}^{0} e^{4t}\, e^{-st}\,dt + \int_{0}^{\infty} e^{-4t} e^{-st}\,dt = \int_{-\infty}^{0} e^{-(s-4)t}\,dt + \int_{0}^{\infty} e^{-(s+4)t}\,dt$$

$$= \left[\frac{e^{-(s-4)t}}{-(s-4)}\right]_{-\infty}^{0} + \left[\frac{e^{-(s+4)t}}{-(s+4)}\right]_{0}^{\infty} = \left[\frac{e^{0}}{-(s-4)} - \frac{e^{(s-4)\infty}}{-(s-4)}\right] + \left[\frac{e^{-(s+4)\infty}}{-(s+4)} - \frac{e^{0}}{-(s+4)}\right]$$

$$= -\frac{1}{s-4} + \frac{e^{(\sigma+j\Omega-4)\infty}}{s-4} - \frac{e^{-(\sigma+j\Omega+4)\infty}}{s+4} + \frac{1}{s+4}$$

$$= -\frac{1}{s-4} + \frac{e^{(\sigma-4)\times\infty}\, e^{j\Omega\times\infty}}{s-4} - \frac{e^{-(\sigma+4)\times\infty}\, e^{-j\Omega\times\infty}}{s+4} + \frac{1}{s+4}$$

Put, $s = \sigma + j\Omega$

When, $\sigma < 4$, $\sigma - 4$ = Negative. $\therefore e^{(\sigma-4)\infty} = e^{-\infty} = 0$

When, $\sigma > 4$, $\sigma - 4$ = Positive. $\therefore e^{(\sigma-4)\infty} = e^{\infty} = \infty$

When, $\sigma < -4$, $\sigma + 4$ = Negative. $\therefore e^{-(\sigma+4)\infty} = e^{\infty} = \infty$

When, $\sigma > -4$, $\sigma + 4$ = Positive. $\therefore e^{-(\sigma+4)\infty} = e^{-\infty} = 0$

Therefore we can say that, X(s) converges when σ lies between −4 and +4.

When σ lies between − 4 and + 4, the X(s) is given by,

$$X(s) = -\frac{1}{s-4} + \frac{e^{(\sigma-4)\times\infty}\, e^{j\Omega\times\infty}}{s-4} - \frac{e^{-(\sigma+4)\times\infty}\, e^{-j\Omega\times\infty}}{s+4} + \frac{1}{s+4}$$

$$= -\frac{1}{s-4} + \frac{e^{-\infty}\, e^{j\Omega\times\infty}}{s-4} - \frac{e^{-\infty}\, e^{-j\Omega\times\infty}}{s+4} + \frac{1}{s+4}$$

$$= -\frac{1}{s-4} + \frac{0\times e^{j\Omega\times\infty}}{s-4} - \frac{0\times e^{-j\Omega\times\infty}}{s+4} + \frac{1}{s+4}$$

$$= -\frac{1}{s-4} + 0 - 0 + \frac{1}{s+4}$$

$$= \frac{1}{s+4} - \frac{1}{s-4} = \frac{s-4-(s+4)}{(s+4)(s-4)} = -\frac{8}{s^2-16}$$

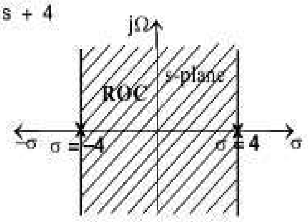


Fig 5 : ROC of $x(t) = e^{-4|t|}$.

$(a+b)(a-b) = a^2 - b^2$

$$\therefore \mathcal{L}\{e^{-4|t|}\} = -\frac{8}{s^2-16}$$; with ROC as all points in s-plane in between the lines passing through $\sigma = -4$ and $\sigma = 4$.

Find Laplace transform of the signals (2) and (3)

1 2 3

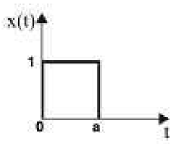


Fig 3.3.2.

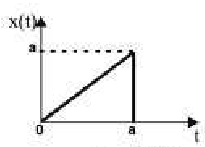


Fig 3.3.3.

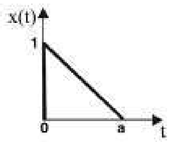


Fig 3.3.4.

2

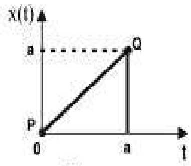


Fig 2

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\frac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \frac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

Coordinates of point - P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point - Q = $[t_2, x(t_2)] = [a, a]$

On substituting the coordinates of points - P and Q in equation - (1) we get,

$$\frac{x(t) - 0}{0 - a} = \frac{t - 0}{0 - a} \Rightarrow \frac{x(t)}{-a} = \frac{t}{-a} \Rightarrow x(t) = t$$

$\therefore x(t) = t$;for $t = 0$ to a

$= 0$;for $t > a$

To Evaluate Laplace transform of x(t)

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_0^a t\, e^{-st}\, dt$$

$$= \left[t \times \frac{e^{-st}}{-s} - \int 1 \times \frac{e^{-st}}{-s}\, dt \right]_0^a = \left[-\frac{t\, e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a$$

$\int uv = u \int v - \int [du \int v]$	
$u = t$	$v = e^{-st}$

$$= \left[-\frac{a\, e^{-sa}}{s} - \frac{e^{-sa}}{s^2} + 0 + \frac{e^0}{s^2} \right] = \frac{1}{s^2} - \frac{e^{-as}}{s^2} - \frac{a e^{-as}}{s}$$

$$= \frac{1}{s^2}\left[1 - e^{-as}(1 + as)\right]$$

3

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\frac{y - y_1}{y_1 - y_2} = \frac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\frac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \frac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

Coordinates of point - P = $[t_1, x(t_1)] = [0, 1]$

Coordinates of point - Q = $[t_2, x(t_2)] = [a, 0]$

On substituting the coordinates of points - P and Q in equation - (1) we get,

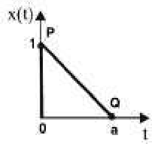


Fig 3

$$\frac{x(t) - 1}{1 - 0} = \frac{t - 0}{0 - a} \Rightarrow x(t) - 1 = -\frac{t}{a} \Rightarrow x(t) = 1 - \frac{t}{a}$$

$\therefore x(t) = 1 - \frac{t}{a}$;for $t = 0$ to a

$= 0$;for $t > a$

To Evaluate Laplace transform of x(t)

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

$$= \int_0^a \left(1-\frac{t}{a}\right) e^{-st}\, dt = \int_0^a e^{-st}\, dt - \frac{1}{a}\int_0^a t\, e^{-st}\, dt$$

$\int uv = u\int v - \int[du \int v]$	
$u = t$	$v = e^{-st}$

$$= \left[\frac{e^{-st}}{-s}\right]_0^a - \frac{1}{a}\left[t \times \frac{e^{-st}}{-s} - \int 1\times \frac{e^{-st}}{-s}\, dt\right]_0^a$$

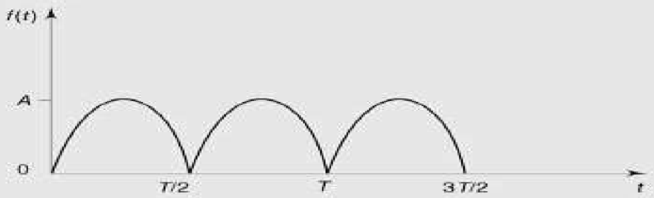
$$= \left[-\frac{e^{-st}}{s}\right]_0^a - \frac{1}{a}\left[-\frac{t\, e^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^a$$

$$= \left[-\frac{e^{-as}}{s} + \frac{e^{0}}{s}\right] - \frac{1}{a}\left[-\frac{a\, e^{-as}}{s} - \frac{e^{-as}}{s^2} + 0 + \frac{e^{0}}{s^2}\right]$$

$$= -\frac{e^{-as}}{s} + \frac{1}{s} + \frac{e^{-as}}{s} + \frac{e^{-as}}{as^2} - \frac{1}{as^2}$$

$$= \frac{1}{s} + \frac{e^{-as}}{as^2} - \frac{1}{as^2} = \frac{1}{as^2}\left[e^{-as} + as - 1\right]$$

Example *Find the Laplace transform of the full wave rectified output as shown in Fig.*



Solution The function for the given waveform is

$$f(t) = A \sin \omega_0 t \text{ for } 0 < t < T/2$$

Hence,

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT/2}} \int_0^{T/2} f(t) e^{-st}\, dt$$

$$= \frac{A}{1-e^{-sT/2}} \int_0^{T/2} \sin \omega_0 t\, e^{-st} dt$$

$$= \frac{A}{1-e^{-sT/2}} \left[\frac{e^{-st}}{s^2+\omega_0^2}(-s \sin \omega_0 t - \omega_0 \cos \omega_0 t)\right]_0^{T/2}$$

$$= \frac{A}{1-e^{-sT/2}} \cdot \frac{1}{(s^2+\omega_0^2)}\left[\omega_0 e^{-sT/2} + \omega_0\right]$$

$$= \frac{A\omega_0}{s^2+\omega_0^2} \frac{(1+e^{-sT/2})}{(1-e^{-sT/2})}$$

$$= \frac{A\omega_0}{s^2+\omega_0^2} \frac{e^{sT/4}+e^{-sT/4}}{e^{sT/4}-e^{-sT/4}}$$

$$= \frac{A\omega_0}{s^2+\omega_0^2} \coth(sT/4)$$

Determine the inverse Laplace transform of $X(s) = \frac{2}{s(s+1)(s+2)}$

Solution

Given that, $X(s) = \frac{2}{s(s+1)(s+2)}$

By partial fraction expansion technique, X(s) can be expressed as,

$$X(s) = \frac{2}{s(s+1)(s+2)} = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2}$$

The residue K_1 is obtained by multiplying X(s) by s and letting s = 0.

$$K_1 = X(s) \times s\Big|_{s=0} = \frac{2}{s(s+1)(s+2)} \times s\Bigg|_{s=0} = \frac{2}{(s+1)(s+2)}\Bigg|_{s=0} = \frac{2}{1 \times 2} = 1$$

The residue K_2 is obtained by multiplying X(s) by (s + 1) and letting s = −1.

$$K_2 = X(s) \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+1)(s+2)} \times (s+1)\Bigg|_{s=-1} = \frac{2}{s(s+2)}\Bigg|_{s=-1} = \frac{2}{-1(-1+2)} = -2$$

The residue K_3 is obtained by multiplying X(s) by (s + 2) and letting s = −2.

$$K_3 = X(s) \times (s+2)\Big|_{s=-2} = \frac{2}{s(s+1)(s+2)} \times (s+2)\Bigg|_{s=-2} = \frac{2}{s(s+1)}\Bigg|_{s=-2} = \frac{2}{-2(-2+1)} = 1$$

$$\therefore X(s) = \frac{2}{s(s+1)(s+2)} = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

$$\text{Now, } x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}\right\}$$

$$= u(t) - 2e^{-t}u(t) + e^{-2t}u(t)$$

$$= (1 - 2e^{-t} + e^{-2t})u(t) = (1 - e^{-t})^2 u(t)$$

$$(x-y)^2 = x^2 - 2xy + y^2$$

2.3.2 Region of Convergence

The Laplace transform of a signal is given by $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$. The values of s for which the integral $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$ converges is called ***Region Of Convergence (ROC)***. The ROC for the following three types of signals are discussed here.

Case i : Right sided (causal) signal
Case ii : Left sided (anticausal) signal
Case iii : Two sided signal.

Case i : Right sided (causal) signal

Let, $x(t) = e^{-at}u(t)$, where $a > 0$

$= e^{-at}$ for $t \geq 0$

Now, the Laplace transform of x(t) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} e^{-at}\, u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{+\infty} e^{-at}\, e^{-st}\, dt = \int_{0}^{+\infty} e^{-(s+a)t}\, dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty}$$

$$= \frac{e^{-(\sigma + j\Omega + a)\infty}}{-(s+a)} - \frac{e^0}{-(s+a)} = -\frac{e^{-(\sigma + a)\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a}$$

Put, $s = \sigma + j\Omega$

$$\therefore \mathcal{L}\{x(t)\} = -\frac{e^{-k\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a}$$

where, $k = \sigma + a = \sigma - (-a)$

When $\sigma > -a$, $k = \sigma - (-a) =$ Positive, $\therefore e^{-k\infty} = e^{-\infty} = 0$

When $\sigma < -a$, $k = \sigma - (-a) =$ Negative, $\therefore e^{-k\infty} = e^{+\infty} = \infty$

Hence we can say that, X(s) converges, when $\sigma > -a$, and does not converge for $\sigma < -a$.

$\therefore$ Abscissa of convergence, $\sigma_c = -a$.

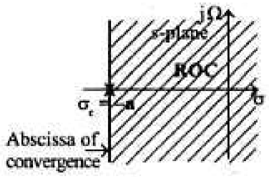


Fig 3.2 : ROC of $x(t) = e^{-at}u(t)$.

When $\sigma > -a$, the X(s) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = -\frac{e^{-k\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} = -\frac{0 \times e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} = \frac{1}{s+a}$$

Therefore for a causal signal the ROC includes all points on the s-plane to the right of abscissa of convergence, $\sigma_c = -a$, as shown in fig 3.2.

***Case ii :** Left sided (anticausal) signal*

Let, $x(t) = e^{-bt}u(-t) = e^{-bt}$ for $t \leq 0$, where $b > 0$

Now, the Laplace transform of x(t) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} e^{-bt}\, u(-t)\, e^{-st}\, dt = \int_{-\infty}^{0} e^{-bt}\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{-(s+b)t}\, dt = \left[\frac{e^{-(s+b)t}}{-(s+b)}\right]_{-\infty}^{0} = \frac{e^{0}}{-(s+b)} - \frac{e^{(\sigma + j\Omega + b)\infty}}{-(s+b)}$$

Put, $s = \sigma + j\Omega$

$$= -\frac{1}{s+b} + \frac{e^{(\sigma+b)\times\infty} e^{j\Omega\times\infty}}{s+b} = -\frac{1}{s+b} + \frac{e^{k\times\infty}\, e^{j\Omega\times\infty}}{s+b}$$

where, $k = \sigma + b = \sigma - (-b)$

When $\sigma > -b$, $k = \sigma - (-b)$ = Positive, $\therefore e^{k\infty} = e^{\infty} = \infty$

When $\sigma < -b$, $k = \sigma - (-b)$ = Negative, $\therefore e^{k\infty} = e^{-\infty} = 0$

Hence we can say that, X(s) converges, when $\sigma < -b$, and does not converge for $\sigma > -b$.

$\therefore$ Abscissa of convergence, $\sigma_c = -b$.

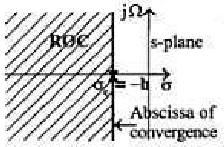


Fig 3.3 : ROC of $x(t) = e^{-bt}u(-t)$.

When $\sigma < -b$, the X(s) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = -\frac{1}{s+b} + \frac{e^{k\times\infty}\, e^{j\Omega\times\infty}}{s+b} = -\frac{1}{s+b} + \frac{0 \times e^{j\Omega\times\infty}}{s+b} = -\frac{1}{s+b}$$

Therefore for an anticausal signal the ROC includes all points on the s-plane to the left of abscissa of convergence, $\sigma_c = -b$, as shown in fig 3.3.

Table : Some Standard Laplace Transform Pairs

Note : σ = Real part of s

x(t)	X(s)	ROC
$\delta(t)$	1	Entire s-plane
u(t)	$\frac{1}{s}$	$\sigma > 0$
t u(t)	$\frac{1}{s^2}$	$\sigma > 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$ where, n = 1, 2, 3, ...	$\frac{1}{s^n}$	$\sigma > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\sigma > -a$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\sigma < -a$
$t^n u(t)$ where, n = 1, 2, 3	$\frac{n!}{s^{n+1}}$	$\sigma > 0$
$t\, e^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\sigma > -a$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t)$ where, n = 1, 2, 3, ...	$\frac{1}{(s+a)^n}$	$\sigma > -a$

$t^n e^{-at} u(t)$ where, n = 1, 2, 3, ...	$\frac{n!}{(s+a)^{n+1}}$	$\sigma > -a$
$\sin\Omega_0 t\, u(t)$	$\frac{\Omega_0}{s^2 + \Omega_0^2}$	$\sigma > 0$
$\cos\Omega_0 t\, u(t)$	$\frac{s}{s^2 + \Omega_0^2}$	$\sigma > 0$
$\sinh\Omega_0 t\, u(t)$	$\frac{\Omega_0}{s^2 - \Omega_0^2}$	$\sigma > \Omega_0$
$\cosh\Omega_0 t\, u(t)$	$\frac{s}{s^2 - \Omega_0^2}$	$\sigma > \Omega_0$
$e^{-at} \sin\Omega_0 t\, u(t)$	$\frac{\Omega_0}{(s+a)^2 + \Omega_0^2}$	$\sigma > -a$
$e^{-at} \cos\Omega_0 t\, u(t)$	$\frac{s+a}{(s+a)^2 + \Omega_0^2}$	$\sigma > -a$

Example

Let, $X(s) = \mathcal{L}\{x(t)\}$. Determine the initial value, x(0) and the final value, x(∞), for the following signals using initial value and final value theorems.

a) $X(s) = \frac{1}{s(s-1)}$ b) $X(s) = \frac{s+1}{s^2 + 2s + 2}$ c) $X(s) = \frac{7s+6}{s(3s+5)}$

d) $X(s) = \frac{s^2+1}{s^2 + 6s + 5}$ e) $X(s) = \frac{s+5}{s^2(s+9)}$

Solution

a) Given that, $X(s) = \frac{1}{s(s-1)}$

$$\text{Initial value, } x(0) = \underset{s\to\infty}{Lt}\ s\,X(s) = \underset{s\to\infty}{Lt}\ s\frac{1}{s(s-1)} = \underset{s\to\infty}{Lt}\ \frac{1}{(s-1)} = \underset{s\to\infty}{Lt}\ \frac{1}{s\left(1-\frac{1}{s}\right)}$$

$$= \underset{s\to\infty}{Lt}\ \frac{1}{s}\ \frac{1}{\left(1-\frac{1}{s}\right)} = \frac{1}{\infty}\ \frac{1}{\left(1-\frac{1}{\infty}\right)} = 0 \times \frac{1}{1-0} = 0$$

$$\text{Final value, } x(\infty) = \underset{s\to 0}{Lt}\ sX(s) = \underset{s\to 0}{Lt}\ s\frac{1}{s(s-1)} = \underset{s\to 0}{Lt}\ \frac{1}{(s-1)} = \frac{1}{0-1} = -1$$

b) Given that, $X(s) = \frac{s+1}{s^2 + 2s + 2}$

$$\text{Initial value, } x(0) = \underset{s\to\infty}{Lt}\ sX(s) = \underset{s\to\infty}{Lt}\ s\frac{s+1}{s^2+2s+2} = \underset{s\to\infty}{Lt}\ s\frac{s\left(1+\frac{1}{s}\right)}{s^2\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}$$

$$= \underset{s\to\infty}{Lt}\ \frac{1+\frac{1}{s}}{1+\frac{2}{s}+\frac{2}{s^2}} = \frac{1+\frac{1}{\infty}}{1+\frac{2}{\infty}+\frac{2}{\infty}} = \frac{1+0}{1+0+0} = 1$$

$$\text{Final value, } x(\infty) = \underset{s\to 0}{Lt}\ sX(s) = \underset{s\to 0}{Lt}\ s\frac{s+1}{s^2+2s+2}$$

$$= 0 \times \frac{0+1}{0+0+2} = 0$$

c) Given that, $X(s) = \frac{7s + 6}{s(3s + 5)}$

$$\text{Initial value, } x(0) = \underset{s\to\infty}{Lt}\ sX(s) = \underset{s\to\infty}{Lt}\ s\frac{7s+6}{s(3s+5)} = \underset{s\to\infty}{Lt}\ s\frac{s\left(7+\frac{6}{s}\right)}{s^2\left(3+\frac{5}{s}\right)}$$

$$= \underset{s\to\infty}{Lt}\ \frac{7+\frac{6}{s}}{3+\frac{5}{s}} = \frac{7+\frac{6}{\infty}}{3+\frac{5}{\infty}} = \frac{7+0}{3+0} = \frac{7}{3}$$

$$\text{Final value, } x(\infty) = \underset{s\to 0}{Lt}\ sX(s) = \underset{s\to 0}{Lt}\ s\frac{7s+6}{s(3s+5)}$$

$$= \underset{s\to 0}{Lt}\ \frac{7s+6}{3s+5} = \frac{0+6}{0+5} = \frac{6}{5}$$

d) Given that, $X(s) = \frac{s^2 + 1}{s^2 + 6s + 5}$

$$\text{Initial value, } x(0) = \underset{s\to\infty}{Lt}\ sX(s) = \underset{s\to\infty}{Lt}\ s\frac{s^2+1}{s^2+6s+5} = \underset{s\to\infty}{Lt}\ s\frac{s^2\left(1+\frac{1}{s^2}\right)}{s^2\left(1+\frac{6}{s}+\frac{1}{s^2}\right)}$$

$$= \underset{s\to\infty}{Lt}\ s\frac{1+\frac{1}{s^2}}{1+\frac{6}{s}+\frac{1}{s^2}} = \infty \times \frac{1+\frac{1}{\infty}}{1+\frac{6}{\infty}+\frac{1}{\infty}} = \infty \times \frac{1+0}{1+0+0} = \infty$$

$$\text{Final value, } x(\infty) = \underset{s\to 0}{Lt}\ sX(s) = \underset{s\to 0}{Lt}\ s\frac{s^2+1}{s^2+6s+5} = 0 \times \frac{0+1}{0+0+5} = 0$$

e) Given that, $X(s) = \frac{s + 5}{s^2(s + 9)}$

$$\text{Initial value, } x(0) = \underset{s\to\infty}{Lt}\ sX(s) = \underset{s\to\infty}{Lt}\ s\frac{s+5}{s^2(s+9)} = \underset{s\to\infty}{Lt}\ s\frac{s\left(1+\frac{5}{s}\right)}{s^3\left(1+\frac{9}{s}\right)}$$

$$= \underset{s\to\infty}{Lt}\ \frac{1}{s}\ \frac{1+\frac{5}{s}}{1+\frac{9}{s}} = \frac{1}{\infty} \times \frac{1+\frac{1}{\infty}}{1+\frac{9}{\infty}} = 0 \times \frac{1+0}{1+0} = 0$$

$$\text{Final value, } x(\infty) = \underset{s\to 0}{Lt}\ sX(s) = \underset{s\to 0}{Lt}\ s\frac{s+5}{s^2(s+9)} = \underset{s\to 0}{Lt}\ \frac{s+5}{s^2+9s} = \frac{0+5}{0+0} = \infty$$

2.3.3 LAPLACE TRANSFORMS OF SOME IMPORTANT FUNCTIONS

I. Unit Step Function

$$f(t) = 1,\ 0 < t < \infty$$

$$\mathcal{L}\{u(t)\} = \int_0^\infty 1 \cdot e^{-st} dt = -\frac{1}{s}\left[e^{-st}\right]_0^\infty = -\frac{1}{s}[0-1] = \frac{1}{s}$$

2. Exponential function

$$f(t) = Ae^{-at}$$

$$\mathcal{L}\{Ae^{-at}\} = \int_0^\infty A\, e^{-at} e^{-st}\, dt$$

$$= A\int_0^\infty e^{-(a+s)t}\, dt$$

$$= -\frac{A}{a+s}\left[e^{-(a+s)t}\right]_0^\infty = \frac{A}{(s+a)}$$

Hence, $\mathcal{L}\{Ae^{-at}\} = \dfrac{A}{(s+a)}$

3. Sine Function

$$f(t) = \sin \omega_0 t$$

Using Euler's identity, we have

$$\sin \omega_0 t = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$$

Hence, $\mathcal{L}\{\sin \omega_0 t\} = \dfrac{1}{2j}\left[\mathcal{L}(e^{j\omega_0 t}) - \mathcal{L}(e^{-j\omega_0 t})\right]$

$$= \frac{1}{2j}\left[\frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0}\right] = \frac{\omega_0}{s^2 + \omega_0^2}$$

Hence, $\mathcal{L}\{\sin \omega_0 t\} = \dfrac{\omega_0}{s^2 + \omega_0^2}$

4. Cosine Function

$$f(t) = \cos \omega_0 t$$

We know that $\cos \omega_0 t = \dfrac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$

$$\mathcal{L}\{\cos \omega_0 t\} = \frac{1}{2}\left[\mathcal{L}(e^{j\omega_0 t}) + \mathcal{L}(e^{-j\omega_0 t})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s - j\omega_0} + \frac{1}{s + j\omega_0}\right] = \frac{s}{s^2 + \omega_0^2}$$

Hence, $\mathcal{L}\{\cos \omega_0 t\} = \dfrac{s}{s^2 + \omega_0^2}$

5. Hyperbolic Sine and Cosine Functions

$$\sinh \omega_0 t = \frac{1}{2}\left[e^{\omega_0 t} - e^{-\omega_0 t}\right]$$

$$\cosh \omega_0 t = \frac{1}{2}\left[e^{\omega_0 t} + e^{-\omega_0 t}\right]$$

$$\mathcal{L}\{\sinh \omega_0 t\} = \frac{1}{2}\left[\mathcal{L}(e^{\omega_0 t}) - \mathcal{L}(e^{-\omega_0 t})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-\omega_0} - \frac{1}{s+\omega_0}\right] = \frac{\omega_0}{s^2-\omega_0^2}$$

$$\mathcal{L}\{\sinh \omega_0 t\} = \frac{\omega_0}{s^2-\omega_0^2}$$

Similarly, $\mathcal{L}\{\cosh \omega_0 t\} = \frac{1}{2}\left[\mathcal{L}(e^{\omega_0 t}) + \mathcal{L}(e^{-\omega_0 t})\right]$

6. Damped Sine and Cosine Functions

$$\mathcal{L}\{e^{-at} \sin \omega_0 t\} = \mathcal{L}\left\{e^{-at}\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right)\right\}$$

$$= \frac{1}{2j}\left[\mathcal{L}\{e^{-(a-j\omega_0)t} - e^{-(a+j\omega_0)t}\}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s+(a-j\omega_0)} - \frac{1}{s+(a+j\omega_0)}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{(s+a)-j\omega_0} - \frac{1}{(s+a)+j\omega_0}\right]$$

$$= \frac{\omega_0}{(s+a)^2+\omega_0^2}$$

Hence, $\mathcal{L}\{e^{-at} \sin \omega_0 t\} = \frac{\omega_0}{(s+a)^2+\omega_0^2}$

Similarly, $\mathcal{L}\{e^{-at} \cos \omega_0 t\} = \mathcal{L}\left\{e^{-at}\left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right)\right\}$

$$= \frac{s+a}{(s+a)^2+\omega_0^2}$$

Hence, $\mathcal{L}\{e^{-at} \cos \omega_0 t\} = \frac{s+a}{(s+a)^2+\omega_0^2}$

7. Damped Hyperbolic Sine and Cosine Functions

$$\mathcal{L}\{e^{-at} \sinh \omega_0 t\} = \mathcal{L}\left\{e^{-at}\left(\frac{e^{\omega_0 t} - e^{-\omega_0 t}}{2}\right)\right\}$$

$$= \frac{1}{2}\left[\mathcal{L}(e^{-(a-\omega_0)t}) - \mathcal{L}(e^{-(a+\omega_0)t})\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+a-\omega_0} - \frac{1}{s+a+\omega_0}\right]$$

$$= \frac{\omega_0}{(s+a)^2-\omega_0^2}$$

Hence, $\mathcal{L}\{e^{-at}\sinh\omega_0 t\} = \dfrac{\omega_0}{(s+a)^2-\omega_0^2}$

Similarly, $\mathcal{L}\{e^{-at}\cosh\omega_0 t\} = \dfrac{s+a}{(s+a)^2-\omega_0^2}$

8. t^n Function

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}\,dt = \int_0^\infty t^n\, d\left(\frac{e^{-st}}{-s}\right)$$

$$= \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\, nt^{n-1}\,dt$$

$$= \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1}\,dt$$

$$= \frac{n}{s}\mathcal{L}(t^{n-1})$$

Similarly, $\mathcal{L}\{t^{n-1}\} = \dfrac{n-1}{s}\mathcal{L}(t^{n-2})$

By taking Laplace transformations of t^{n-2}, t^{n-3}, ... and substituting in the above equation, we get

$$\mathcal{L}\{t^n\} = \frac{n}{s}\frac{n-1}{s}\frac{n-2}{s}\cdots\frac{2}{s}\frac{1}{s}\mathcal{L}(t^{n-n})$$

$$= \frac{n!}{s^n}\mathcal{L}(t^0) = \frac{n!}{s^n}\times\frac{1}{s} = \frac{n!}{s^{n+1}}, \text{ when } n \text{ is a positive integer.}$$

Therefore, $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$

Substituting $n = 1$, we have $\mathcal{L}\{t\} = 1/s^2$

Example Find the Laplace transform of the following functions

(a) $f(t) = t^3 + 3t^2 - 6t + 4$

(b) $f(t) = \cos^3 3t$

(c) $f(t) = \sin at \cos bt$

(d) $f(t) = t \sin at$

(e) $f(t) = \dfrac{1-e^t}{t}$

(f) $f(t) = \delta(t^2 - 3t + 2)$

Solution

(a) $\mathcal{L}\{f(t)\} = \mathcal{L}[t^3 + 3t^2 - 6t + 4]$

$$= \frac{3!}{s^4} + 3\frac{2!}{s^3} - 6\frac{1!}{s^2} + \frac{4}{s}$$

$$= \frac{6}{s^4} + \frac{6}{s^3} - \frac{6}{s^2} + \frac{4}{s}$$

(b) $f(t) = \cos^3 3t$

We know that $\cos 3A = 4\cos^3 A - 3\cos A$

Therefore, $\mathcal{L}\{\cos^3 3t\} = \mathcal{L}\left[\dfrac{\cos 9t + 3\cos 3t}{4}\right]$

$$= \frac{1}{4}\left[\frac{s}{s^2+9^2} + 3\frac{s}{s^2+3^2}\right]$$

$$= \frac{1}{4}\left[\frac{s}{s^2+81} + \frac{3s}{s^2+9}\right]$$

(c) $\mathcal{L}\{\sin at \cos bt\} = \mathcal{L}\left[\dfrac{1}{2}\{\sin (a+b)t + \sin (a-b)t\}\right]$

$$= \frac{1}{2}\left[\frac{a+b}{s^2+(a+b)^2} + \frac{a-b}{s^2+(a-b)^2}\right]$$

(d) $\mathcal{L}\{t \sin at\} = -\dfrac{d}{ds}\mathcal{L}[\sin at]$

$$= -\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]$$

$$= -a\frac{d}{ds}[(s^2+a^2)^{-1}]$$

$$= -a\left[-\frac{1}{(s^2+a^2)^2}\cdot 2s\right]$$

$$= \frac{2as}{(s^2+a^2)^2}$$

(e) $f(t) = \left[\dfrac{1-e^t}{t}\right]$

Here, $\mathcal{L}\{1-e^t\} = \dfrac{1}{s} - \dfrac{1}{(s-1)}$

$$\mathcal{L}\left\{\frac{1-e^t}{t}\right\} = \int_s^\infty \left[\frac{1}{s} - \frac{1}{(s-1)}\right] ds = [\log s - \log (s-1)]_s^\infty$$

$$= \left[\log \frac{s}{s-1}\right]_s^\infty$$

$$= \left[-\log \frac{s-1}{s}\right]_s^\infty$$

$$= \left[-\log\left(1-\frac{1}{s}\right)\right]_s^\infty$$

$$= \log\left(1-\frac{1}{s}\right) = \log\left(\frac{s-1}{s}\right)$$

(f) The given impulse function is $f(t) = \delta(t^2 - 3t + 2)$

$$= \delta[(t-1)(t-2)]$$

$$= \delta(t-1)\,u(t-1) + \delta(t-2)\,u(t-2)$$

$$= \delta(t-1) + \delta(t-2)$$

Therefore, $F(s) = \mathcal{L}\,[\delta(t-1) + \mathcal{L}\,[\delta(t-2)]$

$$= e^{-s} + e^{-2s}$$

Example Determine the Laplace transform of the rectangular pulse shown in Fig. **E3.2**.

Solution $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$

$$= \int_0^T 1 \cdot e^{-st} dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^T$$

$$= \frac{1}{-s}\,[e^{-sT} - 1]$$

$$= \frac{1}{s}\,[1 - e^{-sT}]$$

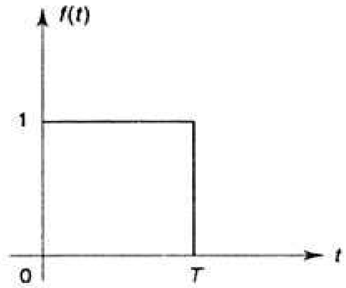


Fig. E3.2

Example For the waveform shown in Fig. E3.6, find the Laplace transform.

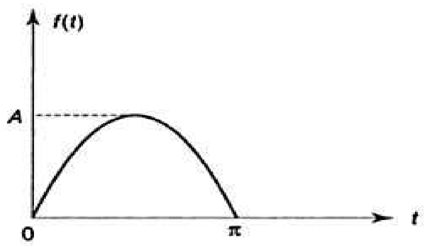


Fig. E3.6

Solution The function for the given waveform is

$f(t) = A \sin t$ for $0 < t < \pi$

$= 0,$ $t > \pi$

By definition, we have

$$\mathcal{L}\{f(t)\} = \int^\infty f(t)\, e^{-st} dt$$

$$= \int_0^{\pi} A \sin t \, e^{-st} dt$$

$$= A \int_0^{\pi} \sin t \, e^{-st} dt$$

$$= \frac{A}{(s^2+1)} [e^{-st}(-s \sin t - \cos t)]_0^{\pi}$$

$$= A \frac{e^{-s\pi}+1}{(s^2+1)}$$

INITIAL AND FINAL VALUE THEOREMS

Initial Value Theorem

If the function $f(t)$ and its derivative $f'(t)$ are Laplace transformable, then

$$\underset{t \to 0^+}{\text{Lt}} f(t) = \underset{s \to \infty}{\text{Lt}} sF(s)$$

Proof We know that

$$\mathcal{L}\{f'(t)\} = s[\mathcal{L}(f(t))] - f(0)$$

By taking the limit $s \to \infty$ on both sides

$$\underset{s \to \infty}{\text{Lt}} \mathcal{L}\{f'(t)\} = \underset{s \to \infty}{\text{Lt}} [sF(s) - f(0)]$$

$$\underset{s \to \infty}{\text{Lt}} \int_0^{\infty} f'(t) e^{-st} dt = \underset{s \to \infty}{\text{Lt}} [sF(s) - f(0)]$$

As $s \to \infty$, the integration of LHS becomes zero

i.e.
$$\int_0^{\infty} \underset{s \to \infty}{\text{Lt}} [f'(t) e^{-st}] dt = 0$$

$$\underset{s \to \infty}{\text{Lt}} sF(s) - f(0) = 0$$

Therefore,
$$\underset{s \to \infty}{\text{Lt}} sF(s) = f(0) = \underset{t \to 0+}{\text{Lt}} f(t)$$

Final Value Theorem

If $f(t)$ and $f'(t)$ are Laplace transformable, then

$$\underset{t \to \infty}{\text{Lt}} f(t) = \underset{s \to 0}{\text{Lt}} sF(s)$$

Proof We know that

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Taking the limit $s \to 0$ on both sides, we get

$$\underset{s \to 0}{\text{Lt}} \mathcal{L}\{f'(t)\} = \underset{s \to 0}{\text{Lt}} [sF(s) - f(0)]$$

$$\underset{s\to 0}{\mathrm{Lt}}\int_0^\infty f'(t)\, e^{-st}\, dt = \underset{s\to 0}{\mathrm{Lt}}\; sF(s) - f(0)]$$

Therefore,
$$\int_0^\infty f'(t) dt = \underset{s\to 0}{\mathrm{Lt}}\; [sF(s) - f(0)]$$

$$[f(t)]_0^\infty = \underset{t\to\infty}{\mathrm{Lt}}\; f(t) - \underset{t\to 0}{\mathrm{Lt}}\; f(t) = \underset{s\to 0}{\mathrm{Lt}}\; sF(s) - f(0)$$

Since $f(0)$ is not a function of s, it gets cancelled from both sides of the above equation.

Therefore,
$$\underset{t\to\infty}{\mathrm{Lt}}\; f(t) = \underset{s\to 0}{\mathrm{Lt}}\; sF(s)$$

2.3.4 LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

The time-shift theorem is useful in determining the transform of periodic time functions. Let function $f(t)$ be a causal periodic waveform which satisfies the condition $f(t) = f(t + nT)$ for all $t > 0$ where T is the period of the function and $n = 0, 1, 2, \ldots$

$$F(s) = \int_0^\infty f(t)e^{-st}\, dt$$

$$= \int_0^T f(t)e^{-st}\, dt + \int_T^{2T} f(t)e^{-st}\, dt + \cdots + \int_{nT}^{(n+1)T} f(t)e^{-st}\, dt + \cdots$$

As $f(t)$ is periodic, the above equation becomes

$$= \int_0^T f(t)e^{-st}\, dt + e^{-sT}\int_0^T f(t)e^{-st}\, dt + \cdots + e^{-nsT}\int_0^T f(t)e^{-st}\, dt + \cdots$$

$$= \left[1 + e^{-sT} + e^{-2sT} + \cdots + e^{-nsT} + \cdots\right]\int_0^T f(t)\, e^{-st}\, dt$$

$$= \left[1 + e^{-sT} + \left(e^{-sT}\right)^2 + \cdots + \left(e^{-sT}\right)^n + \cdots\right] F_1(s)$$

where
$$F_1(s) = \int_0^T f(t)\, e^{-st}\, dt$$

Here, $F_1(s) = \mathcal{L}\{[u(t) - u(t-T)]\, f(t)\}$, which is the transform of the first period of the time function, and $\{[u(t) - u(t-T)]\, f(t)\}$ has non-zero only in the first period of $f(t)$.

When we apply the binomial theorem to the bracketed expression, it becomes $1/(1 - e^{-sT})$

$$F(s) = \frac{1}{1 - e^{-sT}}\int_0^T f(t)\, e^{-st}\, dt = \frac{F_1(s)}{1 - e^{-sT}}$$

Example Find the Laplace transform of the periodic rectangular waveform shown in Fig. E3.13.

Solution Here the period is $2T$

Therefore, $\mathcal{L}\{f(t)\} = \dfrac{1}{1-e^{-2sT}}\left[\int_0^{2T} f(t)\,e^{-st}\,dt\right]$

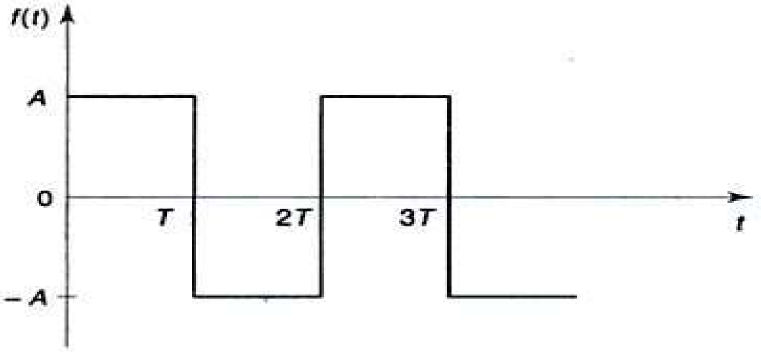


Fig. E3.13

$$= \frac{1}{1-e^{-2sT}}\left[\int_0^{T} A\,e^{-st}\,dt + \int_T^{2T}(-A)\,e^{-st}\,dt\right]$$

$$= \frac{1}{1-e^{-2sT}}\left[\frac{-A}{s}\left(e^{-st}\right)_0^T + \frac{A}{s}\left(e^{-st}\right)_T^{2T}\right]$$

$$= \frac{1}{1-e^{-2sT}}\left[-\frac{A}{s}\left(e^{-sT}-1\right) + \frac{A}{s}\left(e^{-2sT}-e^{-sT}\right)\right]$$

$$= \frac{1}{1-e^{-2sT}}\left[-\frac{A}{s}\left(e^{-sT}-1\right) + \frac{A}{s}\left(e^{-2sT}-e^{-sT}\right)\right]$$

$$= \frac{1}{1-e^{-2sT}}\,\frac{A}{s}\left(1-2e^{-sT}+e^{-2sT}\right)$$

$$= \frac{1}{1-e^{-2sT}}\left[\frac{A}{s}\left(1-e^{-sT}\right)^2\right]$$

$$= \frac{A}{s}\left(\frac{\left(1-e^{-sT}\right)^2}{\left(1-e^{-sT}\right)\left(1+e^{-sT}\right)}\right)$$

$$= \frac{A}{s}\left(\frac{1-e^{-sT}}{1+e^{-sT}}\right) = \frac{A}{s}\tanh\left(\frac{sT}{2}\right)$$

Example Find the Laplace transform of the periodic sawtooth waveform shown in Fig. E3.14.

Solution For the given transform, the period of one cycle is T. Hence,

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\left[\int_0^T f(t)\, e^{-st}\, dt\right], \text{ where } f(t) = \frac{A}{T}t$$

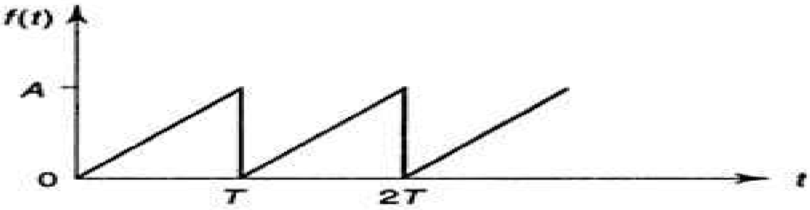


Fig. E3.14

$$= \frac{1}{1-e^{-sT}}\left[\int_0^T \frac{A}{T}\, t\, e^{-st}\, dt\right]$$

$$= \frac{1}{1-e^{-sT}} \cdot \frac{A}{T}\int_0^T t\, e^{-st}\, dt$$

$$= \frac{A}{T} \cdot \frac{1}{1-e^{-sT}}\left[\left\{t\,\frac{e^{-st}}{-s}\right\}_0^T - \left\{\frac{e^{-st}}{s^2}\right\}_0^T\right]$$

$$= \frac{A}{T} \cdot \frac{1}{1-e^{-sT}}\left[T\,\frac{e^{-sT}}{-s} - \frac{e^{-sT}}{s^2} + \frac{1}{s^2}\right]$$

$$= \frac{A}{Ts^2(1-e^{-sT})}(1 - e^{-sT} - sTe^{-sT})$$

Example Find the Laplace transform of the full wave rectified output as shown in Fig. E3.15.

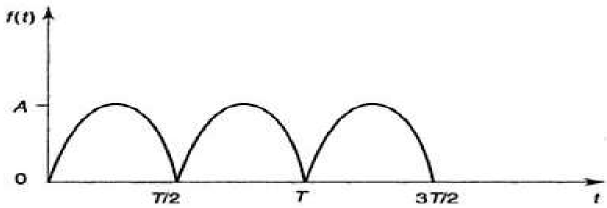


Fig. E3.15

Solution The function for the given waveform is

$$f(t) = A \sin \omega_0 t \quad \text{for} \quad 0 < t < T/2$$

Hence,

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT/2}}\int_0^{T/2} f(t)\, e^{-st}\, dt$$

$$= \frac{A}{1-e^{-sT/2}} \int_0^{T/2} \sin \omega_0 t \, e^{-st} \, dt$$

$$= \frac{A}{1-e^{-sT/2}} \left[\frac{e^{-st}}{s^2+\omega_0^2} (-s \sin \omega_0 t - \omega_0 \cos \omega_0 t) \right]_0^{T/2}$$

$$= \frac{A}{1-e^{-sT/2}} \cdot \frac{1}{(s^2+\omega_0^2)} [\omega_0 e^{-sT/2} + \omega_0]$$

$$= \frac{A\omega_0}{s^2+\omega_0^2} \frac{(1+e^{-sT/2})}{(1-e^{-sT/2})}$$

$$= \frac{A\omega_0}{s^2+\omega_0^2} \frac{e^{sT/4}+e^{-sT/4}}{e^{sT/4}-e^{-sT/4}}$$

$$= \frac{A\omega_0}{s^2+\omega_0^2} \coth (sT/4)$$

Example Obtain the trigonometric Fourier series for the half-wave rectified sine wave shown in Fig. E2.3.

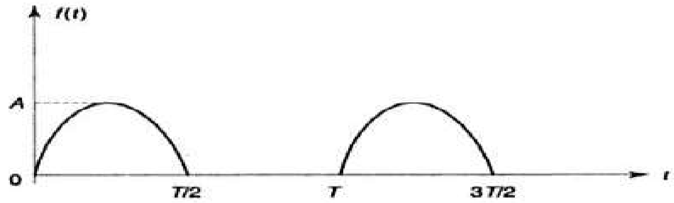


Fig. E2.3

Solution As the waveform shows no symmetry, the series may contain both sine and cosine terms. Here, $f(t) = A \sin \omega_0 t$

To evaluate a_0:

$$a_0 = \frac{2}{T} \int_0^T A \sin \omega_0 t \, dt$$

$$= \frac{2}{T} \int_0^{T/2} A \sin \omega_0 t \, dt$$

$$= \frac{2A}{\omega_0 T} [-\cos \omega_0 t]_0^{T/2} = \frac{2A}{\omega_0 T} [-\cos (\omega_0 T/2) + 1]$$

Substituting $\omega_0 T = 2\pi$, we have $a_0 = \frac{2A}{\pi}$.

To evaluate a_n:

$$a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega_0 t \, dt$$

$$= \frac{2}{T} \int_0^{T/2} A \sin \omega_0 t \cos n\omega_0 t \, dt$$

$$= \frac{2A}{\omega_0 T} \left[\frac{-n \sin \omega_0 t \sin n\omega_0 t - \cos n\omega_0 t \cos \omega_0 t}{-n^2+1} \right]_0^{T/2}$$

Substituting $\omega_0 T = 2\pi$, we have

$$a_n = \frac{A}{\pi(1-n^2)} [\cos n\pi + 1]$$

Hence, $a_n = \frac{2A}{\pi(1-n^2)}$, for n even

$= 0$, for n odd

For $n = 1$, this expression is infinite and hence we have to integrate separately to evaluate a_1.

Therefore, $a_1 = \frac{2}{T}\int_0^{T/2} A \sin \omega_0 t \cos \omega_0 t \, dt$

$$= \frac{A}{T}\int_0^{T/2} \sin 2\omega_0 t \, dt$$

$$= \frac{A}{2\omega_0 T}[-\cos 2\omega_0 t]_0^{T/2}$$

When $\omega_0 T = 2\pi$, we have $a_1 = 0$.
To find b_n:

$$b_n = \frac{2}{T}\int_0^{T} f(t) \sin n\omega_0 t \, dt$$

$$= \frac{2}{T}\int_0^{T/2} A \sin \omega_0 t \sin n\omega_0 t \, dt$$

$$= \frac{2A}{\omega_0 T}\left[\frac{n \sin \omega_0 t \cos n\omega_0 t - \sin n\omega_0 t \cos \omega_0 t}{-n^2+1}\right]_0^{T/2}$$

When $\omega_0 T = 2\pi$, we have $b_n = 0$.
For $n = 1$, the expression is infinite and hence b_1 has to be calculated separately.

$$b_1 = \frac{2A}{T}\int_0^{T/2} \sin^2 \omega_0 t \, dt$$

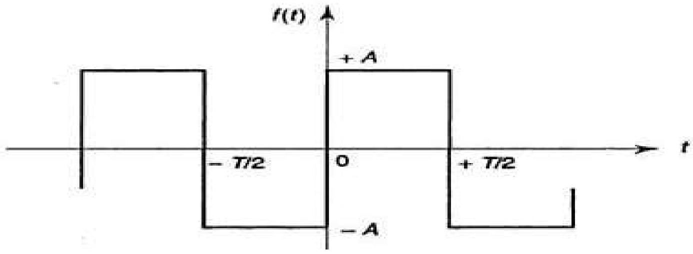
$$= \frac{2A}{\omega_0 T}\left[\frac{\omega_0 t}{2} - \frac{\sin 2\omega_0 t}{4}\right]_0^{T/2}$$

When $\omega_0 T = 2\pi$, we have $b_1 = \frac{A}{2}$.

Substituting the values of the coefficients in Eq. 2.2, we get

$$f(t) = \frac{A}{\pi}\left\{1 + \frac{\pi}{2}\sin \omega_0 t - \frac{2}{3}\cos 2\omega_0 t - \frac{2}{15}\cos 4\omega_0 t - \cdots\right\}$$

Determine the exponential Fourier series and hence find a_n and b_n of the trigonometric series and compare the results.



To evaluate c_n
Since the wave is odd, c_n consists of pure imaginary coefficients. From Eq. 2.7, we have

$$c_n = \frac{1}{T}\int_0^T f(t)\, e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T}\left[\int_{-T/2}^{0} (-A)\, e^{-jn\omega_0 t}\, dt + \int_0^{T/2} A\, e^{-jn\omega_0 t}\, dt\right]$$

$$= \frac{A}{T}\left\{\left[(-1)\frac{1}{(-jn\omega_0)}\, e^{-jn\omega_0 t}\right]_{-T/2}^{0} + \left[\frac{1}{(-jn\omega_0)}\, e^{-jn\omega_0 t}\right]_0^{T/2}\right\}$$

$$= \frac{A}{T}\cdot\frac{1}{(-jn\omega_0)}\left\{-e^0 + e^{jn\omega_0(T/2)} + e^{-jn\omega_0(T/2)} - e^0\right\}$$

When $\omega_0 = \frac{2\pi}{T}$, we get

$$c_n = \frac{A}{T}\cdot\frac{T}{-jn2\pi}\left\{-e^0 + e^{jn(2\pi/T)(T/2)} + e^{-jn(2\pi/2)(T/2)} - e^0\right\}$$

$$= \frac{A}{(-j2\pi n)}\left\{-e^0 + e^{jn\pi} + e^{-jn\pi} - e^0\right\} = j\frac{A}{n\pi}(e^{jn\pi} - 1)$$

Here, $e^{jn\pi} = +1$ for even n and $e^{jn\pi} = -1$ for odd n

Therefore, $c_n = -j\left(\frac{2A}{n\pi}\right)$ for odd n only.

2.4 PARSEVAL'S IDENTITY FOR FOURIER SERIES

A periodic function $f(t)$ with a period T is expressed by the Fourier series as

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

Now, $$[f(t)]^2 = \frac{1}{2}a_0 f(t) + \sum_{n=1}^{\infty}[a_n f(t)\cos n\omega_0 t + b_n f(t)\sin n\omega_0 t]$$

Therefore, $$\frac{1}{T}\int_{-T/2}^{T/2}[f(t)]^2\, dt = \frac{(a_0/2)}{T}\int_{-T/2}^{T/2}[f(t)]\, dt$$

$$+\frac{1}{T}\sum_{n=1}^{\infty}\left[a_n\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\, dt + b_n\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\, dt\right]$$

From Eqns 2.2, 2.3 and 2.4, we have

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\, dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\, dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\, dt$$

Therefore, substituting all these values, we get

$$\frac{1}{T}\int_{-T/2}^{T/2}[f(t)]^2\, dt = \left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

This is the ***Parseval's identity.***

2.5 POWER SPECTRUM OF A PERIODIC FUNCTION

The power of a periodic signal spectrum $f(t)$ in the time domain is defined as

$$P = \frac{1}{T} \int_{-T/2}^{T/2} [f(t)]^2 \, dt$$

The Fourier series for the signal $f(t)$ is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 t}$$

According to Parseval's relation, we have

$$P_{av} = \frac{1}{T} \int_{-T/2}^{T/2} [f(t)]^2 \, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} f(t) \sum_{n=-\infty}^{\infty} c_n \, e^{jn\omega_0 t} \, dt$$

$$= \sum_{n=-\infty}^{\infty} c_n \frac{1}{T} \int_{-T/2}^{T/2} f(t) \, e^{jn\omega_0 t} \, dt$$

$$= \sum_{n=-\infty}^{\infty} c_n \, c_{-n}$$

$$= \sum_{n=-\infty}^{\infty} |c_n|^2 \text{, watts}$$

2.6 Energy Spectrum for a Non-Periodic Function

For a non-periodic energy signal, such as a single pulse, the total energy in $(-\infty, \infty)$ is finite, whereas the average power, i.e. energy per unit time, is zero because $\frac{1}{T}$ tends to zero as T tends to infinity. Hence, the total energy associated with $f(t)$ is given by

$$E = \int_{-\infty}^{\infty} f^2(t) \, dt$$

Since, $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) \, e^{j\omega t} \, d\omega$, we obtain

$$E = \int_{-\infty}^{\infty} f(t) \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) \, e^{j\omega t} \, d\omega \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) \left[\int_{-\infty}^{\infty} f(t) \, e^{j\omega t} \, dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) \, F(-j\omega) \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) \, F^*(j\omega) \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2 \, d\omega$$

$$= \int_{-\infty}^{\infty} |F(f)|^2 \, df, \text{ joules}$$

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} |F(f)|^2 \, df$$

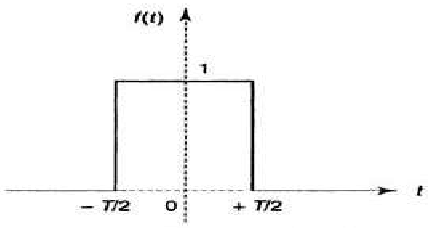
This result is called ***Rayleigh's energy theorem or Parseval's theorem*** for Fourier transform. The quantity $|F(f)|^2$ is referred to as the *energy spectral density*, $S(f)$, which is equal to the energy per unit frequency.

2.7 PROBLEMS

Gate Function

consider the single gate function (rectangular pulse) shown in Fig. It has the analytic expression given by

$$f(t) = \begin{cases} 1, & \text{for } -T/2 < t < T/2 \\ 0, & \text{otherwise} \end{cases}$$



Single Gate Function

The Fourier transform of $f(t)$ is

$$F(j\omega) = \mathcal{F}[f(t)] = \int_{-\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-T/2}^{T/2} 1 \cdot e^{-j\omega t}\, dt$$

$$= \frac{1}{-j\omega}\left[e^{-j\omega t}\right]_{-T/2}^{T/2}$$

$$= \frac{1}{-j\omega}\left[e^{-j\omega T/2} - e^{j\omega T/2}\right]$$

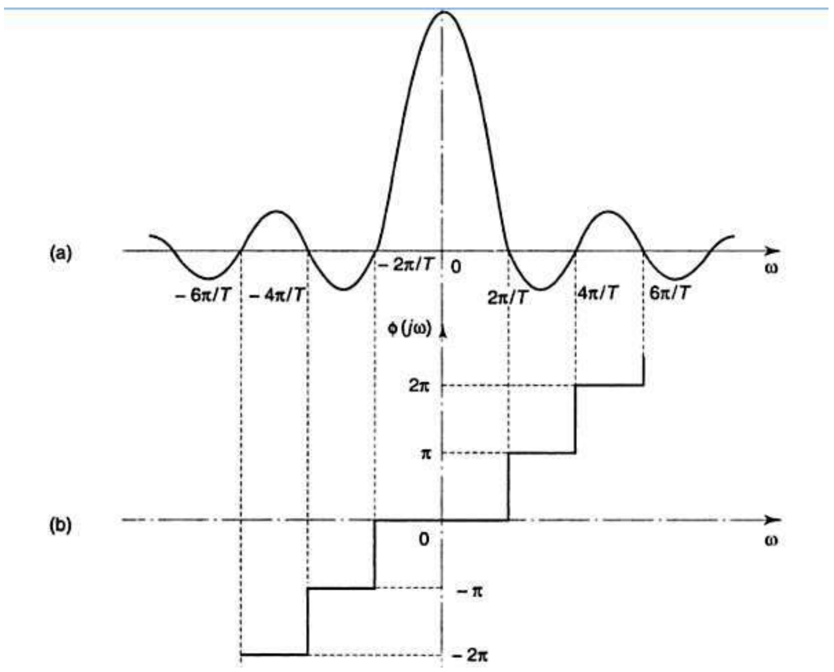
$$= T \cdot \frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)} = T \operatorname{sinc}\left(\frac{\omega T}{2}\right)$$

Hence, the amplitude spectrum is

$$|F(j\omega)| = T\left|\operatorname{sinc}\left(\frac{\omega T}{2}\right)\right|$$

and the phase spectrum is $\lfloor F(\omega) = \begin{cases} 0, & \operatorname{sinc}\left(\frac{\omega T}{2}\right) > 0 \\ \pi & \operatorname{sinc}\left(\frac{\omega T}{2}\right) < 0 \end{cases}$

AMPLITUDE SPECTRA AND PHASE SPECTRA



Rectangular Pulse

Consider the rectangular pulse shown in Fig. The analytic expression for the given pulse is

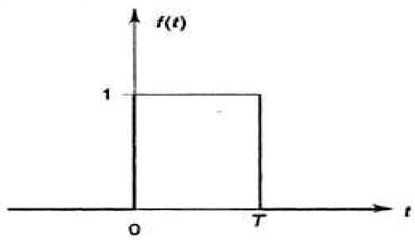


Fig.

$$f(t) = \begin{cases} 1, & \text{for } 0 < t < T \\ 0, & \text{otherwise} \end{cases}$$

The Fourier transform of $f(t)$ becomes

$$F(j\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_0^T e^{-j\omega t}\, dt = \left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^T = \frac{e^{-j\omega T} - 1}{-j\omega}$$

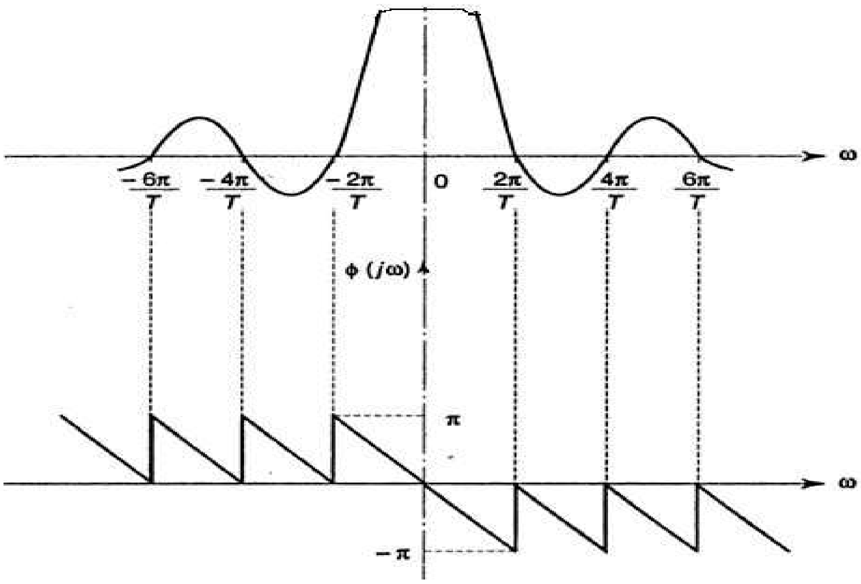
$$= \frac{e^{-j\omega T/2}}{-j\omega}\left[e^{-j\omega T/2} - e^{j\omega T/2}\right]$$

$$= \frac{2e^{-j\omega T/2}}{\omega}\left[\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j}\right]$$

$$= Te^{-j\omega T/2}\left[\frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)}\right]$$

$$= Te^{-j\omega T/2}\ \sin c\left(\frac{\omega T}{2}\right)$$

AMPLITUDE AND PHASE SPECTRA



Example Find the Fourier transform of a rectangular pulse 2 seconds long with a magnitude of 10 volts as shown in Fig. E2.16.

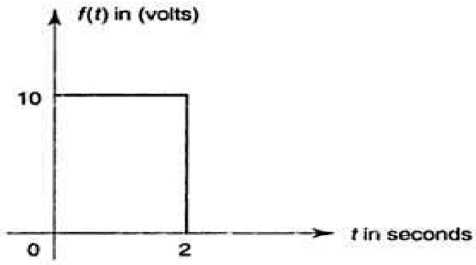


Fig. E2.16

Solution Fourier transform $F(j\omega)$ of the given pulse is given by

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$= \int_0^2 10\, e^{-j\omega t}\, dt = 10\left[\frac{e^{-j\omega t}}{-j\omega}\right]_0^2$$

$$= 10\left(\frac{e^{-j2\omega} - 1}{-j\omega}\right)$$

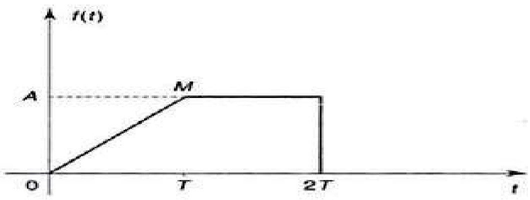
$$= 10\,\frac{e^{-j\omega}}{-j\omega}\left[e^{-j\omega} - e^{j\omega}\right]$$

$$= 20\,\frac{e^{-j\omega}}{\omega}\left[\frac{e^{j\omega} - e^{-j\omega}}{2j}\right]$$

$$= 20\, e^{-j\omega}\,\frac{\sin\omega}{\omega}$$

$$= 20\, e^{-j\omega}\,\text{sinc}\,\omega$$

Example Find the Fourier transform of the signal $f(t)$ shown in Fig.



Solution The equation of the line OM is $\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$

Hence, $\frac{f(t) - 0}{t - 0} = \frac{0 - A}{0 - T}$

$$f(t) = \frac{A}{T}t, \text{ for } 0 < t < T$$
$$= A, \text{ for } T < t < 2T$$

$$F(j\omega) = \int_0^T \frac{A}{T} t e^{-j\omega t}\, dt + \int_T^{2T} A e^{-j\omega t}\, dt$$

$$= \frac{A}{T}\left[t\left\{\frac{e^{-j\omega t}}{-j\omega}\right\} - \left\{\frac{e^{-j\omega t}}{(-j\omega)^2}\right\}\right]_0^T + A\left[\frac{e^{-j\omega t}}{-j\omega}\right]_T^{2T}$$

$$= \frac{A}{T}\left[T\frac{e^{-j\omega T}}{-j\omega} - \frac{e^{-j\omega T}}{-\omega^2} - 0 + \frac{1}{-\omega^2}\right]$$

$$+ A\left[\frac{e^{-j2\omega T}}{-j\omega} - \frac{e^{-j\omega T}}{-j\omega}\right]$$

$$= A\left[-\frac{e^{-j\omega T}}{j\omega} + \frac{e^{-j\omega T}}{T\omega^2} - \frac{1}{T\omega^2} - \frac{e^{-j2\omega T}}{j\omega} + \frac{e^{-j\omega T}}{j\omega}\right]$$

$$= \frac{A}{T\omega^2}\left[e^{-j\omega T} - 1\right] + j\frac{A}{\omega}e^{-j2\omega T}$$

$$= \frac{A}{T\omega^2}e^{-j\omega T/2}\left[e^{-j\omega T/2} - e^{j\omega T/2}\right] + j\frac{A}{\omega}e^{-j2\omega T}$$

$$= -j\frac{2A}{T\omega^2}e^{-j\omega T/2}\sin\frac{\omega T}{2} + j\frac{A}{\omega}e^{-j2\omega T}$$
$$= -j\frac{A}{\omega}e^{-j\omega T/2}\operatorname{sinc}\frac{\omega T}{2} + j\frac{A}{\omega}e^{-j2\omega T}$$
$$= \frac{A}{j\omega}\left[e^{-j\omega T/2}\operatorname{sinc}\frac{\omega T}{2} - e^{-j2\omega T}\right]$$

Example Find the Fourier transform of the signal shown in Fig. E2.28.

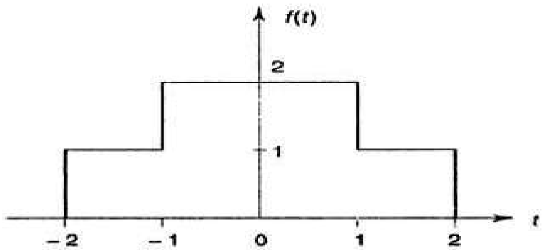


Fig. E 2.28

Solution For the given signal ,

$f(t) = 2,$ for $-1 < t < 1$
$= 1,$ for $-2 < t < -1$ and
$= 1,$ for $1 < t < 2$

$$F(j\omega) = \mathcal{F}[f(t)] = \int_{-1}^{1} 2.e^{-j\omega t}\,dt + \int_{-2}^{-1} e^{-j\omega t}\,dt + \int_{1}^{2} e^{-j\omega t}\,dt$$

$$= \frac{1}{-j\omega}\left\{\left[2e^{-j\omega t}\right]_{-1}^{1} + \left[e^{-j\omega t}\right]_{-2}^{-1} + \left[e^{-j\omega t}\right]_{1}^{2}\right\}$$

$$= \frac{1}{-j\omega}\left\{2\left(e^{-j\omega} - e^{j\omega}\right) + \left(e^{j\omega} - e^{2j\omega}\right) + \left(e^{-j2\omega} - e^{-j\omega}\right)\right\}$$

$$= \frac{1}{\omega}\left\{4\left[\frac{e^{j\omega} - e^{-j\omega}}{2j}\right] + 2\left[\frac{e^{j2\omega} - e^{-j2\omega}}{2j}\right] - 2\left[\frac{e^{j\omega} - e^{-j\omega}}{2j}\right]\right\}$$

$$= \frac{4}{\omega}\left\{\sin\omega + \frac{1}{2}\sin 2\omega - \frac{1}{2}\sin\omega\right\}$$

$$= \frac{4}{\omega}\left\{\frac{1}{2}\sin\omega + \frac{1}{2}\sin 2\omega\right\}$$

$$= 2\frac{\sin\omega}{\omega} + 2\frac{\sin 2\omega}{\omega}$$

$$= 2\operatorname{sinc}\omega + 4\operatorname{sinc}2\omega$$

2.8 Inverse Laplace Transform by Partial Fraction Expansion

2.8.1 Distinct Real Roots

Consider first an example with distinct real roots.

Find the inverse Laplace Transform of:

$$F(s)=\frac{s+1}{s(s+2)}=\frac{A_1}{s}+\frac{A_2}{s+2}$$

Solution:

$$A_1=\left.\frac{s+1}{s(s+2)}\right|_{s=0}=\frac{1}{2}$$

$$A_2=\left.\frac{s+1}{s(s+2)}\right|_{s=-2}=\frac{-1}{-2}=\frac{1}{2}$$

So

$$F(s)=\frac{1}{2}\frac{1}{s}+\frac{1}{2}\frac{1}{s+2}$$

and

$$f(t)=\frac{1}{2}U(t)+\frac{1}{2}e^{-2t}U(t)$$

2.8.2 Repeated Real Roots

Example: Distinct Real Roots

Find the inverse Laplace Transform of the function F(s).

$$F(s)=\frac{s^2+1}{s^2(s+2)}=\frac{A_1}{s+2}+\frac{A_2}{s}+\frac{A_3}{s^2}$$

Solution:.

$$A_1=\left.\frac{s^2+1}{s^2(s+2)}\right|_{s=-2}=\frac{5}{4}$$

$$A_3=\left.\frac{s^2+1}{s^2(s+2)}\right|_{s=0}=\frac{1}{2}$$

$$s^2+1=s^2(s+2)\left(\frac{A_1}{s+2}+\frac{A_2}{s}+\frac{A_3}{s^2}\right)$$
$$=s^2A_1+s(s+2)A_2+(s+2)A_3$$

Equating like powers of "s" gives us:

power of "s"	Equation
s^2	$1=A_1+A_2$
s^1	$0=2A_2+A_3$
s^0	$1=2A_3$

$$F(s)=\frac{5}{4}\frac{1}{s+2}-\frac{1}{4}\frac{1}{s}+\frac{1}{2}\frac{1}{s^2}$$

and

$$f(t)=\frac{5}{4}e^{-2t}-\frac{1}{4}+\frac{1}{2}t$$

2.8.3 Complex Roots

$$F(s)=\frac{s+3}{(s+5)(s^2+4s+5)}$$

Example: Complex Conjugate Roots (Method 1)

.

$$F(s)=\frac{s+3}{(s+5)(s^2+4s+5)}$$

Solution:

$$F(s)=\frac{s+3}{(s+5)(s^2+4s+5)}=\frac{s+3}{(s+5)(s+2-j)(s+2+j)}$$
$$=\frac{A_1}{(s+5)}+\frac{A_2}{(s+2-j)}+\frac{A_3}{(s+2+j)}$$

where

$$A_1 = (s+5)F(s)\big|_{s=-5}$$
$$A_2 = (s+2-j)F(s)\big|_{s=-2+j}$$
$$A_3 = (s+2+j)F(s)\big|_{s=-2-j} = A_2^*$$

$$A_1 = (s+5)F(s)\big|_{s=-5} = \left.\frac{s+3}{(s+5)(s^2+4s+5)}\right|_{s=-5} = -0.2$$

$$A_2 = (s+2-j)F(s)\big|_{s=-2+j} = \left.\frac{s+3}{(s+5)(s+2+j)}\right|_{s=-2+j}$$

$$= \frac{-2+j+3}{(-2+j+5)(-2+j+2+j)} = 0.1-0.2j$$

$$A_3 = (s+2+j)F(s)\big|_{s=-2-j} = A_2^* = 0.1+0.2j$$

so

$$F(s) = \frac{-0.2}{s+5} + \frac{0.1-0.2j}{s+2-j} + \frac{0.1+0.2j}{s+2+j}$$

2.8.4 Example - Combining multiple expansion methods

Find the inverse Laplace Transform of

$$F(s) = \frac{5s^2+8s-5}{s^2(s^2+2s+5)}$$

Solution:

$$F(s) = \frac{5s^2+8s-5}{s^2(s^2+2s+5)} = \frac{A_1}{s} + \frac{A_2}{s^2} + \frac{Bs+C}{s+2s+5}$$

Since we have a repeated root, let's cross-multiply to get

$$5s^2+8s-5 = s^2(s^2+2s+5)\left(\frac{A_1}{s} + \frac{A_2}{s^2} + \frac{Bs+C}{s+2s+5}\right)$$
$$= A_1(s^3+2s^2+5s) + A_2(s^2+2s+5) + Bs^3 + Cs^2$$

Then equating like powers of s

Power of s	Equation
s^3	$0=A_1+B$
s^2	$5=2A_1+A_2+C$
s^1	$8=5A_1+2A_2$
s^0	$-5=5A_2$

Starting at the last equation

$$A_2 = -1$$
$$A_1 = \frac{8+2}{5} = 2$$
$$C = 5 - 4 + 1 = 2$$
$$B = -A_1 = -2$$

So

$$F(s) = \frac{2}{s} - \frac{1}{s^2} + \frac{-2s+2}{s^2+2s+5}$$
$$F(s) = \frac{2}{s} - \frac{1}{s^2} + \frac{-2s+2}{(s+1)^2+4}$$

$$f(t) = 2 - t + e^{-t}\left(-2\cos(2t) + 2\sin(2t)\right)$$

2.8.5 Example: Order of Numerator Equals Order of Denominator

Find the inverse Laplace Transform of the function F(s).

$$F(s) = \frac{3s^2+2s+3}{s^2+3s+2}$$

Solution:

For the fraction shown below, the order of the numerator polynomial is not less than that of the denominator polynomial, therefore we first perform long division

$$\begin{array}{r} 3 \\ s^2+3s+2 \overline{)\,3s^2+2s+3} \\ \underline{3s^2+9s+6} \\ -7s-3 \end{array}$$

Now we can express the fraction as a constant plus a proper ratio of polynomials.

$$F(s) = 3 + \frac{-7s-3}{s^2+3s+2} = 3 + \frac{-7s-3}{(s+1)(s+2)}$$

$$= 3 + \frac{A_1}{s+1} + \frac{A_2}{s+2}$$

Using the cover up method to get A_1 and A_2 we get

$$F(s) = 3 + \frac{4}{s+1} - \frac{11}{s+2}$$

so

$$f(t) = 3\delta(t) + 4e^{-t} - 11e^{-2t}$$

2.8.6 Exponentials in the numerator

Example: Exponentials in the numerator

Find the inverse Laplace Transform of the function F(s).

$$F(s) = \frac{s\left(1 + e^{-1.5s} + e^{-2.2s}\right) + e^{-1.5s}}{s(s+2)}$$

Solution:

The exponential terms indicate a time delay (see the time delay property). The first thing we need to do is collect terms that have the same time delay.

$$F(s) = \frac{s}{s(s+2)} + e^{-1.5s}\frac{s+1}{s(s+2)} + e^{-2.2s}\frac{s}{s(s+2)}$$

$$= \frac{1}{(s+2)} + e^{-1.5s}\frac{s+1}{s(s+2)} + e^{-2.2s}\frac{1}{(s+2)}$$

We now perform a partial fraction expansion for each time delay term (in this case we only need to perform the expansion for the term with the 1.5 second delay), but in general you must do a complete expansion for each term.

$$F(s) = \frac{1}{(s+2)} + e^{-1.5s}\left(\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{(s+2)}\right) + e^{-2.2s}\frac{1}{(s+2)}$$

Now we can do the inverse Laplace Transform of each term (with the appropriate time delays)

$$f(t) = e^{-2t}\gamma(t) + \left(\frac{1}{2} - \frac{1}{2}e^{-2(t-1.5)}\right)\gamma(t-1.5) + e^{-2(t-2.2)}\gamma(t-2.2)$$

SUMMARY

1. The Fourier series is frequency domain representation of periodic signals.
2. The Fourier series exists only if Dirichlet's conditions are satisfied.
3. The signals with negative frequency are required for mathematical representation of real signals in terms of complex exponential signals.
4. In exponential form of Fourier series, $|c_n|$ represents the magnitude of n^{th} harmonic component.
5. In exponential form of Fourier series, $\angle c_n$ represents the phase of the n^{th} harmonic component.
6. The plot of harmonic magnitude/phase versus harmonic number "n" (or harmonic frequency) is called frequency spectrum.
7. The frequency spectrum obtained from Fourier series is also called line spectrum.
8. The plot of magnitude versus n (or $n\Omega_0$) is called magnitude (line) spectrum.
9. The plot of phase versus n (or $n\Omega_0$) is called phase (line) spectrum.
10. For signals with even symmetry, the Fourier coefficients b_n are zero.
11. For signals with odd symmetry, the Fourier coefficients a_0 and a_n are zero.
12. For signals with half wave symmetry, the Fourier series will consists of odd harmonic terms alone.
13. A signal with half wave symmetry, if in addition has even/odd symmetry then it is said to have quarter wave symmetry.
14. For signals with quarter wave symmetry, the Fourier series will consists of either odd harmonics of sine terms or odd harmonics of cosine terms.
15. The Fourier transform has been developed from Fourier series by considering the fundamental period T as infinity.
16. The Fourier transform is used to obtain the frequency domain representation of non-periodic as well as periodic signals.
17. The Fourier transform of a signal exists only if the signal is absolutely integral.
18. The Fourier transform of a signal is also called analysis of the signal.
19. The inverse Fourier transform of a signal is also called synthesis of the signal.
20. The frequency spectrum of non-periodic signals will be continuous, whereas frequency spectrum of periodic signals will be discrete.
21. The magnitude spectrum will have even symmetry and phase spectrum will have odd symmetry.
22. The Fourier transform of a periodic continuous time signal will have impulses located at the harmonic frequencies of the signal.
23. The ratio of Fourier transform of output and input signal of a system is called transfer function in frequency domain.
24. The Fourier transform of impulse response gives the frequency domain transfer function.
25. The Fourier transform is evaluation of Laplace transform along imaginary axis in s-plane.