UNIT-III

LINEAR TIME INVARIANT CONTINUOUS TIME SYSTEMS

INTRODUCTION

In this chapter methods of analysis of LTI continuous-time systems are discussed. First, time domain method is discussed and frequency domain analysis method is introduced later. In this chapter some of the important mathematical techniques, e.g., Convolution integral, Fourier series, Fourier-transform, Laplace-transform, their properties and their application in time and frequency domain analysis of linear time-invariant (LTI) systems including ideal filters are also discussed.

3.1 DIFFERENTIAL EQUATION

Example

Determine the natural response of the system described by the equation,

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 5y(t) = \frac{dx(t)}{dt} + 4x(t)\,; \quad y(0) = 1\,; \quad \left.\frac{dy(t)}{dt}\right|_{t=0} = -2$$

Solution

The natural response is response of the system due to initial conditions and so it is given by zero-input response.

$$\text{Zero-input response}, \quad y_{zi}(t) = y_h(t)\big|_{\text{with constants evaluated using initial conditions}}$$

where, $y_h(t)$ = Homogeneous solution

The given system equation is,

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 5y(t) = \frac{dx(t)}{dt} + 4x(t) \qquad .....(1)$$

Homogeneous Solution

The homogeneous solution is the solution of the system equation when $x(t) = 0$.

On substituting $x(t) = 0$ in system equation (equation (1)) we get,

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 5y(t) = 0 \qquad .....(2)$$

Let, $y(t) = C\,e^{\lambda t}$; $\therefore \frac{d}{dt}y(t) = C\lambda\,e^{\lambda t}$; $\frac{d^2}{dt^2}y(t) = C\lambda^2 e^{\lambda t}$

On substituting the above terms in equation (2) we get,

$$C\lambda^2 e^{\lambda t} + 6\,C\lambda\,e^{\lambda t} + 5\,Ce^{\lambda t} = 0$$

$$\therefore (\lambda^2 + 6\lambda + 5)\,Ce^{\lambda t} = 0$$

The characteristic polynomial of the above equation is,

$$\lambda^2 + 6\lambda + 5 = 0 \quad \Rightarrow \quad (\lambda + 1)(\lambda + 5) = 0 \quad \Rightarrow \quad \lambda = -1, -5$$

Now the homogeneous solution is given by,

$$\text{Homogeneous solution}, \quad y_h(t) = C_1\,e^{\lambda_1 t} + C_2\,e^{\lambda_2 t} = C_1\,e^{-t} + C_2\,e^{-5t}$$

Natural Response (or Zero-input Response)

$$\text{Zero-input response}, \quad y_{zi}(t) = y_h(t)\big|_{\text{with constants evaluated using initial conditions}}$$

$$= C_1\,e^{-t} + C_2\,e^{-5t}\big|_{\text{with } C_1 \text{ and } C_2 \text{ evaluated using initial conditions}}$$

$$\therefore \frac{dy_{zi}(t)}{dt} = -C_1\,e^{-t} - 5C_2\,e^{-5t}$$

At $t = 0$, $y_{zi}(0) = C_1\,e^0 + C_2\,e^0 = C_1 + C_2$

Given that, $y_{zi}(0) = 1$, $\therefore C_1 + C_2 = 1$

At $t = 0$, $\frac{dy_{zi}(t)}{dt} = -C_1\,e^0 - 5C_2\,e^0 = -C_1 - 5C_2$

Given that, $\left.\frac{dy_{zi}(t)}{dt}\right|_{t=0} = -2$, $\therefore -C_1 - 5C_2 = -2$

On adding equations (3) and (4) we get,

$$-4C_2 = -1 \Rightarrow C_2 = \frac{1}{4}$$

From equation (3), $C_1 = 1 - C_2 = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore$ Natural response, $y_{zi}(t) = \frac{3}{4} e^{-t} + \frac{1}{4} e^{-5t}$; $t \geq 0 = \frac{1}{4}\left(3e^{-t} + e^{-5t}\right) u(t)$

Example

Determine the forced response of the system described by the equation,

$$5 \frac{dy(t)}{dt} + 10\, y(t) = 2\, x(t), \quad \text{for the input, } x(t) = 2\, u(t).$$

Solution

The forced response is the response of the system due to input signal with zero initial conditions and so it is given by zero-state response.

Zero state response, $y_{zs}(t) = y_h(t) + y_p(t) \Big|_{\text{with constants evaluated with zero initial conditions}}$

where, $y_h(t)$ = Homogeneous solution and $y_p(t)$ = Particular solution

The given system equation is,

$$5 \frac{dy(t)}{dt} + 10\, y(t) = 2\, x(t) \qquad .....(1)$$

Homogeneous Solution

The homogeneous solution is the solution of the system equation when $x(t) = 0$.

On substituting $x(t) = 0$ in system equation (equation (1)) we get,

$$5 \frac{dy(t)}{dt} + 10\, y(t) = 0 \qquad .....(2)$$

Let, $y(t) = C e^{\lambda t}$; $\therefore \frac{d}{dt} y(t) = C \lambda e^{\lambda t}$

On substituting the above terms in equation (2) we get,

$$5 C \lambda e^{\lambda t} + 10 C e^{\lambda t} = 0$$

$$\therefore (5\lambda + 10) C e^{\lambda t} = 0$$

The characteristic polynomial of the above equation is,

$$5\lambda + 10 = 0 \quad \Rightarrow \quad \lambda + 2 = 0 \quad \Rightarrow \quad \lambda = -2$$

Now the homogeneous solution is given by,

Homogeneous solution, $y_h(t) = C e^{\lambda t} = C e^{-2t}$

Particular Solution

The particular solution is the solution of the system equation (equation (1)) for specific input.

Here input, $x(t) = 2\, u(t)$

Let the particular solution, $y_p(t)$ is of the form,

$$y_p(t) = K\, x(t)$$

$\therefore y_p(t) = 2K\,u(t)\,;\quad \frac{dy_p(t)}{dt} = 2K\,\delta(t)$ $\quad \boxed{\frac{d}{dt}u(t) = \delta(t)}$

On substituting the above terms and the input in system equation (equation (1)) we get,

$$5\,\frac{dy(t)}{dt} + 10\,y(t) = 2\,x(t)$$

$$\Downarrow$$

$$10\,K\,\delta(t) + 20\,K\,u(t) = 4\,u(t)$$

At $t = 1$, $10\,K\,\delta(1) + 20\,K\,u(1) = 4\,u(1) \quad \Rightarrow \quad 20\,K = 4 \quad \Rightarrow \quad K = 1/5$ $\quad \boxed{\delta(1) = 0,\ u(1) = 1}$

$\therefore$ Particular solution, $y_p(t) = \frac{2}{5}\,u(t)$

Forced Response (or Zero-State Response)

Zero state response , $y_{zs}(t) = y_h(t) + y_p(t)\Big|_{\text{with constants evaluated with zero initial conditions}}$

$$= C\,e^{-2t} + \frac{2}{5}\,u(t)$$

At $t = 0$, $\quad y_{zs}(t) = C\,e^0 + \frac{2}{5}\,u(0) = C + \frac{2}{5}$

Since, $\quad y_{zs}(0) = 0, \quad C + \frac{2}{5} = 0 \quad \Rightarrow \quad C = -\frac{2}{5}$

$\therefore$ Forced Response, $\quad y_{zs}(t) = -\frac{2}{5}\,e^{-2t} + \frac{2}{5}\,u(t);\ \text{for } t \geq 0 = \frac{2}{5}\left(1 - e^{-2t}\right)u(t)$

Example

Determine the complete response of the system described by the equation,

$$\frac{d^2y(t)}{dt^2} + 5\,\frac{dy(t)}{dt} + 4\,y(t) = \frac{dx(t)}{dt}\ ;\quad y(0) = 0\ ;\quad \frac{dy(t)}{dt}\Big|_{t=0} = 1,\ \text{ for the input, } x(t) = e^{-2t}\,u(t)$$

Solution

The given system equation is,

$$\frac{d^2y(t)}{dt^2} + 5\,\frac{dy(t)}{dt} + 4\,y(t) = \frac{dx(t)}{dt} \qquad .....(1)$$

Homogeneous Solution

The homogeneous solution is the solution of the system equation when $x(t) = 0$.

On substituting $x(t) = 0$ in system equation (equation (1)) we get,

$$\frac{d^2y(t)}{dt^2} + 5\,\frac{dy(t)}{dt} + 4\,y(t) = 0 \qquad .....(2)$$

Let, $y(t) = C\,e^{\lambda t}\ ;\quad \therefore \frac{d}{dt}y(t) = C\lambda\,e^{\lambda t}$ and $\frac{d^2}{dt}y(t) = C\lambda^2\,e^{\lambda t}$

On substituting the above terms in equation (2) we get,

$$C\lambda^2\,e^{\lambda t} + 5\,C\lambda\,e^{\lambda t} + 4\,Ce^{\lambda t} = 0$$

$$\therefore \left(\lambda^2 + 5\,\lambda + 4\right)Ce^{\lambda t} = 0$$

The characteristic polynomial of the above equation is,

$$\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad (\lambda + 4)(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda = -4, -1$$

Now the homogeneous solution is given by,

Homogeneous solution, $y_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{-4t} + C_2 e^{-t}$

Particular Solution

The particular solution is the solution of the system equation (equation (1)) for specific input.

Here input, $x(t) = e^{-2t} u(t)$

$$\therefore x(t) = e^{-2t} \; ; \; \text{for} \quad t \geq 0$$

$$\therefore \frac{dx(t)}{dt} = -2e^{-2t}$$

Let the particular solution, $y_p(t)$ is of the form,

$$y_p(t) = K\,x(t)$$

$$\therefore y_p(t) = K\,e^{-2t}; \qquad \frac{dy_p(t)}{dt} = -2K\,e^{-2t}; \qquad \frac{d^2y(t)}{dt^2} = 4K\,e^{-2t}$$

On substituting the above terms and the input in system equation (equation (1)) we get,

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4\,y(t) = \frac{dx(t)}{dt}$$

$$\Downarrow$$

$$4K\,e^{-2t} - 10K\,e^{-2t} + 4K\,e^{-2t} = -2\,e^{-2t}$$

On dividing throughout by e^{-2t} we get,

$$4K - 10K + 4K = -2 \quad \Rightarrow \quad -2K = -2 \quad \Rightarrow \quad K = 1$$

$\therefore$ Particular solution, $y_p(t) = e^{-2t}$

Total (or Complete) Response

Total response, $y(t) = y_h(t) + y_p(t)$

$$\therefore y(t) = C_1\,e^{-4t} + C_2\,e^{-t} + e^{-2t}$$

When $t = 0$, $y(t) = y(0) = C_1\,e^0 + C_2\,e^0 + e^0 = C_1 + C_2 + 1$

Given that $y(0) = 0$, $\therefore C_1 + C_2 + 1 = 0$ .....(3)

Here, $\frac{dy(t)}{dt} = -4C_1\,e^{-4t} - C_2\,e^{-t} - 2\,e^{-2t}$

Now, $\left.\frac{dy(t)}{dt}\right|_{t=0} = -4C_1\,e^0 - C_2\,e^0 - 2\,e^0 = -4C_1 - C_2 - 2$

Given that, $\left.\frac{dy(t)}{dt}\right|_{t=0} = 1$; $\quad \therefore -4C_1 - C_2 - 2 = 1$ .....(4)

On adding equation (3) and (4) we get,

$$-3C_1 - 1 = 1 \quad \Rightarrow \quad -3C_1 = 2 \quad \Rightarrow \quad C_1 = -\frac{2}{3}$$

From equation (3), $C_2 = -1 - C_1 = -1 + \frac{2}{3} = -\frac{1}{3}$

$\therefore$ Total Response, $y(t) = -\frac{2}{3}e^{-4t} - \frac{1}{3}e^{-t} + e^{-2t} \quad ; \quad t \geq 0$

(or) $y(t) = \left(e^{-2t} - \frac{2}{3}e^{-4t} - \frac{1}{3}e^{-t}\right)u(t)$

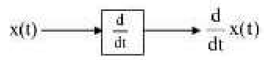
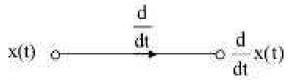
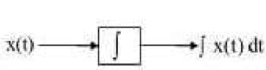
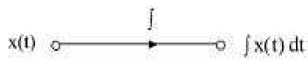
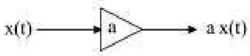
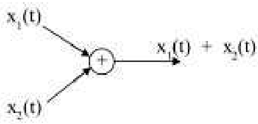
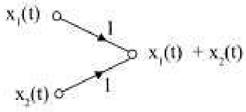
3.2 Block Diagram and Signal Flow Graph Representation of LTI Continuous Time System

Block Diagram

A ***block diagram*** of a system is a pictorial representation of the functions performed by the system. The block diagram of a system is constructed using the mathematical equation governing the system.

The basic elements of a block diagram are Differentiator, Integrator, Constant Multiplier and Signal Adder. The symbols used for the basic elements and their input-ouput relation are listed in table 2.2.

Table 2.2 : Basic Elements of Block Diagram and Signal Flow Graph

Description	Elements of block diagram	Elements of signal flow graph
Differentiator		
Integrator (with zero initial condition)		
Constant Multiplier		
Signal Adder		

Signal Flow Graph

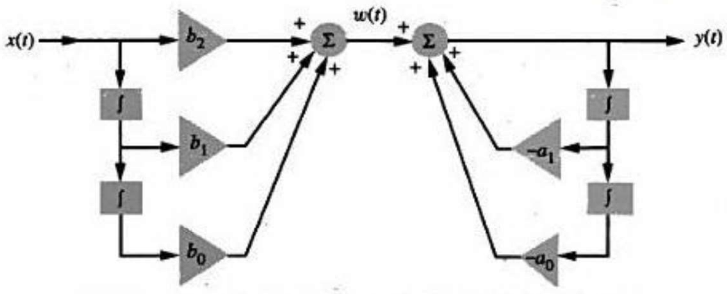
A ***signal flow graph*** of a system is a graphical representation of the functions performed by the system. The signal flow graph shows the flow of signals from one point of a system to another and gives the relationship among the signals. The signal flow graph of a system is constructed using the mathemetical equation governing the system.

The basic elements of a signal flow graph are nodes and directed branches. Each node represents a signal. The signal at a node is given by the sum of all incoming signals. Each branch has an input node and an output node. The direction of signal flow is marked by an arrow on the branch and the operation performed by the signal is indicated by an operator like integrator/differentiator. When the signal passes from the input node to the output node, it is operated by the operation specified by the branch. The basic operations performed by the branches of a signal flow graph are listed in table 2.2.

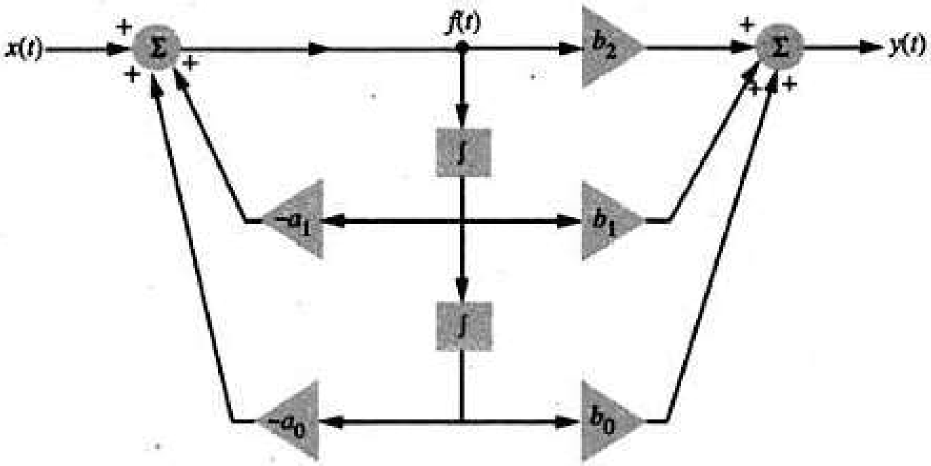
a continuous time LTI system can also be realized in Direct form-I and Direct form-II. A linear constant coefficient differential equation of the form,

$$\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y(t) = b_2\frac{d^2x}{dt^2} + b_1\frac{dx}{dt} + b_0x$$

has the realizations shown in Figs. 2.21 and 2.22, known as *Direct form*-I and II *respectively*.



Direct form-I realization of a continuous-time system.



Direct form-II realization of a continuous-time LTI system.

3.3 System Analysis using Fourier Transform

First Order System
Let us consider the first order system

$$\frac{dy(t)}{dt} + ay(t) = x(t),$$

for some $a > 0$. Applying the CTFT to both sides,

$$\mathcal{F}\left\{\frac{dy(t)}{dt} + ay(t)\right\} = \mathcal{F}\{x(t)\},$$

and use linearity property, and differentiation property of CTFT, we have

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega).$$

Rearranging the terms, we can find the frequency response of the system

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{a + j\omega}.$$

Now, recall the CTFT pair:

$$h(t) = e^{-at}u(t) \Longleftrightarrow H(j\omega) = \frac{1}{a + j\omega},$$

$h(t)$ can be deduced. Just as quick derivation of this equation, we note that

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(j\omega+a)t}dt$$
$$= -\left[\frac{1}{j\omega + a}e^{-(j\omega+a)t}\right]_0^{\infty} = \frac{1}{j\omega + a}.$$

General Systems
In general, we want to study the system

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}.$$

Our objective is to determine $h(t)$ and $H(j\omega)$. Applying CTFT on both sides:

$$\mathcal{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}.$$

Therefore, by linearity and differentiation property, we have

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega).$$

The convolution property gives $Y(j\omega) = X(j\omega)H(j\omega)$, so

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}.$$

Now, $H(j\omega)$ is expressed as a rational function, i.e., a ratio of polynomial. Therefore, we can apply the technique of partial fraction expansion to express $H(j\omega)$ in a form that allows us to determine $h(t)$ by inspection using the transform pair

$$h(t) = e^{-at}u(t) \Longleftrightarrow H(j\omega) = \frac{1}{a + j\omega},$$

and related transform pair, such as

$$te^{-at}u(t) \Longleftrightarrow \frac{1}{(a + j\omega)^2}.$$

Example
Consider the LTI system

$$\frac{d^2y(t)}{dt^2} + 4y(t) + 3y(t) = \frac{dx(t)}{dt} + 2x(t).$$

Taking CTFT on both sides yields

$$(j\omega)^2Y(j\omega) + 4j\omega Y(j\omega) + 3Y(j\omega) = j\omega X(j\omega) + 2X(j\omega).$$

and by rearranging terms we have

$$H(j\omega) = \frac{j\omega + 2}{(j\omega)^2 + 4(j\omega) + 3} = \frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}.$$

Then, by partial fraction expansion we have

$$H(j\omega) = \frac{1}{2}\left(\frac{1}{j\omega + 1}\right) + \frac{1}{2}\left(\frac{1}{j\omega + 3}\right).$$

Thus, $h(t)$ is

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t).$$

Example
If the input signal is $x(t) = e^{-t}u(t)$, what should be the output $y(t)$ if the impulse response of the system is given by $h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$?

Taking CTFT, we know that $X(j\omega) = \frac{1}{j\omega+1}$, and $H(j\omega) = \frac{j\omega+2}{(j\omega+1)(j\omega+3)}$. Therefore, the output is

$$Y(j\omega) = H(j\omega)X(j\omega) = \left[\frac{j\omega + 2}{(j\omega + 1)(j\omega + 3)}\right]\left[\frac{1}{j\omega + 1}\right] = \frac{j\omega + 2}{(j\omega + 1)^2(j\omega + 3)}.$$

By partial fraction expansion, we have

$$Y(j\omega) = \frac{\frac{1}{4}}{j\omega + 1} + \frac{\frac{1}{2}}{(j\omega + 1)^2} - \frac{\frac{1}{4}}{j\omega + 3}.$$

Therefore, the output is

$$y(t) = \left[\frac{1}{4}e^{-t} + \frac{1}{2}te^{-t} - \frac{1}{4}e^{-3t}\right]u(t).$$

3.4 STATE SPACE REPRESENTATION

Example

The state space representation of a continuous time system is given by,

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \; ; \; B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \; ; \; C = \begin{bmatrix} 1 & 3 \end{bmatrix} \; ; \; D = [3]$$

Derive the transfer function of the continuous time system.

Let, **P** be a square matrix.

Now, $P^{-1} = \dfrac{\text{Transpose of Cofactor Matrix of } P}{\text{Determinant of } P}$

If, **P** is a square matrix of size 2 × 2, then its cofactor matrix is obtained by interchanging the elements of main diagonal and changing the sign of other two elements as shown in the following example.

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$\therefore P^{-1} = \frac{1}{\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}} \times \begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix}$$

Solution

Transfer function of a continuous time system is given by,

$$\frac{Y(s)}{X(s)} = C(sI - A)^{-1} B + D$$

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} s-2 & 1 \\ -3 & s-1 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{\begin{vmatrix} s-2 & 1 \\ -3 & s-1 \end{vmatrix}} \begin{bmatrix} s-1 & -1 \\ 3 & s-2 \end{bmatrix} = \frac{1}{(s-2)(s-1) - (-3) \times 1} \begin{bmatrix} s-1 & -1 \\ 3 & s-2 \end{bmatrix}$$

$$= \frac{1}{s^2 - 3s + 5} \begin{bmatrix} s-1 & -1 \\ 3 & s-2 \end{bmatrix} = \begin{bmatrix} \dfrac{s-1}{s^2-3s+5} & \dfrac{-1}{s^2-3s+5} \\ \dfrac{3}{s^2-3s+5} & \dfrac{s-2}{s^2-3s+5} \end{bmatrix}$$

$$\therefore \frac{Y(s)}{X(s)} = C(sI - A)^{-1} B + D$$

$$= \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} \dfrac{s-1}{s^2-3s+5} & \dfrac{-1}{s^2-3s+5} \\ \dfrac{3}{s^2-3s+5} & \dfrac{s-2}{s^2-3s+5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + [3]$$

$$= \begin{bmatrix} \dfrac{s-1+9}{s^2-3s+5} & \dfrac{-1+3(s-2)}{s^2-3s+5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + [3] = \left[\frac{s-1+9}{s^2-3s+5} + \frac{2(-1+3(s-2))}{s^2-3s+5} \right] + [3]$$

$$= \frac{s-1+9 + 2(-1+3(s-2)) + 3(s^2-3s+5)}{s^2 - 3s + 5}$$

$$= \frac{s-1+9-2+6s-12+3s^2-9s+15}{s^2-3s+5} = \frac{3s^2-2s+9}{s^2-3s+5}$$

Example

Find the state transition matrix for the continuous time system parameter matrix, $A = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$

Solution

The state transition matrix, $e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s+3 & 0 \\ 0 & s+2 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{\begin{vmatrix} s+3 & 0 \\ 0 & s+2 \end{vmatrix}} \begin{bmatrix} s+2 & 0 \\ 0 & s+3 \end{bmatrix}$$

$$= \frac{1}{(s+3)(s+2)} \begin{bmatrix} s+2 & 0 \\ 0 & s+3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} & 0 \\ 0 & \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \end{bmatrix} = \begin{bmatrix} e^{-3t}\,u(t) & 0 \\ 0 & e^{-2t}\,u(t) \end{bmatrix}$$

Let, **P** be a square matrix.

Now, $P^{-1} = \frac{\text{Transpose of Cofactor Matrix of } P}{\text{Determinant of } P}$

If, **P** is a square matrix of size 2 × 2, then its cofactor matrix is obtained by interchanging the elements of main diagonal and changing the sign of other two elements as shown in the following example.

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

$$\therefore P^{-1} = \frac{1}{\begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix}} \times \begin{bmatrix} P_{22} & -P_{12} \\ -P_{21} & P_{11} \end{bmatrix}$$

Example

The state equation of an LTI continuous time system is given by, $\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$

Compute the solution of state equation by assuming the initial state vector as, $\begin{bmatrix} q_1(0) \\ q_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution

The solution of state equations are given by,

$$Q(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}\,Q(0)\} + \mathcal{L}^{-1}\{(sI - A)^{-1}\,B\,X(s)\}$$

Here **X**(s) = 0, (because there is no input).

$$\therefore Q(t) = \mathcal{L}^{-1}\{(sI - A)^{-1}\,Q(0)\}$$

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{1}{\begin{vmatrix} s-1 & 0 \\ -1 & s-1 \end{vmatrix}} \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$= \frac{1}{(s-1)^2} \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

Let, **P** be a square matrix.

Now, $P^{-1} = \frac{\text{Transpose of Cofactor Matrix of } P}{\text{Determinant of } P}$

If, **P** is a square matrix of size 2 × 2, then its cofactor matrix is obtained by interchanging the elements of main diagonal and changing the sign of other two elements as shown in the following example.

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

$$\therefore P^{-1} = \frac{1}{\begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix}} \times \begin{bmatrix} P_{22} & -P_{12} \\ -P_{21} & P_{11} \end{bmatrix}$$

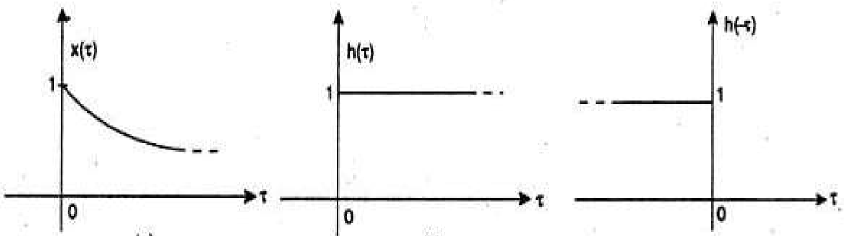
3.5 CONVOLUTION INTEGRAL GRAPHICAL METHOD

The convolution can be performed as follows:

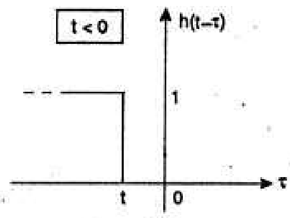
1. Plot the given signal $x(\tau)$ and impulse response $h(\tau)$ by replacing t by a dummy variable τ.
2. Obtain $h(t-\tau)$ by folding $h(\tau)$ about $\tau = 0$ and shifting by time t.
3. Multiply signal $x(\tau)$ and impulse response $h(t-\tau)$ and integrate over the overlapped area to obtain $y(t)$.
4. Increase the value t such that the function of $x(\tau)$ and $h(t-\tau)$ changes. Calculate $y(t)$.
5. Repeat step 4 and 5 for all intervals.

Problem The impulse response of the system is $h(t) = u(t)$. The input signal $x(t) = e^{-at}u(t), |a|>0$. Find the output of the system $y(t)$.

Solution



(a) (b) (c)

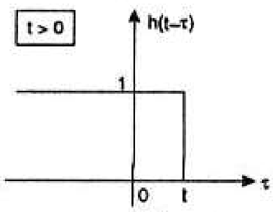


(d)

For $t < 0$

$x(\tau)$ and $h(t-\tau)$ are not overlapped.

Therefore $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau) = 0$

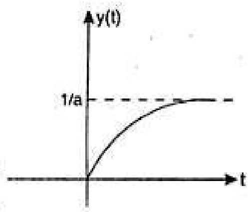


(e)

For $t > 0$

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & \text{otherwise} \end{cases}$$

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t e^{-a\tau}d\tau$$



(f)

The lowest value of $h(t-\tau)$ still exit in the negative axis, where the values of $x(\tau)$ is zero. Hence the lower value of the integration takes the value '0'. Similarly, the upper value can take any value above 0 but less than ∞, hence it is 't'.

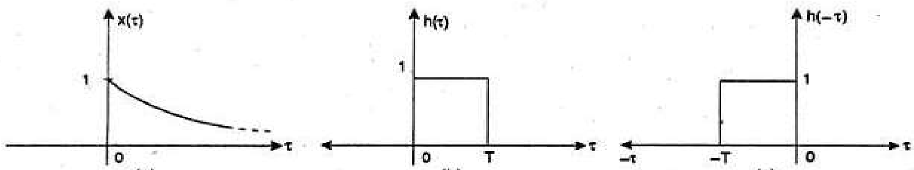
$$y(t) = \frac{-1}{a} e^{-a\tau}\Big|_0^t = -\frac{1}{a}\left[e^{-at} - e^0\right] = \frac{1}{a}(1 - e^{-at})$$

Problem The impulse response is given by

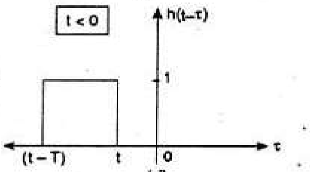
$$h(t) = \begin{cases} u(t), & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

The input signal $x(t) = e^{-at}u(t)$. Find the output of the system $y(t)$.

Solution



(a) (b) (c)



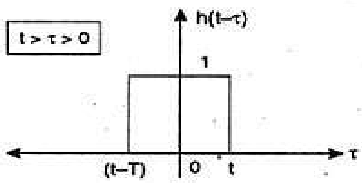
(d)

For $t < 0$

$x(\tau)$ and $h(t-\tau)$ are not overlapped.

Therefore,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0$$



(e)

For $T > t > 0$

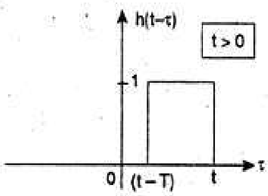
$$x(\tau) = e^{-a\tau}, \tau \ge 0$$

$$h(t-\tau) = u(t-\tau),\ T > t > 0$$

$$y(t) = \int_0^t e^{-a\tau} d\tau = \frac{-1}{a} e^{-a\tau}\Big|_0^t$$

$$y(t) = -\frac{1}{a}(e^{-at} - e^0)$$

$$y(t) = \frac{1}{a}(1 - e^{-at})$$



(f)

For $t > T$

The lowest value of $h(t-\tau)$ crossed '0', hence the lower value of integration becomes $(t-T)$, but the upper value of integration can take any value 't'.

$$y(t) = \int_{t-T}^{t} e^{-a\tau} d\tau = \frac{-1}{a} e^{-a\tau}\Big|_{t-T}^{t}$$

$$y(t) = -\frac{1}{a}\left[e^{-a(t)} - e^{-a(t-T)}\right]$$

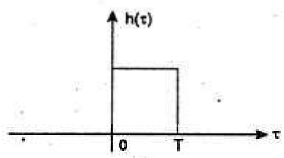
$$y(t) = \frac{e^{-at}}{a}\left[e^{+aT} - 1\right]$$

Problem Using association property, interchange the position of $x(t)$ and $h(t)$ in the convolution product and find the value of $y(t)$ for problem 3.12.

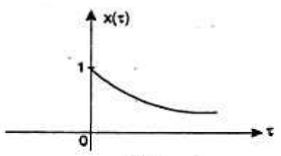
Solution

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

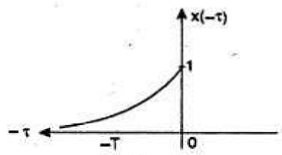
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$



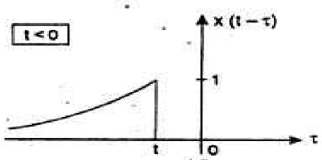
(a)



(b)



(c)

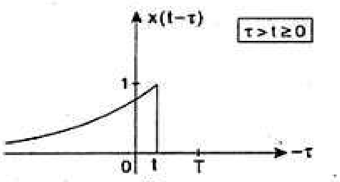


(d)

For $t < 0$

$h(\tau)$ and $x(t-\tau)$ are not overlapped.
Therefore,

$$y(t) = \int_0^t h(\tau)x(t-\tau)\,d\tau = 0$$



(e)

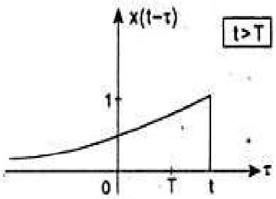
For $T > t \geq 0$

$$h(\tau) = \begin{cases} 1, & T \geq t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$x(\tau) = \begin{cases} e^{-a\tau}, & \tau \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$y(t) = \int_0^t h(\tau)x(t-\tau)\,d\tau = \int_0^t e^{-a(t-\tau)}\,d\tau$$

$$y(t) = \frac{1}{a}e^{-a(t-\tau)}\Big|_0^t = \frac{1}{a}(e^{-0} - e^{-at}) = \frac{1}{a}(1 - e^{-at})$$



(f)

For $t > T$

't' has crossed 'T' which cannot have any value in $x(t-\tau)$ when multiplied with h (τ). Hence, the upper value of integration is limited to T. But lowest value of t is still in negative x-axis where $h(t)$ does not have any value. Hence lower value of integration is '0'.

$$y(t) = \int_0^T e^{-a(t-\tau)}\,d\tau = \frac{1}{a}e^{-a(t-\tau)}\Big|_0^T$$

$$y(t) = \frac{1}{a}\left[e^{-a(t-T)} - e^{-a(t)}\right] = \frac{e^{-at}}{a}\left[e^{aT} - 1\right]$$

CONVOLUTION INTEGRAL

The output of a continuous-time Linear time-invariant (LTI) system can also be determined from the input signal $x(t)$ and the impulse response $h(t)$. Let us consider a basic system wherein input signal $x(t)$ produces an output signal $y(t)$ as it passed through the system as shown in Fig.

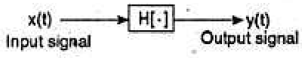


Fig. Basic System

The output response of the continuous-time system is given by

$$y(t) = H[x(t)]$$

Substitute equation (3.5)

$$y(t) = H\left\{\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right\}$$

Using the linear property of the system,

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)H\{\delta(t-\tau)\}d\tau$$

It is fact that operator (in this case H[·] will be operated on the function not on the constant scaling function.

Since, the system is time-invariant,

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

where, $h(t) = H\{\delta(t)\}$

The output $y(t)$ is the weighted superposition of impulse response time shifted by τ, the equation is called convolution integral. The convolution is symbolically represented as '*', i.e.

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

3.5.1 PROPERTIES OF CONVOLUTION INTEGRAL

In order to analyze the properties of the convolution integral, let us consider two LTI systems $h_1(t)$ and $h_2(t)$.

Commutative Property

Let us consider two continuous-time LTI systems, with impulse responses $h_1(t)$ and $h_2(t)$ connected in series as shown in Fig.

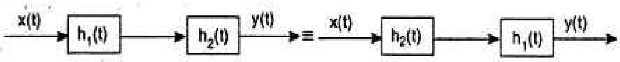


Fig. Systems are Connected in Series

By definition of commutative property,

$$h_1(t) * h_2(t) = h_2(t) * h_1(t)$$

Proof $$h_1(t)*h_2(t)=\int_{\tau=-\infty}^{\infty}h_1(\tau)h_2(t-\tau)d\tau$$

Let $t-\tau=t' \Rightarrow \tau=t-t'$

$$h_1(t)*h_2(t)=\int_{t'=-\infty}^{\infty}h_1(t-t')h_2(t')dt'$$

$$h_1(t)*h_2(t)=\int_{t'=-\infty}^{\infty}h_2(t')h_1(t-t')dt'$$

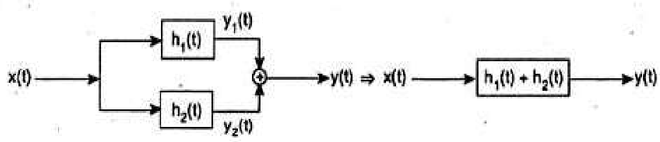
$$h_1(t)*h_2(t)=h_2(t)*h_1(t)$$

The commutative property can also be extended to the signals.

$$y(t)=x(t)*h(t)=h(t)*x(t)$$

Distributive Property

Let us consider two continuous-time LTI systems, with impulse responses $h_1(t)$ and $h_2(t)$ connected in parallel as shown in Fig. 3.4.



Systems Connected in Parallel

By definition of distributive property,

$$x(t)*[h_1(t)+h_2(t)]=x(t)*h_1(t)+x(t)*h_2(t)$$

Proof The output of the first system,

$$y_1(t)=x(t)*h_1(t)$$

Similarly, the output of the second system,

$$y_2(t)=x(t)*h_2(t)$$

The complete output of the system, $y(t)$ is given by

$$y(t)=y_1(t)+y_2(t)$$

$$y(t)=x(t)*h_1(t)+x(t)*h_2(t)$$

$$y(t)=\int_{\tau=-\infty}^{\infty}x(\tau)h_1(t-\tau)\,d\tau+\int_{\tau=-\infty}^{\infty}x(\tau)h_2(t-\tau)\,d\tau$$

$$y(t)=\int_{\tau=-\infty}^{\infty}x(\tau)[h_1(t-\tau)+h_2(t-\tau)]\,d\tau$$

$$y(t)=\int_{\tau=-\infty}^{\infty}x(\tau)h(t-\tau)\,d\tau$$

where, $h(t-\tau)=h_1(t-\tau)+h_2(t-\tau) \Rightarrow h(t)=h_1(t)+h_2(t)$

Therefore, $$y(t)=\int_{\tau=-\infty}^{\infty}x(\tau)h(t-\tau)\,d\tau=x(t)*h(t)$$

If two systems are connected in parallel, then the impulse of the system to the input signal $x(t)$ is equal to the sum of two impulse response.

Associative Property

Let us consider two continuous-time LTI systems, with impulses $h_1(t)$ and $h_2(t)$,connected in series as shown in Fig. 3.5.

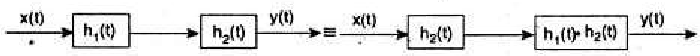


Fig. ***Systems are Connected in Series***

By definition of associative property,

$$[x(t)*h_1(t)]*h_2(t) = x(t)*[h_1(t)*h_2(t)]$$

Proof The output of the first system,

$$y_1(t) = x(t)*h_1(t)$$

$$y_1(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h_1(t-\tau)\,d\tau$$

The output of the second system,

$$y(t) = y_1(t)*h_2(t)$$

$$y(t) = \int_{t=-\infty}^{\infty} y_1(k)h_2(t-k)\,dk$$

$$y(t) = \int_{k=-\infty}^{\infty}\left[\int_{\tau=-\infty}^{\infty} x(\tau)h_1(k-\tau)\,d\tau\right]h_2(t-k)\,dk$$

$$y(t) = \int_{k=-\infty}^{\infty}\int_{\tau=-\infty}^{\infty} x(\tau)h_1(k-\tau)h_2(t-k)\,dk\,d\tau$$

Let $k-\tau = t' \Rightarrow dk = dt'$

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\left[\int_{t'=-\infty}^{\infty} h_1(t')h_2[(t-(t'+\tau))]dt'\right]d\tau$$

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\left[\int_{t'=-\infty}^{\infty} h_1(t')h_2[(t-\tau)-t']dt'\right]d\tau$$

$$y(t) = \int_{t=-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

where, $h(t-\tau) = \int_{t'=-\infty}^{\infty} h_1(t')h_2[(t-\tau)-t']dt' = h_1(t)*h_2(t)$

Therefore, $y(t) = x(t)*h(t)$

$$y(t) = x(t)*[h_1(t)*h_2(t)]$$

Convolution with Impulse Response

The convolution of a signal with a unit impulse is the signal itself.

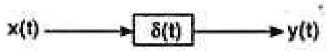


***Fig.* . System with Unit Impulse Response**

$$y(t) = x(t)*\delta(t) = x(t)$$

Proof The output of the system is shown in Fig. 3.6.

$$y(t) = x(t) * \delta(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau = x(t)$$

Hint $\delta(t-T) = \begin{cases} 1, & t = T \\ 0, & \text{elsewhere} \end{cases}$

Convolution with Step Response

The convolution of a unit step signal with an impulse response is given by

$$y(t) = u(t) * h(t)$$

Fig. 3.7 System with Unit Step Impulse Response

Proof The output of the system whose input is unit step in given by

$$y(t) = u(t) * h(t) = h(t) * u(t)$$

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau) u(t-\tau)\, d\tau$$

$$y(t) = \int_{\tau=-\infty}^{0} h(\tau)\, d\tau$$

3.5.2 PROPERTIES OF CONTINUOUS-TIME LTI SYSTEM

Causal System

By definition, for a causal continuous-time LTI system, the impulse response must be zero for $t < 0$. The causality can be extend to convolution integral, i.e.

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

For causal continuous-time LTI system, $h(\tau) = 0$ for $\tau < 0$. Therefore, the output of a causal system must be expressed as

$$y(t) = \int_{\tau=0}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

A causal system cannot generate an output signal before an input signal is applied.

Memoryless System

A memoryless system is one whose output $y(t)$ depends solely on present input. By the definition of convolution integral,

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$$

By commutative property,

$$y(t) = x(t) * h(t) = h(t) * x(t)$$

$$y(t) = \int_{\tau=-\infty}^{\infty} h(t) x(t-\tau)\, d\tau$$

A continuous-time system is memoryless, if and only if

$$h(t) = 0 \text{ for } t \neq 0$$

Such a memoryless LTI system has the form

$$y(t) = K x(t)$$

where K = constant

Such system has the impulse response

$$h(t) = K\delta(t)$$

Invertibility of LTI System

The system is invertible, only if an inverse system exist. When the inverse system is connected in series with the given system, it produces an output equal to the input to the first system. This is shown in Fig.

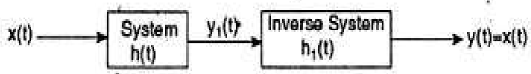


Fig. Invertible System

If the given system is invertible, there exists an LTI inverse. For the invertible system, following condition has to be satisfied.

$$h(t) * h_1(t) = \delta(t)$$

Stability of LTI System

A system is said to be stable, if every bounded input produces a bounded output (BIBO). In order to satisfy the stability condition to LTI system, let us consider a bounded input $x(t)$ such that,

$$|x(t)| < M_X \text{ for all } t$$

If such bounded input $x(t)$ is applied to a LTI system with unit impulse response $h(t)$, then according to convolution integral, we can obtain an output of the system $y(t)$.

$$|y(t)| = \int_{\tau=-\infty}^{\infty} |x(\tau) h(t-\tau)\, d\tau|$$

$$|y(t)| = \int_{\tau=-\infty}^{\infty} |h(\tau) x(t-\tau)\, d\tau|$$

Since $|x(t-\tau)| < M_X$ for all t, then

$$|y(t)| \le M_X \int_{\tau=-\infty}^{\infty} |h(\tau)| \text{ for all } t$$

The system is stable if the impulse response is absolutely integrable, i.e.

$$\int_{\tau=-\infty}^{\infty} |h(\tau)| < \infty$$

The equation (3.20) is the condition for stability in continuous-time LTI system.

Problem The impulse response of the system $h(t) = \frac{1}{k} e^{-t/k} u(t)$. Test whether $h(t)$ is stable or not

Solution By definition of stability,

$$\int_{\tau=-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

given $$h(t) = \frac{1}{k} e^{-t/k} u(t)$$

Therefore, $$\int_{\tau=0}^{\infty} |h(\tau)|\, d\tau = \int_{\tau=0}^{\infty} \left|\frac{1}{k} e^{-\tau/k}\right| d\tau$$

$$\int_{\tau=0}^{\infty}|h(\tau)|\,d\tau = \frac{1}{k}\int_{0}^{\infty} e^{-\tau/k}\,d\tau$$

$$\int_{\tau=0}^{\infty}|h(\tau)|\,d\tau = \frac{1}{k}\left(\frac{-1}{k}\right)e^{-\tau/k}\Big|_0^{\infty}$$

$$\int_{\tau=-\infty}^{\infty}|h(\tau)|\,d\tau = \frac{-1}{k^2}\left[e^{-\infty}-e^{0}\right] = \frac{1}{k^2} < \infty$$

Therefore, the given system is stable.

Problem The impulse response of the system is given by $h(t) = e^{at}u(t),\ a > 1$. Test whether $h(t)$ is stable or not.

Solution By definition of stability,

$$\int_{t=-\infty}^{\infty}|h(t)|\,dt < \infty$$

$$\int_{t=-\infty}^{\infty}|e^{at}u(t)|\,dt < \infty$$

$$\int_{t=0}^{\infty}\left|e^{at}\right|dt = \int_{t=0}^{\infty} e^{at}\,dt$$

$$\int_{t=0}^{\infty}\left|e^{at}\right|dt = \frac{1}{a}e^{at}\Big|_0^{\infty}$$

$$\int_{t=0}^{\infty}\left|e^{at}\right|dt = \frac{1}{a}\left[e^{\infty}-e^{0}\right] = \infty$$

Therefore, the given system is stable.

Problem Impulse response of the system is given by $h(t) = te^{-at}(t), a \neq 0$. Test whether the given $h(t)$ is stable or not.

Solution By definition of stability,

$$\int_{t=-\infty}^{\infty}|h(t)|\,d\tau < \infty$$

$$\int_{t=-\infty}^{\infty}\left|te^{-at}u(t)\right|d\tau < \infty$$

$$\int_{t=0}^{\infty} te^{-at}\,dt = \frac{1}{a^2}e^{-at}\left[(-a)t-1\right]\Big|_0^{\infty}$$

Hint $\int_{t=t_1}^{t_2} te^{-at}\,dt = \frac{1}{a^2}e^{-at}(-at-1)\Big|_{t_1}^{t_2}$

$$\int_{t=0}^{\infty} te^{-at}\,dt = \frac{1}{a^2}\left\{e^{-\infty}(-\infty-1)-e^{0}(0-1)\right\}$$

$$\int_{t=0}^{\infty} te^{-at}\,dt = \frac{1}{a^2}\{0+1\}$$

Hint $e^{o}=1, e^{-\infty}=0$

$$\int_{t=0}^{\infty} te^{-at}\,dt = \frac{1}{a^2} < \infty$$

Therefore, the given system is stable.

Problem The impulse response of the system is given by $h(t) = e^{-at}\sin bt\, u(t)$. Test whether the given $h(t)$ is stable or not.

Solution By definition of stability,

$$\int_{t=-\infty}^{\infty}|h(t)|\,dt < \infty$$

$$\int_{t=-\infty}^{\infty}|e^{-at}\sin bt\, u(t)|\,dt < \infty$$

$$\int_{t=0}^{\infty} te^{-at}dt = \frac{1}{a^2}\left\{e^{-\infty}(-\infty-1)-e^0(0-1)\right\}$$

$$\int_{t=0}^{\infty} te^{-at}dt = \frac{1}{a^2}\{0+1\}$$

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Problem The impulse response of the system is $h(t) = u(t) - u(t-6)$. Find the step response of the system.

Solution The step response is given by

$$s(t) = \int_{\tau=-\infty}^{t} h(\tau)\, d\tau$$

$$s(t) = \int_{\tau=-\infty}^{t} [u(\tau) - u(\tau-6)]\, d\tau$$

$$s(t) = \int_{\tau=0}^{t} 1\, d\tau - \int_{\tau=6}^{t} 1\, d\tau$$

$$s(t) = \tau\big|_0^t - \tau\big|_6^t$$

$$s(t) = (t-0) - (t-6) = 6$$

Problem The impulse response of the system is $h(t) = e^{-at}\cos bt\, u(t)$. Find the step response.

Solution The step response of the system is given by

$$s(t) = \int_{\tau=-\infty}^{t} h(\tau)\, d\tau$$

$$s(t) = \int_{\tau=-\infty}^{t} \left[e^{-a\tau}\cos b\tau\, u(\tau)\right] d\tau = \int_{\tau=0}^{t} e^{-a\tau}\cos b\tau\, d\tau$$

$$s(t) = \frac{e^{-a\tau}}{a^2+b^2}(a\cos b\tau + b\sin b\tau)\Bigg|_0^t$$

Hint $\int_{t_1}^{t_2} e^{-at}\cos bt = \frac{e^{-at}}{a^2+b^2}(a\cos bt + b\sin bt)\Big|_{t_1}^{t_2}$

$$s(t) = \frac{1}{a^2+b^2}\left\{e^{-at}(a\cos bt + b\sin bt) - e^0(a\cos 0 + b\sin 0)\right\}$$

$$s(t) = \frac{1}{a^2+b^2}\left[e^{-at}(a\cos bt + b\sin bt) - a\right]$$

Problem The impulse response of the system is $h(t) = te^{-at}u(t)$. Find the step response of the system.

Solution The step response of the system is given by

$$s(t) = \int_{\tau=-\infty}^{t} h(\tau)\,d\tau$$

$$s(t) = \int_{\tau=-\infty}^{t} e^{-\tau}u(\tau)\,d\tau$$

$$s(t) = \int_{\tau=0}^{t} \tau e^{-\tau}\,d\tau$$

$$s(t) = e^{-\tau}(-\tau-1)\Big|_0^t$$

$$s(t) = e^{-t}[-t-1] - e^{0}[-1]$$

$$s(t) = e^{-t}[-t-1] + 1$$

Hint $\int_{t_1}^{t_2} t\,e^{at}\,dt = \frac{1}{a^2}e^{at}(at-1)\Big|_{t_1}^{t_2}$

SUMMARY

There are many different methods for describing the action of an LTI system on an input signal. In this chapter, we have examined four different descriptions of LTI systems: the impulse response, difference and differential equation, block diagram, and state-variable descriptions. All four descriptions are equivalent in the input–output sense; that is, for a given input, each description will produce the identical output. However, different descriptions offer different insights into system characteristics and use different techniques for obtaining the output from the input. Thus, each description has its own advantages and disadvantages that come into play in solving a particular system problem.

The impulse response is the output of a system when the input is an impulse. The output of an LTI system in response to an arbitrary input is expressed in terms of the impulse response as a convolution operation. System properties, such as causality and stability, are directly related to the impulse response, which also offers a convenient framework for analyzing interconnections among systems. The input must be known for all time in order to determine the output of a system by using the impulse response and convolution.

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The input and output of an LTI system may also be related by either a differential or difference equation. Differential equations often follow directly from the physical principles that define the behavior and interaction of continuous-time system components. The order of a differential equation reflects the maximum number of energy storage devices in the system, while the order of a difference equation represents the system's maximum memory of past outputs. In contrast to impulse response descriptions, the output of a system from a given point in time forward can be determined without knowledge of all past inputs, provided that the initial conditions are known. Initial conditions are the initial values of energy storage or system memory, and they summarize the effect of all past inputs up to the starting time of interest. The solution of a differential or difference equation can be separated into a natural and a forced response. The natural response describes the behavior of the system due to the initial conditions; the forced response describes the behavior of the system in response to the input acting alone.