UNIT IV
ANALYSIS OF DISCREATE TIME SIGNALS

INTRODUCTION

The Laplace transform plays a very important role in the analysis of analog signals or systems and in solving linear constant coefficient differential equations. It transforms the differential equations into the complex s-plane where algebraic operations and inverse transform can be performed to obtain the solution.

Like the Laplace transform, the z-transform provides the solution for linear constant coefficient difference equations, relating the input and output digital signals in the time domain. It gives a method for the analysis of discrete time systems in the frequency domain.

An analog filter can be described by a frequency domain transfer function of the general form

$$H(s) = \frac{K(s-z_1)(s-z_2)(s-z_3)\cdots}{(s-p_1)(s-p_2)(s-p_3)\cdots}$$

where s is the Laplace variable and K is a constant. The poles $p_1, p_2, p_3 \ldots$ and zeros $z_1, z_2, z_3 \ldots$ can be plotted in the complex s-plane.

The transfer function $H(z)$ of a digital filter may be described as

$$H(z) = \frac{K(z-z_1)(z-z_2)(z-z_3)\cdots}{(z-p_1)(z-p_2)(z-p_3)\cdots}$$

Here the variable z is not the same as the variable s. For example, the frequency response of a digital filter is determined by substituting $z = e^{j\omega}$; but the equivalent substitution in the analog case is $s = j\omega$, where ω is the angular frequency in radians per second. Another essential difference is that the frequency response of an analog filter is not a periodic function. The transfer function $H(s)$ is converted into a transfer function $H(z)$, so that the frequency response of the digital filter over the range $0 \le \omega \le \pi$ approximates that of the analog filter over the range $0 \le \omega \le \infty$.

4.1 DEFINITION OF THE z-TRANSFORM

The z-transform of a discrete-time signal $x(n)$ is defined as the power series

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where z is a complex variable. This expression is generally referred to as the *two-sided z-transform*.

If $x(n)$ is a causal sequence, $x(n) = 0$ for $n < 0$, then its z-transform is

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

This expression is called a *one-sided z-transform*.

This causal sequence produces negative powers of z in $X(z)$. Generally we assume that $x(n)$ is a causal sequence, unless it is otherwise stated.

If $x(n)$ is a non-causal sequence, $x(n) = 0$ for $n \ge 0$, then its z-transform is

$$X(z) = \sum_{n=-\infty}^{-1} x(n) z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{-1} x(n) z^{-n}$$

This expression is also called a one-sided z-transform. This non-causal sequence produces positive powers of z in $X(z)$.

Definition of the Inverse z-transform

The inverse z-transform is computed or derived to recover the original time domain discrete signal sequence $x(n)$ from its frequency domain signal $X(z)$. The operation can be expressed mathematically as

$$x(n) = Z^{-1}[X(z)]$$

4.2 Region of Convergence (ROC)

Equation 4.1 gives

$$X(z)\Big|_{z=re^{j\omega}} = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, r^{-n}\, e^{-j\omega n}$$

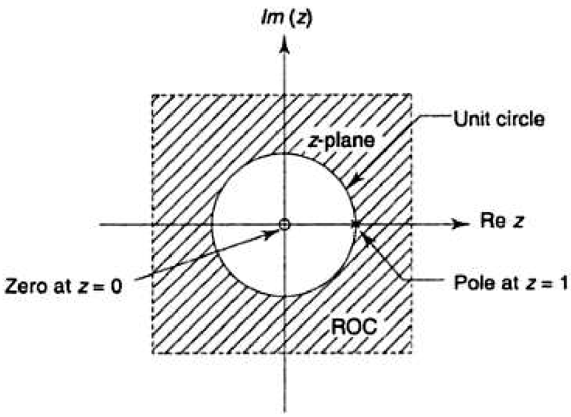
$$X(z)\Big|_{z=re^{j\omega}} = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, r^{-n}\, e^{-j\omega n}$$

which is the Fourier transform of the modified sequence $[x(n)r^{-n}]$. If $r = 1$, i.e. $|z| = 1$, $X(z)$ reduces to its Fourier transform. The series of the above equation converges if $x(n)r^{-n}$ is absolutely summable, i.e.

$$\sum_{n=-\infty}^{\infty} |x(n)\, r^{-n}| < \infty.$$

If the output signal magnitude of the digital signal system, $x(n)$, is to be finite, then the magnitude of its z-transform, $X(z)$, must be finite. The set of z values in the z-plane, for which the magnitude of $X(z)$ is finite, is called the *Region of Convergence* (ROC).

The ROC of a rational z-transform is bounded by the location of its poles. For example, the z-transform of the unit step response $u(n)$ is $X(z) = \dfrac{z}{z-1}$ which has a zero at $z = 0$ and a pole at $z = 1$ and the ROC is $|z| > 1$ and extending all the way to ∞, as shown in Fig. 4.2.



Pole-Zero Plot and ROC of the Unit-Step Response u(n)

Important Properties of the ROC for the z-transform

(i) $X(z)$ converges uniformly if and only if the ROC of the z-transform $X(z)$ of the sequence includes the unit circle. The ROC of $X(z)$ consists of a ring in the z-plane centered about the origin. That is, the ROC of the z-transform of $x(n)$ has values of z for which $x(n)\, r^{-n}$ is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |x(n)\, r^{-n}| < \infty$$

(ii) The ROC does not contain any poles.

(iii) When $x(n)$ is of finite duration, then the ROC is the entire z-plane, except possibly $z = 0$ and /or $z = \infty$.

(iv) If $x(n)$ is a right-sided sequence, the ROC will not include infinity.

(v) If $x(n)$ is a left-sided sequence, the ROC will not include $z = 0$. However, if $x(n) = 0$ for all $n > 0$, the ROC will include $z = 0$.

Example Determine the z-transform of the following finite duration signals.

(a) $x(n) = \{3,\ 1,\ 2,\ \underset{\uparrow}{5},\ 7,\ 0,\ 1\}$

(b) $x(n) = \{2,\ 4,\ \underset{\uparrow}{5},\ 7,\ 0,\ 1,\ 2\}$

(c) $x(n) = \{1, 2, 5, 4, 0, 1\}$

(d) $x(n) = \{0, 0, 1, 2, 5, 4, 0, 1\}$

(e) $x(n) = \delta(n)$

(f) $x(n) = \delta(n - k)$

(g) $x(n) = \delta(n + k)$

Solution

(a) $x(n) = \{3,\ 1,\ 2,\ \underset{\uparrow}{5},\ 7,\ 0,\ 1\}$

Taking z-transform, we get

$$X(z) = 3z^3 + z^2 + 2z + 5 + 7z^{-1} + z^{-3}.$$

ROC: Entire z-plane except $z = 0$ and $z = \infty$.

(b) $x(n) = \{2,\ 4,\ \underset{\uparrow}{5},\ 7,\ 0,\ 1,\ 2\}$

Taking z-transform, we get

$$X(z) = 2z^2 + 4z + 5 + 7z^{-1} + z^{-3} + 2z^{-4}.$$

ROC: Entire z-plane except $z = 0$ and $z = \infty$

(c) $x(n) = \{1, 2, 5, 4, 0, 1\}$

Taking z-transform, we get

$$X(z) = 1 + 2z^{-1} + 5z^{-2} + 4z^{-3} + z^{-5}$$

ROC: Entire z-plane except $z = 0$.

(d) $x(n) = \{0, 0, 1, 2, 5, 4, 0, 1\}$

Taking z-transform, we get

$$X(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 4z^{-5} + z^{-7}.$$

ROC: Entire z-plane except $z = 0$.

(e) $x(n) = \delta(n)$, hence $X(z) = 1$, ROC: Entire z-plane.

(f) $x(n) = \delta(n - k)$, $k > 0$, hence $X(z) = z^{-k}$, ROC: Entire z-plane except $z = 0$

(g) $x(n) = \delta(n + k)$, $k > 0$, hence $X(z) = z^{k}$, ROC: Entire z-plane except $z = \infty$.

Example Determine the z-transform including the region of convergence of

$$x(n) = \begin{cases} a^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Solution The z-transform for the given $x(n)$ is

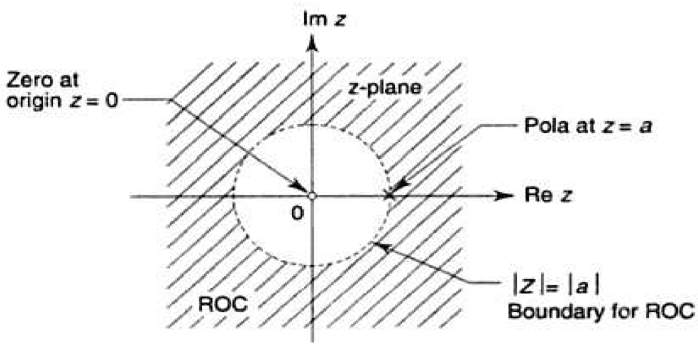
$$X(z) = Z[a^n] = \sum_{n=-\infty}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

We know that $\sum_{0}^{\infty} a^n = \frac{1}{1-a}, |a| < 1$

Hence, $X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$

This converges when $|az^{-1}| < 1$ or $|z| > |a|$. Values of z for which $X(z) = 0$ are called zeros of $X(z)$, and values of z for which $X(z) \to \infty$ are called poles of $X(z)$.

Here the poles are at $z = a$ and zeros at $z = 0$. The region of convergence is shown in Fig. E 4.3.



ROC for the z-transform of x(n) = aⁿ.

Example Determine the z-transform of the signal

$$x(n) = \delta(n+1) + 3\delta(n) + 6\delta(n-3) - \delta(n-4).$$

Solution From the linearity property, we have

$$X(z) = Z\{\delta(n+1)\} + 3Z\{\delta(n)\} + 6Z\{\delta(n-3)\} - Z\{\delta(n-4)\}$$

Using the z-transform pairs, we obtain

$$X(z) = z + 3 + 6z^{-3} - z^{-4}$$

Therefore, $x(n) = \left\{ 1, \underset{\uparrow}{3}, 0, 0, 6, -1 \right\}$

The ROC is the entire z-plane except $z = 0$ and $z = \infty$.

The same result can be obtained by using the definition of the transform.

Example Find the z-transform of $x(n) = \cos \omega_0 n$ for $n \geq 0$

Solution $x(n) = \cos \omega_0 n = \frac{1}{2}[e^{j\omega_0 n} + e^{-j\omega_0 n}]$

Using the transform, for $n \geq 0$,

$$Z[a^n] = \frac{1}{1 - az^{-1}}, |z| > a$$

Therefore, for $n \geq 0$, $Z[(e^{j\omega_0})^n] = \frac{1}{1 - e^{j\omega_0} z^{-1}}, \ |z| > 1$

Similarly for $n \geq 0$, $Z[(e^{-j\omega_0})^n] = \frac{1}{1 - e^{-j\omega_0} z^{-1}}, |z| > 1$

Therefore,

$$X(z) = Z[\cos \omega_0 n] = Z\left[\frac{1}{2}\left(e^{j\omega_0 n} + e^{-j\omega_0 n}\right)\right]$$

$$= \frac{\frac{1}{2}}{1 - e^{j\omega_0} z^{-1}} + \frac{\frac{1}{2}}{1 - e^{-j\omega_0} z^{-1}}$$

$$= \frac{1 - \frac{1}{2}\left[e^{j\omega_0} + e^{-j\omega_0}\right] z^{-1}}{(1 - e^{j\omega_0} z^{-1})(1 - e^{-j\omega_0} z^{-1})}$$

$$= \frac{1 - z^{-1} \cos \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}}$$

$$= \frac{z(z - \cos \omega_0)}{z^2 - 2z \cos \omega_0 + 1}, |z| > 1$$

Similarly, we can find $Z[\sin \omega_0 n]$ using the property of linearity, i.e.

$$Z[\sin \omega_0 n] = Z\left[\frac{1}{2j}\left(e^{j\omega_0 n} - e^{-j\omega_0 n}\right)\right]$$

$$= \frac{z^{-1} \sin \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}} = \frac{z \sin \omega_0}{z^2 - 2z \cos \omega_0 + 1}, \ |z| > 1$$

Example By applying the time shifting property, determine the z-transform of the signal

$$X(z) = \frac{z^{-1}}{1 - 3z^{-1}}$$

Solution

$$X(z) = \frac{z^{-1}}{1 - 3z^{-1}} = z^{-1} X_1(z)$$

where $X_1(z) = \frac{1}{1 - 3z^{-1}}$

Here, from the time shifting property, we have $k = 1$ and $x(n) = (3)^n u(n)$

Hence $x(n) = (3)^{n-1} u(n - 1)$

Example Find $x(n)$

if $$X(z) = \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

Solution Given $$X(z) = \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}} = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

Therefore, $$x(n) = Z^{-1}\left[\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{2}\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}\right]$$

$$= \left(\frac{1}{2}\right)^n u(n) + \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

$$= \left(\frac{1}{2}\right)^n [u(n) + u(n-1)]$$

$$= \left(\frac{1}{2}\right)^n [u(n) - u(n-1) + 2u(n-1)]$$

$$= \left(\frac{1}{2}\right)^n [\delta(n) + 2u(n-1)]$$

Example Using scaling property, determine the z-transform of

(i) $a^n \cos \omega_0 n$ and (ii) $a^n \sin \omega_0 n$

Solution

(i) We know that $$Z[\cos \omega_0 n] = \frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0 + z^{-2}}$$

$$= \frac{z(z-\cos\omega_0)}{z^2 - 2z\cos\omega_0 + 1}$$

Using the scaling property, we obtain

$$Z[a^n \cos \omega_0 n] = \frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0 + a^2 z^{-2}} = \frac{z(z-a\cos\omega_0)}{z^2 - 2za\cos\omega_0 + a^2}$$

Similarly, we obtain

$$Z[a^n \sin \omega_0 n] = \frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0 + a^2 z^{-2}} = \frac{za\sin\omega_0}{z^2 - 2za\cos\omega_0 + a^2}$$

Example Find the z-transform of $x(n) = 2^n u(n-2)$

Solution

Given $$x(n) = 2^n u(n-2)$$

$$Z[u(n)] = \frac{1}{1-z^{-1}}$$

Hence, $$Z[u(n-2)] = \frac{z^{-2}}{1-z^{-1}}$$

Therefore $$Z[2^n u(n-2)] = \left.\frac{z^{-2}}{1-z^{-1}}\right|_{z^{-1}\to 2z^{-1}}$$

$$= \frac{(2z^{-1})^2}{1-2z^{-1}} = \frac{4z^{-2}}{1-2z^{-1}}$$

Solution Taking z-transform of the given two sequences $x(n)$ and $h(n)$, we get

$$X(z) = 2 + z^{-1} + 0.5z^{-3}$$
$$H(z) = 2 + 2z^{-1} + z^{-2} + z^{-3}$$
$$Y(z) = X(z)\,H(z) = (2 + z^{-1} + 0.5z^{-3})(2 + 2z^{-1} + z^{-2} + z^{-3})$$
$$= 4 + 6z^{-1} + 4z^{-2} + 4z^{-3} + 2z^{-4} + 0.5z^{-5} + 0.5z^{-6}$$

Taking inverse z-transform, we get

$$y(n) = \left\{\underset{\uparrow}{4}, 6, 4, 4, 2, 0.5, 0.5\right\}$$

Alternate method

Let $y(n) = x(n) * h(n)$

The convolution can be obtained easily using the convolution table.

$y(n)\downarrow$ \ $x(n)\rightarrow$	2	1	0	0.5
2	4	2	0	1
2	4	2	0	1
1	2	1	0	0.5
1	2	1	0	0.5

Thus $y(n)$ can be obtained by adding elements of sequence along the slant lines.

$$y(n) = \left\{\underset{\uparrow}{4}, 6, 4, 4, 2, 0.5, 0.5\right\}$$

Example Find $x(n)$ by using convolution for

$$X(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$

Solution

Given $$X(z) = \frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}$$
$$= X_1(z)\cdot X_2(z)$$

where $X_1(z) = \dfrac{1}{\left(1-\frac{1}{2}z^{-1}\right)}$ and $X_2(z) = \dfrac{1}{\left(1+\frac{1}{4}z^{-1}\right)}$

Taking inverse z-transform, we get

$$x_1(n) = z^{-1}\left[\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)}\right] = \left(\frac{1}{2}\right)^n u(n)$$

$$x_2(n) = z^{-1}\left[\frac{1}{\left(1+\frac{1}{4}z^{-1}\right)}\right] = \left(-\frac{1}{4}\right)^n u(n)$$

Therefore, $x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(n-k)x_2(k)$

$$= \sum_{k=0}^{n}\left(\frac{1}{2}\right)^{n-k}\left(\frac{-1}{4}\right)^k$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \left[\frac{(-1/4)}{(1/2)}\right]^k$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \left(-\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^n \frac{1-\left(-\frac{1}{2}\right)^{n+1}}{1-\left(-\frac{1}{2}\right)}$$

$$= \left(\frac{1}{2}\right)^n \cdot \frac{2}{3}\left[1-\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)^n\right] = \left[\frac{2}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{-1}{4}\right)^n\right]u(n)$$

Example If $X(z) = 2 + 3z^{-1} + 4z^{-2}$, find the initial and final values of the corresponding sequence, $x(n)$.

Solution

$$x(0) = \lim_{|z|\to\infty}\left[2 + 3z^{-1} + 4z^{-2}\right] = 2 + \frac{3}{\infty} + \frac{4}{\infty} = 2$$

$$x(\infty) = \lim_{|z|\to 1}\left[(1-z^{-1})(2+3z^{-1}+4z^{-2})\right]$$

$$= \lim_{|z|\to 1}\left[2 + z^{-1} + z^{-2} - 4z^{-3}\right] = 2 + 1 + 1 - 4 = 0$$

Also, by inspection, the above results are confirmed that the initial value is two as it is the coefficient of z^0 and the final value is 0 as the sequence is a finite one.

4.3 EVALUATION OF THE INVERSE z-TRANSFORM

The inverse z-transform was defined in Section 4.2.1. The three basic methods of performing the inverse z-transform, viz. (i) long division method, (ii) partial fraction expansion method and (iii) residue method are discussed in this section.

4.3.1 Long Division Method

The z-transform of a signal or system which is expressed as the ratio of two polynomials in z, is simply divided out to produce a power series in the form of an equation.

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n},$$

Example A system has an impulse response $h(n) = \{1, 2, 3\}$ and output response

$y(n) = \{1, 1, 2, -1, 3\}$. Determine the input sequence $x(n)$.

Solution Performing the z-transform of $h(n)$ and $y(n)$, we have

$$H(z) = Z[h(n)] = Z[1, 2, 3] = 1 + 2z^{-1} + 3z^{-2}$$

$$Y(z) = Z[y(n)] = Z[1, 1, 2, -1, 3] = 1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4}$$

We know that $H(z) = \dfrac{Y(z)}{X(z)}$

Therefore, $X(z) = \dfrac{Y(z)}{H(z)} = \dfrac{1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4}}{1 + 2z^{-1} + 3z^{-2}}$

$$\begin{array}{r|l} & 1 - z^{-1} + z^{-2} \\ \hline 1 + 2z^{-1} + 3z^{-2} & 1 + z^{-1} + 2z^{-2} - z^{-3} + 3z^{-4} \\ & 1 + 2z^{-1} + 3z^{-2} \\ \hline & -z^{-1} - z^{-2} - z^{-3} \\ & -z^{-1} - 2z^{-2} - 3z^{-3} \\ \hline & z^{-2} + 2z^{-3} + 3z^{-4} \\ & z^{-2} + 2z^{-3} + 3z^{-4} \\ \hline & 0 \end{array}$$

Therefore, $X(z) = 1 - z^{-1} + z^{-2}$
Taking inverse z-transform, we get

$$x(n) = \left\{ \underset{\uparrow}{1}, -1, 1 \right\}$$

Example Using long division, determine the inverse z-transform of

$$X(z) = \frac{1}{1 - (3/2)z^{-1} + (1/2)z^{-2}}$$

when (a) ROC: $|z| > 1$ and (b) ROC: $|z| < \frac{1}{2}$

Solution (a) Since the ROC : $|z| > 1$ is the exterior of a circle, $x(n)$ is a causal signal. Thus we seek a power series expansion in negative powers of z. By dividing the numerator of $X(z)$ by its denominator, we obtain

$$\begin{array}{r|l} & 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \frac{31}{16}z^{-4} + \cdots \\ \hline 1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} & 1 \\ & 1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \\ \hline & \frac{3}{2}z^{-1} - \frac{1}{2}z^{-2} \\ & \frac{3}{2}z^{-1} - \frac{9}{4}z^{-2} + \frac{3}{4}z^{-3} \\ \hline & \frac{7}{4}z^{-2} - \frac{3}{4}z^{-3} \\ & \frac{7}{4}z^{-2} - \frac{21}{8}z^{-3} + \frac{7}{8}z^{-4} \\ \hline & \frac{15}{8}z^{-3} - \frac{7}{8}z^{-4} \\ & \frac{15}{8}z^{-3} - \frac{45}{16}z^{-4} + \frac{15}{16}z^{-5} \\ \hline & \frac{31}{16}z^{-4} - \frac{15}{16}z^{-5} \end{array}$$

Therefore, $X(z) = 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \frac{31}{16}z^{-4} + \cdots$

Taking inverse z-transform, we obtain $x(n) = \left\{ \underset{\uparrow}{1}, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \cdots \right\}$

(b) In this case the ROC: $|z| < 0.5$ is the interior of a circle. Consequently the signal $x(n)$ is anti-causal. To obtain a power series expansion in positive powers of z, we perform the long division in the following way.

$$
\begin{array}{r|l}
 & 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \ldots \\
\hline
\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & 1 \\
 & 1 - 3z + 2z^2 \\
\hline
 & 3z - 2z^2 \\
 & 3z - 9z^2 + 6z^3 \\
\hline
 & 7z^2 - 6z^3 \\
 & 7z^2 - 21z^3 + 14z^4 \\
\hline
 & 15z^3 - 14z^4 \\
 & 15z^3 - 45z^4 + 30z^5 \\
\hline
 & 31z^4 - 30z^5
\end{array}
$$

Therefore, $X(z) = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \ldots$

Taking inverse z-transform, we get $x(n) = \left\{\cdots, 62, 30, 14, 6, 2, 0, \underset{\uparrow}{0}\right\}$

4.3.2 Partial Fraction Method

Example By using partial fraction expansion method, find the inverse z-transform of

$$H(z) = \frac{-4 + 8z^{-1}}{1 + 6z^{-1} + 8z^{-2}}$$

Solution

$$H(z) = \frac{-4 + 8z^{-1}}{1 + 6z^{-1} + 8z^{-2}} = \frac{-4 + 8z^{-1}}{(1 + 4z^{-1})(1 + 2z^{-1})}$$

$$H(z) = \frac{A_1}{1 + 4z^{-1}} + \frac{A_2}{1 + 2z^{-1}}$$

$$A_1 = \left.\frac{-4 + 8z^{-1}}{1 + 2z^{-1}}\right|_{\text{at } z^{-1} = -1/4} = \frac{-6}{1/2} = -12$$

$$A_2 = \left.\frac{-4 + 8z^{-1}}{1 + 4z^{-1}}\right|_{\text{at } z^{-1} = -1/2} = \frac{-8}{-1} = 8$$

$$\text{Therefore, } H(z) = \frac{-12}{1 + 4z^{-1}} + \frac{8}{1 + 2z^{-1}}$$

Taking inverse z-transform, we get

$$h(n) = [-12(-4)^n + 8(-2)^n]\, u(n)$$

Example Determine the causal signal $x(n)$ having the z-transform

$$X(z) = \frac{1}{(1 + z^{-1})(1 - z^{-1})^2}$$

Solution Expanding the given $X(z)$ in terms of the positive powers of z.

$$X(z) = \frac{z^3}{(z + 1)(z - 1)^2}$$

$$\text{Hence } F(z) = \frac{X(z)}{z} = \frac{z^2}{(z + 1)(z - 1)^2} = \frac{A_1}{(z + 1)} + \frac{A_2}{(z - 1)} + \frac{A_3}{(z - 1)^2}$$

$$A_3 = (z-1)^2 F(z)|_{z=1} = \frac{z^2}{(z+1)}|_{z=1} = \frac{1}{2}$$

$$A_2 = \frac{d}{dz}\left[\frac{z^2}{(z+1)}\right]_{z=1} = \frac{(z+1)2z - z^2}{(z+1)^2}\Bigg|_{z=1} = \frac{3}{4}$$

Therefore, $F(z) = \frac{1}{4}\frac{1}{(z+1)} + \frac{3}{4}\frac{1}{(z-1)} + \frac{1}{2}\frac{1}{(z-1)^2}$

Therefore, $X(z) = \frac{1}{4}\frac{z}{(z+1)} + \frac{3}{4}\frac{z}{(z-1)} + \frac{1}{2}\frac{z}{(z-1)^2}$

Taking inverse z-transform of $X(z)$, we obtain

$$x(n) = \frac{1}{4}(-1)^n u(n) + \frac{3}{4}u(n) + \frac{1}{2}nu(n)$$

$$= \left[\frac{1}{4}(-1)^n + \frac{3}{4} + \frac{1}{2}n\right]u(n)$$

Example Determine the inverse z-transform of the following $X(z)$ by the partial fraction expansion method

$$X(z) = \frac{z+2}{2z^2 - 7z + 3}$$

if the ROCs are (a) $|z| > 3$, (b) $|z| < 1/2$ and (c) $1/2 < |z| < 3$.

Solution We desire the partial fraction expansion of $X(z)/z$, which is

$$F(z) = \frac{X(z)}{z} = \frac{z+2}{z(2z^2 - 7z + 3)} = \frac{z+2}{2z\left(z - \frac{1}{2}\right)(z-3)}$$

$$= \frac{A_0}{z} + \frac{A_1}{z - \frac{1}{2}} + \frac{A_2}{z-3}$$

where, $A_0 = zF(z)|_{z=0} = \frac{z+2}{2\left(z-\frac{1}{2}\right)(z-3)}\Bigg|_{z=0} = \frac{2}{3}$

$$A_1 = \left(z - \frac{1}{2}\right)F(z)\Big|_{z=\frac{1}{2}} = \frac{z+2}{2z(z-3)}\Bigg|_{z=\frac{1}{2}} = -1$$

$$A_2 = (z-3)F(z)|_{z=3} = \frac{z+2}{2z\left(z-\frac{1}{2}\right)}\Bigg|_{z=3} = \frac{1}{3}$$

Hence, by multiplying $X(z)/z$ by z, we obtain

$$X(z) = \frac{2}{3} - \frac{z}{z - \frac{1}{2}} + \frac{z/3}{z-3}$$

Here, the given function $X(z)$ has two poles, $p_1 = \frac{1}{2}$ and $p_2 = 3$ and the following three inverse transforms.

(a) In the region $|z| > 3$, all poles are interior, i.e. the signal $x(n)$ is causal, and therefore,

$$x(n) = \frac{2}{3}\delta(n) - \left(\frac{1}{2}\right)^n u(n) + \frac{1}{3}(3)^n u(n).$$

(b) In the region $|z| < \frac{1}{2}$, both the poles are exterior, i.e. $x(n)$ is anti-causal and hence

$$x(n) = \frac{2}{3}\delta(n) - \left(\frac{1}{2}\right)^n u(-n-1) - \frac{1}{3}(3)^n u(-n-1).$$

(c) In the region $\frac{1}{2} < |z| < 3$, $\left[\text{i.e.} |z| < 3 \text{ anti-causal and} |z| > \frac{1}{2} \text{ causal}\right]$, the pole $p_1 = \frac{1}{2}$ is interior and $p_2 = 3$ is exterior, and hence

$$x(n) = \frac{2}{3}\delta(n) - \left(\frac{1}{2}\right)^n u(n) - \frac{1}{3}(3)^n u(-n-1).$$

Example Determine the inverse z-transform of

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

if the regions of convergence are (a) $|z| > 1$, (b) $|z| < \frac{1}{3}$ and (c) $\frac{1}{3} < |z| < 1$

Solution The partial fraction expansion of $X(z)$ yields

$$F(z) = \frac{X(z)}{z} = \frac{1}{3z^2 - 4z + 1} = \frac{1}{3(z-1)\left(z - \frac{1}{3}\right)}$$

$$= \frac{A_1}{(z-1)} + \frac{A_2}{\left(z - \frac{1}{3}\right)}$$

$$A_1 = F(z)(z-1)|_{z=1} = \frac{1}{3\left(z - \frac{1}{3}\right)}\Big|_{z=1} = \frac{1}{2}$$

$$A_2 = F(z)\left(z - \frac{1}{3}\right)\Big|_{z=\frac{1}{3}} = \frac{1}{3(z-1)}\Big|_{z=\frac{1}{3}} = -\frac{1}{2}$$

$$\frac{X(z)}{z} = \frac{\frac{1}{2}}{(z-1)} + \frac{-\frac{1}{2}}{\left(z - \frac{1}{3}\right)}$$

$$X(z) = \frac{\frac{1}{2}z}{(z-1)} - \frac{\frac{1}{2}z}{\left(z - \frac{1}{3}\right)}$$

(a) When the ROC is $|z| > 1$, the signal $x(n)$ is causal and both terms are causal.

Therefore, $x(n) = \frac{1}{2}(1)^n u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$

$$= \frac{1}{2}\left[1 - \left(\frac{1}{3}\right)^n\right] u(n)$$

(b) When the ROC is $|z| < \frac{1}{3}$, the signal $x(n)$ is anti-causal, i.e. the inverse gives negative time sequences. Therefore,

$$x(n) = \left[-\frac{1}{2}(1)^n + \frac{1}{2}\left(\frac{1}{3}\right)^n\right] u(-n-1)$$

(c) Here the ROC $\frac{1}{3} < |z| < 1$ is a ring, which implies that the signal $x(n)$ is two-sided. Therefore one of the terms corresponds to a causal signal and the other to an anti-causal signal, obviously the given ROC is the overlapping of the regions $|z| > \frac{1}{3}$ and $|z| < 1$. The partial fraction expansion remains the same; however because of ROC, the pole at 1/3 provides the causal part corresponding to positive time and the pole at 1 is the anti-causal part corresponding to negative time. Therefore, $x(n)$ becomes

$$x(n) = -\frac{1}{2}(1)^n u(-n-1) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

Example Find $x(n)$ using (i) long division and (ii) partial fraction for $X(z)$ given by

$$X(z) = \frac{2+3z^{-1}}{(1+z^{-1})\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

Also verify the results in each case for $0 \le n \le 3$.

Solution

(i) Long Division Method

$$X(z) = \frac{2+3z^{-1}}{1+\frac{5}{4}z^{-1}+\frac{1}{8}z^{-2}-\frac{1}{8}z^{-3}}$$

$$\begin{array}{r|l}
 & 2+\frac{1}{2}z^{-1}-\frac{7}{8}z^{-2}+\frac{41}{32}z^{-3} \\
1+\frac{5}{4}z^{-1}+\frac{1}{8}z^{-2}-\frac{1}{8}z^{-3} & 2+3z^{-1} \\
 & 2+\frac{5}{4}z^{-1}+\frac{1}{4}z^{-2}-\frac{1}{4}z^{-3} \\ \hline
 & \frac{1}{2}z^{-1}-\frac{1}{4}z^{-2}+\frac{1}{4}z^{-3} \\
 & \frac{1}{2}z^{-1}+\frac{5}{8}z^{-2}+\frac{1}{16}z^{-3}-\frac{1}{16}z^{-4} \\ \hline
 & -\frac{7}{8}z^{-2}+\frac{3}{16}z^{-3}+\frac{1}{16}z^{-4} \\
 & -\frac{7}{8}z^{-2}+\frac{35}{32}z^{-3}-\frac{7}{64}z^{-4}+\frac{7}{64}z^{-5} \\ \hline
 & \frac{41}{32}z^{-3}+\cdots
\end{array}$$

Therefore, $X(z) = 2+\frac{1}{2}z^{-1}-\frac{7}{8}z^{-2}+\frac{41}{32}z^{-3}$

Taking inverse z-transform, we get $x(n) = \left[\underset{\uparrow}{2}, \frac{1}{2}, -\frac{7}{8}, \frac{41}{32}, \cdots\right]$

(ii) Partial Fraction Expansion Method

$$X(z) = \frac{2+3z^{-1}}{(1+z^{-1})\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

$$= \frac{A_1}{1+z^{-1}} + \frac{A_2}{1+\frac{1}{2}z^{-1}} + \frac{A_3}{1-\frac{1}{4}z^{-1}}$$

$$A_1 = \left.\frac{2+3z^{-1}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}\right|_{z^{-1}=-1} = -\frac{8}{5}$$

$$A_2 = \left.\frac{2+3z^{-1}}{(1+z^{-1})\left(1-\frac{1}{4}z^{-1}\right)}\right|_{z^{-1}=-2} = \frac{8}{3}$$

$$A_3 = \left.\frac{2+3z^{-1}}{(1+z^{-1})\left(1+\frac{1}{2}z^{-1}\right)}\right|_{z^{-1}=4} = \frac{14}{15}$$

$$X(z) = \frac{\left(-\frac{8}{5}\right)}{1+z^{-1}} + \frac{\left(\frac{8}{3}\right)}{1+\frac{1}{2}z^{-1}} + \frac{\left(\frac{14}{15}\right)}{1-\frac{1}{4}z^{-1}}$$

Hence $x(n) = Z^{-1}[X(z)] = \left[-\frac{8}{5}(-1)^n + \frac{8}{3}\left(-\frac{1}{2}\right)^n + \frac{14}{15}\left(\frac{1}{4}\right)^n\right]u(n)$

When $n = 0$, $x(0) = 2$
When $n = 1$, $x(1) = 1/2$
When $n = 2$, $x(2) = -7/8$
When $n = 3$, $x(3) = 41/32$

Therefore, $x(n) = \left[\underset{\uparrow}{2}, \frac{1}{2}, -\frac{7}{8}, \frac{41}{32}, \ldots\right]$

Therefore, the results obtained in each case are identical.

4.3.3 Residue Method

This method is useful in determining the time-signal $x(n)$ by summing residues of $[X(z)z^{n-1}]$ at all poles. This is expressed mathematically as

$$x(n) = \sum_{\substack{\text{all poles}\\ X(z)}} \text{residues of } [X(z)\, z^{n-1}]$$

where the residue for a pole of order m at $z = \alpha$ is

$$\text{Residue} = \frac{1}{(m-1)!}\lim_{z\to\alpha}\left\{\frac{d^{m-1}}{dz^{m-1}}\left[(z-\alpha)^m X(z)\, z^{n-1}\right]\right\}$$

Example Using the residue method, determine $x(n)$ for

$$X(z) = \frac{z}{(z-1)(z-2)}.$$

Solution $X(z)$ has two poles of order $m = 1$ at $z = 1$ and at $z = 2$. The corresponding residues can be obtained as follows:

For poles at $z = 1$

$$\text{Residue} = \frac{1}{0!}\lim_{z\to 1}\left\{\frac{d^0}{dz^0}\left[(z-1)^1 \frac{z\cdot z^{n-1}}{(z-1)(z-2)}\right]\right\}$$

$$= \lim_{z\to 1}\left[\frac{z}{z-2}z^{n-1}\right] = -1,$$

For poles at $z = 2$

$$\text{Residue} = \frac{1}{0!}\lim_{z \to 2}\left\{\frac{d^0}{dz^0}\left[(z-2)^1 \frac{z \cdot z^{n-1}}{(z-1)(z-2)}\right]\right\}$$

$$= \lim_{z \to 2}\left[\frac{z}{z-1} z^{n-1}\right] = 2 \cdot 2^{n-1} = 2^n$$

Therefore, $x(n) = (-1 + 2^n)\, u(n)$

Example Using the residue method, find the inverse z-transform of

$$X(z) = \frac{1}{(z-0.25)(z-0.5)}, \text{ ROC: } |z| > 0.5$$

Solution

Let $X_0(z) = X(z)z^{n-1}$

$$= \frac{z^{n-1}}{(z-0.25)(z-0.5)}$$

For convenience let us write this as

$$X_0(z) = \frac{z^n}{z(z-0.25)(z-0.5)}$$

This has poles at $z = 0$, $z = 0.25$ and $z = 0.5$

By the residue theorem

$$x(n) = \underset{z=0}{Res}[X_0(z)] + \underset{z=0.25}{Res}[X_0(z)] + \underset{z=0.5}{Res}[X_0(z)]$$

$$= \underset{z=0}{Res}\frac{z^n}{z(z-0.25)(z-0.5)} + \underset{z=0.25}{Res}\frac{z^n}{z(z-0.25)(z-0.5)}$$

$$+ \underset{z=0.5}{Res}\frac{z^n}{z(z-0.25)(z-0.5)}$$

(a) For pole at $z = 0$

$$\underset{z=0}{Res} X_0(z) = zX_0(z)\,|\, z = 0$$

$$= \left.\frac{z^n}{(z-0.25)(z-0.5)}\right|_{z=0}$$

$$= 0$$

(b) For pole at $z = 0.25$

$$\underset{z=0.25}{Res} X_0(z) = (z-0.25)X_0(z)\,|_{z=0.25}$$

$$= \left.\frac{(z-0.25)\,z^n}{z(z-0.25)(z-0.5)}\right|_{z=0.25}$$

$$= \left.\frac{z^{n-1}}{z-0.5}\right|_{z=0.25}$$

$$= \left.\frac{z^{-1} z^{n}}{z-0.5}\right|_{z=0.25}$$

$$= \frac{4(1/4)^n}{-(1/4)} = -16(1/4)^n$$

(c) For pole at $z = 0.5$

$$\underset{z=0.5}{Res}\, X_0(z) = (z-0.5)\, X_0(z)\,|_{z=0.5}$$

$$= \left.\frac{(z-0.5)\, z^n}{z(z-0.25)(z-0.5)}\right|_{z=0.5}$$

$$= \left.\frac{z^{n-1}}{z-0.25}\right|_{z=0.5}$$

$$= \frac{2(1/2)^n}{1/4} = 8(1/2)^n$$

Therefore, $x(n) = [8(1/2)^n) - 16(1/4)^n]u(n)$

4.4 Problems

Example

Find the one sided Z-transform of the following discrete time signals.

a) $x(n) = n\,a^{(n-1)}$ b) $x(n) = n^2$

Solution

a) Given that, $x(n) = n\,a^{(n-1)}$

Let, $x_1(n) = a^n$

By definition of one sided Z-transform,

$$X_1(z) = \sum_{n=0}^{\infty} x_1(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(a z^{-1}\right)^n$$

$$= \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

Using infinite geometric series sum formula

Let, $x_1(n-1) = a^{n-1}$

By shifting property,

$$Z\{x_1(n-1)\} = z^{-1} X_1(z) = z^{-1} \frac{z}{z-a} = \frac{1}{z-a}$$

Given that, $x(n) = n\,a^{n-1}$

$$Z\{x(n)\} = Z\{n\,a^{n-1}\} = Z\{n\, x_1(n-1)\} = -z \frac{d}{dz} X_1(z)$$

If $Z\{x(n)\} = X(z)$

then $Z\{n\,x(n)\} = -z\frac{d}{dz} X(z)$

b) Given that, $x(n) = n^2$

Let us multiply the given discrete time signal by a discrete unit step signal,

$\therefore\ x(n) = n^2\,u(n)$

***Note :** Multiplying a one sided sequence by u(n) will not alter its value.*

By the property of Z-transform, we get,

$$\mathcal{Z}\{n^m u(n)\} = \left(-z\frac{d}{dz}\right)^m U(z)$$

$$\text{where, } U(z) = \mathcal{Z}\{u(n)\} = \frac{z}{z-1}$$

$$\therefore\ -z\frac{d}{dz}U(z) = -z\left[\frac{d}{dz}\left(\frac{z}{z-1}\right)\right] = -z\left[\frac{z-1-z}{(z-1)^2}\right] = \frac{z}{(z-1)^2} \qquad \boxed{d\frac{u}{v} = \frac{v\,du - u\,dv}{v^2}}$$

$$\left(-z\frac{d}{dz}\right)^2 U(z) = -z\frac{d}{dz}\left[-z\frac{d}{dz}U(z)\right]$$

$$= -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) = -z\left(\frac{(z-1)^2 - z \times 2(z-1)}{(z-1)^4}\right)$$

$$= -z\left(\frac{(z-1)(z-1-2z)}{(z-1)^4}\right) = -z\left(\frac{-(z+1)}{(z-1)^3}\right) = \frac{z(z+1)}{(z-1)^3}$$

$$\therefore\ \mathcal{Z}\{x(n)\} = \mathcal{Z}\{n^2u(n)\} = \left(-z\frac{d}{dz}\right)^2 U(z) = \frac{z(z+1)}{(z-1)^3}$$

Example

Find the one sided Z-transform of the discrete time signals generated by mathematically sampling the following continuous time signals.

a) t^2 b) $\sin\Omega_0 t$ c) $\cos\Omega_0 t$

Solution

a) Given that, $x(t) = t^2$

The discrete time signals is generated by replacing t by nT, where T is the sampling time period.

$\therefore\ x(n) = (nT)^2 = n^2T^2 = n^2\,g(n)$

where, $g(n) = T^2$

By the definition of one sided Z-transform we get,

$$G(z) = \mathcal{Z}\{g(n)\} = \mathcal{Z}\{T^2\} = \sum_{n=0}^{\infty} T^2 z^{-n} = T^2\sum_{n=0}^{\infty}(z^{-1})^n = T^2\left(\frac{1}{1-z^{-1}}\right) = \frac{T^2 z}{z-1}$$

By the property of Z-transform we get,

$$X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{n^2 g(n)\} = \left(-z\frac{d}{dz}\right)^2 G(z) = -z\frac{d}{dz}\left(-z\frac{d}{dz}G(z)\right)$$

$$= -z\frac{d}{dz}\left(-z\frac{d}{dz}\frac{T^2z}{z-1}\right) = -z\frac{d}{dz}\left(-z \times \frac{(z-1)T^2 - T^2z}{(z-1)^2}\right)$$

$$= -z\frac{d}{dz}\left(\frac{zT^2}{(z-1)^2}\right) = -z \times \frac{(z-1)^2T^2 - zT^2 \times 2(z-1)}{(z-1)^4}$$

$$= -z \times \frac{(z-1)(zT^2 - T^2 - 2zT^2)}{(z-1)^4} = -z \times \frac{-zT^2 - T^2}{(z-1)^3} = \frac{zT^2(z+1)}{(z-1)^3}$$

b) Given that, $x(t) = \sin\Omega_0 t$

The discrete time signals is generated by replacing t by nT, where T is the sampling time period.

$\therefore \quad x(n) = \sin(\Omega_0 nT) = \sin\omega n\ ; \quad$ where $\omega = \Omega_0 T$

By the definition of one sided Z-transform,

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} \sin\omega n \times z^{-n}$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \sum_{n=0}^{\infty} \frac{e^{j\omega n} - e^{-j\omega n}}{2j} z^{-n} = \frac{1}{2j}\sum_{n=0}^{\infty} e^{j\omega n} z^{-n} - \frac{1}{2j}\sum_{n=0}^{\infty} e^{-j\omega n} z^{-n}$$

$$= \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{j\omega} z^{-1}\right)^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{-j\omega} z^{-1}\right)^n$$

$$= \frac{1}{2j}\,\frac{1}{1 - e^{j\omega} z^{-1}} - \frac{1}{2j}\,\frac{1}{1 - e^{-j\omega} z^{-1}}$$

Using infinite geometric series sum formula

$$= \frac{1}{2j}\,\frac{z}{z - e^{j\omega}} - \frac{1}{2j}\,\frac{z}{z - e^{-j\omega}}$$

$$= \frac{z(z - e^{-j\omega}) - z(z - e^{j\omega})}{2j(z - e^{j\omega})(z - e^{-j\omega})} = \frac{z^2 - z e^{-j\omega} - z^2 + z e^{j\omega}}{2j(z^2 - z e^{-j\omega} - z e^{j\omega} + e^{j\omega} e^{-j\omega})}$$

$$= \frac{z(e^{j\omega} - e^{-j\omega})/2j}{z^2 - z(e^{j\omega} + e^{-j\omega}) + 1}$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}} \qquad \boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1}\ ; \quad \text{where } \omega = \Omega_0 T$$

c) Given that, $x(t) = \cos\Omega_0 t$

The discrete time signal is generated by replacing t by nT, where T is the sampling time period.

$\therefore\ x(n) = \cos(\Omega_0 nT) = \cos\omega n\ ;$ where $\omega = \Omega_0 T$

By the definition of one sided Z-transform,

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} \cos\omega n \times z^{-n}$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \sum_{n=0}^{\infty} \frac{e^{j\omega n} + e^{-j\omega n}}{2} z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty} e^{j\omega n} z^{-n} + \frac{1}{2}\sum_{n=0}^{\infty} e^{-j\omega n} z^{-n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{j\omega} z^{-1}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-j\omega} z^{-1}\right)^n$$

$$= \frac{1}{2}\,\frac{1}{1 - e^{j\omega} z^{-1}} + \frac{1}{2}\,\frac{1}{1 - e^{-j\omega} z^{-1}}$$

Using infinite geometric series sum formula

$$= \frac{1}{2}\,\frac{z}{z - e^{j\omega}} + \frac{1}{2}\,\frac{z}{z - e^{-j\omega}}$$

$$= \frac{z(z - e^{-j\omega}) + z(z - e^{j\omega})}{2(z - e^{j\omega})(z - e^{-j\omega})} = \frac{z^2 - z e^{-j\omega} + z^2 - z e^{j\omega}}{2(z^2 - z e^{-j\omega} - z e^{j\omega} + e^{j\omega} e^{-j\omega})}$$

$$= \frac{2z^2 - z(e^{j\omega} + e^{-j\omega})}{2[z^2 - z(e^{j\omega} + e^{-j\omega}) + 1]} = \frac{z^2 - z(e^{j\omega} + e^{-j\omega})/2}{z^2 - z(e^{j\omega} + e^{-j\omega}) + 1}$$

$$= \frac{z(z - \cos\omega)}{z^2 - 2z\cos\omega + 1}\ ; \quad \text{where } \omega = \Omega_0 T$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

Example

Find the one sided Z-transform of the discrete time signals generated by mathematically sampling the following continuous time signals.

a) $e^{-at}\cos\Omega_0 t$ b) $e^{-at}\sin\Omega_0 t$

Solution

a) Given that, $x(t) = e^{-at}\cos\Omega_0 t$

The discrete time signal x(n) is generated by replacing t by nT, where T is the sampling time period.

$\therefore\ x(n) = e^{-anT}\cos\Omega_0 nT = e^{-anT}\cos\omega n$; where $\omega = \Omega_0 T$

By the definition of one sided Z-transform we get,

$$X(z) = Z\{x(n)\} = \sum_{n=0}^{\infty} e^{-anT}\cos\omega n\, z^{-n} = \sum_{n=0}^{\infty} e^{-anT}\left(\frac{e^{j\omega n}+e^{-j\omega n}}{2}\right)z^{-n}$$

$$\boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-aT}e^{j\omega}z^{-1}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(e^{-aT}e^{-j\omega}z^{-1}\right)^n$$

Using infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

$$= \frac{1}{2}\,\frac{1}{1-e^{-aT}e^{j\omega}z^{-1}} + \frac{1}{2}\,\frac{1}{1-e^{-aT}e^{-j\omega}z^{-1}}$$

$$= \frac{1}{2}\,\frac{1}{1-e^{j\omega}/ze^{aT}} + \frac{1}{2}\,\frac{1}{1-e^{-j\omega}/ze^{aT}}$$

$$= \frac{1}{2}\left[\frac{ze^{aT}}{ze^{aT}-e^{j\omega}} + \frac{ze^{aT}}{ze^{aT}-e^{-j\omega}}\right]$$

$$= \frac{1}{2}\left[\frac{ze^{aT}\left(ze^{aT}-e^{-j\omega}\right) + ze^{aT}\left(ze^{aT}-e^{j\omega}\right)}{\left(ze^{aT}-e^{j\omega}\right)\left(ze^{aT}-e^{-j\omega}\right)}\right]$$

$$= \frac{ze^{aT}}{2}\left[\frac{ze^{aT}-e^{-j\omega}+ze^{aT}-e^{j\omega}}{\left(ze^{aT}\right)^2 - ze^{aT}e^{-j\omega} - ze^{aT}e^{j\omega} + e^{j\omega}e^{-j\omega}}\right]$$

$$= \frac{ze^{aT}}{2}\left[\frac{2ze^{aT}-\left(e^{j\omega}+e^{-j\omega}\right)}{z^2e^{2aT} - ze^{aT}\left(e^{j\omega}+e^{-j\omega}\right)+1}\right]$$

$$= \left[\frac{ze^{aT}\left(ze^{aT}-\cos\omega\right)}{z^2e^{2aT}-2ze^{aT}\cos\omega+1}\right] \quad ; \quad \text{where } \omega = \Omega_0 T$$

$$\boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}}$$

b) Given that, $x(t) = e^{-at} \sin\Omega_0 t$

The discrete time signal x(n) is generated by replacing t by nT, where T is the sampling time period.

$\therefore\ x(n) = e^{-anT} \sin\Omega_0 nT = e^{-anT} \sin\omega n$; where $\omega = \Omega_0 T$

By the definition of one sided Z-transform we get,

$$X(z) = Z\{x(n)\} = \sum_{n=0}^{\infty} e^{-anT} \sin\omega n\, z^{-n} = \sum_{n=0}^{\infty} e^{-anT} \left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right) z^{-n}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{-aT}e^{j\omega}z^{-1}\right)^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{-aT}e^{-j\omega}z^{-1}\right)^n$$

Infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$

$$= \frac{1}{2j}\,\frac{1}{1 - e^{-aT}e^{j\omega}z^{-1}} - \frac{1}{2j}\,\frac{1}{1 - e^{-aT}e^{-j\omega}z^{-1}}$$

$$= \frac{1}{2j}\,\frac{1}{1 - e^{j\omega}/z\,e^{aT}} - \frac{1}{2j}\,\frac{1}{1 - e^{-j\omega}/z\,e^{aT}}$$

$$= \frac{1}{2j}\,\frac{z\,e^{aT}}{z\,e^{aT} - e^{j\omega}} - \frac{1}{2j}\,\frac{z\,e^{aT}}{z\,e^{aT} - e^{-j\omega}}$$

$$= \frac{1}{2j}\left[\frac{z\,e^{aT}\left(z\,e^{aT} - e^{-j\omega}\right) - z\,e^{aT}\left(z\,e^{aT} - e^{j\omega}\right)}{\left(z\,e^{aT} - e^{j\omega}\right)\left(z\,e^{aT} - e^{-j\omega}\right)}\right]$$

$$= \frac{1}{2j}\left[\frac{\left(z\,e^{aT}\right)\left[z\,e^{aT} - e^{-j\omega} - z\,e^{aT} + e^{j\omega}\right]}{\left(z\,e^{aT}\right)^2 - z\,e^{aT}e^{-j\omega} - z\,e^{aT}e^{j\omega} + e^{j\omega}e^{-j\omega}}\right]$$

$$= \left[\frac{z\,e^{aT}\left[e^{j\omega} - e^{-j\omega}\right]/2j}{z^2 e^{2aT} - z\,e^{aT}\left(e^{j\omega} + e^{-j\omega}\right) + 1}\right]$$

$$= \frac{z\,e^{aT}\sin\omega}{z^2 e^{2aT} - 2z e^{aT}\cos\omega + 1} \quad ; \quad \text{where } \omega = \Omega_0 T$$

4.5 Representation of Poles and Zeros in z-Plane

The complex variable, z is defined as,

$z = u + jv$

where, u = Real part of z

v = Imaginary part of z

Hence the z-plane is a complex plane, with u on real axis and v on imaginary axis (Refer fig 7.1 in section 7.1). In the z-plane, the zeros are marked by small circle "o" and the poles are marked by letter "x".

For example consider a rational function of z shown below.

$$X(z) = \frac{1.25 - 1.25z^{-1} + 0.2z^{-2}}{2 + 2z^{-1} + z^{-2}}$$

$$= \frac{1.25\left(1 - z^{-1} + \frac{0.2}{1.25}z^{-2}\right)}{2\left(1 + z^{-1} + \frac{1}{2}z^{-2}\right)} = \frac{0.625\,(1 - z^{-1} + 0.16z^{-2})}{(1 + z^{-1} + 0.5z^{-2})}$$

$$= \frac{0.625z^{-2}(z^2 - z + 0.16)}{z^{-2}(z^2 + z + 0.5)} = \frac{0.625\,(z - 0.8)(z - 0.2)}{(z + 0.5 + j0.5)(z + 0.5 - j0.5)}$$

The roots of quadratic, $z^2 - z + 0.16 = 0$ are,

$$z = \frac{1 \pm \sqrt{1 - 4 \times 0.16}}{2} = \frac{1 \pm 0.6}{2} = 0.8, 0.2$$

$$\therefore \quad z^2 - z + 0.16 = (z - 0.8)(z - 0.2)$$

The roots of quadratic, $z^2 + z + 0.5 = 0$ are,

$$z = \frac{-1 \pm \sqrt{1 - 4 \times 0.5}}{2} = \frac{-1 \pm j}{2} = -0.5 \pm j0.5$$

$$\therefore \quad z^2 + z + 0.5 = (z + 0.5 + j0.5)(z + 0.5 - j0.5)$$

The zeros of X(z) are roots of numerator polynomial, which has two roots.

Therefore, the zeros of X(z) are,

$$z_1 = 0.8, \; z_2 = 0.2$$

The poles of X(z) are roots of denominator polynomial, which has two roots.

Therefore, the poles of X(z) are,

$$p_1 = -0.5 - j0.5, \quad p_2 = -0.5 + j0.5$$

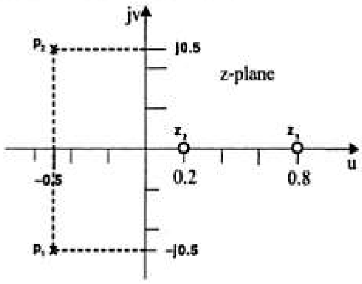


Fig 7.5 : Pole-zero plot of X(z) of equation (7.37).

The pole-zero plot of X(z) is shown in fig 7.5.

Example

Determine the inverse z-transform of the following z-domain functions.

a) $X(z) = \frac{3z^2 + 2z + 1}{z^2 + 3z + 2}$ b) $X(z) = \frac{z - 0.4}{z^2 + z + 2}$ c) $X(z) = \frac{z - 4}{(z - 1)(z - 2)^2}$

Solution

a) Given that, $X(z) = \frac{3z^2 + 2z + 1}{z^2 + 3z + 2}$

$$X(z) = \frac{3z^2 + 2z + 1}{z^2 + 3z + 2}$$

$$\therefore \frac{X(z)}{z} = \frac{3z^2 + 2z + 1}{z(z^2 + 3z + 2)} = \frac{3z^2 + 2z + 1}{z(z + 1)(z + 2)}$$

By partial fraction expansion technique $\frac{X(z)}{z}$ can be expressed as,

$$\frac{X(z)}{z} = \frac{3z^2 + 2z + 1}{z(z + 1)(z + 2)} = \frac{A_1}{z} + \frac{A_2}{z + 1} + \frac{A_3}{z + 3}$$

$$A_1 = z\frac{X(z)}{z}\Big|_{z=0} = z\frac{3z^2 + 2z + 1}{z(z + 1)(z + 2)}\Big|_{z=0} = \frac{1}{1 \times 2} = 0.5$$

$$A_2 = (z + 1)\frac{X(z)}{z}\Big|_{z=-1} = (z + 1)\frac{3z^2 + 2z + 1}{z(z + 1)(z + 2)}\Big|_{z=-1} = \frac{3(-1)^2 + 2(-1) + 1}{-1 \times (-1 + 2)} = -2$$

$$A_3 = (z + 2)\frac{X(z)}{z}\Big|_{z=-2} = (z + 2)\frac{3z^2 + 2z + 1}{z(z + 1)(z + 2)}\Big|_{z=-2} = \frac{3(-2)^2 + 2(-2) + 1}{-2 \times (-2 + 1)} = 4.5$$

$$\therefore \frac{X(z)}{z} = \frac{0.5}{z} - \frac{2}{z + 1} + \frac{4.5}{z + 2}$$

$$\therefore X(z) = 0.5 - \frac{2z}{z + 1} + \frac{4.5z}{z + 2}$$

$$= 0.5 - 2\frac{z}{z - (-1)} + 4.5\frac{z}{z - (-2)}$$

$Z\{\delta(n)\} = 1$
$Z\{a^n u(n)\} = \frac{z}{z - a}$

On taking inverse z-transform of X(z), we get,

$$x(n) = 0.5\,\delta(n) - 2\,(-1)^n u(n) + 4.5\,(-2)^n u(n) = 0.5\,\delta(n) + [-2\,(-1)^n + 4.5\,(-2)^n]\,u(n)$$

b) Given that, $X(z) = \dfrac{z - 0.4}{z^2 + z + 2}$

The roots of the quadratic $z^2 + z + 2 = 0$ are,

$$z = \frac{-1 \pm \sqrt{1 - 4 \times 2}}{2} = -0.5 \pm j\sqrt{7}/2$$

$$X(z) = \frac{z - 0.4}{z^2 + z + 2} = \frac{z - 0.4}{(z + 0.5 - j\sqrt{7}/2)\,(z + 0.5 + j\sqrt{7}/2)}$$

By partial fraction expansion we get,

$$X(z) = \frac{A}{z + 0.5 - j\sqrt{7}/2} + \frac{A^*}{z + 0.5 + j\sqrt{7}/2}$$

$$A = (z + 0.5 - j\sqrt{7}/2)\left.\frac{z - 0.4}{(z + 0.5 - j\sqrt{7}/2)\,(z + 0.5 + j\sqrt{7}/2)}\right|_{z = -0.5 + j\sqrt{7}/2}$$

$$= \left.\frac{z - 0.4}{(z + 0.5 + j\sqrt{7}/2)}\right|_{z = -0.5 + j\sqrt{7}/2} = \frac{-0.5 + j\sqrt{7}/2 - 0.4}{-0.5 + j\sqrt{7}/2 + 0.5 + j\sqrt{7}/2}$$

$$= \frac{-0.9 + j\sqrt{7}/2}{j\sqrt{7}} = \frac{-0.9}{j\sqrt{7}} + \frac{j\sqrt{7}/2}{j\sqrt{7}} = 0.5 + \frac{j0.9}{\sqrt{7}}$$

$$\therefore\ A^* = \left(0.5 + \frac{j0.9}{\sqrt{7}}\right)^* = \left(0.5 - \frac{j0.9}{\sqrt{7}}\right)$$

$$\therefore\ X(z) = \frac{0.5 + j0.9/\sqrt{7}}{z + 0.5 - j\sqrt{7}/2} + \frac{0.5 - j0.9/\sqrt{7}}{z + 0.5 + j\sqrt{7}/2}$$

Multiply and dvide by z

$$= (0.5 + j0.9/\sqrt{7})\,\frac{1}{z}\,\frac{z}{z + 0.5 - j\sqrt{7}/2} + (0.5 - j0.9/\sqrt{7})\,\frac{1}{z}\,\frac{z}{z + 0.5 + j\sqrt{7}/2}$$

$$= (0.5 + j0.9/\sqrt{7})\,z^{-1}\,\frac{z}{z - (-0.5 + j\sqrt{7}/2)} + (0.5 - j0.9/\sqrt{7})\,z^{-1}\,\frac{z}{z - (-0.5 - j\sqrt{7}/2)}$$

On taking inverse z-transform of X(z) we get,

$$x(n) = (0.5 + j0.9/\sqrt{7})\,(-0.5 + j\sqrt{7}/2)^{(n-1)}u(n-1) + (0.5 - j0.9/\sqrt{7})\,(-0.5 - j\sqrt{7}/2)^{(n-1)}u(n-1)$$

If $Z\{a^n u(n)\} = \dfrac{z}{z - a}$

then by time shifting property,

$$Z\{a^{(n-1)} u(n-1)\} = z^{-1}\,\frac{z}{z - a}$$

Determine the inverse Z-transform of the following function.

a) $X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$ b) $X(z) = \frac{z^2}{z^2 - z + 0.5}$

c) $X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$ d) $X(z) = \frac{1}{(1 + z^{-1})(1 - z^{-1})^2}$

Solution

a) Given that, $X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} = \frac{1}{1 - \frac{1.5}{z} + \frac{0.5}{z^2}} = \frac{z^2}{z^2 - 1.5z + 0.5} = \frac{z^2}{(z-1)(z-0.5)}$$

$$\therefore \frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)}$$

By partial fraction expansion, X(z)/z can be expressed as,

$$\frac{X(z)}{z} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

$$A_1 = (z-1)\frac{X(z)}{z}\Big|_{z=1} = (z-1)\frac{z}{(z-1)(z-0.5)}\Big|_{z=1} = \frac{1}{1-0.5} = 2$$

$$A_2 = (z-0.5)\frac{X(z)}{z}\Big|_{z=0.5} = (z-0.5)\frac{z}{(z-1)(z-0.5)}\Big|_{z=0.5} = \frac{0.5}{0.5-1} = -1$$

$$\therefore \frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

$$\therefore X(z) = \frac{2z}{z-1} - \frac{z}{z-0.5}$$

> $Z\{a^n u(n)\} = \frac{z}{z-a}$; ROC $|z| > |a|$
>
> $Z\{u(n)\} = \frac{z}{z-1}$; ROC $|z| > 1$

On taking inverse Z-transform of X(z), we get,

$$x(n) = 2\,u(n) - 0.5^n u(n) = [2 - 0.5^n]\,u(n)$$

---

b) Given that, $X(z) = \frac{z^2}{z^2 - z + 0.5}$

$$X(z) = \frac{z^2}{z^2 - z + 0.5} = \frac{z^2}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}$$

$$\therefore \frac{X(z)}{z} = \frac{z}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}$$

> The roots of quadratic $z^2 - z + 0.5 = 0$ are,
>
> $z = \frac{1 \pm \sqrt{1 - 4 \times 0.5}}{2}$
>
> $= 0.5 \pm j0.5$

By partial fraction expansion, we can write,

$$\frac{X(z)}{z} = \frac{A}{z - 0.5 - j0.5} + \frac{A^*}{z - 0.5 + j0.5}$$

$$A = (z - 0.5 - j0.5)\frac{X(z)}{z}\Big|_{z = 0.5 + j0.5}$$

$$= (z - 0.5 - j0.5)\frac{z}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}\Big|_{z = 0.5 + j0.5}$$

$$\therefore A = \frac{0.5 + j0.5}{0.5 + j0.5 - 0.5 + j0.5} = \frac{0.5 + j0.5}{j1.0} = 0.5 - j0.5$$

$$\therefore A^* = (0.5 - j0.5)^* = 0.5 + j0.5$$

$$\therefore \frac{X(z)}{z} = \frac{0.5 - j0.5}{z - 0.5 - j0.5} + \frac{0.5 + j0.5}{z - 0.5 + j0.5}$$

$$X(z) = \frac{(0.5 - j0.5)\,z}{z - (0.5 + j0.5)} + \frac{(0.5 + j0.5)\,z}{z - (0.5 - j0.5)}$$

$z\{a^n u(n)\} = \frac{z}{z-a}$; ROC $\lvert z\rvert > \lvert a\rvert$

On taking inverse z-transform of X(z) we get,

$$x(n) = (0.5 - j0.5)(0.5 + j0.5)^n u(n) + (0.5 + j0.5)(0.5 - j0.5)^n u(n)$$

Alternatively the above result can be expressed as shown below.

Here, $0.5 + j0.5 = 0.707 \angle 45^\circ = 0.707 \angle 0.25\pi$

$0.5 - j0.5 = 0.707 \angle -45^\circ = 0.707 \angle -0.25\pi$

$$\therefore x(n) = [0.707 \angle -0.25\pi][0.707 \angle 0.25\pi]^n u(n) + [0.707 \angle 0.25\pi][0.707 \angle -0.25\pi]^n u(n)$$

$$= [0.707 \angle -0.25\pi][0.707^n \angle 0.25\pi n] u(n) + [0.707 \angle 0.25\pi][0.707^n \angle -0.25\pi n] u(n)$$

$$= 0.707^{(n+1)} \angle (0.25\pi (n-1)) u(n) + 0.707^{(n+1)} \angle (-0.25\pi(n-1)) u(n)$$

$$= 0.707^{(n+1)} [1 \angle 0.25\pi (n-1) + 1 \angle -0.25\pi(n-1)] u(n)$$

$$= 0.707^{(n+1)} [\cos(0.25\pi (n-1)) + j \sin(0.25\pi(n-1)) + \cos(0.25\pi(n-1)) - j \sin(0.25\pi(n-1))] u(n)$$

$$= 0.707^{(n+1)}\, 2 \cos(0.25\pi (n-1))\, u(n)$$

c) Given that, $X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}} = \frac{z^{-1}(z+1)}{z^{-2}(z^2 - z + 0.5)}$$

$$= \frac{z(z+1)}{(z^2 - z + 0.5)} = \frac{z(z+1)}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}$$

The roots of the quadratic $z^2 - z + 0.5 = 0$ are, $z = \frac{1 \pm \sqrt{1 - 4 \times 0.5}}{2} = 0.5 \pm j0.5$

By partial fraction expansion, we can write,

$$\frac{X(z)}{z} = \frac{(z+1)}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)} = \frac{A}{z - 0.5 - j0.5} + \frac{A^*}{z - 0.5 + j0.5}$$

$$A = (z - 0.5 - j0.5) \left.\frac{X(z)}{z}\right|_{z = 0.5 + j0.5}$$

$$= (z - 0.5 - j0.5) \left.\frac{(z+1)}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}\right|_{z = 0.5 + j0.5}$$

$$= \frac{0.5 + j0.5 + 1}{0.5 + j0.5 - 0.5 + j0.5} = \frac{1.5 + j0.5}{j1} = -j1.5 + 0.5 = 0.5 - j1.5$$

$$A^* = (0.5 - j1.5)^* = 0.5 + j1.5$$

$$\therefore \frac{X(z)}{z} = \frac{0.5 - j1.5}{z - 0.5 - j0.5} + \frac{0.5 + j1.5}{z - 0.5 + j0.5}$$

$$X(z) = (0.5 - j1.5)\frac{z}{z - (0.5 + j0.5)} + (0.5 + j1.5)\frac{z}{z - (0.5 - j0.5)}$$

$z\{a^n u(n)\} = \frac{z}{z-a}$

On taking inverse Z-transform of X(z) we get,

$$x(n) = (0.5 - j1.5)(0.5 + j0.5)^n u(n) + (0.5 + j1.5)(0.5 - j0.5)^n u(n)$$

Alternatively the above result can be expressed as shown below.

Here, $0.5 - j1.5 = 1.581\ \angle -71.6^\circ = 1.581\ \angle -0.4\pi$

$0.5 + j1.5 = 1.581\ \angle 71.6^\circ = 1.581\ \angle 0.4\pi$

$0.5 + j0.5 = 0.707\ \angle 45^\circ = 0.707\ \angle 0.25\pi$

$0.5 - j0.5 = 0.707\ \angle -45^\circ = 0.707\ \angle -0.25\pi$

$$\therefore x(n) = [1.581\ \angle -0.4\pi][0.707\ \angle 0.25\pi]^n u(n) + [1.581\ \angle 0.4\pi][0.707\ \angle -0.25\pi]^n u(n)$$

$$= [1.581\ \angle -0.4\pi][0.707^n\ \angle 0.25\pi n] u(n) + [1.581\ \angle 0.4\pi][0.707^n\ \angle -0.25\pi n] u(n)$$

$$= 1.581\ (0.707)^n [1\ \angle \pi(0.25n - 0.4) + 1\ \angle -\pi(0.25n - 0.4)] u(n)$$

$$= 1.581\ (0.707)^n [\cos(\pi(0.25n - 0.4)) + j\sin(\pi(0.25n - 0.4)) + \cos(\pi(0.25n - 0.4)) - j\sin(\pi(0.25n - 0.4))] u(n)$$

$$= 1.581\ (0.707)^n\ 2\cos(\pi(0.25n - 0.4))\ u(n)$$

$$= 3.162\ (0.707)^n \cos(\pi(0.25n - 0.4))\ u(n)$$

d) Given that, $X(z) = \dfrac{1}{(1 + z^{-1})(1 - z^{-1})^2}$

$$X(z) = \frac{1}{(1 + z^{-1})(1 - z^{-1})^2} = \frac{1}{z^{-1}(z + 1)\,z^{-2}(z - 1)^2} = \frac{z^3}{(z + 1)(z - 1)^2}$$

$$\therefore \frac{X(z)}{z} = \frac{z^2}{(z + 1)(z - 1)^2}$$

By partial fraction expansion, we can write,

$$\frac{X(z)}{z} = \frac{A_1}{z + 1} + \frac{A_2}{(z - 1)^2} + \frac{A_3}{z - 1}$$

$$A_1 = (z + 1)\left.\frac{X(z)}{z}\right|_{z=-1} = (z + 1)\left.\frac{z^2}{(z + 1)(z - 1)^2}\right|_{z=-1} = \left.\frac{z^2}{(z - 1)^2}\right|_{z=-1} = \frac{(-1)^2}{(-1 - 1)^2} = 0.2$$

$$A_2 = (z - 1)^2\left.\frac{X(z)}{z}\right|_{z=1} = (z - 1)^2\left.\frac{z^2}{(z + 1)(z - 1)^2}\right|_{z=1} = \left.\frac{z^2}{z + 1}\right|_{z=1} = \frac{1}{1 + 1} = 0.5$$

$$A_3 = \frac{d}{dz}\left[(z - 1)^2 \frac{X(z)}{z}\right]\Bigg|_{z=1} = \frac{d}{dz}\left[(z - 1)^2 \frac{z^2}{(z + 1)(z - 1)^2}\right]\Bigg|_{z=1}$$

$$= \frac{d}{dz}\left[\frac{z^2}{z + 1}\right]\Bigg|_{z=1} = \left.\frac{(z + 1)\,2z - z^2}{(z + 1)^2}\right|_{z=1} = \frac{(1 + 1) \times 2 - 1}{(1 + 1)^2} = \frac{3}{4} = 0.75$$

$$\therefore \frac{X(z)}{z} = \frac{0.25}{z + 1} + \frac{0.5}{(z - 1)^2} + \frac{0.75}{z - 1}$$

$$X(z) = 0.25\frac{z}{z + 1} + 0.5\frac{z}{(z - 1)^2} + 0.75\frac{z}{z - 1}$$

$$= 0.25\frac{z}{z - (-1)} + 0.5\frac{z}{(z - 1)^2} + 0.75\frac{z}{z - 1}$$

On taking inverse Z-transform of X(z) we get,

$$x(n) = 0.25(-1)^n + 0.5n\,u(n) + 0.75\,u(n)$$

$$= [0.25(-1)^n + 0.5n + 0.75]\,u(n)$$

$Z\{a^n u(n)\} = \dfrac{z}{z - a}$

$Z\{n\,u(n)\} = \dfrac{z}{(z - 1)^2}$

$Z\{u(n)\} = \dfrac{z}{z - 1}$

$$\therefore \frac{X(z)}{z} = \frac{1.5}{z - 0.6} - \frac{0.5}{z - 0.2}$$

$$\therefore X(z) = 1.5\frac{z}{z - 0.6} - 0.5\frac{z}{z - 0.2}$$

Now, the poles of X(z) are $p_1 = 0.6$, $p_2 = 0.2$

a) ROC is |z| > 0.6

The specified ROC is exterior of the circle whose radius corresponds to the largest pole, hence x(n) will be a causal (or right sided) signal.

$$\therefore x(n) = 1.5(0.6)^n u(n) - 0.5\,(0.2)^n u(n)$$

$Z\{a^n u(n)\} = \frac{z}{z - a}$; ROC $\lvert z\rvert > \lvert a\rvert$

b) ROC is |z| < 0.2

The specified ROC is interior of the circle whose radius corresponds to the smallest pole, hence x(n) will be an anticausal (or left sided) signal.

$$\therefore x(n) = 1.5(-(0.6)^n u(-n-1)) - 0.5\,(-(0.2)^n u(-n-1))$$
$$= -1.5\,(0.6)^n u(-n-1) + 0.5\,(0.2)^n u(-n-1)$$

$Z\{-a^n u(-n-1)\} = \frac{z}{z - a}$; ROC $\lvert z\rvert < \lvert a\rvert$

c) ROC is 0.2 < |z| < 0.6

The specified ROC is the region in between two circles of radius 0.2 and 0.7. Hence the term corresponds to the pole, $p_1 = 0.6$ will be anticausal signal (because $|z| < 0.6$) and the term corresponds to the pole, $p_2 = 0.2$, will be a causal signal (because $|z| > 0.2$).

$$\therefore x(n) = 1.5(-(0.6)^n u(-n-1)) - 0.5\,(0.2)^n u(n)$$
$$= -1.5(0.6)^n u(-n-1)) - 0.5\,(0.2)^n u(n)$$

Example

Determine the impulse response h(n) for the system described by the second order difference equation,

$y(n) - 4y(n-1) + 4y(n-2) = x(n-1)$.

Solution

The difference equation governing the system is,

$$y(n) - 4y(n-1) + 4y(n-2) = x(n-1)$$

Let us take Z-transform of the difference equation governing the system with zero initial conditions.

$$\therefore Z\{y(n) - 4y(n-1) + 4y(n-2)\} = Z\{x(n-1)\}$$
$$Z\{y(n)\} - 4\,Z\{y(n-1)\} + 4\,Z\{y(n-2)\} = Z\{x(n-1)\}$$
$$Y(z) - 4z^{-1}Y(z) + 4z^{-2}Y(z) = z^{-1}X(z)$$
$$(1 - 4z^{-1} + 4z^{-2})\,Y(z) = z^{-1}X(z)$$

If $Z\{x(n)\} = X(z)$ then by shifting property $Z\{x(n-m)\} = z^{-m}X(z)$
If $Z\{y(n)\} = Y(z)$ then by shifting property $Z\{y(n-m)\} = z^{-m}Y(z)$

$$\therefore \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - 4z^{-1} + 4z^{-2}}$$

We know that, $\frac{Y(z)}{X(z)} = H(z)$

$$\therefore H(z) = \frac{z^{-1}}{1 - 4z^{-1} + 4z^{-2}} = \frac{z^{-1}}{z^{-2}(z^2 - 4z + 4)} = \frac{z}{(z-2)^2} = \frac{1}{2}\frac{2z}{(z-2)^2}$$

The impulse response h(n) is given by inverse Z-transform of H(z).

$$\text{Impulse response, } h(n) = Z^{-1}\{H(z)\} = Z^{-1}\left\{\frac{1}{2}\frac{2z}{(z-2)^2}\right\}$$
$$= (1/2)\,n2^n u(n) = n2^{(n-1)} u(n)$$

$Z\{na^n u(n)\} = \frac{az}{(z - a)^2}$

Example

Find the transfer function and unit sample response of the second order difference equation with zero initial condition,

$y(n) = x(n) - 0.25y(n-2)$.

Solution

The difference equation governing the system is,

$$y(n) = x(n) - 0.25\, y(n-2)$$

Let us take Z-transform of the difference equation governing the system with zero initial condition.

$$Z\{y(n)\} = Z\{x(n - 0.25\, y(n-2)\}$$

$$Z\{y(n)\} = Z\{x(n)\} - 0.25\, Z\{y(n-2)\}$$

$$Y(z) = X(z) - 0.25\, z^{-2}\, Y(z)$$

$$Y(z) + 0.25z^{-2}\, Y(z) = X(z)$$

$$(1 + 0.25z^{-2})\, Y(z) = X(z)$$

$Z\{x(n)\} = X(z)$
$Z\{y(n)\} = Y(z)$ $Z\{y(n-2)\} = z^{-2}\, Y(z)$ (Using shifting property)

$\therefore$ Transfer function, $$\frac{Y(z)}{X(z)} = \frac{1}{1 + 0.25z^{-2}}$$

Let us assume that the initial conditions are zero.

On taking Z-transform of the difference equation governing the system we get,

$Z\{x(n)\} = X(z)\ ,\ \therefore\ Z\{ax(n-m)\} = az^{-m}X(z)$
$Z\{y(n)\} = Y(z)\ ,\ \therefore\ Z\{ay(n-m)\} = az^{-m}Y(z)$

$$Z\{y(n) - 2y(n-1) + y(n-2)\} = Z\{x(n) + 3x(n-3)\}$$

$$Z\{y(n)\} - 2\, Z\{y(n-1)\} + Z\{y(n-2)\} = Z\{x(n)\} + 3\, Z\{x(n-3)\}$$

$$Y(z) - 2z^{-1}\, Y(z) + z^{-2}\, Y(z) = X(z) + 3z^{-3}\, X(z)$$

$$(1 - 2z^{-1} + z^{-2})\, Y(z) = (1 + 3z^{-3})\, X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1 + 3z^{-3}}{1 - 2z^{-1} + z^{-2}}$$

$Z\{u(n)\} = \frac{z}{z-1}$
$Z\{n\, u(n)\} = \frac{z}{(z-1)^2}$
$Z\{(n+1)\, u(n+1)\} = z\frac{z}{(z-1)^2}$
$Z\{(n-2)\, u(n-2)\} = z^{-2}\frac{z}{(z-1)^2}$

We know that, $$\frac{Y(z)}{X(z)} = H(z)$$

$$\therefore H(z) = \frac{1 + 3z^{-3}}{1 - 2z^{-1} + z^{-2}} = \frac{1 + 3z^{-3}}{z^{-2}(z^2 - 2z + 1)} = \frac{z^2 + 3z^{-1}}{(z-1)^2}$$

$$= \frac{z^2}{(z-1)^2} + \frac{3z^{-1}}{(z-1)^2} = z\frac{z}{(z-1)^2} + 3z^{-2}\frac{z}{(z-1)^2}$$

The impulse response is obtained by taking inverse Z-transform of H(z).

$$\therefore \text{Impulse response, } h(n) = Z^{-1}\{H(z)\} = Z^{-1}\left\{z\frac{z}{(z-1)^2} + 3z^{-2}\frac{z}{(z-1)^2}\right\}$$

$$= Z^{-1}\left\{z\frac{z}{(z-1)^2}\right\} + 3Z^{-1}\left\{z^{-2}\frac{z}{(z-1)^2}\right\}$$

$$= (n+1)\, u(n+1) + 3(n-2)\, u(n-2)$$

Example 7.15

Find the impulse response of the system described by the difference equation,
$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$.

Solution

The difference equation governing the LTI system is,

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

$\mathcal{Z}\{y(n)\} = Y(z)$; ∴	$\mathcal{Z}\{y(n-m)\} = z^{-m}\,Y(z)$
$\mathcal{Z}\{x(n)\} = X(z)$; ∴	$\mathcal{Z}\{x(n-m)\} = z^{-m}\,X(z)$

On taking Z-transform we get,

$$Y(z) - 3z^{-1}Y(z) - 4z^{-2}Y(z) = X(z) + 2z^{-1}X(z)$$

$$(1 - 3z^{-1} - 4z^{-2})\,Y(z) = (1 + 2z^{-1})\,X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}}$$

The roots of the quadratic, $z^2 - 3z - 4 = 0$ are,

$$z = \frac{3 \pm \sqrt{3^2 + 4 \times 4}}{2} = 4 \text{ or } -1$$

We know that $\frac{Y(z)}{X(z)} = H(z)$

$$\therefore H(z) = \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}} = \frac{z^{-2}(z^2 + 2z)}{z^{-2}(z^2 - 3z - 4)} = \frac{z^2 + 2z}{(z-4)(z+1)}$$

Example

Obtain and sketch the impulse response of shift invariant system described by,
$y(n) = 0.4\,x(n) + x(n-1) + 0.6\,x(n-2) + x(n-3) + 0.4\,x(n-4)$.

Solution

The difference equation governing the system is,

$$y(n) = 0.4\,x(n) + x(n-1) + 0.6\,x(n-2) + x(n-3) + 0.4\,x(n-4)$$

If $\mathcal{Z}\{x(n)\} = X(z)$ then by shifting property $\mathcal{Z}\{x(n-k)\} = z^{-k}\,X(z)$

On taking Z-transform we get,

$$Y(z) = 0.4X(z) + z^{-1}X(z) + 0.6z^{-2}X(z) + z^{-3}X(z) + 0.4z^{-4}X(z)$$

$$Y(z) = [0.4 + z^{-1} + 0.6z^{-2} + z^{-3} + 0.4z^{-4}]\,X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \left[0.4 + z^{-1} + 0.6z^{-2} + z^{-3} + 0.4z^{-4}\right]$$

We know that $\frac{Y(z)}{X(z)} = H(z)$;

$$\therefore H(z) = 0.4 + z^{-1} + 0.6z^{-2} + z^{-3} + 0.4z^{-4} \quad .....(1)$$

By the definition of one sided Z-transform we get,

$$H(z) = \sum_{n=0}^{+\infty} h(n)z^{-n}$$

$$= h(0)\,z^0 + h(1)\,z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3} + h(4)\,z^{-4} + ... \quad .....(2)$$

On comparing equations (1) & (2) we get,

$h(0) = 0.4$	$h(3) = 1$
$h(1) = 1$	$h(4) = 0.4$
$h(2) = 0.6$	$h(n) = 0$; for $n < 0$ and $n > 4$

∴ Impulse response, $h(n) = \{0.4, 1.0, 0.6, 1.0, 0.4\}$
↑

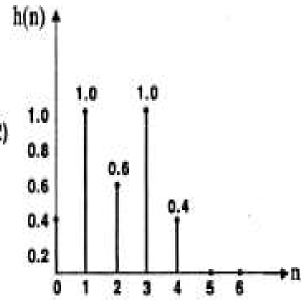


Fig 1 : Graphical representation of impulse response h(n).

Example

Determine the response of discrete time LTI system governed by the difference equation $y(n) = -0.5\,y(n-1) + x(n)$, when the input is unit step and initial condition, a) $y(-1) = 0$ and b) $y(-1) = 1/3$.

Solution

Given that, $x(n) = u(n)$; $\therefore X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{u(n)\} = \frac{z}{z-1}$ .....(1)

Given that, $y(n) = -0.5\,y(n-1) + x(n)$

$\therefore y(n) + 0.5\,y(n-1) = x(n)$

On taking $\mathcal{Z}$-transform of above equation we get,

If $\mathcal{Z}\{y(n)\} = Y(z)$ then $\mathcal{Z}\{y(n-1)\} = z^{-1}Y(z) - y(-1)$

$$Y(z) + 0.5\left[z^{-1}Y(z) + y(-1)\right] = X(z)$$

$$Y(z)\left[1 + 0.5z^{-1}\right] + 0.5y(-1) = \frac{z}{z-1} \quad \text{Using equation (1)}$$

$$Y(z)\left(1 + \frac{0.5}{z}\right) = \frac{z}{z-1} - 0.5y(-1)$$

$$Y(z)\left(\frac{z+0.5}{z}\right) = \frac{z}{z-1} - 0.5y(-1)$$

$$\therefore Y(z) = \frac{z^2}{(z-1)(z+0.5)} - \frac{1}{2}\frac{zy(-1)}{z+0.5}$$

Let, $P(z) = \frac{z^2}{(z-1)(z+0.5)} \Rightarrow \frac{P(z)}{z} = \frac{z}{(z-1)(z+0.5)}$

Let, $\frac{z}{(z-1)(z+0.5)} = \frac{A}{z-1} + \frac{B}{z+0.5}$

$$A = \frac{z}{(z-1)(z+0.5)} \times (z-1)\Big|_{z=1} = \frac{1}{1+0.5} = \frac{1}{1.5} = \frac{2}{3}$$

$$B = \frac{z}{(z-1)(z+0.5)} \times (z+0.5)\Big|_{z=-0.5} = \frac{-0.5}{-0.5-1} = \frac{-0.5}{-1.5} = \frac{5}{15} = \frac{1}{3}$$

$$\therefore \frac{P(z)}{z} = \frac{2}{3}\frac{1}{z-1} + \frac{1}{3}\frac{1}{z+0.5} \Rightarrow P(z) = \frac{2}{3}\frac{z}{z-1} + \frac{1}{3}\frac{z}{z+0.5}$$

$$\therefore Y(z) = \frac{2}{3}\frac{z}{z-1} + \frac{1}{3}\frac{z}{z+0.5} - \frac{1}{2}\frac{zy(-1)}{z+0.5} \quad .....(2)$$

a) When $y(-1) = 0$

From equation (2), when $y(-1) = 0$, we get,

$$Y(z) = \frac{2}{3}\frac{z}{z-1} + \frac{1}{3}\frac{z}{z+0.5}$$

$$\therefore \text{Response, } y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{\frac{2}{3}\frac{z}{z-1} + \frac{1}{3}\frac{z}{z+0.5}\right\} = \frac{2}{3}u(n) + \frac{1}{3}(-0.5)^n u(n) = \frac{1}{3}\left[2 + (-0.5)^n\right]u(n)$$

Using $\mathcal{Z}$-transform perform deconvolution of response, $y(n) = 2\,(0.4)^n\,u(n) - (0.2)^n\,u(n)$ and impulse response, $h(n) = 0.4^n\,u(n)$, to extract the input $x(n)$.

Solution

Given that, $y(n) = 2\,(0.4)^n\,u(n) - (0.2)^n\,u(n)$

$$\therefore Y(z) = \mathcal{Z}\{y(n)\} = \mathcal{Z}\{2\,(0.4)^n u(n) - (0.2)^n\,u(n)\}$$

$$= \frac{2z}{z-0.4} - \frac{z}{z-0.2} = \frac{2z(z-0.2) - z(z-0.4)}{(z-0.4)\,(z-0.2)} = \frac{2z^2 - 0.4z - z^2 + 0.4z}{(z-0.4)\,(z-0.2)} = \frac{z^2}{(z-0.4)\,(z-0.2)}$$

Given that, $h(n) = 0.4^n\,u(n)$

$$\therefore H(z) = \mathcal{Z}\{h(n)\} = \mathcal{Z}\{0.4^n u(n)\} = \frac{z}{z-0.4}$$

We know that, $H(z) = \frac{Y(z)}{X(z)}$

$$\therefore X(z) = \frac{Y(z)}{H(z)} = Y(z) \times \frac{1}{H(z)} = \frac{z^2}{(z-0.4)\,(z-0.2)} \times \frac{z-0.4}{z} = \frac{z}{z-0.2}$$

$$\therefore \text{Input, } x(n) = \mathcal{Z}^{-1}\{X(z)\} = \mathcal{Z}^{-1}\left\{\frac{z}{z-0.2}\right\} = 0.2^n\,u(n)$$

4.6 DISCRETE-TIME FOURIER TRANSFORM (DTFT)

The **discrete-time Fourier transform** (DTFT) or, simply, the Fourier transform of a discrete time sequence $x(n)$ is represented by the complex exponential sequence $[e^{-j\omega n}]$ where ω is the real frequency variable. This transform is useful to map the time-domain sequence into a continuous function of a frequency variable. The DTFT and the z-transform are applicable to any arbitrary sequences, whereas the DFT can be applied only to finite length sequences.

Definition

The discrete-time Fourier transform $X(e^{j\omega})$ of a sequence $x(n)$ is defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

This equation represents the Fourier series representation of the periodic function $X(e^{j\omega})$. Hence, the Fourier coefficients $x(n)$ can be determined from $X(e^{j\omega})$ using the Fourier integral expressed by

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega$$

called the inverse discrete-time Fourier transform (IDTFT).

4.7 Transfer Function of LTI Discrete Time System in Frequency Domain

The ratio of Fourier transform of output and the Fourier transform of input is called ***transfer function*** of LTI discrete time system in frequency domain.

Let, $x(n)$ = Input to the discrete time system

$y(n)$ = Output of the discrete time system

$\therefore$ $X(e^{j\omega})$ = Fourier transform of $x(n)$

$Y(e^{j\omega})$ = Fourier transform of $y(n)$

Now, Transfer function $= \dfrac{Y(e^{j\omega})}{X(e^{j\omega})}$

The transfer function of an LTI discrete time system in frequency domain can be obtained from the difference equation governing the input-output relation of the LTI discrete time system given below, (refer chapter-6, equation (6.17)).

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

On taking Fourier transform of above equation and rearraging the resultant equation as a ratio of $Y(e^{j\omega})$ and $X(e^{j\omega})$, the transfer function of LTI discrete time system in frequency domain is obtained.

4.8 Response of LTI Discrete Time System using Discrete Time Fourier Transform

Consider the transfer function of LTI discrete time system.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

Now, response in frequency domain, $Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$

On taking inverse Fourier transform of equation (8.38) we get,

$$y(n) = \mathcal{F}^{-1}\{X(e^{j\omega})\, H(e^{j\omega})\}$$

From the equation (8.39) we can say that the output $y(n)$ is given by the inverse Fourier transform of the product of $X(e^{j\omega})$ and $H(e^{j\omega})$.

Example

Find the Fourier transform of x(n), where $x(n) = 1 \; ; \; 0 \le n \le 4$
$= 0 \; ; \;$ otherwise

Solution

By the definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{4} x(n)\, e^{-j\omega n} = \frac{1 - e^{-j5\omega}}{1 - e^{-j\omega}}$$

Using finite geometric series sum formula, $\sum_{n=0}^{N-1} C^n = \frac{1 - C^N}{1 - C}$

$$= \frac{1 - e^{\frac{-j5\omega}{2}} e^{\frac{-j5\omega}{2}}}{1 - e^{\frac{-j\omega}{2}} e^{\frac{-j\omega}{2}}} = \frac{\left(e^{\frac{j5\omega}{2}} - e^{\frac{-j5\omega}{2}}\right) e^{\frac{-j5\omega}{2}}}{\left(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right) e^{\frac{-j\omega}{2}}}$$

$$= \left(\frac{2j \sin\frac{5\omega}{2}}{2j \sin\frac{\omega}{2}}\right) e^{\frac{-j5\omega}{2} + \frac{j\omega}{2}} = \frac{\sin\frac{5\omega}{2}}{\sin\frac{\omega}{2}} e^{\frac{-j4\omega}{2}} = \frac{\sin\frac{5\omega}{2}}{\sin\frac{\omega}{2}} e^{-j2\omega}$$

$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

Example

Determine the Fourier transform of the signal $x(n) = a^{|n|} \; ; \; -1 < a < 1$

Solution

The signal x(n) can be expressed as sum of two signals $x_1(n)$ and $x_2(n)$ as shown below.

$\therefore x(n) = x_1(n) + x_2(n)$

where, $x_1(n) = a^n \; ; \; n \ge 0$ and $x_2(n) = a^{-n} \; ; \; n < 0$
$= 0 \; ; \; n < 0$ $\qquad = 0 \; ; \; n \ge 0$

Let, $X_1(e^{j\omega})$ = Fourier transform of $x_1(n)$ and $X_2(e^{j\omega})$ = Fourier transform of $x_2(n)$.

By definition of Fourier transform,

$$X_1(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} = \sum_{n=0}^{+\infty} a^n e^{-j\omega n} = \sum_{n=0}^{+\infty} (a e^{-j\omega})^n = \frac{1}{1 - a e^{-j\omega}}$$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1 - C}$

$(a e^{j\omega})^0 = 1$

By definition of Fourier transform,

$$X_2(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} = \sum_{n=-\infty}^{-1} (a\, e^{j\omega})^{-n} = \sum_{n=1}^{+\infty} (a e^{j\omega})^n = \sum_{n=0}^{+\infty} (a e^{j\omega})^n - 1$$

$$= \frac{1}{1 - a e^{j\omega}} - 1 = \frac{1 - 1 + a e^{j\omega}}{1 - a e^{j\omega}} = \frac{a e^{j\omega}}{1 - a e^{j\omega}}$$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1 - C}$

Let $X(e^{j\omega})$ = Fourier transform of x(n).

By property of linearity,

$$X(e^{j\omega}) = X_1(e^{j\omega}) + X_2(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}} + \frac{a e^{j\omega}}{1 - a e^{j\omega}}$$

$$= \frac{1 - a e^{j\omega} + a e^{j\omega}(1 - a e^{-j\omega})}{(1 - a e^{-j\omega})(1 - a e^{j\omega})} = \frac{1 - a e^{j\omega} + a e^{j\omega} - a^2}{1 - a e^{-j\omega} - a e^{j\omega} + a^2}$$

$$= \frac{1 - a^2}{1 - 2a \cos\omega + a^2}$$

$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$

Example

Find $X(e^{j\omega})$, if $x(n) = \frac{1}{2}\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] u(n)$

Solution

Given that, $x(n) = \frac{1}{2}\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] u(n)$; for all n

$$\therefore x(n) = \frac{1}{2}\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] = \frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{1}{2}\left(\frac{1}{4}\right)^n \; ; \text{ for } n \geq 0$$

By definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{+\infty} \left[\frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{1}{2}\left(\frac{1}{4}\right)^n\right] e^{-j\omega n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n e^{-j\omega n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n e^{-j\omega n} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{-j\omega}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}e^{-j\omega}\right)^n$$

$$= \frac{1}{2}\,\frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{1}{2}\,\frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$

$$= \frac{1}{2}\left[\frac{1-\frac{1}{4}e^{-j\omega} + 1 - \frac{1}{2}e^{-j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{2-\frac{3}{4}e^{-j\omega}}{\left(1-\frac{1}{4}e^{-j\omega}-\frac{1}{2}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}\right)}\right] = \frac{1-0.375\,e^{-j\omega}}{1-0.75\,e^{-j\omega}+0.125\,e^{-j2\omega}}$$

Example

Compute the Fourier transform and sketch the magnitude and phase function of causal three sample sequence given by,

$$x(n) = \frac{1}{3} \; ; 0 \leq n \leq 2$$
$$= 0 \; ; \text{ else}$$

Solution

Let, $X(e^{j\omega})$ be Fourier transform of x(n).

Now by definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{2} x(n)\, e^{-j\omega n}$$

$$= x(0)\,e^0 + x(1)\,e^{-j\omega} + x(2)\,e^{-j2\omega} = \frac{1}{3} + \frac{1}{3}e^{-j\omega} + \frac{1}{3}e^{-j2\omega}$$

$$= \frac{1}{3} + \frac{1}{3}(\cos\omega - j\sin\omega) + \frac{1}{3}(\cos 2\omega - j\sin 2\omega)$$

$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

$$= \frac{1}{3}(1 + \cos\omega + \cos 2\omega) - j\frac{1}{3}(\sin\omega + \sin 2\omega)$$

Example

Find the inverse Fourier transform of the frequency response of first order system, $H(e^{j\omega}) = (1 - a\,e^{-j\omega})^{-1}$.

Solution

Given that, $H(e^{j\omega}) = (1 - a\,e^{-j\omega})^{-1} = \frac{1}{1-a\,e^{-j\omega}}$

Using Taylor series expansion, the above equation of $H(e^{j\omega})$ can be expanded as shown below.

$$H(e^{j\omega}) = 1 + a\,e^{-j\omega} + a^2\,e^{-j2\omega} + \ldots\ldots + a^k\,e^{-jk\omega} + \ldots\ldots \qquad \ldots\ldots(1)$$

Let, h(n) = Inverse Fourier transform of $H(e^{j\omega})$.

By definition of Fourier transform we get,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

$$= \ldots\ldots + h(-2)\,e^{j2\omega} + h(-1)\,e^{j\omega} + h(0) + h(1)\,e^{-j\omega} + h(2)\,e^{-j2\omega} + \ldots\ldots \qquad \ldots\ldots(2)$$

On comparing the two expressions for $H(e^{j\omega})$ [equation (1) and (2)] we can say that the samples of h(n) are the coefficients of $e^{-j\omega n}$.

$$\therefore h(n) = \left\{\underset{\uparrow}{1}, a, a^2, \ldots\ldots, a^k, \ldots\ldots\right\}$$

$$h(n) = \begin{cases} a^n & ; \; n \geq 0 \\ 0 & ; \; n < 0 \end{cases} \quad \Rightarrow \quad h(n) = a^n u(n)$$

Example

Determine the output sequence from the output spectrum $Y(e^{j\omega})$, where $Y(e^{j\omega}) = \frac{1}{3}\frac{e^{j\omega}+1+e^{-j\omega}}{1-ae^{-j\omega}}$

Solution

The output sequence y(n) is obtained by taking inverse Fourier transform of $Y(e^{j\omega})$.

$$Y(e^{j\omega}) = \frac{1}{3}\frac{e^{j\omega}+1+e^{-j\omega}}{1-ae^{-j\omega}} = \frac{1}{3}\left[\frac{e^{j\omega}}{1-ae^{-j\omega}}+\frac{1}{1-ae^{-j\omega}}+\frac{e^{-j\omega}}{1-ae^{-j\omega}}\right]$$

$$Y(e^{j\omega}) = \frac{1}{3}\left[Y_1(e^{j\omega})+Y_2(e^{j\omega})+Y_3(e^{j\omega})\right]$$

where, $Y_1(e^{j\omega}) = \frac{e^{j\omega}}{1-ae^{-j\omega}}$; $Y_2(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$ and $Y_3(e^{j\omega}) = \frac{e^{-j\omega}}{1-ae^{-j\omega}}$

Let, $y_1(n) = \mathcal{F}^{-1}\{Y_1(e^{j\omega})\}$; $y_2(n) = \mathcal{F}^{-1}\{Y_2(e^{j\omega})\}$; $y_3(n) = \mathcal{F}^{-1}\{Y_3(e^{j\omega})\}$

By Taylor's series expansion we get,

$$Y_2(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}} = 1 + ae^{-j\omega} + a^2e^{-j2\omega} + a^3e^{-j3\omega} + ...$$

$$= \sum_{n=0}^{+\infty} a^n e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} a^n u(n) e^{-j\omega n} \quad .....(1)$$

$u(n) = 1$ for $n \geq 0$
$= 0$ for $n < 0$

By definition of Fourier transform we can write,

$$Y_2(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y_2(n)\, e^{-j\omega n} \quad .....(2)$$

By comparing equations (1) and (2) we can write,

$$y_2(n) = a^n u(n)$$

Here, $Y_1(e^{j\omega}) = \frac{e^{j\omega}}{1-ae^{-j\omega}} = e^{j\omega}\, Y_2(e^{j\omega})$

$\therefore y_1(n) = a^{(n+1)} u(n+1)$ — Using shifting property

Here, $Y_3(e^{j\omega}) = \frac{e^{-j\omega}}{1-ae^{-j\omega}} = e^{-j\omega}\, Y_2(e^{j\omega})$

$\therefore y_3(n) = a^{(n-1)} u(n-1)$ — Using shifting property

Let, y(n) = Inverse Fourier transform of $Y(e^{j\omega})$.

$$\therefore y(n) = \mathcal{F}^{-1}\{Y(e^{j\omega})\} = \mathcal{F}^{-1}\left\{\frac{1}{3}\left[Y_1(e^{j\omega})+Y_2(e^{j\omega})+Y_3(e^{j\omega})\right]\right\}$$

$$= \frac{1}{3}\left[\mathcal{F}^{-1}\{Y_1(\omega)\} + \mathcal{F}^{-1}\{Y_2(\omega)\} + \mathcal{F}^{-1}\{Y_3(\omega)\}\right]$$

$$= \frac{1}{3}\left[y_1(n) + y_2(n) + y_3(n)\right]$$

$$= \frac{1}{3}\left[a^{(n+1)}u(n+1) + a^n u(n) + a^{(n-1)}u(n-1)\right]$$

Example

If $X(e^{j\omega}) = e^{-j1.5\omega}$; $|\omega| \le 1$

$= 0$; $1 \le \omega \le \pi$, Find x(n) and plot.

Solution

The x(n) is obtained by taking inverse Fourier transform of $X(e^{j\omega})$.

By definition of inverse Fourier transform,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-1}^{+1} e^{-j1.5\omega}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{+1} e^{j\omega(n-1.5)}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-1.5)}}{j(n-1.5)}\right]_{-1}^{1} = \frac{1}{j2\pi(n-1.5)}\left[e^{j(n-1.5)} - e^{-j(n-1.5)}\right]$$

$$= \frac{1}{\pi(n-1.5)}\left[\frac{e^{j(n-1.5)} - e^{-j(n-1.5)}}{2j}\right] = \frac{1}{\pi(n-1.5)}\sin(n-1.5)$$

$$= \frac{\sin(n-1.5)}{\pi(n-1.5)} \quad ; \quad \text{for all } n$$

Example

If $H(e^{j\omega}) = 1$; $\omega \le \omega_0$

$= 0$; $|\omega_0| < \omega \le \pi$, Find the impulse response h(n).

Solution

The impulse response h(n) can be obtained by taking inverse Fourier transform of $H(e^{j\omega})$.

By definition of inverse Fourier transform,

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_0}^{\omega_0}$$

$$= \frac{1}{j2\pi n}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right] = \frac{1}{\pi n}\left[\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right] = \frac{\sin \omega_0 n}{\pi n} \quad \text{except when } n = 0$$

Example

Find the transfer function of the second order recursive filter in frequency domain whose impulse response is $h(n) = r^n \cos(\omega_0 n)\, u(n)$ for all n.

Solution

The transfer function of a system is the Fourier transform of impulse response.

By definition of Fourier transform,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = \sum_{n=0}^{+\infty} r^n \cos\omega_0 n\; e^{-j\omega n}$$

$u(n) = 1$ for $n \geq 0$ $= 0$ for $n < 0$

$$= \sum_{n=0}^{+\infty} r^n \left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right] e^{-j\omega n} = \frac{1}{2}\sum_{n=0}^{+\infty}\left[r^n e^{j\omega_0 n} e^{-j\omega n} + r^n e^{-j\omega_0 n} e^{-j\omega n}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{+\infty}\left[r\, e^{j\omega_0} e^{-j\omega}\right]^n + \frac{1}{2}\sum_{n=0}^{\infty}\left[r\, e^{-j\omega_0} e^{-j\omega}\right]^n$$

For $|r| < 1$, we can apply the infinite geometric series sum formula to give,

$$H(e^{j\omega}) = \frac{1}{2}\,\frac{1}{1 - r e^{j\omega_0} e^{-j\omega}} + \frac{1}{2}\,\frac{1}{1 - r e^{-j\omega_0} e^{-j\omega}} = \frac{1}{2}\left[\frac{1 - r e^{-j\omega_0} e^{-j\omega} + 1 - r e^{j\omega_0} e^{-j\omega}}{\left(1 - r e^{j\omega_0} e^{-j\omega}\right)\left(1 - r e^{-j\omega_0} e^{-j\omega}\right)}\right]$$

$$= \frac{1}{2}\,\frac{2 - r e^{-j\omega}\left(e^{-j\omega_0} + e^{j\omega_0}\right)}{1 - r e^{-j\omega_0} e^{-j\omega} - r e^{j\omega_0} e^{-j\omega} + r^2 e^{-j2\omega}} = \frac{1}{2}\,\frac{2 - r e^{-j\omega}\left(e^{j\omega_0} + e^{-j\omega_0}\right)}{1 - r e^{-j\omega}\left(e^{j\omega_0} + e^{-j\omega_0}\right) + r^2 e^{-j2\omega}}$$

$$= \frac{1}{2}\,\frac{2 - r e^{-j\omega}\, 2\cos\omega_0}{1 - r e^{-j\omega}\, 2\cos\omega_0 + r^2 e^{-j2\omega}} = \frac{1 - r\cos\omega_0\, e^{-j\omega}}{1 - 2r\cos\omega_0\, e^{-j\omega} + r^2 e^{-j2\omega}}$$

Example

Determine the impulse response and frequency response of the LTI system defined by, $y(n) = x(n) + b\,y(n-1)$.

Solution

a) Impulse Response

The impulse response h(n) is given by inverse Z-transform of H(z), where, $H(z) = \frac{Y(z)}{X(z)}$.

Given that, $y(n) = x(n) + b\,y(n-1)$. .....(1)

On taking Z-transform of equation (1) we get,

$$Y(z) = X(z) + b\,z^{-1}\,Y(z) \quad\Rightarrow\quad Y(z) - b\,z^{-1}\,Y(z) = X(z) \quad\Rightarrow\quad Y(z)\,(1 - b\,z^{-1}) = X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - b\,z^{-1}} \qquad .....(2)$$

On taking inverse Z-transform of equation (2) we get,

$$h(n) = Z^{-1}\{H(z)\} = b^n\,u(n)$$

The impulse response, $h(n) = b^n\,u(n)$, for all n.

b) Frequency Response

The frequency response $H(e^{j\omega})$ is obtained by evaluating H(z) when $z = e^{j\omega}$.

$$\therefore \text{Frequency response, } H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{1}{1 - bz^{-1}}\bigg|_{z=e^{j\omega}} = \frac{1}{1 - be^{-j\omega}}$$

The magnitude function of $H(e^{j\omega})$ is defined as,

$$|H(e^{j\omega})| = \sqrt{H(e^{j\omega})\,H^*(e^{j\omega})}\text{, where } H^*(e^{j\omega}) = \text{Conjugate of } H(e^{j\omega}).$$

$$\therefore \text{Magnitude function, } |H(e^{j\omega})| = \left[\frac{1}{1-b\,e^{-j\omega}} \times \frac{1}{1-b\,e^{j\omega}}\right]^{\frac{1}{2}} = \left[\frac{1}{1-b\,e^{j\omega}-b\,e^{-j\omega}+b^2}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{1+b^2-b\left(e^{j\omega}+e^{-j\omega}\right)}\right]^{\frac{1}{2}} \quad \frac{1}{\left(1+b^2-2b\cos\omega\right)^{\frac{1}{2}}} \qquad \boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}}$$

$$\text{The phase function, } \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right]$$

where, $H_i(e^{j\omega})$ = Imaginary part of $H(e^{j\omega})$ and $H_r(e^{j\omega})$ = Real part of $H(e^{j\omega})$

To separate the real parts and imaginary parts of $H(e^{j\omega})$, multiply the numerator and denominator by the complex conjugate of the denominator.

$$\therefore \quad H(e^{j\omega}) = \frac{1}{1-be^{-j\omega}} \times \frac{1-b\,e^{j\omega}}{1-b\,e^{j\omega}} = \frac{1-b\,e^{j\omega}}{1-b\,e^{j\omega}-b\,e^{-j\omega}+b^2}$$

$$= \frac{1-b(\cos\omega+j\sin\omega)}{1+b^2-b\left(e^{j\omega}+e^{-j\omega}\right)} = \frac{1-b\cos\omega-jb\sin\omega}{1+b^2-2b\cos\omega}$$

$$= \frac{1-b\cos\omega}{1+b^2-2b\cos\omega} - j\frac{b\sin\omega}{1+b^2-2b\cos\omega}$$

$$\therefore \quad H_i(e^{j\omega}) = \frac{-b\sin\omega}{1+b^2-2b\cos\omega} \quad \text{and} \quad H_r(e^{j\omega}) = \frac{1-b\cos\omega}{1+b^2-2b\cos\omega}$$

$$\text{Phase function, } \angle H(e^{j\omega}) = \tan^{-1}\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})} = \tan^{-1}\left[\frac{-b\sin\omega}{1-b\cos\omega}\right]$$

Example

The impulse response of an LTI system is given by $h(n) = 0.6^n\,u(n)$. Find the frequency response.

Solution

The frequency response $H(e^{j\omega})$ is obtained by taking Fourier transform of the impulse response h(n).

Given that, impulse response, $h(n) = 0.6^n\,u(n)$ for all n.

On taking Fourier transform we get,

$$H(e^{j\omega}) = \mathcal{F}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\,e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} 0.6^n\,u(n)\,e^{-j\omega n} = \sum_{n=0}^{\infty} 0.6^n\,e^{-j\omega n} = \sum_{n=0}^{\infty}\left(0.6\,e^{-j\omega}\right)^n \qquad \boxed{\begin{aligned} u(n) &= 1;\ n \ge 0 \\ &= 0;\ n < 0 \end{aligned}}$$

$$= \frac{1}{1-0.6\,e^{-j\omega}} \qquad \boxed{\text{Using infinite geometric series sum formula } \sum_{n=0}^{\infty} C^n = \frac{1}{1-C} \text{ when } |C| < 1}$$

Here $H(e^{j\omega})$ is a complex function of ω. To separate real and imaginary parts of $H(e^{j\omega})$, multiply the numerator and denominator by the complex conjugate of the denominator.

$$\therefore H(e^{j\omega}) = \frac{1}{1-0.6\,e^{-j\omega}} \times \frac{1-0.6\,e^{j\omega}}{1-0.6\,e^{j\omega}}$$

$$= \frac{1-0.6\,e^{j\omega}}{1-0.6\,e^{j\omega}-0.6\,e^{-j\omega}+0.36} = \frac{1-0.6(\cos\omega+j\sin\omega)}{1-0.6\left(e^{j\omega}+e^{-j\omega}\right)+0.36}$$

$$= \frac{1-0.6\cos\omega-j0.6\sin\omega}{1.36-1.2\cos\omega} = \frac{1-0.6\cos\omega}{1.36-1.2\cos\omega} - j\left(\frac{0.6\sin\omega}{1.36-1.2\cos\omega}\right)$$

The magnitude function of $H(e^{j\omega})$ is defined as, $|H(e^{j\omega})| = \left[H_r^2(e^{j\omega}) + H_i^2(e^{j\omega})\right]^{\frac{1}{2}}$

$\therefore$ Magnitude function, $$|H(e^{j\omega})| = \left[\left(\frac{1-0.6\cos\omega}{1.36-1.2\cos\omega}\right)^2 + \left(\frac{-0.6\sin\omega}{1.36-1.2\cos\omega}\right)^2\right]^{\frac{1}{2}}$$

$$= \left[\left(\frac{(1-0.6\cos\omega)^2 + (-0.6\sin\omega)^2}{(1.36-1.2\cos\omega)^2}\right)\right]^{\frac{1}{2}}$$

$$= \frac{(1+0.36\cos^2\omega - 1.2\cos\omega + 0.36\sin^2\omega)^{\frac{1}{2}}}{1.36-1.2\cos\omega}$$

$\therefore$ Magnitude function, $$|H(e^{j\omega})| = \frac{(1+0.36(\sin^2\omega+\cos^2\omega) - 1.2\cos\omega)^{\frac{1}{2}}}{1.36-1.2\cos\omega}$$

$$= \frac{(1+0.36-1.2\cos\omega)^{\frac{1}{2}}}{1.36-1.2\cos\omega} = \frac{1}{(1.36-1.2\cos\omega)^{\frac{1}{2}}}$$

The phase function is defined as,

$$\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right] = \tan^{-1}\left[\frac{-0.6\sin\omega}{1-0.6\cos\omega}\right]$$

Find the Fourier transform of x(n) = { 2, 1, 2 }.

Solution

By definition of Fourier transform,

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\,e^{-j\omega n} = \sum_{n=0}^{2} x(n)\,e^{-j\omega n} = x(0)\,e^{0} + x(1)\,e^{-j\omega} + x(2)\,e^{-j2\omega}$$

$$= 2 + e^{-j\omega} + 2e^{-j2\omega} = 2e^{-j\omega}(e^{j\omega}+e^{-j\omega}) + e^{-j\omega}$$

$$= 4\cos\omega\,e^{-j\omega} + e^{-j\omega} = (1+4\cos\omega)\,e^{-j\omega}$$

Determine the Fourier transform of x(n) = u(n) − u(n−N).

Solution

x(n) can be expressed as, x(n) = 1 ; for n = 0 to N−1.

By definition of Fourier transform,

Using finite geometric series sum formula $$\sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\,e^{-j\omega n} = \sum_{n=0}^{N-1} 1\times e^{-j\omega n} = \sum_{n=0}^{N-1}\left(e^{-j\omega}\right)^n = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

$$= \frac{1-\left(e^{\frac{-j\omega N}{2}}e^{\frac{-j\omega N}{2}}\right)}{1-\left(e^{\frac{-j\omega}{2}}e^{\frac{-j\omega}{2}}\right)} = \frac{e^{\frac{-j\omega N}{2}}\left[e^{\frac{j\omega N}{2}} - e^{\frac{-j\omega N}{2}}\right]}{e^{\frac{-j\omega}{2}}\left[e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right]} = e^{-j\omega\left(\frac{N}{2}-\frac{1}{2}\right)}\left[\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}\right] = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}\right]$$

Find the Fourier transform of , x(n) = −aⁿ u(−n −1), where |a| < 1.

Solution

By definition of Fourier transform,

when n = 0; $a^{-n}e^{j\omega n} = 1$

Using finite geometric series sum formula $$\sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\,e^{-j\omega n} = \sum_{n=-\infty}^{-1} -a^n e^{-j\omega n} = \sum_{n=1}^{\infty} -a^{-n} e^{j\omega n} = 1 - \sum_{n=0}^{\infty} a^{-n} e^{j\omega n} = 1 - \sum_{n=0}^{\infty}\left(a^{-1}e^{j\omega}\right)^n$$

$$= 1 - \frac{1}{1-a^{-1}e^{j\omega}} = 1 - \frac{a}{a-e^{j\omega}} = \frac{a-e^{j\omega}-a}{a-e^{j\omega}} = \frac{-e^{j\omega}}{a-e^{j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-a}$$

Find the discrete time Fourier transform of the signal, $x(n)=(0.2)^n\ u(n)\ +(0.2)^{-n}\ u(-n-1)$.

Solution

By definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\ e^{-j\omega n} = \sum_{n=-\infty}^{\infty} (0.2)^n\ u(n)\ e^{-j\omega n} + \sum_{n=-\infty}^{\infty} (0.2)^{-n}\ u(-n-1)\ e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.2\ e^{-j\omega})^n + \sum_{n=-\infty}^{-1} (0.2\ e^{j\omega})^{-n} = \sum_{n=0}^{\infty} (0.2\ e^{-j\omega})^n + \sum_{n=1}^{\infty} (0.2\ e^{j\omega})^n$$

when n = 0; $(0.2\ e^{j\omega})^n = 1$

$$= \sum_{n=0}^{\infty} (0.2\ e^{-j\omega})^n + \sum_{n=0}^{\infty} (0.2\ e^{j\omega})^n - 1 = \frac{1}{1-0.2\ e^{-j\omega}} + \frac{1}{1-0.2\ e^{j\omega}} - 1$$

$$= \frac{1-0.2\ e^{j\omega} + 1 - 0.2\ e^{-j\omega} - (1-0.2\ e^{-j\omega})(1-0.2\ e^{j\omega})}{(1-0.2\ e^{-j\omega})(1-0.2\ e^{j\omega})}$$

$$= \frac{1-0.2\ e^{j\omega} + 1 - 0.2\ e^{-j\omega} - (1-0.2\ e^{j\omega} - 0.2\ e^{-j\omega} + 0.04)}{1-0.2\ e^{j\omega} - 0.2\ e^{-j\omega} + 0.04}$$

$$= \frac{1-0.04}{1-0.2\ (e^{j\omega}+e^{-j\omega}) + 0.04} = \frac{0.96}{1.04 - 0.4\cos\omega}$$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$ when $|C| < 1$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

Find the inverse Fourier transform of the rectangular pulse spectrum defined as,

$X(e^{j\omega}) = 1\ ;\ |\omega| \le W$

$\quad = 0\ ;\ W \le |\omega| \le \pi$

Solution

By definition inverse Fourier transform,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\ e^{j\omega n}\ d\omega = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}\ d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-W}^{W} = \frac{1}{2\pi}\left[\frac{e^{jWn}}{jn} - \frac{e^{-jWn}}{jn}\right] = \frac{1}{\pi n}\left[\frac{e^{jWn}-e^{-jWn}}{2j}\right]$$

$$= \frac{\sin Wn}{\pi n} = \frac{W}{\pi}\frac{\sin Wn}{Wn} = \frac{W}{\pi}\ \text{sinc}\ Wn$$

$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

$\frac{\sin\theta}{\theta} = \text{sinc}\,\theta$