UNIT V

LINEAR TIME INVARIANT-DISCREATE TIME SYSTEMS

INTRODUCTION

Linearity and time-invariant properties are considered to be important in analyzing and realizing a system. In this chapter, we develop the relation between input and output that satisfies the linearity and time-invariant property of the system. In this chapter, we introduce the term "convolution sum", which gives the mathematical relationship for the input-output. The input-output relation is explicitly discussed both in discrete- and continuous-time.

5.1 SOLUTION OF LINEAR CONSTANT CO-EFFICIENT EQUATION

Our aim is to determine the output $y(n)$, $n \geq 0$ of the system given a specific input $x(n)$, $n \geq 0$ and a set of initial condition. The basic idea is to obtain two sets of solutions to the difference equation, a homogeneous solution and a particular solution.

The homogeneous solution is obtained by setting terms involving the input $x(n)$ to zero and finding outputs that are possible with zero inputs. The initial conditions are used to determine the arbitrary co-efficients in the homogeneous solution. The particular solution is obtained by guessing a sequence $y(n)$ that would be obtained with the given input sequence $x(n)$.

Thus the total solution can be written as,

$$y(n) = y_h(n) + y_p(n)$$

where, the solution $y_h(n)$, with $x(n) = 0$ is known as homogeneous solution and $y_p(n)$ is called the particular solution.

5.1.1 The Homogeneous Solution of a Difference Equation (The Natural Response)

To obtain the solution of homogeneous equation, we start the problem of solving the linear constant-co-efficient difference equation given by equation

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{m=0}^{M} b_m\, x(n-m)$$

$$\sum_{k=0}^{N} a_k\, y_h(n-k) = 0, \qquad a_0 = 1$$

i.e., by assuming that the input $x(n) = 0$.

To solve eqn. (3.69), we assume a solution of form,

$$y_h(n) = \alpha^n.$$

Eqn. can be written as

$$\therefore \quad \sum_{k=0}^{N} a_k\, \alpha^{(n-k)} = 0$$

$$\alpha^n + a_1\alpha^{n-1} + a_2\alpha^{n-2} + \ldots\ldots + a_N\alpha^{n-N} = 0$$

$$\alpha^n\,[1 + a_1\alpha^{-1} + a_2\alpha^{-2} + \ldots\ldots + a_N\alpha^{-N}] = 0.$$

We are interested in non-trivial solution, $y_h(n) \neq 0$, therefore α must be a root of equation.

$$1 + a_1\alpha^{-1} + a_2\alpha^{-2} + \ldots\ldots + a_N\alpha^{-N} = 0$$

$$\alpha^N + a_1\alpha^{N-1} + \ldots\ldots + a_{N-1}\alpha + a_N = 0$$

Eqn. is usually referred as the characteristic equation.

Problem *Determine the homogeneous solution of the system described by the first order difference equation.*

$$y(n) + 3y(n-1) = x(n), \text{ with initial condition } y(-1) = 1.$$

Sol. For the homogeneous solution, $x(n) = 0$ thus, $y_h(n) + 3y_h(n-1) = 0$.

We assume solution of the form of

$$y_h(n) = \alpha^n$$

$$\therefore \quad \alpha^n + 3\alpha^{n-1} = 0.$$

$$\alpha^{n-1}[\alpha + 3] = 0$$

$$\boxed{\alpha = -3}$$

Thus, the general form of solution of homogeneous difference equation is,

$$y_h(n) = A\alpha^n$$

$$= A(-3)^n.$$

Using the initial condition $y(-1) = 1$. We have,

$$y_h(n) = -3y_h(n-1)$$

Put $n = 0$ $\quad y_h(0) = -3y_h(0-1) = -3\,;\, y_h(0) = A$

$$A = -3$$

Therefore the homogeneous solution is given by,

$$y_h(n) = 3(-3)^n = (-3)^{n+1}.$$

Problem *Determine the homogeneous solution of 2nd order difference equatic*

$$y(n) - y(n-1) - y(n-2) = 0$$

with initial condition $y(0) = 0$, $y(1) = 1$.

Sol. Let us assume the solution of the form

$$y_h(n) = \alpha^n$$

$$\therefore \quad \alpha^n - \alpha^{n-1} - \alpha^{n-2} = 0.$$

$$\alpha^{n-2}[\alpha^2 - \alpha - 1] = 0$$

Therefore, the roots are,

$$\alpha = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}.$$

The general solution to the homogeneous eqn. is.

$$y_h(n) = A_1\left(\frac{1+\sqrt{5}}{2}\right)^n + A_2\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Using the initial condition

$y(0) = 0, y(1) = 1$, we have,

$$0 = A_1 + A_2$$

$$1 = A_1\left(\frac{1+\sqrt{5}}{2}\right) + A_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$A_1 = \frac{1}{\sqrt{5}}, A_2 = -\frac{1}{\sqrt{5}}$$

$$y_h(n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Repeated roots. If the characteristic equation has repeated roots α_1 is repeated m times, then the general solution of homogeneous equation contains the term,

$$\alpha_1^n (A_{11} + A_{12} n + A_{13} n^2 + \dots\dots + A_{1m} n^{m-1})$$

e.g. $y(n) - 9y(n-1) + 9y(n-2) = 0$

We put $y_h(n) = \alpha^n$

$$\alpha^n - 9\alpha^{n-1} + 9\alpha^{n-2} = 0$$

$$\alpha^{n-2}[\alpha^2 - 9\alpha + 9] = 0.$$

This eqn. has two roots $\alpha = 3$.

$$y_h(n) = 3^n (A_{11} + A_{12}{}^n).$$

5.1.2 The Particular Solution of the Difference Eqn. or the Forced Response

The particular solution $y_p(n)$ is required to satisfy the difference eqn. $\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{m=0}^{M} b_m\, y(n-m)$ for the specified input sequence $x(n)$, $n \geq 0$. The method to be used is known as the method of undetermined co-efficient.

Problem *Determine the particular solution of the 1st order difference equation.*

$$y(n) - ay(n-1) = x(n). \quad a \neq 1$$

when the input $x(n)$ is a unit step sequence i.e.,

$$x(n) = u(n).$$

Sol. Assumed solution of the difference equation

$$y_p(n) = A\,u(n)$$

$$A\,u(n) - a\,Au(n-1) = u(n).$$

$$A - aA = 1 \quad a \neq 1.$$

$$A = \frac{1}{1-a}$$

Therefore the particular solution,

$$y_p(n) = \frac{1}{1-a}\,u(n) \qquad a \neq 1.$$

5.1.3 The Total Solution of the Difference Equation

The linearity property of the linear constant co-efficient difference equation allows us to add the homogeneous solution and the particular solution to obtain the total solution.

Thus, $y(n) = u_h(n) + u_p(n)$

Problem *Determine the total solution y(n), $n \geq 0$ to the difference equation*

$$y(n) - ay(n-1) = x(n) \quad a \neq 1.$$

when $x(n) = u(n)$ and $(y-1) = 0$.

Sol. Assumed solution of the difference eqn.

$$y_p(n) = Ax(n)$$

$$y_p(n) = A\,u(n), x(n) = u(n).$$

$$\therefore \quad Au(n) - aAu(n-1) = u(n).$$

To find the value of A, we must solve this eqn. for any $n \geq 1$, where none of the terms vanish, thus

$$A - aA = 1, \ a \neq 1.$$

$$A = \frac{1}{1-a}$$

Therefore, the particular solution,

$$y_p(n) = \frac{1}{1-a}u(n) \qquad a \neq 1$$

$$\boxed{y_p(n) = \frac{1}{1-a}} \qquad n \geq 0, a \neq 1.$$

If we wish to find the complete solution, we must also find $y_h(n)$, for $x(n) = 0$. We assume that solution of form,

$$y_h(n) = \alpha^n.$$

Substituting $y_h(n)$ is the given eqn. for $x(n) = 0$.

$$\alpha^n - a\,\alpha^{n-1} = 0$$

$$\alpha^{n-1}[\alpha - a] = 0$$

$$\boxed{\alpha = a}$$

Thus the solution of homogeneous equation is

$$y_h(n) = A_1\,\alpha^n.$$

Consequently, the total solution is,

$$y(n) = y_h(n) + y_p(n)$$

$$\boxed{y(n) = A_1\,a^n + \frac{1}{1-a}} \qquad n \geq 0, a \neq 1.$$

where the constant A_1 is determined to satisfy the initial condition $y(-1)$.

To evaluate the A_1, we must evaluate the given eqn. at $n = 0$, *i.e.*,

$$y(0) - ay(-1) = 1$$

$$y(0) = 1.$$

On the other hand, eqn. (1) is evaluated at $n = 0$ yields,

$$y(0) = A_1 + \frac{1}{1-a}$$

$$1 = A_1 + \frac{1}{1-a}$$

$$A_1 = 1 - \frac{1}{1-a}$$

$$\boxed{A_1 = \frac{-a}{1-a}}$$

Therefore, the total solution,

$$y(n) = \frac{-a}{1-a}a^n + \frac{1}{1-a}, \quad n \geq 0$$

$$= \frac{1-a.a^n}{1-a}$$

$$\boxed{y(n) = \frac{1-a^{n+1}}{1-a}} \qquad n \geq 0.$$

Problem *Determine the response y(n), $n \ge 0$ of the system described by difference equation,*

$$y(n) - y(n-1) - 2y(n-2) = x(n) + 2x(n-1)$$

when the input sequence is, $x(n) = 2^n u(n)$ with $y(-1) = y(-2) = 0$.

Sol. Let us first determine the homogeneous solution of the given difference equation. We assume the solution to be the form of

$$y_h(n) = \alpha^n$$

$$\therefore \quad \alpha^n - \alpha^{n-1} - 2\alpha^{n-2} = 0$$

$$\alpha^{n-2}[\alpha^2 - \alpha - 2] = 0$$

$$\alpha = +2, -1.$$

$$y_h(n) = A_1 2^n + A_2(-1)^n.$$

The particular solution is assumed to be an exponential sequence of the same form as $x(n)$. Let us assume a solution of the form,

$$y_p(n) = A\,2^n u(n).$$

Since the particular solution $y_p(n)$ is already contained in homogeneous solution, so that this particular solution is redundant. We treat this situation if the characteristics equation has multiple roots. Therefore, we assume,

$$y_p(n) = An\,2^n u(n).$$

Substituting this in the given eqn., we obtain,

$$An2^n u(n) - A(n-1)2^{n-1}u(n) - 2A(n-2)2^{n-2}u(n)$$
$$= 2^n u(n) + 2(2)^{n-1} u(n-1).$$

To determine the value of A we evaluate this eqn. for $n \ge 2$, where none of the term vanish.

If we select $n = 2$, then,

$$A\,2(4) - A(1)(2) = 4 + 2(2)$$

$$8A - 2A = 8$$

$$6A = 8 \qquad \boxed{A = \frac{4}{3}}$$

Therefore, $\quad y_p(n) = \frac{4}{3} n\,2^n u(n).$

The total solution is,

$$\boxed{y(n) = A_1 2^n + A_2(-1)^n + \frac{4}{3} n 2^n u(n)}$$

where the constant A_1 and A_2 are determined such that the initial conditions are satisfied. To accomplish this, evaluate the given eqn. at $n \ge 0, 1$.

$$y(0) - y(-1) - 2y(-2) = x(0) = 1.$$

$$\boxed{y(0) = 1}$$

and at $n = 1$.

$$y(1) - y(0) - 2y(-1) = x(1) + 2x(0) = 2 + 2 = 4.$$

$$\boxed{y(1) = 5}$$

Using the value of $y(0)$ and $y(1)$ in eqn. (1), we have

$$A_1 + A_2 = 1$$

$$2A_1 - A_2 + \frac{8}{3} = 5.$$

These two eqn. give $A_1 = \frac{1}{9}, A_2 = \frac{10}{9}$. Thus the final solution for $x(n) = 2^n u(n)$ is given by,

$$\boxed{y(n) = -\frac{1}{9}2^n + \frac{10}{9}(-1)^n + \frac{4}{3}n2^n.}$$

5.2 BASIC STRUCTURES FOR IIR SYSTEMS

Causal IIR systems are characterised by the constant coefficient difference equation of Eq. 9.1 or equivalently, by the real rational transfer function of Eq. 9.2. From these equations, it can be seen that the realisation of infinite duration impulse response (IIR) systems involves a recursive computational algorithm. In this section, the most important filter structures namely direct Forms I and II, cascade and parallel realisations for IIR systems are discussed.

5.2.1 Direct Form Realisation of IIR Systems

Equation 9.2 is the standard form of the system transfer function. By inspection of this equation, the block diagram representation can be drawn directly for the **direct form realisation**. The multipliers in the feed forward paths are the numerator coefficients and the multipliers in the feedback paths are the negatives of the denominator coefficients. Since the multiplier coefficients in the structures are exactly the coefficients of the transfer function, they are called direct form structures.

Direct Form I

The digital system structure determined directly from either Eq. 9.1 or Eq. 9.2 is called the direct form I. In this case, the system function is divided into two parts connected in cascade, the first part containing only the zeros, followed by the part containing only the poles. An intermediate sequence $w(n)$ is introduced. A possible IIR system direct form I realisation is shown in Fig. 9.3, in which $w(n)$ represents the output of the first part and input to the second.

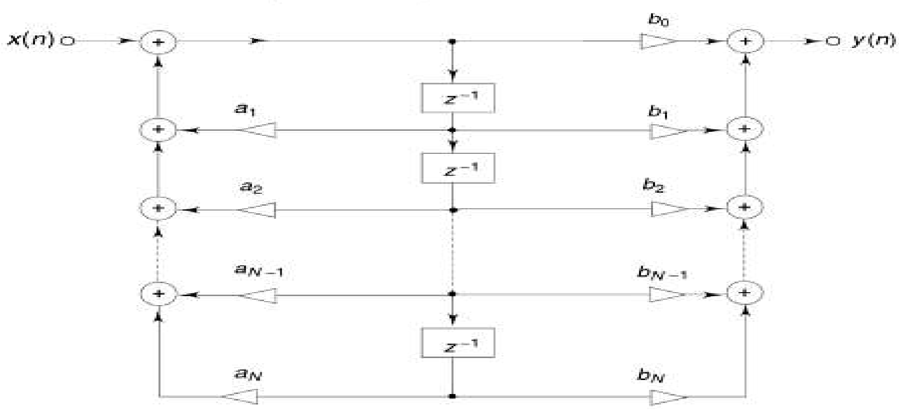
Direct Form II

Since we are dealing with linear systems, the order of these parts can be interchanged. This property yields a second direct form realisation. In direct Form II, the poles of $H(z)$ are realised first and the zeros second. Here, the transfer function $H(z)$ is broken into a product of two transfer functions $H_1(z)$ and $H_2(z)$, where $H_1(z)$ has only poles and $H_2(z)$ contains only the zeros as given below:

$$H(z) = H_1(z).H_2(z)$$

where

$$H_1(z) = \frac{1}{\left(1 - \sum_{k=1}^{N} a_k z^{-k}\right)} \quad \text{and} \quad H_2(z) = \sum_{k=0}^{M} b_k z^{-k}$$



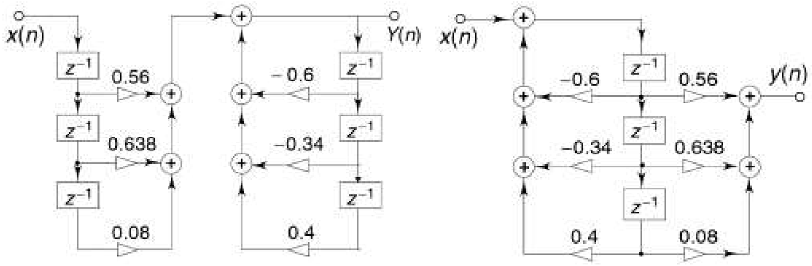
Example *Determine the direct Forms I and II realisations for a third-order IIR transfer function.*

$$H(z)=\frac{0.28z^2+0.319z+0.04}{0.5z^3+0.3z^2+0.17z-0.2}$$

Solution Multiplying the transfer function numerator and denominator by $2z^{-3}$, we obtain the standard form of the transfer function.

$$H(z)=\frac{0.56z^{-1}+0.638z^{-2}+0.08z^{-3}}{1+0.6z^{-1}+0.34z^{-2}-0.4z^{-3}}$$

The direct Forms I and II realisations of the above transfer function are shown in Fig. E9.2(a) and (b) respectively.



Example *Determine the direct Forms I and II for the second-order filter given by*
$y(n) = 2b \cos \omega_0 y(n-1) - b^2 y(n-2) + x(n) - b \cos \omega_0 x(n-1)$

Solution Taking z-transform for the given function, we get

$$Y(z)=2b\cos\omega_0 z^{-1}Y(z)-b^2z^{-2}Y(z)+X(z)-b\cos\omega_0 z^{-1}X(z)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1-b\cos\omega_0 z^{-1}}{1-2b\cos\omega_0 z^{-1}+b^2z^{-2}}$$

Direct Form I

$$H(z) = H_1(z)\, H_2(z)$$

Therefore, $Y(z) = H_1(z)\, H_2(z) X(z)$

In this form, the intermediate sequence $w(n)$ is introduced between $H_1(n)$ and $H_2(n)$

Let $H_1(z)=1-b\cos\omega_0 z^{-1}=\dfrac{W(z)}{X(z)}$

Therefore, $X(z)\,(1-b\cos\omega_0 z^{-1}) = W(z)$

$x(n) - b\cos\omega_0\, x(n-1) = w(n)$

and

$$H_2(z)=(1-2b\cos\omega_0 z^{-1}+b^2z^{-2})^{-1}=\frac{Y(z)}{W(z)}$$

$$Y(z) = \frac{W(z)}{(1-2b\cos\omega_0 z^{-1} + b^2 z^{-2})}$$
$$Y(z)[1-2b\cos\omega_0 z^{-1} + b^2 z^{-2}] = W(z)$$

The inverse z-transform of this function is

$$y(n) - 2\,b\cos\omega_0\, y(n-1) + b^2\, y(n-2) = w(n)$$

The direct Form I realisation structure of the above function is shown in Fig.

Direct Form II

$$Y(z) = H_2(z)\, H_1(z)\, X(z)$$

$$H_2(z) = (1-2\,b\cos\omega_0 z^{-1} + b^2 z^{-2})^{-1} = \frac{U(z)}{X(z)}$$

Let $$H_1(z) = 1 - b\cos\omega_0 z^{-1} = \frac{Y(z)}{U(z)}$$

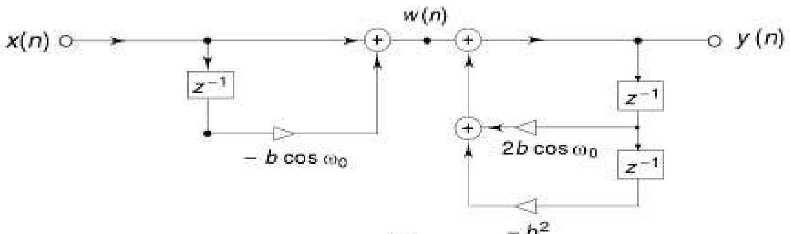
$$X(z) = U(z)\{1-2\ b\cos\omega_0 z^{-1} + b^2 z^{-2}\}$$

Hence, $$x(n) = u(n) - 2\,b\cos\omega_0 u(n-1) + b^2 u(n-2)$$

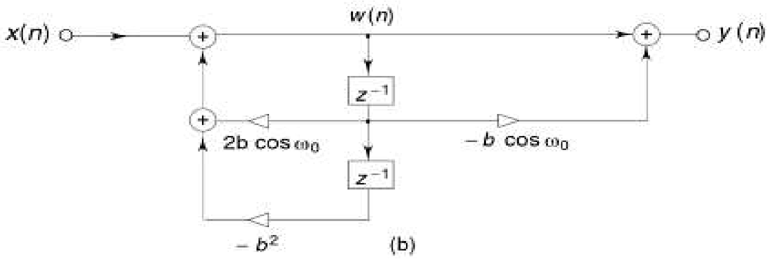
$$Y(z) = U(z)\{1 - b\cos\omega_0 z^{-1}\}$$

Hence, $$y(n) = u(n) - b\cos\omega_0 u(n-1)$$

The Direct Form II realisation structure of the above function is shown in Fig.



(a)



(b)

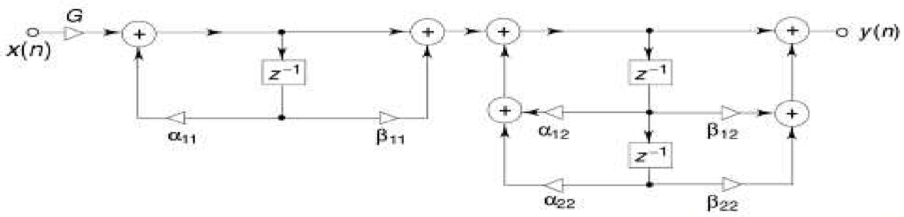
5.2.2 Cascade Realisation of IIR Systems

In cascade realisation, the transfer function $H(z)$ is broken into a product of transfer functions $H_1(z)$, $H_2(z)$,..., $H_k(z)$. Factoring the numerator and denominator polynomials of the transfer function $H(z)$, we obtain

$$H(z) = G\cdot\frac{\prod_{k=1}^{M_1}(1-g_k z^{-1})\ \prod_{k=1}^{M_2}(1-h_k z^{-1})(1-h_k^* z^{-1})}{\prod_{k=1}^{N_1}(1-c_k z^{-1})\ \prod_{k=1}^{N_2}(1-d_k z^{-1})(1-d_k^* z^{-1})}$$

Generally, the numerator and denominator polynomials of the transfer function $H(z)$ are factored into a product of first-order and second-order polynomials. Here $H(z)$ can be expressed as

$$H(z) = G \prod_{k=1}^{[(N+1)/2]} \frac{1+\beta_{1k}z^{-1}+\beta_{2k}z^{-2}}{1-\alpha_{1k}z^{-1}-\alpha_{2k}z^{-2}}$$



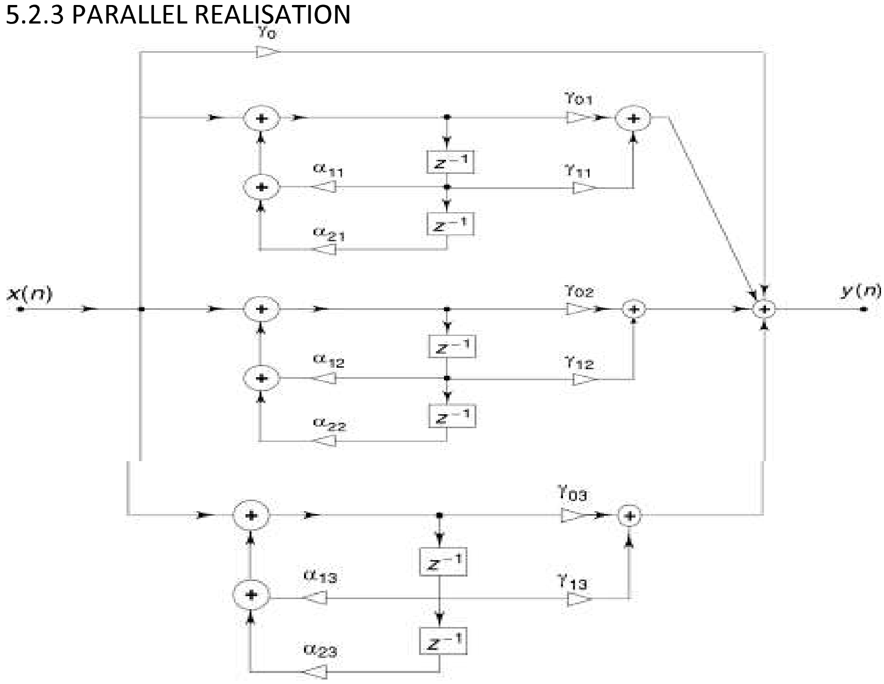
Cascade Structure With Direct Form II Realisation of a Third-Order IIR Transfer Function

Example *Obtain a cascade realisation of the system characterised by the transfer function*

$$H(z) = \frac{2(z+2)}{z(z-0.1)(z+0.5)(z+0.4)}$$

Solution Multiplying the transfer function numerator and denominator by z^{-4}, we obtain the standard form of the transfer function given by

5.2.3 PARALLEL REALISATION



Parallel Form Realisation Structure with the Real and Complex Poles Grouped in Pairs

Example *Determine the parallel realisation of the IIR digital filter transfer functions*

(a) $H(z) = \dfrac{3(2z^2+5z+4)}{(2z+1)(z+2)}$

(b) $H(z) = \dfrac{3z(5z-2)}{\left(z+\frac{1}{2}\right)(3z-1)}$

Solution (a) In order to find the parallel realisation, the partial fraction expansion of $H(z)/z$ is first determined, just as we did for inverse z-transforms. This gives

$$F(z) = \frac{H(z)}{z} = \frac{\frac{3}{2}(2z^2+5z+4)}{z\left(z+\frac{1}{2}\right)(z+2)} = \frac{A_1}{z} + \frac{A_2}{\left(z+\frac{1}{2}\right)} + \frac{A_3}{(z+2)}$$

where
$$A_1 = zF(z)\big|_{z=0} = \frac{\frac{3}{2}(2z^2+5z+4)}{\left(z+\frac{1}{2}\right)(z+2)}\Bigg|_{z=0} = 6$$

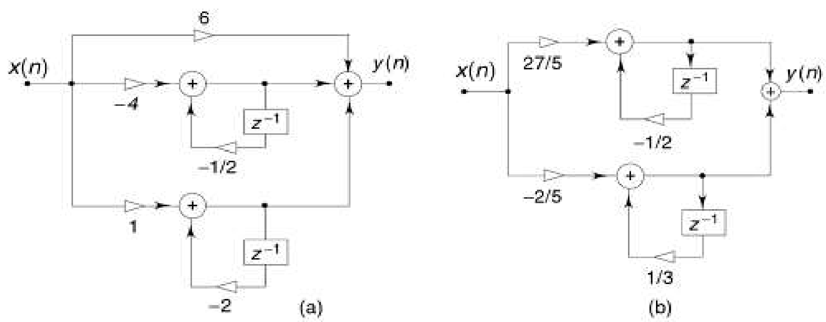
$$A_2 = \left(z+\frac{1}{2}\right)F(z)\Bigg|_{z=-\frac{1}{2}} = \frac{\frac{3}{2}(2z^2+5z+4)}{z(z+2)}\Bigg|_{z=-\frac{1}{2}} = -4$$

$$A_3 = (z+2)F(z)\big|_{z=-2} = \frac{\frac{3}{2}(2z^2+5z+4)}{z\left(z+\frac{1}{2}\right)}\Bigg|_{z=-2} = 1$$

Therefore,
$$\frac{H(z)}{z} = \frac{6}{z} - \frac{4}{z+\frac{1}{2}} + \frac{1}{(z+2)}$$

Hence,
$$H(z) = 6 - \frac{4z}{z+\frac{1}{2}} + \frac{z}{(z+2)} = 6 - \frac{4}{\left(1+\frac{1}{2}z^{-1}\right)} + \frac{1}{1+2z^{-1}}$$

The parallel realisation of this transfer function is shown in Fig. E9.5(a).



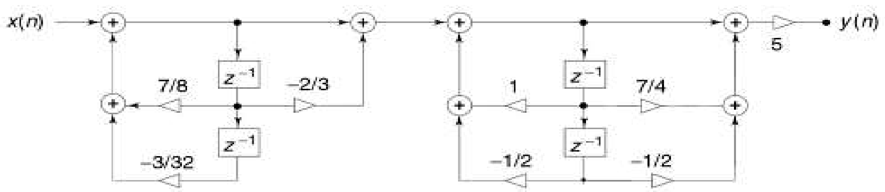
(b)
$$H(z)=\frac{3z(5z-2)}{\left(z+\frac{1}{2}\right)(3z-1)}=\frac{z(5z-2)}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}$$
$$F(z)=\frac{H(z)}{z}=\frac{5z-2}{\left(z+\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}=\frac{A_1}{z+\frac{1}{2}}+\frac{A_2}{\left(z-\frac{1}{3}\right)}$$

where
$$A_1=F(z)\left(z+\frac{1}{2}\right)\Big|_{z=-\frac{1}{2}}=\frac{5z-2}{\left(z-\frac{1}{3}\right)}\Big|_{z=-\frac{1}{2}}=\frac{27}{5}$$
$$A_2=F(z)\left(z-\frac{1}{3}\right)\Big|_{z=\frac{1}{3}}=\frac{5z-2}{\left(z+\frac{1}{2}\right)}\Big|_{z=\frac{1}{3}}=-\frac{2}{5}$$

Therefore,
$$H(z)=\frac{27}{5}\frac{z}{\left(z+\frac{1}{2}\right)}-\frac{2}{5}\frac{z}{\left(z-\frac{1}{3}\right)}$$
$$=\frac{27}{5}\frac{1}{\left(1+\frac{1}{2}z^{-1}\right)}-\frac{2}{5}\frac{1}{\left(1-\frac{1}{3}z^{-1}\right)}$$

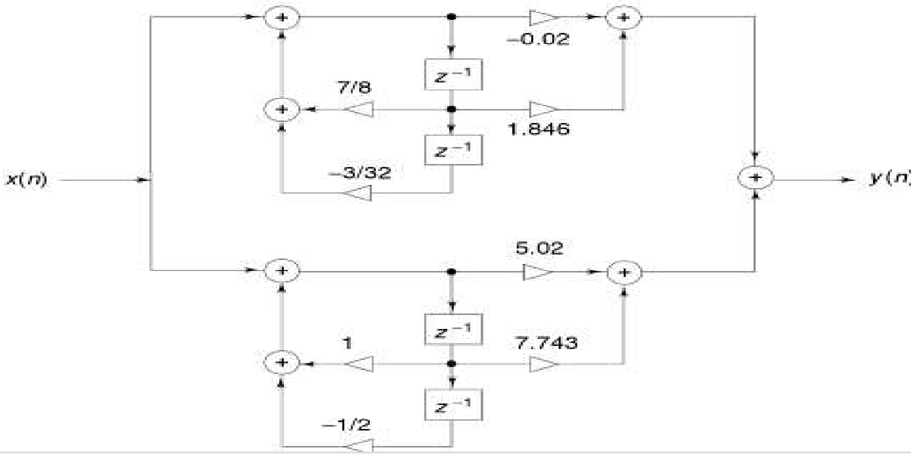
The difference equations for $H_1(z)$ and $H_2(z)$ are
$$y_1(n)=\frac{7}{8}y_1(n-1)-\frac{3}{32}y_1(n-2)+x_1(n)-\frac{2}{3}x_1(n-1)$$
$$y_2(n)=y_2(n-1)-\frac{1}{2}y_2(n-2)+x_2(n)+\frac{7}{4}x_2(n-1)-\frac{1}{2}x_2(n-2)$$

These functions are realised in cascade form as shown in Fig. E9.7(a).



Parallel Realisation To obtain the parallel form realisation, $H(z)$ must be expanded in partial fractions. Thus we have
$$H(z)=\frac{A_1}{1-\frac{3}{4}z^{-1}}+\frac{A_2}{1-\frac{1}{8}z^{-1}}+\frac{A_3}{1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}}+\frac{A_3^*}{1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}}$$

Upon solving, we find $A_1=2.933$, $A_2=-2.947$, $A_3=2.507-j10.45$ and $A_3^*=2.507+j10.45$



Problem *Find the direct form I and direct form II realizations of a discrete time system represented by the transfer function.*

$$H(z) = \frac{8z^3 - 4z^2 + 11z - 2}{\left(z - \frac{1}{4}\right)\left(z^2 - z + \frac{1}{2}\right)}.$$

Sol. Let $H(z) = \frac{Y(z)}{X(z)}$

Direct form-I :

$$\therefore \quad \frac{Y(z)}{X(z)} = \frac{8z^3 - 4z^2 + 11z - 2}{\left(z - \frac{1}{4}\right)\left(z^2 - z + \frac{1}{2}\right)} = \frac{8z^3 - 4z^2 + 11z - 2}{z^3 - z^2 + \frac{1}{2}z - \frac{1}{4}z^2 + \frac{1}{4}z - \frac{1}{8}}$$

$$\frac{Y(z)}{X(z)} = \frac{z^3[8 - 4z^{-1} + 11z^{-2} - 2z^{-3}]}{z^3\left[1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}\right]}.$$

$$\frac{Y(z)}{X(z)} = \frac{8 - 4z^{-1} + 11z^{-2} - 2z^{-3}}{1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}}.$$

On cross multiplying eqn. (8.16), we get,

$$Y(z) - \frac{5}{4}z^{-1}Y(z) + \frac{3}{4}z^{-2}y(z) - \frac{1}{8}z^{-3}Y(z)$$
$$= 8X(z) - 4z^{-1}X(z) + 11z^{-2}X(z) - 2z^{-3}X(z)$$
$$Y(z) = 8X(z) - 4z^{-1}X(z) + 11z^{-2}X(z) - 2z^{-3}X(z)$$
$$+ \frac{5}{4}z^{-1}Y(z) - \frac{3}{4}z^{-2}Y(z) + \frac{1}{8}z^{-3}Y(z)$$

The direct form I structure can be obtained from eqn. as shown in Fig.

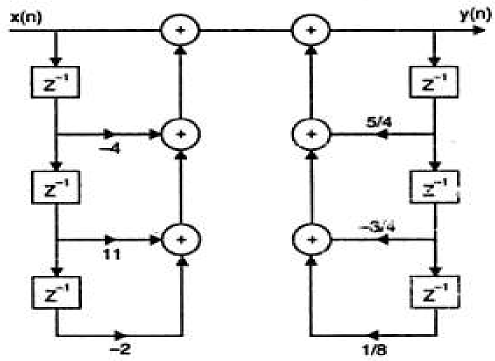


Fig. . Direct form I realization of problem

Problem . *Using first order section, obtain a cascade realization for*

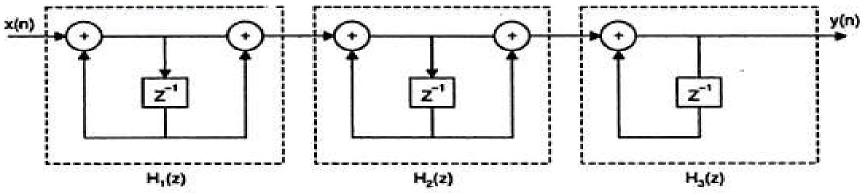
$$H(z) = \frac{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{4}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{10}z^{-1}\right)}.$$

Sol. The H(z) can be decomposed into the three section as $H(z) = H_1(z)\, H_2(z)\, H_3(z)$.

where,
$$H_1(z) = \frac{1+\frac{1}{8}z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$H_2(z) = \frac{1+\frac{1}{4}z^{-1}}{1-\frac{1}{4}z^{-1}} \quad \text{and} \quad H_3(z) = \frac{1}{1-\frac{1}{10}z^{-1}}.$$

The cascade form structure is shown in Fig.



Problem . *Obtain the cascade form structure for the system characterized by*

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n) + \frac{1}{8}x(n-1).$$

Sol. The system function of above system is given by

$$H(z) = \frac{\left(1+\frac{1}{8}z^{-1}\right)}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}.$$

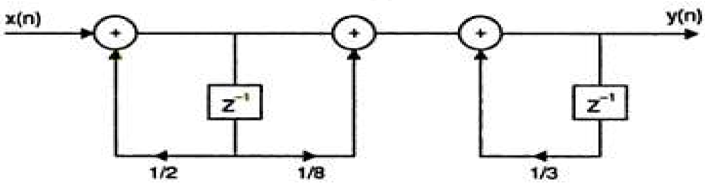
The above equation can be decomposed into two parts as

$$H(z) = \frac{1+\frac{1}{8}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$$H(z) = H_1(z)\, H_2(z)$$

where,
$$H_1(z) = \frac{1+\frac{1}{8}z^{-1}}{1-\frac{1}{2}z^{-1}}\ ; H_2(z) = \frac{1}{1-\frac{1}{3}z^{-1}}.$$

The cascade form structure is shown in Fig.



Parallel-Form Structure

The last common form of IIR structures is the parallel realization. This form of IIR system can be realized by performing partial fraction expansion of H(z). By partial fractional expansion the transfer function H(z) of the system can be expressed as a sum of first and second order sections.

$$H(z) = \frac{Y(z)}{X(z)} = C + \sum_{k=1}^{N} H_k(z).$$

where,
$$H_k(z) = \sum_{k=1}^{N} \frac{A_k}{1+p_k z^{-1}}.$$

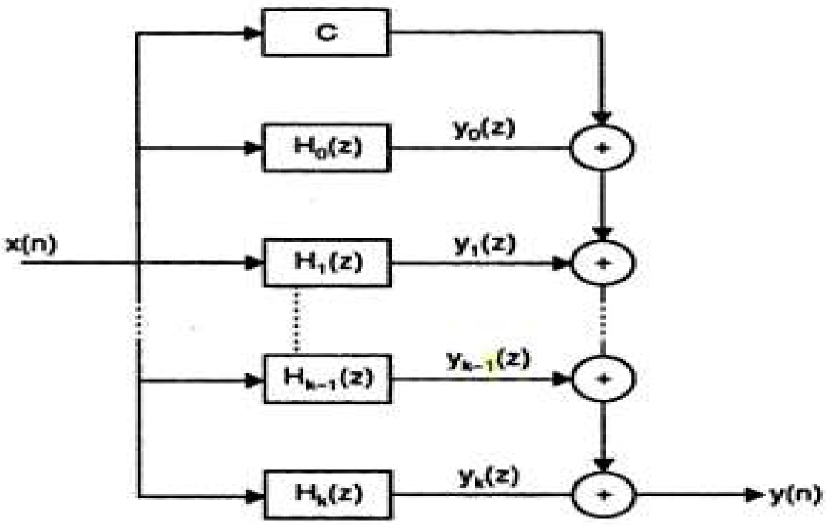
C is constant and defined as $C = \frac{b_N}{a_N}$,

$\{p_k\}$ are the poles and

$\{A_k\}$ are the co-efficients in the partial fraction expansion.

$$\therefore \quad H(z) = C + \sum_{k=1}^{N} \frac{A_k}{1+p_k z^{-1}}$$

The structure given by eqn. is shown in Fig.



The individual first and second order section can be realized either in direct form I or direct form II structures. The overall system is obtained by connecting the individual sections in parallel.

Problem . *Obtain the parallel form realizations of the LTI system governed by the equation.*

$$y(n) = -\frac{3}{8}y(n-1) + \frac{3}{22}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2).$$

Sol. Given that

$$y(n) = -\frac{3}{8}y(n-1) + \frac{3}{22}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2)$$

On taking z-transform of the above equation, we get,

$$Y(z) = -\frac{3}{8}z^{-1}Y(z) + \frac{3}{22}z^{-2}Y(z) + \frac{1}{64}z^{-3}Y(z) + X(z) + 3z^{-1}X(z) + 2z^{-2}X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1+3z^{-1}+2z^{-2}}{1+\frac{3}{8}z^{-1}-\frac{3}{22}z^{-2}-\frac{1}{64}z^{-3}}$$

$$= \frac{(1+z^{-1})(1+2z^{-1})}{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}.$$

By partial fraction expansion,

$$H(z) = \frac{A}{1+\frac{1}{8}z^{-1}} + \frac{B}{1+\frac{1}{2}z^{-1}} + \frac{C}{1-\frac{1}{4}z^{-1}}.$$

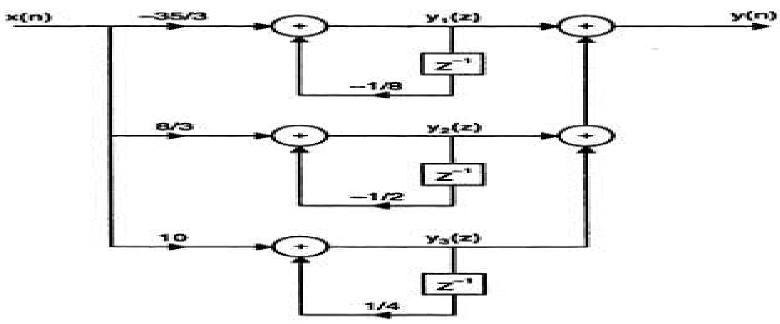
$$A = \left.\frac{(1+z^{-1})(1+2z^{-1})}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}\right|_{z^{-1}=-8} = \frac{-35}{3}.$$

$$B = \left.\frac{(1+z^{-1})(1+2z^{-1})}{\left(1+\frac{1}{8}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}\right|_{z^{-1}=-2} = \frac{8}{3}$$

$$C = \left.\frac{(1+z^{-1})(1+2z^{-1})}{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)}\right|_{z^{-1}=4} = 10$$

$$H(z) = \frac{-35/3}{1+1/8z^{-1}} + \frac{8/3}{1+1/2z^{-1}} + \frac{10}{1-1/4z^{-1}}.$$

$$Y_1(z) = -\frac{1}{8}z^{-1} - \frac{35}{3}X(z) \qquad Y(z) = Y_1(z) + Y_2(z) + Y_3(z).$$



$$Y_2(z) = -\frac{1}{2}z^{-1} + \frac{8}{3}X(z)$$

$$Y_3(z) = \frac{1}{4}z^{-1} + 10X(z)$$

Problem *Obtain the parallel form structure for the system function*

$$H(z) = \frac{1+2z^{-1}+z^{-2}}{1-0.75z^{-1}+0.125z^{-2}}.$$

Sol. The above equation can be factorised into the following form :

$$H(z) = 8 + \frac{-7+8z^{-1}}{1-0.75z^{-1}+0.125z^{-2}}$$

$$= C + H_1(z).$$

The parallel form realization for this example with second order system is shown in Fig.

$$H(z) = 8 + \frac{A_1}{1-0.5z^{-1}} + \frac{A_2}{1-0.25z^{-1}}.$$

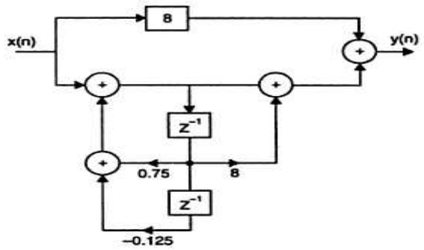
After some arithmetic calculation we find that,

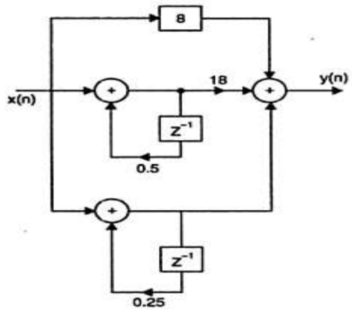
$$A_1 = 18, A_2 = -25,$$

Thus,
$$H(z) = 8 + \frac{18}{1-0.5z^{-1}} - \frac{25}{1-0.25z^{-1}}$$

$$= 8 + H_1(z) + H_2(z).$$

The resulting parallel from structure with first order sections is shown in Fig.





Problem . *Obtain direct form I, direct form II, cascade structure for the system described by*

$$y(n) = -0.1\, y(n-1) + 0.72\, y(n-2) + 0.7x(n) - 0.252\, x(n-2).$$

Sol. We take z-transform both sides of difference eqn., we have

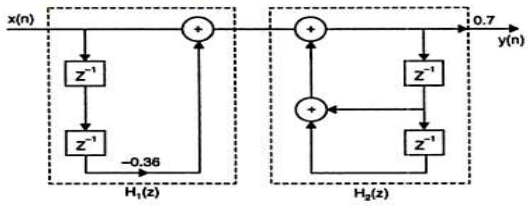
$$Y(z) = -0.1\, z^{-1}\, Y(z) + 0.72\, z^{-2}\, Y(2) + 0.7X(z) - 0.252z^{-2}X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.7(1-0.36z^{-2})}{1+0.1z^{-1}-0.72z^{-2}} = H_1(z)\, H_2(z)$$

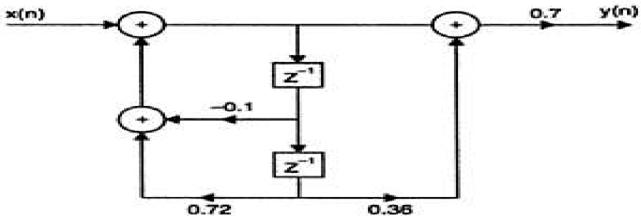
$$H_1(z) = (1 - 0.36\, z^{-2}) \quad \text{and} \quad H_2(z) = \frac{0.7}{1+0.1z^{-1}-0.72z^{-2}}.$$

(*a*) **Direct form I**

Fig. (8.18) illustrates direct form I realization.



The direct form II realization is shown in Fig

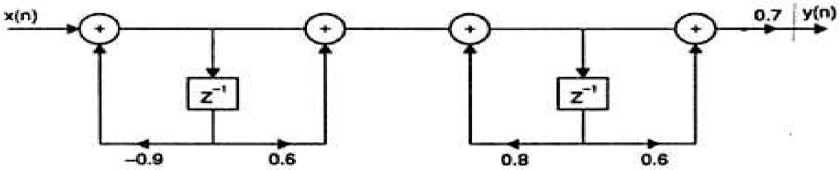


(c) **Cascade form structure**

$$H(z) = \frac{0.7(1-0.36z^{-2})}{1+0.1z^{-1}-0.72z^{-2}}.$$

$$= \frac{0.7(1-0.6z^{-1})(1+0.6z^{-1})}{(1+0.9z^{-1})(1-0.8z^{-1})} = 0.7\, H_1(z)\, H_2(z).$$

The cascade form structure is shown in Fig.



Problem *Realize the system function*

$$H(z) = 1 + \frac{2}{4}z^{-1} + \frac{3}{8}z^{-2} + \frac{3}{4}z^{-3} + \frac{7}{2}z^{-4}$$

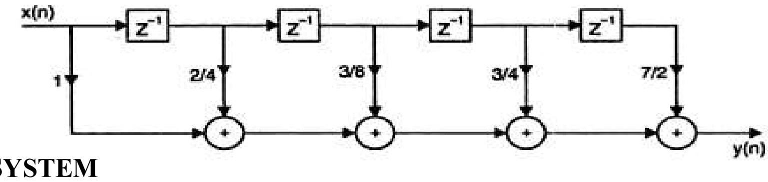
by using direct form structure.

Sol. Given that

$$H(z) = 1 + \frac{2}{4}z^{-1} + \frac{3}{8}z^{-2} + \frac{3}{4}z^{-3} + \frac{7}{2}z^{-4},$$

$$h(0) = 1,\ h(1) = \frac{2}{4},\ h(2) = \frac{3}{8},\ h(3) = \frac{3}{4},\ h(4) = \frac{7}{2}.$$

The realization is shown in Fig.



5.3 FIR SYSTEM

5.3.1 Direct Form Realisation of FIR Systems

The convolution sum relationship gives the system response as

$$y(n) = \sum_{K=0}^{M-1} h(k)\,x(n-k) \tag{9.8}$$

where $y(n)$ and $x(n)$ are the output and input sequence, respectively. This equation gives the input-output relation relation of the FIR filter in the time-domain. This equation can be obtained from Eq. 9.1 by setting $b_k = h(k)$, $a_k = 0$, $k = 1, 2, \ldots, M$.

The direct form realisation for Eq. 9.8 is shown in Fig. 9.15. This is similar to that of Fig. 9.7 when all the coefficients $a_k = 0$. Hence, the direct form realisation structure for FIR system is the special case of the direct form realisation structure for IIR system.

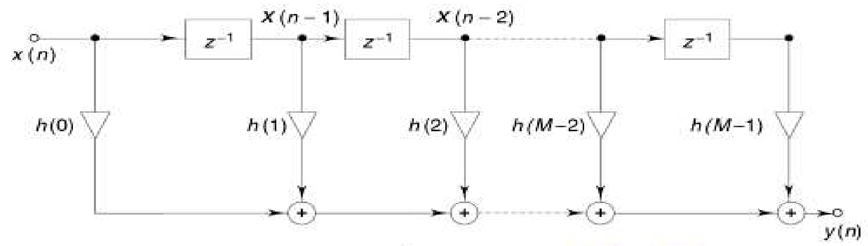


Fig. *Direct Form Realisation Structure of an FIR System*

The transposed structure shown in Fig. is the second direct form structure. Both of these direct form structures are canonic with respect to delays.

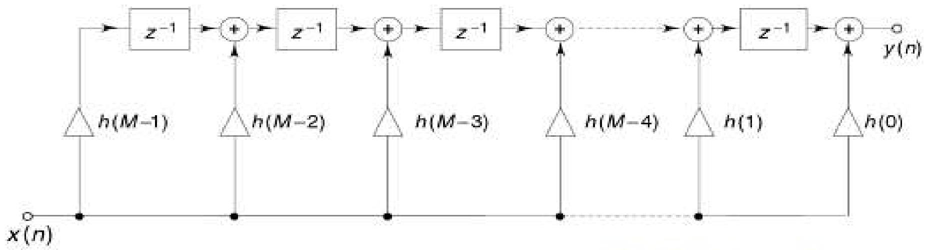


Fig. *Transposed Direct Form Realisation Structure of an FIR System of Fig.*

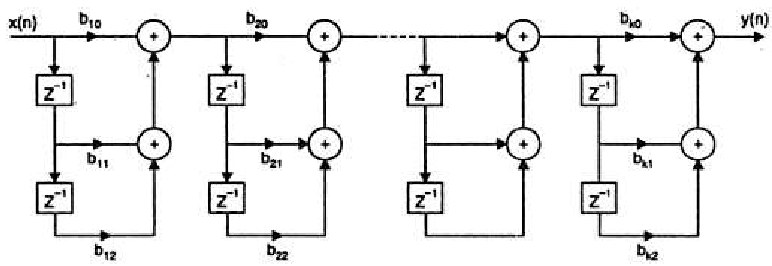
5.3.2 Cascade Form FIR Structure

A higher order FIR system can also be realized as cascade of FIR sections with each section characterized by either first order or second order transfer function. This realization is obtained from the factored form of H(z), which we write in form of

$$H(z) = \prod_{k=1}^{k} (b_{k0} + b_{k1}z^{-1} + b_{k2}\, z^{-2}).$$

where, k is the largest integer less than or equal to N/2. If N is odd then $k = \left(\frac{N-1}{2}\right)$ and if N is even then k = N/2 with $b_{k2} = 0$

The realization is shown in Fig.



Problem *Obtain cascade form realization of the following system function*

$$H(z) = \left[1 + \frac{1}{4}z^{-1} + \frac{z^{-2}}{2}\right]\left[1 + \frac{1}{8}z^{-1} + \frac{z^{-2}}{2}\right].$$

Sol. The system function H(z) can be written as,

$$H(z) = H_1(z)\, H_2(z).$$

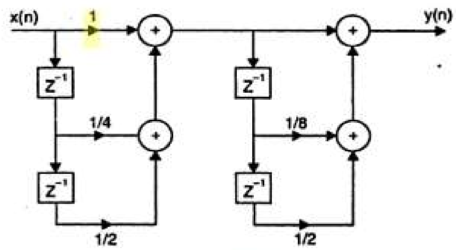
where,

$$H_1(z) = 1 + \frac{1}{4}z^{-1} + \frac{1}{2}z^{-2}.$$

$$H_2(z) = 1 + \frac{1}{8}z^{-1} + \frac{1}{2}z^{-2}.$$

The cascade form realization is shown in Fig.

The cascade form realization is shown in Fig.



Cascade form realization.

5.3.3 Linear Phase FIR Structure

When the FIR system has linear phase, the unit sample response of the system satisfies the symmetric condition is given by

$$h(n) = h(N-1-n)$$

or antisymmetric condition.

$$h(n) = -h(N-1-n)$$

Using the symmetry condition, we can write

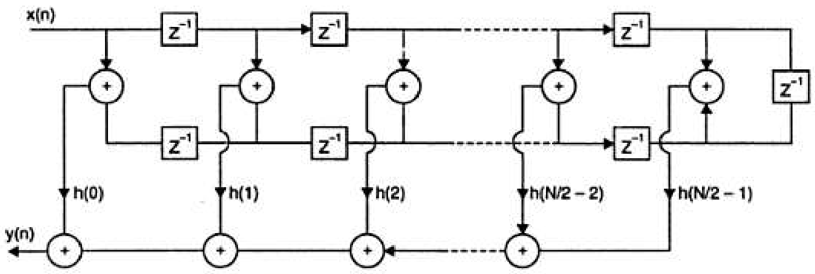
$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}.$$

$$H(z) = \sum_{n=0}^{N/2-1} h(n)\left[z^{-n} + z^{-(N-1-n)}\right]$$

for N is even, and $\quad H(z) = h\left(\frac{N-1}{2}\right) z^{-(N-1)/2} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\left[z^{-n} + z^{-(N-1-n)}\right]$

for N is odd.

The realizations of eqns. (8.34) and (8.35) are shown in Fig. (8.27) and (8.28), respectively.



Direct form realization of linear phase FIR system for N even.

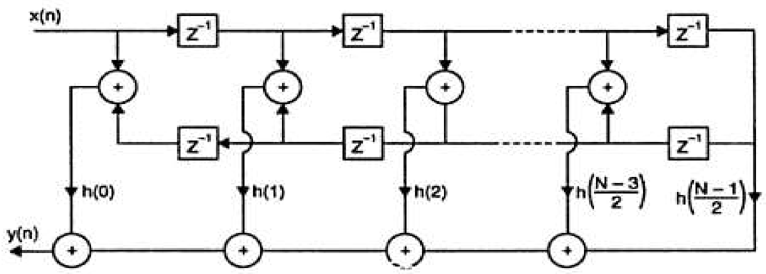


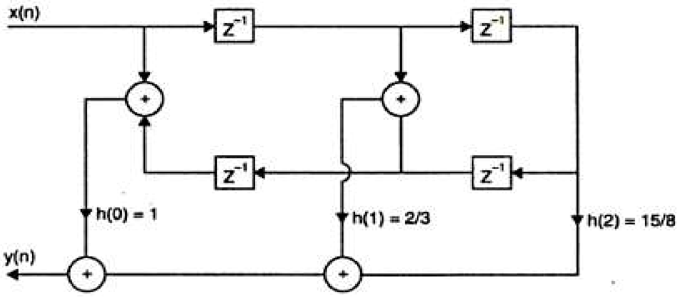
Fig. Direct form realization of linear phase FIR system for N odd.

Problem *Obtain the direct form realization of linear phase FIR system given by*

$$H(z) = 1 + \frac{2}{3}z^{-1} + \frac{15}{8}z^{-2} + \frac{2}{3}z^{-3} + z^{4}.$$

Sol. We have N = 5 (odd). Therefore the above equation can be written as,

$$H(z) = (1 + z^{-4}) + \frac{2}{3}(z^{-1} + z^{-3}) + \frac{15}{8}z^{-2}$$



Here, we have $h(0) = 1 = h(4)$

$$h(1) = \frac{2}{3}$$

$$h(2) = \frac{15}{8}.$$

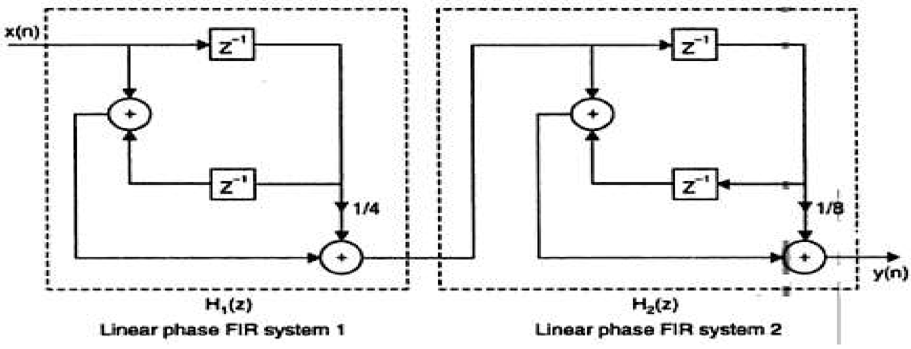
Problem *Obtain a cascade realization using minimum number of multiplications for the system*

$$H(z) = \left(1 + \frac{1}{4}z^{-1} + z^{-2}\right)\left(1 + \frac{1}{8}z^{-1} + z^{-2}\right).$$

Sol. We can consider that H(z) is product of factor

$$H_1(z) = 1 + \frac{1}{4}z^{-1} + z^{-2} \quad \text{and} \quad H_2(z) = 1 + \frac{1}{8}z^{-1} + z^{-2}$$

These two factors $H_1(z)$ and $H_2(z)$ are having linear phase symmetry. The realization is shown in Fig.



5.4 CONVOLUTION SUM

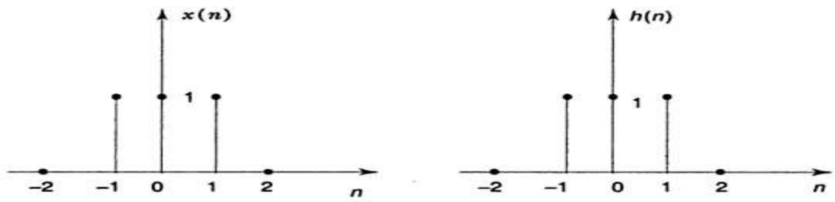
Example Find the convolution of two finite duration sequences

$$x(n) = \begin{cases} 1, & -1 \le n \le +1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad h(n) = \begin{cases} 1, & -1 \le n \le +1 \\ 0, & \text{otherwise} \end{cases}$$

Solution The convolution of two finite duration sequences is given by

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad \text{or} \quad y(n) = \sum_{k=-\infty}^{\infty} x(n-k)\, h(k)$$

Step 1 Plot the given sequence, as shown in Fig. E6.3(a).



The given problem can be solved by using the graphical method as shown in Fig. We know that $y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$.

When $n = 0$, $y(0) = \sum_{k=-\infty}^{\infty} y_0(k) = \sum_{k=-\infty}^{\infty} x(k)\, h(-k) = 3$

When $n = 1$, $y(1) = \sum_{k=-\infty}^{\infty} y_1(k) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k) = 2$

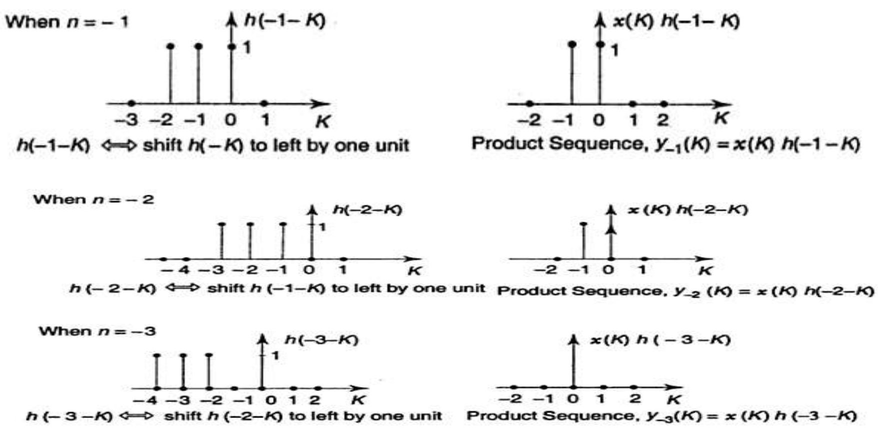
When $n = 2$, $y(2) = \sum_{k=-\infty}^{\infty} y_2(k) = \sum_{k=-\infty}^{\infty} x(k)\, h(2-k) = 1$

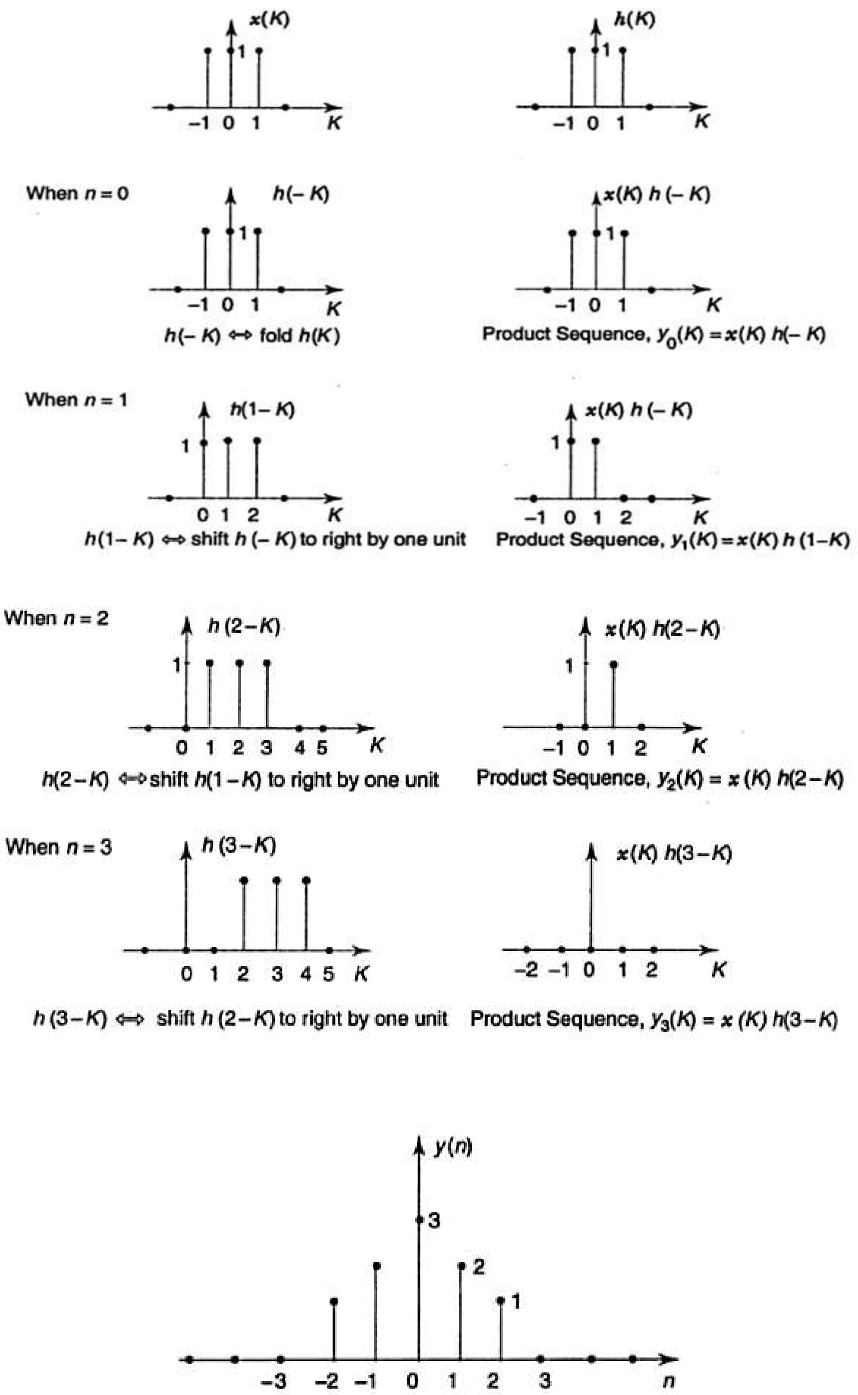
When $n = 3$, $y(3) = \sum_{k=-\infty}^{\infty} y_3(k) = \sum_{k=-\infty}^{\infty} x(k)\, h(3-k) = 0$

When $n = -1$, $y(-1) = \sum_{k=-\infty}^{\infty} y_{-1}(k) = \sum_{k=-\infty}^{\infty} x(k)\, h(-1-k) = 2$

When $n = -2$, $y(-2) = \sum_{k=-\infty}^{\infty} y_{-2}(k) = \sum_{k=-\infty}^{\infty} x(k)\, h(-2-k) = 1$

When $n = -3$, $y(-3) = \sum_{k=-\infty}^{\infty} y_{-3}(k) = \sum_{k=-\infty}^{\infty} x(k)\, h(-3-k) = 0$

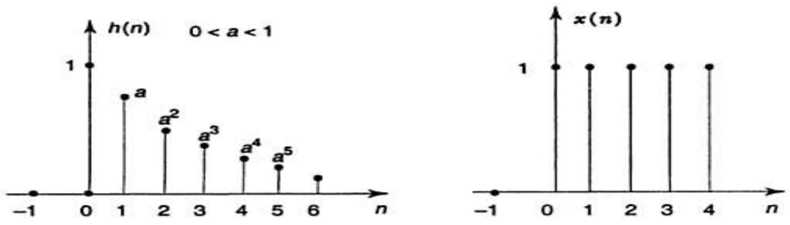




x(K)
1
–1 0 1 K
h(K)
1
–1 0 1 K
When n = 0
h(– K)
1
–1 0 1 K
h(– K) ⇔ fold h(K)
x(K) h (– K)
1
–1 0 1 K
Product Sequence, y0(K) = x(K) h(– K)
When n = 1
h(1– K)
1
0 1 2 K
h(1– K) ⇔ shift h (– K) to right by one unit
x(K) h (– K)
1
–1 0 1 2 K
Product Sequence, y1(K) = x(K) h (1–K)
When n = 2
h (2–K)
1
0 1 2 3 4 5 K
h(2–K) ⇔ shift h(1 – K) to right by one unit
x(K) h(2–K)
1
–1 0 1 2 K
Product Sequence, y2(K) = x (K) h(2–K)
When n = 3
h (3–K)
0 1 2 3 4 5 K
h (3–K) ⇔ shift h (2–K) to right by one unit
x(K) h(3–K)
–2 –1 0 1 2 K
Product Sequence, y3(K) = x (K) h(3–K)
y(n)
3
2
1
–3 –2 –1 0 1 2 3 n

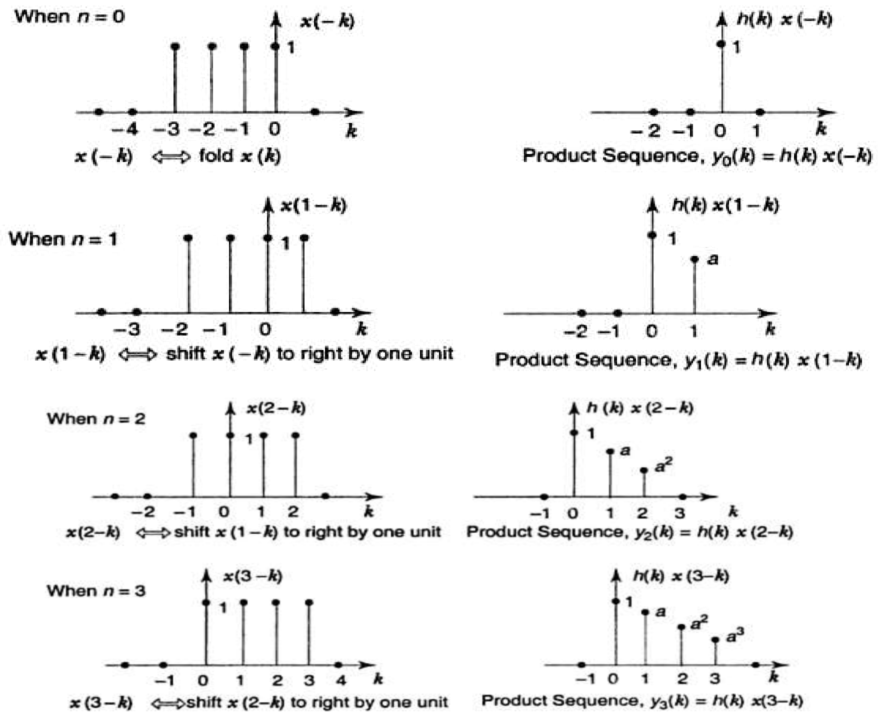
Example Find the convolution of the two signals $x(n) = u(n)$ and $h(n) = a^n u(n)$, $ROC: |a| < 1$; $n \geq 0$.

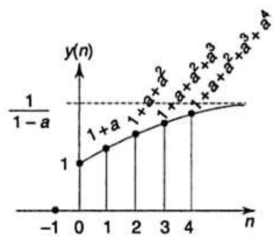
Solution Plot the sequence values of the two signals as shown in Fig. E6.4(a).
The convolution of two signals is given by



Graphical Convolution

The convolution of the given two signals can be determined by using the graphical method as shown in Fig.





Example Compute (a) linear and (b) circular periodic convolutions of the two sequences $x_1(n) = \{1, 1, 2, 2\}$ and $x_2(n) = \{1, 2, 3, 4\}$. (c) Also find circular convolution using the DFT and IDFT.

Solution (a) Using matrix representation given as follows, the linear convolution of the two sequences can be determined.

$x_2(n)$ \ $x_1(n)$	1	1	2	2
1	1	1	2	2
2	2	2	4	4
3	3	3	6	6
4	4	4	8	8

Hence, $x_3(n) = x_1(n) * x_2(n) = \{1, 3, 7, 13, 14, 14, 8\}$

5.5 LINEAR CONSTANT COEFFICIENT DIFFERENCE EQUATION

Problem Determine the impulse response, and frequency response, of the given linear constant coefficient difference equation.

$$y(n)-\frac{5}{6}y(n-1)+\frac{1}{6}y(n-2)=x(n)-\frac{1}{2}x(n-1)$$

Solution

$$y(n)-\frac{5}{6}y(n-1)+\frac{1}{6}y(n-2)=x(n)-\frac{1}{2}x(n-1)$$

Taking Z-Transform,

$$Y(Z)-\frac{5}{6}Y(Z)Z^{-1}+\frac{1}{6}Y(Z)Z^{-2}=X(Z)-\frac{1}{2}X(Z)Z^{-1}$$

$$Y(Z)\left[1-\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}\right]=X(Z)\left(1-\frac{1}{2}Z^{-1}\right)$$

The frequency response,

$$H(Z)=\frac{Y(Z)}{X(Z)}=\frac{\left(1-\frac{1}{2}Z^{-1}\right)}{1-\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}}$$

$$H(Z)=\frac{\left(1-\frac{1}{2}Z^{-1}\right)}{\left(1-\frac{1}{2}Z^{-1}\right)\left(1-\frac{1}{3}Z^{-1}\right)}$$

$$H(Z)=\frac{1}{1-\frac{1}{3}Z^{-1}}$$

Taking inverse Z-Transform,

$$h(n)=\left(\frac{1}{3}\right)^n u(n),\ |Z|>1/3$$

Problem Determine the unit step response of the system, whose linear constant coefficient difference equation, is given by

$$y(n)-0.5y(n-1)+0.06y(n-2)=x(n)-x(n-1)$$

if, $y(-1)=1; y(-2)=2$.

Solution

$$y(n)-0.5y(n-1)+0.06y(n-2)=x(n)-x(n-1)$$

Taking Z-Transform,

$$Y(Z)-0.5[Y(Z)Z^{-1}+y(-1)]+0.06\ [Y(Z)Z^{-2}+Z^{-1}y(-1)+y(-2)]=X(Z)-Z^{-1}X(Z)$$

$$Y(Z)-0.5Z^{-1}Y(Z)-0.5(1)+0.06Z^{-2}Y(Z)+0.06Z^{-1}(1)+0.06(2)=X(Z)(1-Z^{-1})$$

$$Y(Z)[1-0.5Z^{-1}+0.06Z^{-2}]-0.38+0.06Z^{-1}=X(Z)(1-Z^{-1})$$

The input to the system is unit step, $x(n)=u(n)$

$$X(Z)=\sum_{n=0}^{\infty}(1)Z^{-n}$$

$$X(Z)=\frac{1}{1-Z^{-1}}$$

$$Y(Z)[1-0.5Z^{-1}+0.06Z^{-2}]=\left[\frac{1}{1-Z^{-1}}\right](1-Z^{-1})+0.38-0.06Z^{-1}$$

$$Y(Z)[1-0.5Z^{-1}+0.06Z^{-2}]=1.38-0.06Z^{-1}$$

$$Y(Z)=\frac{1.38-0.06Z^{-1}}{(1-0.2Z^{-1})(1-0.3Z^{-1})}$$

On partial fraction expansion,

$$Y(Z)=\frac{A}{(1-0.2Z^{-1})}+\frac{B}{(1-0.3Z^{-1})}$$

$$A=(1-0.2Z^{-1})Y(Z)\Big|_{Z^{-1}=5}=\frac{1.38-0.06(5)}{1-0.3(5)}=-2.16$$

$$B=(1-0.3Z^{-1})Y(Z)\Big|_{Z^{-1}=3.333}=\frac{1.38-(0.06)(3.333)}{1-0.2(3.33)}=3.54$$

$$Y(Z)=\frac{-2.16}{1-0.2Z^{-1}}+\frac{3.54}{1-0.3Z^{-1}}$$

Taking inverse Z-Transform,

$$y(n)=-2.16(0.2)^n u(n)+3.54(0.3)^n u(n)$$

Problem Find the constant coefficient difference equation, if the input to the system is $x(n)=\left(\frac{1}{2}\right)^n u(n)$ and the output of the system is

$$y(n)=\left(\frac{1}{2}\right)^n u(n)+2\left(\frac{1}{3}\right)^n u(n)$$

Solution The input to the system is,

$$x(n)=\left(\frac{1}{2}\right)^n u(n)$$

Taking Z-Transform,

$$X(Z)=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n Z^{-n}=\frac{1}{1-\left(\frac{1}{2}\right)Z^{-1}}$$

The output of the system is,

$$y(n)=\left(\frac{1}{2}\right)^n u(n)+2\left(\frac{1}{3}\right)^n u(n)$$

Taking Z-Transform,

$$Y(Z)=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n Z^{-n}+2\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n Z^{-n}$$

$$Y(Z)=\frac{1}{1-\left(\frac{1}{2}\right)Z^{-1}}+\frac{2}{1-\left(\frac{1}{3}\right)Z^{-1}}$$

The system function is given by,

$$H(Z)=\frac{Y(Z)}{X(Z)}=\frac{\dfrac{1}{1-\left(\frac{1}{2}\right)Z^{-1}}+\dfrac{2}{1-\left(\frac{1}{3}\right)Z^{-1}}}{\dfrac{1}{1-\left(\frac{1}{2}\right)Z^{-1}}}$$

$$\frac{Y(Z)}{X(Z)}=\frac{1-\frac{1}{3}Z^{-1}+2\left[1-\left(\frac{1}{2}\right)Z^{-1}\right]}{\left[1-\left(\frac{1}{2}\right)Z^{-1}\right]\left[1-\left(\frac{1}{3}\right)Z^{-1}\right]}\times\left[1-\left(\frac{1}{2}\right)Z^{-1}\right]$$

$$\frac{Y(Z)}{X(Z)}=\frac{\left(3-\frac{4}{3}Z^{-1}\right)\left(1-\frac{1}{2}Z^{-1}\right)}{\left(1-\frac{1}{2}Z^{-1}\right)\left(1-\frac{1}{3}Z^{-1}\right)}=\frac{3-\frac{17}{6}Z^{-1}+\frac{2}{3}Z^{-2}}{1-\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}}$$

$$Y(Z)\left[1-\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}\right]=X(Z)\left[3-\frac{17}{6}Z^{-1}+\frac{2}{3}Z^{-2}\right]$$

$$Y(Z)-\frac{5}{6}Z^{-1}Y(Z)+\frac{1}{6}Z^{-2}Y(Z)=3X(Z)-\frac{17}{6}Z^{-1}X(Z)+\frac{2}{3}Z^{-2}X(Z)$$

Taking inverse Z-Transform,

$$y(n)-\frac{5}{6}y(n-1)+\frac{1}{6}y(n-2)=3x(n)-\frac{17}{6}x(n-1)+\frac{2}{3}x(n-2)$$

5.6 RELATIONSHIP BETWEEN Z-TRANSFORM AND FOURIER TRANSFORM

Let us derive a relation between Z-Transform and the Fourier transform, by considering a system function.

The Fourier transform representation of a system is given by

$$H(e^{j\omega})=\sum_{n=-\infty}^{\infty}h(n)e^{-j\omega n}$$

The Z-Transform representation of a system is given by

$$H(Z)=\sum_{n=-\infty}^{\infty}h(n)Z^{-n}$$

On comparing equations (11.39), and (11.40), we can conclude that,

$$Z=re^{+j\omega}$$

The unit value of r represent the locus of points in the Z-plane. Hence, $H(e^{j\omega})$ is equal to $H(Z)$, which is evaluated along the unit circle.

$$H(e^{j\omega})=H(Z)\big|_{z=e^{j\omega}}$$

5.7 PROPERTIES OF LTI SYSTEM

Distributive Property

Let us consider two LTI systems with impulse responses $h_1(n)$ and $h_2(n)$ connected parallelly as shown in Fig.

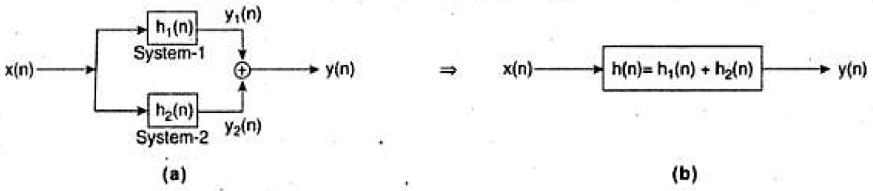


Fig. (a) Systems Connected in Parallel (b) Equivalent Representation

By definition of the distributive property,

$$x(n)*[h_1(n)+h_2(n)] = x(n)*h_1(n)+x(n)*h_2(n)$$

Proof The output of the first system is

$$y_1(n) = x(n)*h_1(n)$$

Similarly, the output of the second system is

$$y_2(n) = x(n)*h_2(n)$$

The overall output of the system $y(n)$ is given by

$$y(n) = y_1(n)+y_2(n)$$

$$y(n) = x(n)*h_1(n)+x(n)*h_2(n)$$

By the definition of convolution from equation (3.11)

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h_1(n-k)+\sum_{k=-\infty}^{\infty} x(k)h_2(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)[h_1(n-k)+h_2(n-k)] = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

where $h(n-k) = h_1(n-k)+h_2(n-k) \Rightarrow h(n) = h_1(n)+h_2(n)$

Therefore, $y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n)*h(n)$

If two systems $h_1(n)$ and $h_2(n)$ are connected in parallel, then the impulse response of the system to the input signal $x(n)$ is equal to sum of the two impulse responses.

Associative Property

Let us consider two LTI systems with impulse responses $h_1(n)$ and $h_2(n)$ connected in series as shown in Fig.

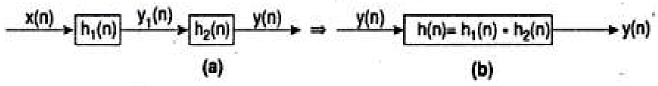


Fig. (a) Systems Connected in Series (b) Equivalent Circuit

By definition of the associative property,

$$[x(n)*h_1(n)]*h_2(n) = x(n)*[h_1(n)*h_2(n)] \tag{3.13}$$

Proof The output of the first system

$$y_1(n) = x(n)*h_1(n) \tag{3.14}$$

Similarly, the output of the second system

$$y(n) = y_1(n)*h_2(n) \tag{3.15}$$

Substitute equation (3.14) in equation (3.15),

$$y(n) = [x(n)*h_1(n)]*h_2(n) = x(n)*[h_1(n)*h_2(n)]$$

Commutative Property

Let us consider two LTI systems with impulse responses $h_1(n)$ and $h_2(n)$ connected in series as shown in Fig.

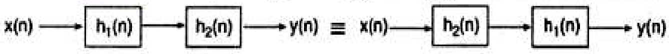


Fig. Systems Connected in Series

By definition of the commutative property,

$$h_1(n)*h_2(n) = h_2(n)*h_1(n) \tag{3.16}$$

$$h_1(n)*h_2(n) = \sum_{k=-\infty}^{\infty} h_1(k)\,h_2(n-k)$$

Proof

Let $n-k=m \Rightarrow k=n-m$

$$h_1(n)*h_2(n) = \sum_{m=-\infty}^{\infty} h_1(n-m)h_2(m)$$

$$h_1(n)*h_2(n) = \sum_{m=-\infty}^{\infty} h_2(m)h_1(n-m)$$

$$h_1(n)*h_2(n) = h_2(n)*h_1(n)$$

Equation (3.16) finds application in solving convolution problems. This property can also be extended to signals, i.e.

$$y(n) = x(n)*h(n) = h(n)*x(n) \tag{3.17}$$

5.8 PROPERTIES OF DISCRETE-TIME LTI SYSTEM

LTI System With and Without Memory

A system is memoryless if the output at any time depends only on the present input
This is true for the LTI system if and only if

$$h(n) = 0, \quad n \neq 0$$

Let us consider the impulse response of the form

$$h(n) = k\,\delta(n)$$

where $k = h(0)$, is a constant

The output of such a system is given by

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,k\delta(n-k)$$

$$y(n) = k\,x(k) \tag{3.18}$$

Hint		
	$\delta(n)=1,$	$n=0$
	$\delta(n-k)=1,$	$n=k$

Equation (3.18) is a memoryless LTI system.

If $h(n) \neq 0, n \neq 0$, then the LTI system is called a memory system.

Invertibility of LTI System

A system is invertible only if an inverse system exists. Similarly, an LTI system is invertible only if an inverse LTI system exists.

Let us consider the following figure.

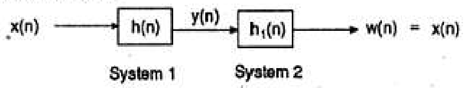


Fig. *Invertibility of LTI System*

The system response $h(n)$ results in an output $y(n)$ and the output of system 1 is given to system 2, whose response $h_1(n)$ results in an output $w(n)$, which is equal to the original input $x(n)$. This is possible if

$$h(n) * h_1(n) = \delta(n) \tag{3.19}$$

Stability for LTI System

A system is said to be stable if every bounded input produces a bounded output. The statement can be extended to LTI systems also.

Let us consider a bounded input $x(n)$, i.e.

$$|x(n)| < M_X < \infty \text{ for all } n \tag{3.20}$$

Suppose the bounded input is applied to an LTI system with unit impulse response $h(n)$, then using convolution sum, we obtain an expression for the output $y(n)$, i.e.

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} x(n)\, h(n-k) \right| \tag{3.21}$$

By the inequality relation, the magnitude of the sum of a set of numbers is no longer larger than the sum of the magnitudes of the numbers, i.e.

$$|y(n)| \le \sum_{k=-\infty}^{\infty} |x(k)||h(n-k)|$$

$$|y(n)| \le \sum_{k=-\infty} |h(k)||x(n-k)| \tag{3.22}$$

From equation (3.20), $|x(n)| < M_X < \infty$

Therefore, $|x(n-k)| < M_X < \infty$ for all n and k.

Substitute the equivalent relation of equation (3.20) in (3.22)

$$|y(n)| \le M_X \sum_{k=-\infty}^{\infty} |h(k)| \text{ for all } k \tag{3.23}$$

The impulse response $h(k)$ is absolutely summable if

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \tag{3.24}$$

then the output of the LTI system $y(n)$ is stable (bounded output). If the impulse response $h(k)$ is not absolutely summable, then the system is a 'nonstable system'.

SOLVED PROBLEMS

Problem Find whether the system with impulse response $h(n) = 2e^{-2|n|}$ is stable or not.

Solution The condition for stability is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty}|h(n)| = \sum_{n=-\infty}^{\infty} 2e^{-2|-n|} = 2\left(\sum_{n=-\infty}^{-1} e^{-2(-n)} + \sum_{n=0}^{\infty} e^{-2(n)}\right)$$

$$\sum_{n=-\infty}^{\infty}|h(n)| = 2\left(\sum_{n=1}^{\infty} e^{-2n} + \sum_{n=0}^{\infty} e^{-2n}\right)$$

$$\sum_{n=-\infty}^{\infty}|h(n)| = 2\left(\frac{e^{-2}}{1-e^{-2}} + \frac{1}{1-e^{-2}}\right)$$

$$\sum_{n=-\infty}^{\infty}|h(n)| = 2\left(\frac{1+e^{-2}}{1-e^{-2}}\right) < \infty$$

Hint $\sum_{n=k}^{\infty}\beta^k = \frac{\beta^k}{1-\beta}, |\beta| < 1$

$\sum_{n=0}^{\infty}\beta^k = \frac{1}{1-\beta}, |\beta| < 1$

Therefore, the system is stable.

Problem Find whether the system with impulse response $h(n) = e^{2n} u(n)$ is stable or not.

Solution The condition for stability is

$$\sum_{n=-\infty}^{\infty}|h(n)| < \infty$$

$$\sum_{n=-\infty}^{\infty}|h(n)| = \sum_{n=0}^{\infty} e^{2n} = 1 + e^2 + e^4 + e^6 + ... = \infty$$

Therefore, the system is unstable.

Causal System

By definition, for a discrete-time causal LTI system, the impulse response $h(n)$ must be zero for $n < 0$. The causality can be extended to convolution sum as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{3.25}$$

For a causal discrete-time LTI system, $h(n) = 0$ for $n < 0$. Therefore, the output of a causal system must be expressed as

$$y(n) = \sum_{k=0}^{\infty} x(k)h(n-k) \tag{3.26}$$

A causal system cannot generate an output before an input is given to the system.

Problem The impulse response of an LTI system is $h(n) = \left(\frac{1}{3}\right)^n u(n)$. Determine the output of the system $y(n)$ at

(i) $n = -2$ (ii) $n = 2$ and (iii) $n = +4$, when input signal $x(n) = u(n)$.

Solution By convolution sum,

$$y(n) = h(n) * x(n) = x(n) * h(n) \text{ (Commutative property)}$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(n)h(n-k)$$

$$y(n)=\sum_{k=-\infty}^{\infty}u(n)\left[\left(\frac{1}{3}\right)^{n-k}u(n-k)\right]$$

$$y(n)=\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^{n-k}$$

For $n=-2$ $\quad y(-2)=0$

For $n=2$

$$y(2)=\sum_{k=0}^{2}\left(\frac{1}{3}\right)^{2-k}=\frac{1}{9}\sum_{k=0}^{2}3^{k}$$

$$y(2)=\frac{1}{9}\left[\frac{1-(3)^3}{1-3}\right]=\frac{13}{9}$$

Hint $\sum_{k=0}^{N}\alpha^{n}=\frac{1-\alpha^{N+1}}{1-\alpha}$

For $n=4$

$$y(4)=\sum_{k=0}^{4}\left(\frac{1}{3}\right)^{4-k}$$

$$y(4)=\left(\frac{1}{3}\right)^{4}\sum_{k=0}^{4}3^{k}=\frac{1}{81}\left(\frac{1-3^5}{1-3}\right)=\frac{121}{81}$$

STEP RESPONSE

By using the convolution sum, we can easily represent the step response in terms of the impulse response. Let us consider the output response of the system $y(n)$ as,

$$y(n)=x(n)*h(n)=\sum_{k=-\infty}^{\infty}x(k)h(n-k)=\sum_{k=-\infty}^{\infty}h(k)x(n-k) \tag{3.27}$$

where, $x(n)$ is input signal and $h(n)$ is impulse response of the system.

The step response of the system, means, applying a unit step function as a signal to the system, that is, $x(n)=u(n)$

where $\quad u(n)=\begin{cases}1, & n\geq 0\\ 0, & \text{otherwise}\end{cases} \qquad u(n-k)=\begin{cases}1, & n\geq k\\ 0, & \text{otherwise}\end{cases}$

Then, step response $\quad S(n)=\sum_{k=0}^{\infty}h(k)u(n-k)$

$$S(n)=\sum_{k=0}^{n}h(k) \tag{3.28}$$

Equation (3.28) explains that the step response is the impulse response.

SOLVED PROBLEMS

Problem Find the step response of the system if the impulse response is $h(n)=\alpha^{n}u(n), 0<\alpha<1$.

Solution The step response of the system is given by,

$$S(n)=h(n)*u(n)$$

$$S(n)=\sum_{k=-\infty}^{\infty}h(k)u(n-k)$$

$$S(n)=\sum_{k=-\infty}^{\infty}[\alpha^{k}u(k)]u(n-k)$$

$$S(n)=\sum_{k=0}^{n}\alpha^{k}=\frac{1-\alpha^{n+1}}{1-\alpha}$$

Problem Find the step response of the system if the impulse response $h(n)=\delta(n-2)-\delta(n-1)$.

Solution The step response, $\quad \delta(n)=h(n)*u(n)$

$$S(n)=[\delta(n-2)-\delta(n-1)]*u(n)$$

$$S(n)=[\delta(n-2)*u(n)]-[\delta(n-1)*u(n)]$$

$$S(n)=u(n-2)-u(n-1)$$