MA8351 DISCRETE MATHEMATICS

UNIT I LOGIC AND PROOFS

Propositional logic – Propositional equivalences – Predicates and quantifiers – Nested quantifiers – Rules of inference – Introduction to proofs – Proof methods and strategy.

UNIT II COMBINATORICS

Mathematical induction – Strong induction and well ordering – The basics of counting – The pigeonhole principle – Permutations and combinations – Recurrence relations – Solving linear recurrence relations – Generating functions – Inclusion and exclusion principle and its applications

UNIT III GRAPHS

Graphs and graph models – Graph terminology and special types of graphs – Matrix representation of graphs and graph isomorphism – Connectivity – Euler and Hamilton paths.

UNIT IV ALGEBRAIC STRUCTURES

Algebraic systems – Semi groups and monoids – Groups – Subgroups – Homomorphism‵s – Normal subgroup and cosets –
Lagrange‵s theorem – Definitions and examples of Rings and Fields.

UNIT V LATTICES AND BOOLEAN ALGEBRA

Partial ordering – Posets – Lattices as posets – Properties of lattices – Lattices as algebraic systems – Sub lattices – Direct product and homomorphism – Some special lattices – Boolean algebra.

Unit1 - LOGIC AND PROOFS

TOPIC 1:
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PROPOSITIONS

A declarative sentence (or assertion) which is true or false, but not both, is called a proposition (or statement). Sentences which are exclamatory, interrogative or imperative in nature are not propositions. Lower case letterssuch as p, q, r . . . are used to denote propositions. For example, we consider the following sentences:

l. Chennai is the capital of Tamilnadu.
2. How beautiful is Rose?
3.2+2=4
4. What time is it?
5. x+y=z

In the given statements, (2) and (4) are obviously not propositions as they are not declarative in nature. (l) and (3) are propositions, but (5) is not, since (1) is true, (3) is false and (5) is neither true nor false as the values of x, y and z are not assigned.

If a proposition is true, we say that the truth value of' that proposition is true, denoted by T or 1. If a proposition is false, the truth value is said to be false, denoted by F or 0.

Definition: Atomic Statement

An atomic statement is a type of declarative sentence which cannot be broken down into other simpler sentence.

Example : It is raining.

Definition: Molecular Statement

Mathematical statements which can be constructed by combining one or more atomic statements using connectives are called molecular or compound statement.

Example :It is raining and it is wet.

CONNECTIVE

Definition-Conjunction

When p and q are any two propositions, the proposition "p and q" denoted by $p \wedge q$ and called the conjunction of p and q is defined as the compound proposition that is true when both p and q are true and is false otherwise.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition-Disjunction

When p and q are any two propositions, the propositions "p or q" denoted by $p \vee q$ and called the disjunction of p and q is defined as the compound proposition that is false when both p and q are false and is true otherwise.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition-Negation

Given any proposition p, another proposition formed by writing "It is not the case that" or "It is false that" before p or by inserting the word 'not' suitably in p is called the negation of p and denoted by ~p (read as 'not p'). ~p is also denoted ¬P.
It p is true, then ~p is false and if p is false, then ~p is true.

p	¬p
T	F
F	T

Above table is the truth table for the negation of p.

For example, if p is the statement "New Delhi is in India", then ¬P is given by¬P:It is not the case that New Delhi is in India.

Conditional Statement: [If... then]

Let p and q be any two statements. Then the statement p →q is called a conditional statement (read as if p then q).p→ q has a truth value F if p has the truth value T and F has the truth value F. In all the remaining cases it has the truth value T.

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Example:

p: Ram is a Computer Science student

q: Ram study DS

p→q: If Ram is a Computer Science student, then he will study DS.

The different situations where the conditional statements applied are listed below.

a. If p then q
b. p only if q
c. q whenever p
d. q is necessary for p
e. q follows from p
f. q when p
g. p is sufficient for q
h. p implies q

Converse, Contrapositive & Inverse Statements

If p→ q is a conditional statement, then

a. q→ p is called converse of p→q
b. ¬q→¬p is called contrapositive of p→ q
c. ¬p→¬q is called inverse of p→q

Example: Write are the contrapositive, the converse and the inverse of the implication "The home team wins whenever it is raining".

Solution: p →q: If it is raining then the home team wins.

Contra positive (¬q →¬p): If the home team does not win then it is not raining.

Converse (q →p) : If the home team wins then it is raining.

Inverse (¬p→¬q) : If it is not raining then the home team does not win.

BICONDITIONAL PROPOSITION

If p and q are two propositions, then the proposition p if and only if q, denoted by $p \leftrightarrow q$ is called the biconditional statement and is defined by the following truth table.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: $p \leftrightarrow q$ is True if both p and q have the same truth values. Otherwise $p \leftrightarrow q$ is False

Example

p: You can take the flight

q: You can buy a ticket

$p \leftrightarrow q$: You can take the flight if and only if you buy a ticket

PROBLEMS:

Part A (2 marks)

1) Symbolize the Statements using Logical Connectives

The automated reply can be sent when the file system is full.

p: The automated reply can be sent

q: The file system is full

Solution: Symbolic form: $q \rightarrow \neg p$

2) Write the symbolized form of the statement. If either Ram takes C ++ or Kumar takes Pascal, then Latha will take Lotus.

R: Ram takes C ++

K: Kumar takes Pascal

L: Latha takes Lotus

Solution: Symbolic form: $(R \vee K) \rightarrow L$

3)Let p, q, r represents the following propositions,

p: It is raining q: The sun is shining r: There are clouds in the sky

Symbolize the following statements.

If it is raining, then there are clouds in the sky
If it is not raining, then the sun is not shining and there are clouds in the sky.

The sun is shining if and only if it is not raining.

Solution: a)$p \rightarrow q$ b) $\neg p \rightarrow (\neg q \wedge r)$ c) $q \leftrightarrow \neg r$

4. Symbolize the following statements:
   (i) If the moon is out and it is not snowing, then Ram goes out for a walk.
   (ii) If the moon is out, then if it is not snowing, Ram goes out for a walk.
   (iii) It is not the case that Ram goes out for a walk if and only if it is not snowing or the moon is out.

Solution: Let the propositions be,

p: The moon is out.

q: It is snowing.

r: Ram goes out for a walk.

Symbolic form:

(i) $(p \wedge \neg q) \rightarrow r$

(ii) $p \rightarrow (\neg q \rightarrow r)$

(iii) $\neg(r \leftrightarrow (\neg q \vee p))$

5) Write are the contrapositive, the converse and the inverse of the implication "The home team wins whenever it is raining".

Solution:

$p \rightarrow q$: If it is raining then the home team wins.

Contra positive ($\neg q \rightarrow \neg p$): If the home team does not win then it is not raining

Converse ($q \rightarrow p$): If the home team wins then it is raining.

Inverse ($\neg p \rightarrow \neg q$) : If it is not raining then the home team does not win.

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Topic 2:

Construction of truth tables

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Truth Tables:

The truth value of a proposition is either true (T) or false (F)

A truth table is a table that shows the truth value of a compound proposition for all possible values.

Problems:

1. Show that the truth values of the formula $p \wedge (p \rightarrow q) \rightarrow q$ are independent of their components

   Solution: The truth table for the formula is

P	Q	$p\rightarrow q$	$p \wedge (p\rightarrow q)$	$(p \wedge (p\rightarrow q))\rightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

2. Show that the Truth value of $(p\rightarrow q) \leftrightarrow (\neg P \vee q)$ is independent of their component

Solution:

P	Q	p→q	⌐p ∨ q	Ans
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

3. Construct a truth table for (q∧ ((p →q))→ p

Solution:

P	Q	p→q	q∧ (p→q)	q∧ (p→q)→p
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

4. Construct a TRUTH table for ⌐ (p ∨ (q∧R)) ↔((p ∨ q)∧(P∨R))

Solution:

P	Q	r	q∧ r	p∨ (q^r)	(p∨ q)	(p∨ r)	(p∨q)∧ (p∨r)	⌐(p ∨ (q ∧ r))	Ans
T	T	T	T	T	T	T	T	F	F
T	T	F	F	T	T	T	T	F	F
T	F	T	F	T	T	T	T	F	F
T	F	F	F	T	T	T	T	F	F
F	T	T	T	T	T	T	T	F	F
F	T	F	F	F	T	F	F	T	F
F	F	F	F	F	F	T	F	T	F
F	F	F	F	F	F	F	F	T	F

TAUTOLOGY AND CONTRADICTION

A statement formula which is always true regardless of the truth values of the variables in it is called a **Tautology**

If a given formula is a tautology then its truth values are all T whatever be the truth values of components. Therefore the last column of the truth table of the given formula has truth values T only.

A statement formula which is false always for the truth values of the components is called a **contradiction**.

The last column of the truth table of the contradiction has only the truth value F for all cases.

Problems:

1. Prove that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

Proof:

P	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Since the truth value of given formula are all, true, the given formula is tautology.

2. Prove that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

Proof:

P	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Since the truth value of given formula are all, true, the given formula is tautology.

3. Verify whether $(p \wedge (p \rightarrow q)) \rightarrow q$ is a tautology

Proof:

p	Q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$(p \wedge (p \rightarrow q)) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Since the truth value of given formula are all, true, the given formula is tautology.

4. Prove that $(\neg p \wedge p) \wedge q$ is a contradiction.

Proof:

p	q	$\neg q \wedge p$	$(\neg p \wedge p) \wedge q$
T	T	F	F
T	F	T	F
F	T	F	F
F	F	F	F

Since the truth value of given formula are all, FALSE the given formula is contradiction.

LOGICAL EQUIVALENCES

Let p and q be two statement formulas, p is said to be logically equivalent to q if p and q have the same set of truth values or equivalently p and q are logically equivalent if p $\Leftrightarrow$ q is a tautology.

Hence, p $\Leftrightarrow$ q if and only if p $\Leftrightarrow$ q is a tautology.

Notation:

1. p $\Leftrightarrow$ q   2. p≡q

Problems

1. Prove that p→q is logically equivalent to ¬p $\vee$ q (i.e. p →q $\Leftrightarrow$ ¬p $\vee$q)

Proof:

P	Q	p→q	¬p$\vee$ q
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

NOTE: Since p → q & ¬q→¬p has same truth values we observe that p→q $\Leftrightarrow$ ¬q→¬p.

2. Prove that p↔q$\Leftrightarrow$ (p → q) $\wedge$ (q→ p)

Proof:

P	Q	p→q	p↔q	q→p	(p→q) $\wedge$ (q→p)
T	T	T	T	T	T
T	F	F	F	T	F
F	T	T	F	F	F
F	F	T	T	T	T

From Truth table we see that p ↔q, (p→ q) $\wedge$ (q→ p), have same truth values.
Hence, p ↔q $\Leftrightarrow$ (p → q) $\wedge$ (q → p).

3. Prove that $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

Proof:

P	q	$p \wedge q$	$\neg p \wedge \neg q$	$(p \vee q) \vee (\neg p \wedge \neg q)$	$p \leftrightarrow q$
T	T	F	F	T	T
T	F	F	F	F	F
F	T	F	F	F	F
F	F	T	T	T	T

Since the truth values are same hence,

$p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

4. State and prove DE Morgan's law

DE Morgan's laws

i) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

ii) $\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$

Proof: i) $\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$

P	Q	$\neg (p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

ii) $\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$

P	Q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	F	T	T
F	F	T	T	T

LOGICAL IMPLICATION

A statement formula A logically implies another, statement formula B if and only if A → B is a tautology.

Therefore, A=>B (A logically implies B) if and only if A→ B is a tautology.

Problems

1. Prove that $(p \wedge q) \Rightarrow (p \vee q)$

Proof:

To Prove: $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

P	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

The last column shows that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology. Therefore, $(p \wedge q) \Rightarrow (p \vee q)$

2. Show that $(p \wedge q) \Rightarrow (p \rightarrow q)$

Proof:To prove: $(p \wedge q) \rightarrow (p \rightarrow q)$

P	Q	$p \wedge q$	$p \rightarrow q$	$(p \wedge q) \rightarrow (p \rightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

The last column shows that $(p \wedge q) \rightarrow (p \rightarrow q)$ is a tautology. Hence $(p \wedge q) \Rightarrow (p \rightarrow q)$.

3. Prove that $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$

Proof:Let S: $(p\rightarrow q) \wedge (q\rightarrow r)\rightarrow (p\rightarrow r)$

To Prove: S is a tautology.

P	Q	R	$(p\rightarrow q)$	$(q\rightarrow r)$	$(p\rightarrow r)$	S
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

The last column shows that S: $(p\rightarrow q) \wedge (q\rightarrow r) \rightarrow (p\rightarrow r)$ is a tautology.
Therefore, $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p\rightarrow r$.

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TOPIC 3:

NORMAL FORMS

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To find PDNF & PCNF using Truth table method:

If we write given statement formula in terms of $\wedge$, $\vee$ and $\neg$ then it is called **Normal form/ Canonical form**.

Principal Disjunctive Normal Form (PDNF)	Principal Conjunctive Normal Form (PCNF)

PDNF: Sum of min terms Example: $(p \wedge q) \vee (\neg p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$	**PCNF:** Product of max terms Example: $(p \vee q) \wedge (\neg p \vee q) \wedge (p \vee \neg q) \wedge (\neg p \vee \neg q)$

Problems

1. Find PDNF and PCNF of the following compound proposition using truth table and Laws of proposition: $(\neg p \vee \neg q) \rightarrow (p \leftrightarrow \neg q)$.

   Solution: Using Truth Table: Let A=$(\neg p \vee \neg q) \rightarrow (p \leftrightarrow \neg q)$.

P	Q	$\neg p$	$\neg q$	$\neg p \vee \neg Q$	$p \leftrightarrow \neg q$	A	Min terms	Max terms
T	T	F	F	F	F	T	$p \wedge q$	-
T	F	F	T	T	T	T	$p \wedge \neg q$	-
F	T	T	F	T	T	T	$\neg p \wedge q$	-
F	F	T	T	T	F	F	-	$p \vee q$

Therefore, PDNF: Sum of Min terms $(p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q)$.

PCNF: $(p \vee q)$.

2. Obtain PCNF and hence PDNF of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

Solution: Using Truth Table: Let $S = (P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

P	Q	R	$P \wedge Q$	$\neg P \wedge R$	$Q \wedge R$	S	Min terms	Max terms
T	T	T	T	F	T	T	$P \wedge Q \wedge R$	
T	T	F	T	F	F	T	$P \wedge Q \wedge \neg R$	
T	F	T	F	F	F	F		$\neg P \vee Q \vee \neg R$
T	F	F	F	F	F	F		$\neg P \vee Q \vee R$
F	T	T	F	T	T	T	$\neg P \wedge Q \wedge R$	
F	T	F	F	F	F	F		$P \vee \neg Q \vee R$
F	F	T	F	T	F	T	$\neg P \wedge \neg Q \wedge R$	
F	F	F	F	F	F	F		$P \vee Q \vee R$

The PDNF of S is, $(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$.

The PCNF of S is, $(\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \vee R)$.

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Topic 4:

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LOGICAL LAWS:

S.No	Primal	Dual	Name of the law
1	$p \vee q \Leftrightarrow q \vee p$	$p \wedge q \Leftrightarrow q \wedge p$	Commutative laws
2	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$	$p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	Associative laws
3	$p \wedge (q \vee r) \Leftrightarrow (p \wedge v) \vee (p \wedge r)$	$p \vee (q \wedge r) \Leftrightarrow (p \vee r) \wedge (p \vee r)$	Distributive laws
4	$\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$	$\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$	De Morgan's laws
5	$p \wedge p \Leftrightarrow p$	$p \vee p \Leftrightarrow p$	Idempotent laws
6	$p \wedge (p \vee q) \Leftrightarrow p$	$p \vee (p \wedge q) \Leftrightarrow p$	Absorption law
7	$p \wedge T \Leftrightarrow p$	$p \vee F \Leftrightarrow p$	Identity law
8	$p \wedge F \Leftrightarrow F$	$p \vee T \Leftrightarrow T$	Dominance law
9	$p \wedge \neg p \Leftrightarrow F$	$p \vee \neg p \Leftrightarrow T$	Negation law
10	$\neg(\neg p) \Leftrightarrow p$		Double Negation law

11	$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$	$q \rightarrow p \Leftrightarrow \neg p \rightarrow \neg q$	Contra positive law
12	$p \rightarrow q \Leftrightarrow \neg p \vee q$	$\neg p \rightarrow q \Leftrightarrow p \vee q$	Conditional as disjunction law
13	$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$	$q \leftrightarrow p \Leftrightarrow (q \rightarrow p) \wedge (p \rightarrow q)$	Biconditional as disjunction law

Problems:

1. Negate and simplify the compound statement $(p \wedge q) \rightarrow r$

   Solution: We know that

   $\neg((p \vee q) \rightarrow r) \Leftrightarrow \neg(\neg(p \vee q) \vee r)$ (by conditional as disjunction))

   $\Leftrightarrow \neg(\neg(p \vee q)) \wedge \neg r$ (by Demorgans law)

   $\Leftrightarrow (p \vee q) \wedge \neg r$ (by double negation law)

2. Show that $(p \vee q) \wedge \neg(\neg p \wedge q) \Leftrightarrow p$

   Proof: $(p \vee q) \wedge (\neg(\neg p) \vee \neg q)$ (by De Morgan's laws)

   $\Leftrightarrow (p \vee q) \wedge (p \vee \neg q)$ (by double negation law)

   $\Leftrightarrow p \vee (q \wedge \neg q)$ (by Distributive laws)

   $\Leftrightarrow p \vee F$ (by negation law)

   $\Leftrightarrow p$ (by Identity law)

3. Show that $p \rightarrow (q \rightarrow r) \Leftrightarrow p \rightarrow (\neg q \vee r) \Leftrightarrow (p \wedge q) \rightarrow r$

   Proof: Consider $p \rightarrow (q \rightarrow r)$

   $\Leftrightarrow p \rightarrow (\neg q \vee r)$ (by conditional as disjunction)

   $\Leftrightarrow \neg p \vee (\neg q \vee r)$ (by conditional as disjunction)

   $\Leftrightarrow (\neg p \vee \neg q) \vee r$ (by associative law)

   $\Leftrightarrow \neg(p \wedge q) \vee r$ (by De Morgan's laws)

   $\Leftrightarrow (p \wedge q) \rightarrow r$ (by conditional as disjunction)

   Therefore, $p \rightarrow (q \rightarrow r) \Leftrightarrow p \rightarrow (\neg q \vee r) \Leftrightarrow (p \wedge q) \rightarrow r$

4. Show that $(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$ using laws of logic.

   Proof:

   Consider $(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r)$

   $\Leftrightarrow ((\neg p \wedge \neg q) \wedge r) \vee ((q \vee p) \wedge r)$ (by associative,distributive law)

   $\Leftrightarrow (\neg(p \vee q) \wedge r) \vee ((p \vee q) \wedge r)$ (by demorgans, commutative)

   $\Leftrightarrow (\neg(p \vee q) \vee (p \vee q)) \wedge r$ (by distributive law)

   $\Leftrightarrow (\neg A \vee A) \wedge r$ Where $A = (p \vee q)$

   $\Leftrightarrow T \wedge r$ (by negation law)

   $\Leftrightarrow r$ (by identity law)

Therefore, $(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$

5. Show that $\neg(p \wedge q) \rightarrow (\neg p \vee (\neg p \vee q)) \Leftrightarrow \neg p \vee q$

Proof:

$\neg(p \wedge q) \rightarrow (\neg p \vee (\neg p \vee q)) \Leftrightarrow \neg(\neg(p \wedge q)) \vee (\neg p \vee (\neg p \vee q))$

$\Leftrightarrow (p \wedge q) \vee (\neg p \vee \neg p \vee q)$ ---------------- (1)

$\Leftrightarrow (p \wedge q) \vee (\neg p \vee q)$

$\Leftrightarrow (p \vee \neg p \vee q) \wedge (q \vee \neg p \vee q)$

$\Leftrightarrow (T \vee q) \wedge (q \vee \neg p)$

$\Leftrightarrow T \wedge (q \vee \neg p)$

$\Leftrightarrow \neg p \vee q$

6. Show that $((p \vee q) \wedge (\neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is tautology.

Proof:Consider $(\neg p \wedge \neg q) \vee (\neg p \wedge \neg r) \Leftrightarrow \neg p \wedge (\neg q \vee \neg r)$

$\Leftrightarrow \neg p \wedge \neg(q \wedge r)$

$\Leftrightarrow \neg(p \vee (q \wedge r))$ ---------------- (1)

Consider, $((p \vee q) \wedge (\neg(\neg p \wedge (\neg q \vee \neg r))) \Leftrightarrow (p \vee q) \wedge \neg(\neg p \wedge \neg(q \wedge r))$

$\Leftrightarrow (p \vee q) \wedge (p \vee (q \wedge r))$

$\Leftrightarrow (p \vee q) \wedge (p \vee q) \wedge (p \vee r)$

$\Leftrightarrow (p \vee q) \wedge (p \vee r)$

$\Leftrightarrow (p \vee (q \wedge r)$ -----------------(2)

From (1) & (2)

$((p \vee q) \wedge (\neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$

$\Leftrightarrow \neg(p \vee (q \wedge r)) \vee p \vee (q \wedge r)$

$\Leftrightarrow T$

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Topic 5 & 6:

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NORMAL FORMS:

To find PDNF & PCNF using logical law method.

Working rule to obtain PDNF	**Working rule to obtain PCNF**
Step 1: Write the given statement in terms of $\wedge$, $\vee$ and $\neg$. Step 2: Apply (each term) $\wedge$ T. Step 3: Instead of T, apply $p \vee \neg p$. Step 4: Apply Distributive Law. Step 5 : Apply idempotent law Step 6: Apply Commutative Law.	Step 1: Write the given statement in terms of $\wedge$, $\vee$ and $\neg$. Step 2: Apply each term $\vee$ F. Step 3: Instead of F, apply $p \wedge \neg p$. Step 4: Apply Distributive Law. Step 5 : Apply idempotent law Step 6: Apply Commutative Law.

PROBLEMS: (using logical law method)

1. Obtain PCNF and hence PDNF of $(P \wedge Q) \vee (\neg P \wedge Q) \vee (Q \wedge R)$.

Solution: Let

$S \equiv (P \wedge Q) \vee (\neg P \wedge Q) \vee (Q \wedge R)$

$\equiv (P \wedge Q \wedge T) \vee (\neg P \wedge Q \wedge T) \vee (Q \wedge R \wedge T) \qquad [\therefore P \wedge T \equiv P]$

$\equiv (P \wedge Q \wedge (R \vee \neg R)) \vee (\neg P \wedge Q \wedge (R \vee \neg R)) \vee (Q \wedge R \wedge (P \vee \neg P))$

$\equiv (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R)$

$\vee (P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R)$

PDNF of $S \equiv (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R)$

PDNF of $\neg S \equiv (\neg P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R)$

PCNF of $S \equiv \neg$(PDNF of $\neg$S) $\equiv (P \vee Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R)$.

2. Without constructing the truth table obtain the product-of-sums canonical form of the formula $(\neg P \to R) \wedge (Q \leftrightarrow P)$. Hence find the sum-of products canonical form.

Solution: Let

$S \equiv (\neg P \to R) \wedge (Q \leftrightarrow P)$

$\equiv \left(\neg(\neg P) \vee R\right) \wedge \left((Q \to P) \wedge (P \to Q)\right)$

$\equiv \left(P \vee R\right) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q)$

$\equiv \left((P \vee R) \vee F\right) \wedge \left((\neg Q \vee P) \vee F\right) \wedge \left((\neg P \vee Q) \vee F\right)$

$\equiv \left((P \vee R) \vee (Q \wedge \neg Q)\right) \wedge \left((\neg Q \vee P) \vee (R \wedge \neg R)\right) \wedge \left((\neg P \vee Q) \vee (R \wedge \neg R)\right)$

$\equiv (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (\neg Q \vee P \vee R) \wedge (\neg Q \vee P \vee \neg R) \wedge$
$(\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

PCNF of
$S \equiv (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$

PCNF of $\neg S \equiv (P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$

PDNF of $S \equiv \neg$(PCNF of $\neg S$) $\equiv (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$.

x...x

Topic 7 & 9: RULES OF INFERENCE

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RULES	Premises
1	$P, P \to Q => Q$
2	$P \to Q, Q \to R => P \to R$
3	$\neg P, P \to Q => \neg Q$
4	$\neg Q, P \to Q => \neg P$
5	$\neg P, P \vee Q => Q$
6	$P, Q => P \wedge Q$
7	$P \wedge Q => P$, $P \wedge Q => Q$

Validity using Rules of Inference:

Rule P: A Premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is tautologically implied by any one or more of the preceeding formulas in the derivation.

Rule CP (Conditional Proof): If we can derive S from R and a set of premises, then we can derive $R \to S$ from the set of premises alone. It is also called deduction theorem. In such cases R is taken as additional premise and S is derived from the given premises and R.

Direct Method of Proof: -

When a conclusion is derived from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called a direct proof.

PROBLEMS :

1. Demonstrate that R is a valid inference from the premises
   $P \rightarrow Q$, $Q \rightarrow R$and P.

   Solution: Premises: $P \rightarrow Q$, $Q \rightarrow R$, P

   Conclusion: R

Steps	Premise	Reason
1	$P \rightarrow Q$	Rule P
2	$Q \rightarrow R$	Rule P
3	$P \rightarrow R$	Rule T: $P \rightarrow Q, Q \rightarrow R => P \rightarrow R$ {1,2}
4	P	Rule P
5	R	Rule T: P , $P \rightarrow Q$ = > Q {3,4}

2. Prove that 't' is a valid conclusion from the premises
   $p \rightarrow q$, $q \rightarrow r$, $r \rightarrow s$, $\neg s$ and $p \vee t$.

   Solution: Premises: $p \rightarrow q$, $q \rightarrow r$, $r \rightarrow s$, $\neg s$ and $p \vee t$.

   Conclusion: t

Steps	Premise	Reason
1	$p \rightarrow q$	Rule P
2	$q \rightarrow r$	Rule P
3	$p \rightarrow r$	Rule T: $P \rightarrow Q, Q \rightarrow R => P \rightarrow R$ {1,2}
4	$r \rightarrow s$	Rule P
5	$p \rightarrow s$	Rule T: $P \rightarrow Q, Q \rightarrow R => P \rightarrow R$ {3,4}
6	$\neg s$	Rule P
7	$\neg p$	Rule T: $\neg Q, P \rightarrow Q => \neg P$ {5,6}
8	$p \vee t$	Rule P
9	T	Rule T: $\neg P, P \vee Q$= > Q {7,8}

3. Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$.

(or)

Prove that $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S) \Rightarrow S \vee R$.

Solution:

Premises: $(P \vee Q)$, $(P \rightarrow R)$, $(Q \rightarrow S)$.

Conclusion: $S \vee R$

Steps	Premise	Reason
1	$P \vee Q$	Rule P
2	$\neg P \rightarrow Q$	Rule T: $P \rightarrow Q \Leftrightarrow \neg P \vee Q$ (conditional as disjunction)
3	$Q \rightarrow S$	Rule P
4	$\neg P \rightarrow S$	Rule T: $P \rightarrow Q$, $Q \rightarrow R \Rightarrow P \rightarrow R$ {2,3}
5	$P \rightarrow R$	Rule P
6	$\neg R \rightarrow \neg P$	Rule T: Contra-Positive {5}
7	$\neg R \rightarrow S$	Rule T: $P \rightarrow Q$, $Q \rightarrow R \Rightarrow P \rightarrow R$ {6,4}
8	$R \vee S$	Rule T: $\neg R \rightarrow S \Leftrightarrow R \vee S$ (conditional as disjunction)
9	$S \vee R$	Rule T: Commutative law {8}

$\therefore S \vee R$ is a valid conclusion.

4. Show that $R \wedge (P \vee Q)$ is valid conclusion from the premises $(P \vee Q)$, $(Q \rightarrow R)$, $(P \rightarrow M)$ and $\neg M$.

Solution: Premises: $(P \vee Q)$, $(Q \rightarrow R)$, $(P \rightarrow M)$ and $\neg M$.

Conclusion: $R \wedge (P \vee Q)$

Steps	Premise	Reason
1	$P \rightarrow M$	Rule P
2	$\neg M$	Rule P
3	$\neg P$	Rule T: $\neg Q$, $P \rightarrow Q \Rightarrow \neg P$ {1,2}

4	$P \vee Q$	Rule P
5	Q	Rule T: $\neg P, P \vee Q => Q$ {3,4}
6	$Q \rightarrow R$	Rule P
7	R	Rule T: $P, P \rightarrow Q => Q$ {5,6}
8	$R \wedge (P \vee Q)$	Rule T: $P, Q => P \wedge Q$ {7,4}

$\therefore R \wedge (P \vee Q)$ is a valid conclusion.

5. Prove that $(P \rightarrow Q) \wedge (R \rightarrow S)$, $(Q \wedge M) \wedge (S \rightarrow N)$, $\neg(M \wedge N)$ and $(P \rightarrow R) \Rightarrow \neg P$.

Solution: Premises: $(P \rightarrow Q) \wedge (R \rightarrow S)$, $(Q \wedge M) \wedge (S \rightarrow N)$, $\neg(M \wedge N)$ and $(P \rightarrow R)$

Conclusion: $\neg P$

Steps	Premise	Reason
1	$(P \rightarrow Q) \wedge (R \rightarrow S)$	Rule P
2	$P \rightarrow Q$	Rule T: $P \wedge Q => P$ {1}
3	$R \rightarrow S$	Rule T: $P \wedge Q => Q$ {1}
4	$(Q \wedge M) \wedge (S \rightarrow N)$	Rule P
5	$Q \wedge M$	Rule T: $P \wedge Q => P$ {4}
6	$S \rightarrow N$	Rule T: $P \wedge Q => Q$ {4}
7	$P \rightarrow R$	Rule P
8	$P \rightarrow S$	Rule T: $P \rightarrow Q, Q \rightarrow R => P \rightarrow R$ {7,3}
9	$P \rightarrow N$	Rule T: $P \rightarrow Q, Q \rightarrow R => P \rightarrow R$ {8,6}
10	$\neg(M \wedge N)$	Rule P
11	$\neg M \vee \neg N$	Rule T: De Morgan's law {10}

12	$\neg N \vee \neg M$	Rule T: Commutative law {11}
13	$N \rightarrow \neg M$	Rule T: $P \rightarrow Q \Leftrightarrow \neg P \vee Q$ (conditional as disjunction) {11}
14	$P \rightarrow \neg M$	Rule T: $P \rightarrow Q, Q \rightarrow R => P \rightarrow R$ {9,13}
15	$M \rightarrow \neg P$	Rule T: Contra-Positive {14}
16	M	Rule T: $Q \wedge M => Q$ {5}
17	$\neg P$	Rule T: $P, P \rightarrow Q => Q$ {15,16}

$\therefore \neg P$ is a valid conclusion.

6)Show that the following argument is valid. If today is Tuesday, I have a test in Mathematics or Economics. If my Economics professor is sick, I will not have a test in Economics. Today is Tuesday and my Economics professor is sick.ThereforeI have a test in Mathematics.

Solution: Let the proposition be

P: Today is Tuesday

q: I have a test in Mathematics

r: I have a test in Economics

$\neg r$: I have not a test in Economics

s: My Economics professor is sick

Premises: $p \rightarrow (q \vee r)$, $s \rightarrow \neg r$, $p \wedge s$

Conclusion: q

Steps	Premise	Reason
1	$p \wedge s$	Rule P
2	p	Rule T: $P \wedge Q => P$ {1}
3	s	Rule T: $P \wedge Q => Q$ {1}
4	$p \rightarrow (q \vee r)$	Rule P
5	$q \vee r$	Rule T: $P, P \rightarrow Q => Q$ {2,4}
6	$s \rightarrow \neg r$	Rule P
7	$\neg r$	Rule T: $P, P \rightarrow Q => Q$ {3,6}
8	q	Rule T: $\neg P, P \vee Q => Q$ {5,7}

7)Show that the following hypotheses 'It is not sunny this afternoon and it is colder than yesterday', 'We will go swimming only if it is sunny'. If we do not go swimming, then we will take a canoe trip' and 'if we take a canoe trip, then we will be home by sunset' lead to the conclusion 'we will be home by sunset'.

Solution: Let the proposition be

P: It is sunny this afternoon

$\neg p$: It is not sunny this afternoon

q: Iit is colder than yesterday

r: We will go swimming

$\neg r$: We will not go swimming

s: We take a canoe trip

t: we will be home by sunset'

Premises: $\neg p \wedge q$, $r \to p$, $\neg r \to s$ and $s \to t$

Conclusion: t

Steps	Premise	Reason
1	$\neg p \wedge q$	Rule P
2	$\neg p$	Rule T: $P \wedge Q \Rightarrow P$ {1}
3	$r \to p$	Rule P
4	$\neg r$	Rule T: $\neg Q, P \to Q \Rightarrow \neg P$ {2,3}
5	$\neg r \to s$	Rule P
6	s	Rule T: $P, P \to Q \Rightarrow Q$ {4,5}
7	$s \to t$	Rule P
8	t	Rule T: $P, P \to Q \Rightarrow Q$ {6,7}

x...x

Topic 8: Inconsistent & Indirect method

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Inconsistency of Premises:

A set of formulas $H_1, H_2 ..., H_m$ is said to be **inconsistent** if their conjunction implies contradiction.i.e., $H_1 \wedge H_2 \wedge ....\wedge H_m \Leftrightarrow F$

Indirect method of proof:

The notion of inconsistency is used in a procedure called indirect method of proof.

Working Rule:

1.Introduce Negation of desired conclusion as a new premise

2.From the new premise together with the given premises derive a contradiction.

3.Assert the desired conclusion as a logical inference from the premises.

PROBLEMS:

1).Prove that the premises $P \to Q,\ Q \to R,\ R \to S,\ S \to \neg R$ and $P \wedge S$ are inconsistent.

Solution: To prove inconsistency we have to derive a contradiction.

Steps	Premise	Reason
1	$P \to Q$	Rule P
2	$Q \to R$	Rule P
3	$P \to R$	Rule T: $P \to Q,\ Q \to R \Rightarrow P \to R$ {1,2}
4	$R \to S$	Rule P
5	$P \to S$	Rule T: $P \to Q,\ Q \to R \Rightarrow P \to R$ {3,4}
6	$S \to \neg R$	Rule P
7	$P \to \neg R$	Rule T: $P \to Q,\ Q \to R \Rightarrow P \to R$ {5,6}
8	$P \wedge S$	Rule P
9	P	Rule T: $P \wedge Q \Rightarrow P$ {8}
10	S	Rule T: : $P \wedge Q \Rightarrow Q$ {8}
11	$\neg R$	Rule T: $P,\ P \to Q \Rightarrow Q$ {7,9}
12	R	Rule T: $P,\ P \to Q \Rightarrow Q$ {3,9 }

13	$\neg R \wedge R$	Rule T: P,Q = > $P \wedge Q$ {11,12}
14	F	Rule T: Negation law {13}

$\therefore$ The given set of premises are inconsistent.

2.Prove that the premises
$a \to (b \to c)$, $d \to (b \wedge \neg c)$, and $a \wedge d$ are inconsistent.

Solution: To prove inconsistency we have to derive a contradiction.

Steps	Premise	Reason
1	$a \wedge d$	Rule P
2	a	Rule T: $P \wedge Q \Rightarrow P$ {1}
3	d	Rule T: $P \wedge Q \Rightarrow Q$ {1}
4	$a \to (b \to c)$	Rule P
5	$b \to c$	Rule T: $P, P \to Q \Rightarrow Q$ {2,4 }
6	$d \to (b \wedge \neg c)$	Rule P
7	$b \wedge \neg c$	Rule T: $P, P \to Q \Rightarrow Q$ {3,7 }
8	b	Rule T: $P \wedge Q \Rightarrow P$ {7}
9	$\neg c$	Rule T: $P \wedge Q \Rightarrow Q$ {7}
10	c	Rule T: $P, P \to Q \Rightarrow Q$ {8,5}
11	$\neg c \wedge c$	Rule T: $P, Q \Rightarrow P \wedge Q$ {9,10}
12	F	Rule T: Negation law{11}

$\therefore$ The given set of premises are inconsistent.

3.Show that the following premises are inconsistent.

- If Jack misses many classes through illness, then he fails high school.
- If Jack fails high school, then he is uneducated.
- If Jack reads a lot of books, then he is not uneducated.
- If Jack misses many classes through illness and reads a lot of books.

Solution: Let the proposition be

P: Jack misses many classes through illness

q: Jack fails high school.

r: Jack is uneducated

$\neg r$: Jack is not uneducated

S: Jack reads a lot of books

Premises: $p \to q$, $q \to r$, $s \to \neg r$ and $p \wedge s$

Conclusion: False

Steps	Premise	Reason
1	$p \to q$	Rule P
2	$q \to r$	Rule P
3	$p \to r$	Rule T: $P \to Q, Q \to R \Rightarrow P \to R$ {1,2}
4	$s \to \neg r$	Rule P
5	$r \to \neg s$	Rule T: Contra-Positive{4}
6	$p \to \neg s$	Rule T: $P \to Q, Q \to R \Rightarrow P \to R$ {3,5}
7	$p \wedge s$	Rule P
8	p	Rule T: $P \wedge Q \Rightarrow P$ {7}
9	s	Rule T: $P \wedge Q \Rightarrow Q$ {7}
10	$\neg s$	Rule T: $P, P \to Q \Rightarrow Q$ {7,9}
11	$\neg s \wedge s$	Rule T: $P, Q \Rightarrow P \wedge Q$ {9,10}
12	F	Rule T: Negation law {11}

$\therefore$ The given set of premises are inconsistent.

4. Using indirect method of proof,
$p \to \neg s$ from the premises $p \to (q \vee r)$,
$q \to \neg p$, $s \to \neg r$ and p.

Solution:

Premises: $p \to (q \vee r)$, $q \to \neg p$, $s \to \neg r$ and p
Conclusion: $p \to \neg s$

Additional premises: $\neg(\text{conclusion})$

$$= \neg(p \to \neg s)$$

$$= \neg(\neg p \vee \neg s)$$

$$= p \wedge s$$

Steps	Premise	Reason
1	$p \wedge s$	Rule P
2	p	Rule T: $P \wedge Q \Rightarrow P$ {1}

3	s	Rule T: $P \wedge Q \Rightarrow Q$ {1}
4	$p \to (q \vee r)$	Rule P
5	$q \vee r$	Rule T: $P, P \to Q \Rightarrow Q$ {2,4 }
6	$s \to \neg r$	Rule P
7	$\neg r$	Rule T: $P, P \to Q \Rightarrow Q$ {3,6 }
8	q	Rule T: $\neg P, P \vee Q \Rightarrow Q$ {5,7}
9	$q \to \neg p$	Rule P
10	$\neg p$	Rule T: $P, P \to Q \Rightarrow Q$ {8,9}
11	p	Rule P
12	$p \wedge \neg p$	Rule T: $P, Q \Rightarrow P \wedge Q$ {10,11}
13	F	Rule T: Negation law {11}

$\therefore p \to \neg s$ is a valid conclusion.

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Topic 10: CP Rule

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1) . Show that $R \to S$ can be derived from the premises $P \to (Q \to S)$, $\neg R \vee P$ and Q.

Solution: Premises: $P \to (Q \to S)$, $\neg R \vee P$ and Q

Conclusion: $R \to S$

Additional premises: R

Steps	Premise	Reason
1	R	Rule P (Additional premise)
2	$\neg R \vee P$	Rule P
3	$R \to P$	Rule T: Conditional as disjunction law {2}
4	P	Rule T: $P, P \to Q => Q$ {1,3}
5	$P \to (Q \to S)$	Rule P
6	$Q \to S$	Rule T: $P, P \to Q => Q$ {4,5}

7	Q	Rule P
8	S	Rule T: $P, P\rightarrow Q => Q$ {6,7}
9	$R\rightarrow S$	Rule CP

$\therefore R \rightarrow S$ is a valid conclusion.

2.Show that the hypothesis 'If you send me an e-mail message then I will finish writing the program', If you do not send me an e-mail message, then I will go to sleep early' and if I go to sleep early, then I will wake feeling refreshed' lead to the conclusion 'if I do not finish writing the program, then I will wake feeling refreshed'.

Solution: Let the proposition be

P: you send me an e-mail message

$\neg p$: you do not send e-mail message

q: I will finish writing the program

$\neg q$: I will not finish writing the program

r: I will go to sleep early

s: I will wake up feeling refreshed

Premises: $p\rightarrow q, \neg p\rightarrow r, r\rightarrow \neg s$

Conclusion: $\neg q\rightarrow s$

Additional premises: $\neg q$

Steps	Premise	Reason
1	$\neg q$	Rule P (Additional premise)
2	$p \rightarrow q$	Rule P
3	$\neg P$	Rule T: $\neg Q, P \rightarrow Q => \neg P$ {1,2}
4	$\neg p \rightarrow r$	Rule P
5	r	Rule T: $P, P\rightarrow Q => Q$ {3,4}
6	$r\rightarrow \neg s$	Rule P
7	$\neg s$	Rule T: $P, P\rightarrow Q => Q$ {5,6}
8	$p \rightarrow \neg s$	Rule CP

$\therefore p \rightarrow \neg s$ is a valid conclusion.

3.Show that the following argument is valid. If Mohan is a lawyer then he is ambitious. If Mohan is early riser then he does not like idlies. If Mohan is ambitious, then he is an early riser. Then 'if Mohan is a lawyer, then he does not likeidlies.

Solution: Let the proposition be

P: Mohan is a lawyer

q: Mohan is ambitious

r: Mohan is an early riser

s: Mohan like idlies

$\neg s$: Mohan does not like idlies

Premises: $p \to q,\ r \to \neg s,\ q \to r$

Conclusion: $p \to \neg s$

Additional premises: p

Steps	Premise	Reason
1	p	Rule P (Additional premise)
2	$p \to q$	Rule P
3	q	Rule T: $P, P \to Q => Q$ {1,2}
4	$q \to r$	Rule P
5	r	Rule T: $P, P \to Q => Q$ {3,4}
6	$r \to \neg s$	Rule P
7	$\neg s$	Rule T: $P, P \to Q => Q$ {5,6}
8	$p \to \neg s$	Rule CP

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Topic 11,12,13 : PREDICATES & QUANTIFIERS

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Predicate

A part of a declarative sentence that attributes a property to the subject. In otherwords, A predicate is a sentence depending on variables which becomes a statement upon substituting values in the domain.

Propositional function

The combination of a variable and a predicate is called propositional function and it is denoted by $P(x)$

Once a value has been assigned to a variable x then $P(x)$ becomes a proposition and has a truth value.

Example 1. Let $P(x)$ denote the statement " x >3 ". What are the truth values of $P(4)$ and $P(2)$

Solution $P(x)$: x >3

$P(4)$: 4 >3 which is true

$P(2)$: 2 >3 which is false

Example 2. Let $Q(x, y)$ denote "$x = y + 3$" what are the truth values of the proposition $Q(1,2)$ and $Q(3,0)$

Solution $Q(x, y)$: $x = y + 3$

$Q(1,2)$: $1 = 2 + 3 \quad \Longrightarrow 1 = 5$ which is false

$Q(3,0)$: $3 = 0 + 3 \quad \Longrightarrow 3 = 3$ which is true

Example 3. Let $R(x, y, z)$ denote "x + y = z" what are the truth values for the proposition $R\ 1,2,3$, $R(0,1,1)$, $R(-2, -1,4)$

Solution $R(x, y, z)$: x + y = z

$R(1, 2, 3)$: $1+ 2 = 3 \implies 3 = 3$ which is true

$R(0,1, 1)$: $0+ 1 = 1 \implies 1 = 1$ which is true

$R(-2, -1, 4)$: $-2 - 1 = 4 \implies -3 = 4$ which is false

Quantifier

There are two types namely (i) Universal quantifier and (ii) Existential quantifier

(i) Universal Quantifier

Let $P(x)$ be a propositional function. "For every x" or "for all x" is called the universal quantifier and is denoted by ∀x or x . ∀ x$P(x)$means that the proposition is true for all x .

The notation ∀ x$P(x)$ denote universal quantifier of $P(x)$. In English the word All, for all, each, every, everything are used in universal quantification

(ii) Existential Quantifier

Let $P(x)$ be a propositional function. "There exists a x" or "There exists some x" is called as the existential quantifier and is denoted by ∃x. ∃x$P(x)$means that the proposition is true for some *x*.

The notation ∃x$P(x)$ denote existential quantifier of $P(x)$.

In English the word some, few, there is, there exist atleast one are used in universal quantifier.

Example1. Consider "all scents have pleasant fragrance".

Solution:

Let $S(x)$: x is a scent.

$F(x)$: x has pleasant fragrance.

$\forall x\, S(x) \longrightarrow F(x)$.

Example2. Some students are intelligent.

Solution Let $S(x)$: x is a student.

$F(x)$: x is intelligent.

$\exists x\, S(x) \wedge F(x)$ is the symbolic representation of the given statement.

Note: $\exists x$ is the negation of $\forall x$ and

$\forall x$ is the negation of $\exists x$.

S. No	Rule	Rule
1	Universal Specification (US)	$\forall x\, p(x) => P(y)$
2	Existential Specification (ES)	$\exists x\, p(x) => P(y)$
3	UniversalGeneralization (UG)	$P(x) => \forall y\, p(y)$
4	Existential Generalization (EG)	$P(x) => \exists y\, p(y)$

Problems:

1. Show that "All men are mortal", "Socrates is a man" . Therefore Socrates is a mortal

Solution Let us use the notations

$H(x)$: x is a man $M(x)$: x is a mortal

s: Socrates

With these symbolic notations, the problem becomes

$\forall x\,((H(x) \to M(x))\,,\, H(s) \Rightarrow M(s)$

Step	Statement	Reason
1	$\forall x\,((H(x) \to M(x))$	Rule P
2	$H(s) \to M(s)$	Rule US
3	$H(s)$	Rule P
4	$M(s)$	Rule T: P , $P \to Q => Q$ {2,3}

2. Show, by indirect method $\forall x\;(P(x) \vee Q(x)) \Rightarrow \forall x P(x) \vee \exists x Q(x)$

Let us assume that $\neg(\forall x P(x) \vee \exists x Q(x))$ as an additional premise and prove a contradiction

Solution

Step	Statement	Reason
1	$\neg(\forall x\, P(x) \vee \exists x\, Q(x))$	Rule P(additional)
2	$\neg \forall x\, P(x) \wedge \neg \exists x\, Q(x)$	Rule T :{1}, De Morgan's law
3	$\neg \forall x\, P(x)$	{2}, $P \wedge Q => P$
4	$\neg \exists x\, Q(x)$	{2}, : $P \wedge Q => Q$
5	$\exists x \neg P(x)$	{3}, negation of quantifier
6	$\forall x \neg Q(x)$	{4}, negation of quantifier
7	$\neg P(y)$	{5}, Rule ES
8	$\neg Q(y)$	{6}, Rule US
9	$\neg P(y) \wedge \neg Q(y)$	{7,8}, $P,Q => P \wedge Q$
10	$\neg(P(y) \vee Q(y))$	{9}, De Morgan's law
11	$\forall x\,(P(x) \vee Q(x))$	Rule P
12	$P(y) \vee Q(y)$	{11}, Rule US
13	$P(y) \vee Q(y) \wedge \neg(P(y) \vee Q(y))$	{10,12},P,Q => $P \wedge Q$
14	False	{13}, Negation law

3. Prove that $\forall x\, P(x) \to Q(y) \wedge R(x)$, $\exists x P(x) \Rightarrow Q(y) \wedge \exists x\, P(x) \wedge R(x)$

Solution

Step	Statement	Reason
1	$\forall x\, P(x) \to Q(y) \wedge R(x)$	Rule P
2	$P(z) \to Q(y) \wedge R(z)$	{1}, Rule US
3	$\exists x P(x)$	Rule P
4	$P(z)$	{3}, Rule ES
5	$Q(y) \wedge R(z)$	{2,4}, : P , P→ Q = > Q
6	$Q(y)$	{5}, P ∧ Q => P
7	$R(z)$	{5}, P ∧ Q => Q
8	$P(z) \wedge R(z)$	{4, 7}, P,Q => P ∧ Q
9	$\exists x\, P(x) \wedge R(x)$	{8}, Rule EG
10	$Q(y) \wedge \exists x\, P(x) \wedge R(x)$	{6,9}, P,Q => P ∧ Q

4. Show that the conclusion $\forall x\, P(x) \to \neg\, Q(x)$ from premisis

$\exists x\, P(x) \wedge Q(x) \to \forall y\, R(y) \to S(y)$ and $\exists y\, R(y) \wedge \neg S(y)$

Solution

Step	Statement	Reason
1	$\exists y\, R(y) \wedge \neg S(y)$	Rule P
2	$R(z) \wedge \neg S(z)$	{1}, Rule ES
3	$\neg[\neg R(z) \vee S(z)]$ $\neg[R(z) \to S(z)]$	{2}, demorgan & conditional as disjunction law
4	$\exists y\, \neg [R(y) \to S(y)]$	{3}, Rule EG
5	$\neg \forall y\, [R(y) \to S(y)]$	{4}, negation of quantifier
6	$\exists x\, P(x) \wedge Q(x) \to \forall y\, R(y) \to S(y)$	Rule P
7	$\neg \exists x\, P(x) \wedge Q(x)$	{5,6}, ¬Q, P → Q=>¬P
8	$\forall x\, \neg [P(x) \wedge Q(x)]$	{7}, negation of quantifier
9	$\neg\, [P(z) \wedge Q(z)]$	{8}, Rule US
10	$\neg P(z) \vee \neg Q(z)$	{9}, De Morgan's law
11	$P(z) \to \neg Q(z)$	{10}, conditional as disjunction law
12	$\forall x\, P(x) \to \neg\, Q(x)$	{11}, Rule UG

5. Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and " Mala is a student in this class imply the conclusion " Mala has taken a course in computer science"

Solution

$P(x)$: x is a student in this discrete mathematics class

$P(m)$: Mala is a student in this class

$Q(x)$: x has taken a course in computer science

$Q(m)$: Mala has taken a courseincomputerscience

Given premises $\forall x\, P(x) \rightarrow Q(x)$, $P(m)$

Conclusion: $Q(m)$

Step	Statement	Reason
1	$\forall x\, P(x) \rightarrow Q(x)$	Rule P
2	$P(m) \rightarrow Q(m)$	{1}, Rule US
3	$P(m)$	Rule P
4	$Q(m)$	{2,3}, : P , P→ Q = > Q

6. Show that $\forall x\, P(x) \longrightarrow Q(x) \wedge \forall x\, Q(x) \longrightarrow R(x) \Longrightarrow \forall x\, P(x) \longrightarrow R(x)$

Solution

Rewrite the question as

$\forall x\, P(x) \longrightarrow Q(x)$, $\forall x\, Q(x) \longrightarrow R(x) \Longrightarrow \forall x\, P(x) \longrightarrow R(x)$

Step	Statement	Reason
1	$\forall x\, P(x) \longrightarrow Q(x)$	Rule P
2	$P(y) \longrightarrow Q(y)$	Rule US, {1}
3	$\forall x\, Q(x) \longrightarrow R(x)$	Rule P
4	$Q(y) \longrightarrow R(y)$	Rule US, {3}
5	$P(y) \longrightarrow R(y)$	{2,4}, P → Q, Q →R=> P →R
6	$\forall x\, P\, x \longrightarrow R(x)$	{5}, Rule UG

7. Using CP or otherwise obtain the following implication

$\forall x\, P(x) \longrightarrow Q(x)$, $\forall x\, R(x) \longrightarrow \neg Q(x) \Longrightarrow \forall x\, R(x) \longrightarrow \neg P(x)$

Solution

Step	Statement	Reason
1	$\forall x\, P(x) \longrightarrow Q(x)$	Rule P
2	$P(y) \longrightarrow Q(y)$	{1}, Rule US
3	$\forall x\, R(x) \longrightarrow \neg Q(x)$	Rule P
4	$R(y) \longrightarrow \neg Q(y)$	{3}, Rule US
5	$Q(y) \longrightarrow \neg R(y)$	{4}, contrapositive law
6	$P(y) \longrightarrow \neg R(y)$	{2,5}, $P \rightarrow Q,\ Q \rightarrow R \Rightarrow P \rightarrow R$
7	$R(y) \longrightarrow \neg P(y)$	{6}, contrapositive law
8	$\forall x\, R(x) \longrightarrow \neg P(x)$	{7}, Rule UG

8. Prove that $\exists x\, M(x)$ follows logically from the premises $\forall x\ H(x) \longrightarrow M(x)$ and $\exists x\, H(x)$

Solution

Step	Statement	Reason
1	$\exists x\, H(x)$	Rule P
2	$H(y)$	{1}, Rule ES
3	$\forall x\, H(x) \longrightarrow M(x)$	Rule P
4	$H(y) \longrightarrow M(y)$	{3}, Rule US
5	$M(y)$	{2,4}, $P,\ P \rightarrow Q \Rightarrow Q$
6	$\exists x\, M(x)$	{5}, Rule EG

9. Prove that $\exists x\, P(x) \wedge Q(x) \Longrightarrow \exists x\, P(x) \wedge \exists x\, Q(x)$

Solution

Step	Statement	Reason
1	$\exists x\, P(x) \wedge Q(x)$	Rule P
2	$P(y) \wedge Q(y)$	{1},Rule ES
3	$P(y)$	{2}, $P \wedge Q \Rightarrow P$
4	$Q(y)$	{3}, $P \wedge Q \Rightarrow Q$
5	$\exists x\, P(x)$	{3}, Rule EG
6	$\exists x\, Q(x)$	{4}, Rule EG
7	$\exists x\, P(x) \wedge \exists x\, Q(x)$	{ 5,6}, P,Q => $P \wedge Q$

10. Prove that $\exists x\, P(x) \wedge S(x)$, $\forall x\, P(x) \longrightarrow R(x) \Longrightarrow \exists x\, R(x) \wedge S(x)$

Solution

Step	Statement	Reason
1	$\exists x\, P(x) \wedge S(x)$	Rule P
2	$P(y) \wedge S(y)$	{1}, Rule ES
3	$S(y)$	{2}, $P \wedge Q \Rightarrow Q$
4	$\forall x\, P(x) \longrightarrow R(x)$	Rule P
5	$P(y) \longrightarrow R(y)$	{4}, Rule US
6	$P(y)$	{2}, $P \wedge Q \Rightarrow P$
7	$R(y)$	{5,6}, $P, P \rightarrow Q \Rightarrow Q$
8	$R(y) \wedge S(y)$	{3,7}, P,Q => $P \wedge Q$
9	$\exists x\, R(x) \wedge S(x)$	{8}, Rule EG

11. By indirect method, prove that $\forall x\, P(x) \to Q(x)$, $\exists x\, P(x) \Longrightarrow \exists x\, Q(x)$

Solution

We assume that contrary and come to contradiction

Assume $\neg\ (\exists x\, Q(x))$

Step	Statement	Reason
1	$\exists x\, P(x)$	Rule P
2	$P(y)$	{1},Rule ES
3	$\neg\ (\exists x\, Q(x))$	Rule P
4	$\forall x \neg Q(x)$	{3}, negation of quantifier
5	$\neg Q(y)$	{4}, Rule US
6	$\forall x\, P(x) \to Q(x)$	Rule P
7	$P(y) \to Q(y)$	{6}, Rule US
8	$P(y) \wedge \neg Q(y)$	{2,5}, P,Q => $P \wedge Q$
9	$\neg\ (\neg P(y) \vee Q(y))$	{7,8}, demorgan law
10	$\neg\ (P(y) \to Q(y))$	{9}, conditional as disjunction law
11	$(P(y) \to Q(y) \wedge \neg\ (P(y) \to Q(y))$	{7,10}, P,Q => $P \wedge Q$
12	False	{11}, negation law

Thus we come to a contradiction.

12. Show that the premises " One student in this class knows how to write programs in Java" and "everyone who knows how to write programs in Java can get high paying job" imply the conclusion someone in this class can get a high paying job.

Solution

Let $C\,x$: x is in the class

$J\,x$: x knows Java programming

$H\,x$: x can get a high paying Job

Then the given premises are $\exists x\, C(x) \wedge J(x))$ and $\forall x\, J(x) \to H(x)$

Therefore the conclusion is $\exists x\, C(x) \wedge H(x)$

Step	Statement	Reason
1	$\exists x\, C(x) \wedge J(x))$	Rule P
2	$C\,(y) \wedge J(y)$	{1}, Rule ES
3	$C(y)$	{2} $P \wedge Q \Rightarrow P$
4	$J(y)$	{2}, $P \wedge Q \Rightarrow Q$
5	$\forall x\, J(x) \rightarrow H(x))$	Rule P
6	$J(y) \rightarrow H(y)$	{5},Rule US
7	$H(y)$	{4,6} $P, P \rightarrow Q \Rightarrow Q$
8	$C(y) \wedge H(y)$	{3,7} $P, Q \Rightarrow P \wedge Q$
9	$\exists x\, C(x) \wedge H(x))$	{8}, Rule EG

13. Show that $\sim P\,(a, b)$ follows logically from $\forall x\, \forall y\, P(x, y) \rightarrow W(x, y)$ and $\sim\ (a,b)$ Note: [negation : $\sim$ or $\neg$]

Solution

Step	Statement	Reason
1	$\forall x\, \forall y\, P(x, y) \rightarrow W(x, y)$	Rule P
2	$\forall y\, P(a, y) \rightarrow W(a, y)$	{1},Universal Specification
3	$P(a, b) \rightarrow W(a, b)$	{2}, Universal Specification
4	$\sim W(a, b) \rightarrow \sim P(a, b)$	{3},Contrapositive law
5	$\sim W(a, b)$	Rule P
6	$\sim P(a, b)$	Rule T,{4,5}, $P, P \rightarrow Q \Rightarrow Q$

14. Show that $\forall x\, P(x) \vee Q(x))$, $\forall x \neg P(x) \Longrightarrow \exists x\, Q(x)$

Solution

Step	Statement	Reason
1	$\forall x\, P(x) \vee Q(x))$	Rule P
2	$P(y) \vee Q(y)$	{1},Rule US
3	$\forall x \neg P(x)$	Rule P
4	$\neg\ P(y)$	{3}, Rule US
5	$Q(y)$	{2, 4}, $\neg P, P \vee Q \Rightarrow Q$
6	$\exists x\, Q(x)$	{5},Rule EG

15. To show that the premises " A student in this class has not read the book" and "Everyone in this class passed the first exam" imply the conclusion "Someone Who passed the first exam has not read the book"

Solution

$P(x)$: x is a student in this class

$Q(x)$: x has read the book

$R(x)$: x passed the first exam

Premises: $\exists x\, P(x) \wedge \neg Q(x))$, $\forall x\, P(x) \rightarrow R(x)$

Conclusion: $\exists x\, R(x) \wedge \neg Q(x))$

Step	Statement	Reason
1	$\exists x\, P(x) \wedge \neg Q(x))$	Rule P
2	$P\,(y) \wedge \neg Q(y)$	{1}, Rule ES
3	$P\,(y)$	{2}, $P \wedge Q \Rightarrow P$
4	$\forall x\, P(x) \rightarrow R(x)$	Rule P
5	$P(y) \rightarrow R(y)$	{4},Rule US
6	$R(y)$	{3,5}, $P,\ P \rightarrow Q \Rightarrow Q$
7	$\neg Q(y)$	{2}, $P \wedge Q \Rightarrow Q$
8	$R\,(y) \wedge \neg Q(y)$	{6,7}, $P, Q \Rightarrow P \wedge Q$
9	$\exists x\, R(x) \wedge \neg Q(x)$	{8}, Rule EG

x..x

Introduction to Proofs:

Proof: A proof is a valid argument that establishes the truth of a mathematical statement.

TYPES OF PROOFS:

Direct Proofs:

A direct proof shows that a condiditonal statement $p \to q$ is true by showing that if p is true then q must also be true

Indirect proof:

In indirect proof of $p \to q$ we take $\neg q$ as a hypothesis and using axioms, definition together with rules of reference show that $\neg p$ must follow

Vacuous Proof

To show that p is false that proof is called vacuous proof of the conditional statement.

Trivial Proof:

A proof of $p \to q$ that uses the fact q is true is called a trivial proof

Proof by contradiction:

In proof by contraiction of $p \to q$ assume $\neg q \to \neg p$ to show that $\neg p$ is true

Problems

1. Prove $\sqrt{2}$ is irrational by giving a proof by contradiction

Solution: : $\sqrt{2}$ is irrational

Assume $\sqrt{2}$ is rational

If $\sqrt{2}$ is rational , then there exist integers 'a' and 'b' the $\sqrt{2} = a/b$

where a and b do not have common factor...(1)

Now $\sqrt{2} = \frac{a}{b}$

On squaring , $2b^2 = a^2$ which gives a^2 is even

a^2 is even implies a is even then $a = 2c$

$2b^2 = 4c^2 \Longrightarrow b^2 = 2c^2$ that means b^2 is even

Again using the fact that if the square of an integer is even then the integer must be even $\therefore$ b is even

$$\sqrt{2} = \frac{a}{b}$$

'a' and 'b' have common number '2' which gives the contradiction to (1)

$\therefore$ our assumption $\sqrt{2}$ is rational is wrong. Hence $\sqrt{2}$ is irrational

2. Prove that if n is a positive integer then n is odd if and only if 5n + 6 is odd.

Solution:

Case (i): Assume n is odd.

Let n = 2k + 1 where k is a positive integer

$\therefore$ 5n + 6 = 5(2k + 1) + 6

= 10k + 11 = 2(5k + 5) + 1 which is an odd number.

Hence if n is odd then 5n + 6 is odd

Case (ii): Assume n is even.

Let n be even. i.e n = 2k where k is a positive integer.

Then 5n + 6 = 5(2k) + 6 = 2 (5k + 3) which is always even.

Thus 5n + 6 is odd if and only if n is odd**.**

3. Prove that square of an even number is an even number by (i) direct method (ii) indirect method and (iii) proof by contradiction

Solution:

(i)Direct proof: ($p \rightarrow q$)

Let n be even i.e. n = 2k, where k is an integer.

$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$ = an even number.

(ii)Indirect proof: ($\neg q \rightarrow \neg p$)

To prove : if n is odd then n^2 is odd

Let n be odd. i.e. $n = 2k - 1$

$\therefore n^2 = (2k-1)^2 = 4k^2 - 4k + 1$

$= 2(2k^2 - 2k) + 1$ = odd number

Hence if n is odd then n^2 is odd (Or) if n is even then n^2 is even.

(iii) Proof by contradiction:

Let q be F then if $\neg p \rightarrow q$ is T implies $\neg p$ is F or p is T.

Assume n^2 to be even when n is odd.

But if n is even we have proved that n^2 is even by the indirect method. Hence if n^2 is even then n is even and our assumption that n^2 is even when n is odd is wrong. So, if n is even then n^2 is even.

x..x