UNIT-3 GRAPH THEORY

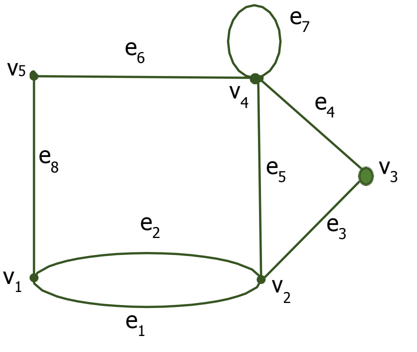
Topic 1: BASIC DEFINITIONS

1) Define GRAPH with example.

A graph G = (V, E) consists of a non-empty set V, called the set of **vertices** (nodes, points) and a set E of ordered or unordered pairs of elements of V, called the set of **edges**, such that there is a mapping from the set E to the set of ordered or unordered pairs of elements of V.

Example:

Figure 1.



1) Define SELF LOOP with example :

An edge of a graph that joins a vertex to itself is called a loop. The direction of a loop is not significant, as the initial and terminal nodes are one and the same.

Example: In Figure 1, the edge e_7 is called Self loop or simply loop.

2) Define PARALLEL EDGES with example :

If, in a directed or undirected graph, certain pairs of vertices are joined by more than one edge, such edges are called parallel edges. In the case of directed edges, the two possible edges between a pair of vertices which are opposite in direction are considered distinct.

Example: In Figure 1, the edges e_1 and e_2 are called parallel edges.

3) Define ADJACENT EDGES AND VERTICES with example:

Two edges are said to be adjacent if they are incident on a common vertex.
Example: In Figure 1, the edge e_6 and e_8 are adjacent and the vertices v_1 and v_5 are adjacent vertices.

4) Define ISOLATED VERTEX & PENDANT VERTEX with example:

A node of a graph which is not adjacent to any other node (viz., which is not connected by an edge to any other node) is called an **isolated vertex (or) isolated node**.

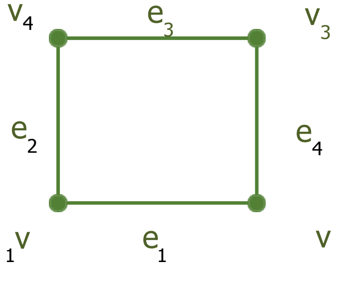
Example: In Figure 1 the vertex v_6 is isolated vertex.
A vertex of a graph is called a **pendant vertex** if only one edge is incident with it.

6. Explain the TYPES OF GRAPHS with example :

SIMPLE GRAPH

A graph, in which there is only one edge between a pair of vertices (neither self-loop nor parallel edges), is called a simple graph.

Example



PSEUDOGRAPH

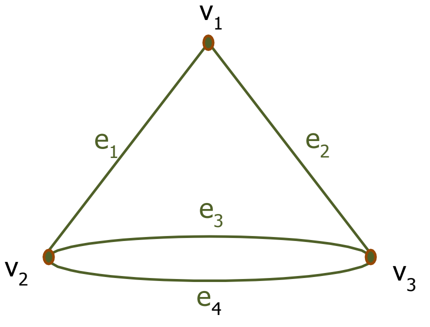
A graph in which loops and parallel edges are allowed is called a Pseudo graph. [Here e_7 is a Self-loop, edges e_1 and e_2 are parallel edges]

Example : look at the Figure 1

MULTIGRAPH

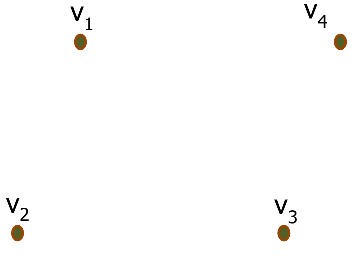
A graph which contains some parallel edges is called a multigraph.

Example



NULL GRAPH

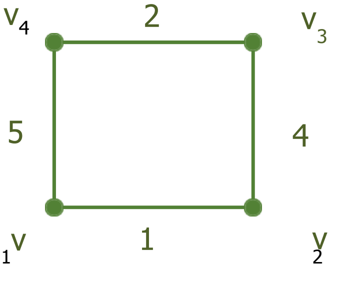
A graph containing only isolated nodes (viz. no edges) is called a **null graph.**
Example:



WEIGHTED GRAPH

Graphs in which a number (weight) is assigned to each edge are called weighted graphs.

Example



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Topic 2: Degree of the vertex

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1) Define DEGREE OF A VERTEX (Undirected Graph)with example:

Solution:

The degree of a vertex in an undirected graph is the number of edges incident with it, with self loop counted twice .The degree of a vertex v is denoted by deg (v). Clearly the degree of an **isolated vertex** is zero. If the degree of a vertex is one, it is called a **pendant vertex.**

In Figure 1

deg (v_1) = 2,
deg (v_2) =4,
deg (v_3)=2 ,deg (v_4) = 5,
deg (v_5) =4,
deg (v_6) = 0
We note that v_6 is an isolated vertex

Example:

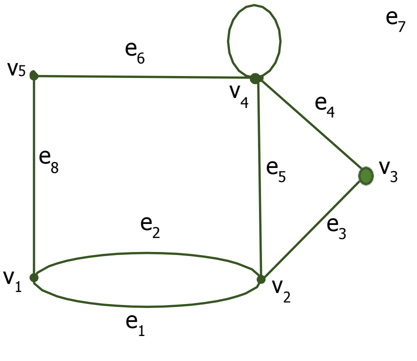


Figure 1.

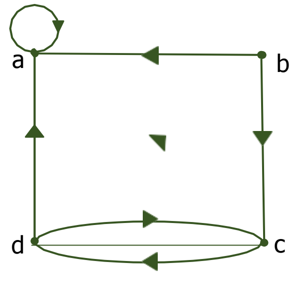
2) Define INDEGREE & OUT DEGREE (Directed Graph or Digraph) with example:

Solution:

In a directed graph, the number of edges with v as their terminal vertex (i.e., the number of edges that converge at v) is called the **in-degree** of v and denoted as $\deg^{-}(v)$.

The number of edges with v as their initial vertex, (i.e., the number of edges that emanate from v) is called the **out-degree** of v and is denoted as $\deg^{+}(v)$.

Let us consider the following directed graph.



In degree of v	Out degree of v	Total degree of v
$\deg^{-}(a)$ =3	$\deg^{+}(a)$ =1	deg(a)=4
$\deg^{-}(b)$=1	$\deg^{+}(b)$ =2	deg(b)=3
$\deg^{-}(c)$ =2	$\deg^{+}(c)$ =1	deg(c)=3
$\deg^{-}(d)$ =1	$\deg^{+}(d)$ =3	deg(d)=4

x..x

Topic 3: Theorems on Graphs

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Theorem 1 (The Handshaking theorem)

Statement:

If G = (V, E) is an undirected graph with e edges, then

$$\sum_i \deg(v_i) = 2e .$$

(i.e.) the sum of the degrees of all the vertices of an undirected graph is twice the number of edges of the graph and hence even.

Proof:

Since every edge is incident with exactly two vertices, every edge contributes 2 to the sum of the degree of the vertices.

$\therefore$ All the e edges contribute (2e) to the sum of the degrees of the vertices (i.e.)

$$\sum_i \deg(v_i) = 2e .$$

Theorem 2: The number of vertices of odd degree in an undirected graph is even.

Proof:

Let G = (V, E) be the undirected graph.

Let V_1 and V_2 and be the sets of vertices of G of even and odd degrees respectively.

Then, by the previous theorem, (The Handshaking theorem)

$$\sum_{v_i \in V_1} \deg(v_i) + \sum_{v_j \in V_2} \deg(v_j) = 2e \text{-------------------------(1)}$$

Since each deg(V_i) is even, $\sum_{v_i \in V_1} \deg(v_i)$ is even = 2k

As the L.H.S. of (I) is even, we get $\sum_{v_j \in V_2} \deg(v_j)$=2e -2k =2(e-k)

Since each deg(V_j) is odd, the number of terms contained in $\sum_{v_j \in V_2} \deg(v_j)$

or in V_2 is even. Hence the number of vertices of odd degree is even.

Theorem 3: The maximum number of edges in a simple graph with 'n' vertices is $\frac{n(n-1)}{2}$.

Proof: Given : G is a simple graph with |V| = n.

We know that for a simple graph with n vertices , deg (v) ≤ n – 1 for every v ∈ V.

By hand shaking theorem,

$$\sum_{i=1}^{n} \deg(v_i) = 2e$$

$$=> 2e = \deg v_1 + \deg v_2 + \ldots\ldots\ldots + \deg v_n$$

$$\leq (n-1) + (n-1) + \ldots\ldots\ldots + (n-1) \text{ (n times)}$$
$$= (n-1)n$$

$$\therefore 2e \leq (n-1)n$$

$$=> e \leq \frac{(n-1)n}{2}$$

Therefore the max. number of edges of G is $\frac{n(n-1)}{2}$

Problems:

1) How many edges are there in a graph with ten vertices each of degree six.

Solution: Let e be the number of edges of the graph.

2e = sum of all degrees = 10x6 = 60

2e = 60

i.e, e = 30 $\therefore$ There are 30 edges.

2) Can a simple graph exist with 15 vertices each of degree 5.

Solution: We know that

2e = sum of all degrees of vertices =15x5 = 75

$\therefore$ e = 75/2 , which is not possible

$\therefore$ Such a graph does not exist.

(or)

By theorem 2, in a graph the number of odd degree vertices is even. Therefore, it is not possible to have 15 vertices, which is of odd degree.

$\therefore$ Such a graph does not exist.

3) For the following degree sequences, 4, 4, 4, 3, 2 find if there exists a graph or not.

Solution: We know that

Sum of the degree of all vertices = 4+4+4+3+2 = 17 which is an odd number.

$\therefore$ Such a graph does not exist.

4) Let $\delta(G)$ and $\Delta(G)$ denote minimum and maximum degrees of all the vertices of G respectively. Then show that for a non-directed graph G is

$$\delta(G) \leq \frac{2|E|}{|V|} \leq \Delta(G).$$

Solution: Given

$\delta(G)$ = Min-degree of all vertices of G.

$\Delta(G)$= Max-degree of all vertices of G.

|V|= Number of vertices of graph G.

|E|= Number of edges of graph G.

We know δ (G) ≤ deg (v) ≤ Δ (G) for every v ∈ V

∴ deg (v) ≥ δ (G) for every v ∈ V…………………(1)
and deg (v) ≤ Δ (G) for every v ∈ V…………….(2)

Given : G is a simple graph with |V| = n and |E| = e edges.

By hand shaking theorem,

$$\sum_{i=1}^{n} \deg(v_i) = 2e$$

= > 2e = deg v_1 + deg v_2 +………………………+deg v_n

≥ δ (G) + δ (G) + δ (G) +………..+ δ (G) (n times) [Using 1]
= n δ (G)

∴ 2e ≥ n δ (G)

= > δ (G) ≤ $\frac{2e}{n}$ ………………..(3)

Also,

= > 2e = deg v_1 + deg v_2 +………………………+deg v_n

≤ Δ (G) +Δ (G)+ Δ (G)+……………..+ Δ (G) (n times) [Using 2]
= n Δ (G)

∴ 2e ≤ nΔ (G)

= > $\frac{2e}{n}$ ≤ Δ (G) ………………..(4)

From (3) & (4) , δ (G) ≤ $\frac{2e}{n}$ ≤ Δ (G)

5) If all the vertices of an undirected graph are each of degree k, show that the number of edges of the graph is a multiple of k.

Solution: Let 2n be the number of vertices of the given graph.

Let |E|= the number of edges of the given graph.

By Handshaking theorem, (Proof should be included here)

$\sum \deg(v_i) = 2 \ |E|$

Therefore $2nk = 2|E| \Rightarrow |E| = nk$

$\Rightarrow$Number of edges = Multiple of k.

Therefore, the number of edges of the given graph is a multiple of k.

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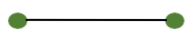
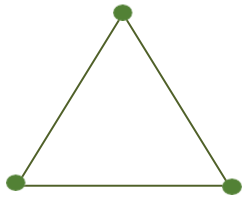
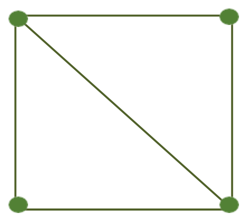
Topic :4 SPECIAL TYPES OF GRAPHS

Explain SPECIAL TYPES OF GRAPHS with Example:

1. Complete Graph:

If a vertex is connected to all other vertices in a graph, then it is called a complete graph and it is denoted by K_n (where n is a number of vertices in a graph).

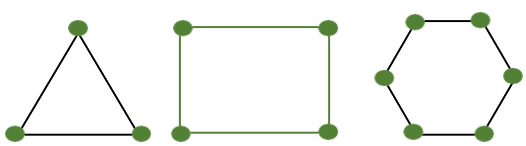
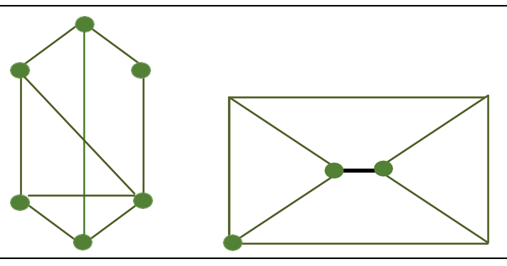
Example:

K_2	
K_3	
K_4	

2. Regular Graph:

If every vertex of a simple graph has the same degree then the graph is called regular graph (or) number of vertices and number of edges is same.

Example:

2-Regular Graphs	
3-Reguar Graphs	

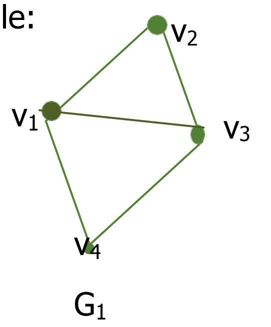
3. Sub graph:

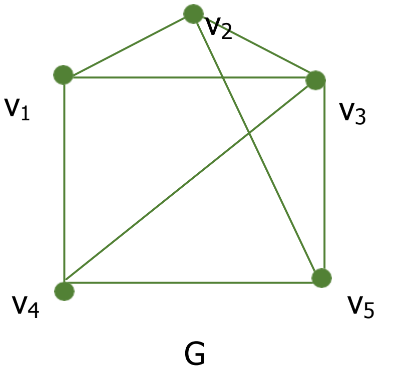
Given two graphs G and G_1, we say that G_1 is a subgraph of G if the following conditions hold.

(i) All the vertices and all the edges of G_1 are in G.

(ii) Each edge of G_1 has the same end vertices in G as in G_1.

Example:



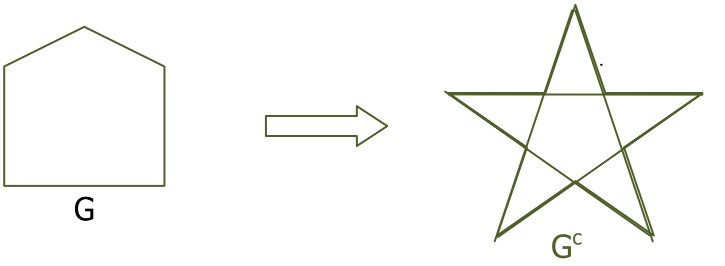


G_1 is a subgraph of G since all the vertices and all the edges of the graph G_1 are in the graph of G and that every edge in G_1 has the same end vertices in G as in G_1.

4. Complementary Graph:

The complement $\overline{G}$ of a simple graph G, has the same vertices as G. Two vertices are adjacent in $\overline{G}$ iff they are not adjacent in G.

Example:

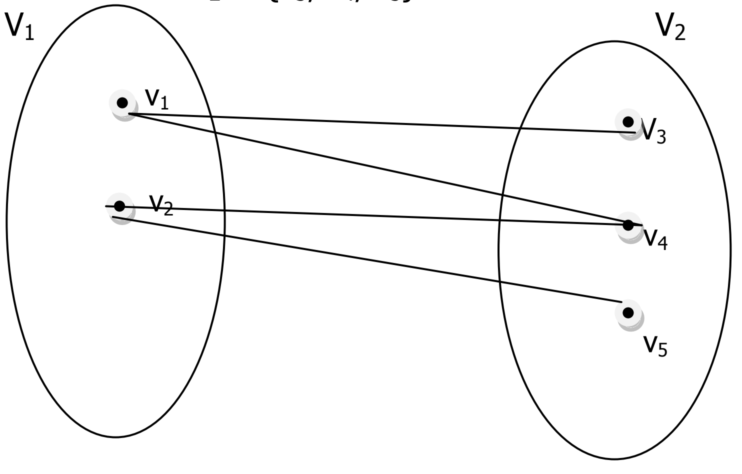


5.Bipartite Graph:

A bipartite graph is one whose vertices, V, can be divided into two independent sets, V_1 and V_2, and every edge of the graph connects one vertex in V_1 to one vertex in V_2.

Example:

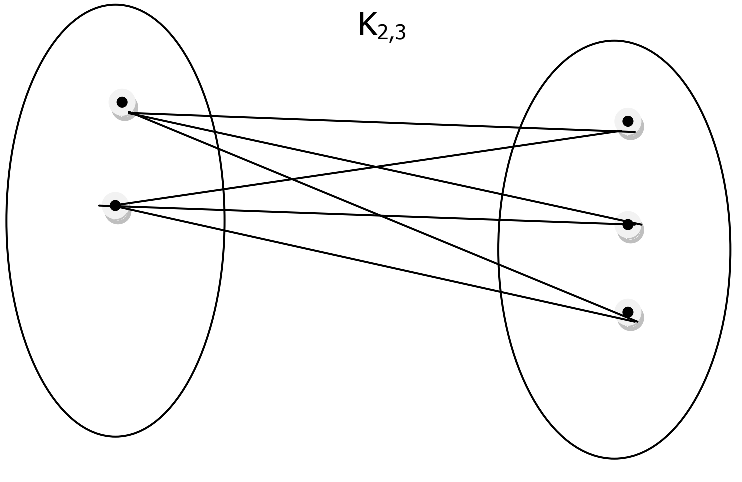
$V = \{V_1, V_2\}$
$V_1 = \{v_1, v_2\}$
$V_2 = \{v_3, v_4, v_5\}$



5. Complete Bipartite Graph:

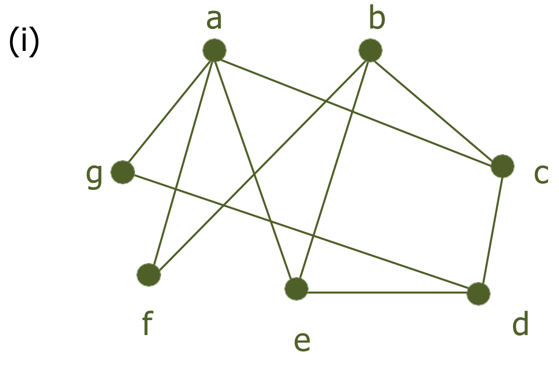
A complete bipartite graph is a bipartite graph in which every vertex of V_i is connected to every vertex of V_j. A complete bipartite graph with 'm' and 'n' vertices is denoted by $K_{m,n}$

$K_{2,3}$

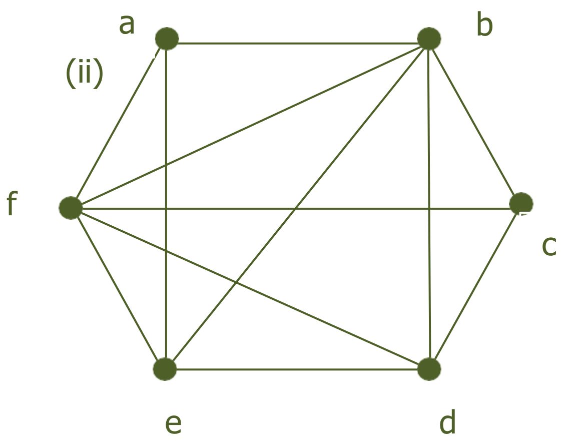


Problems:

1. Are the following graphs G and H are Bipartite?



(G)



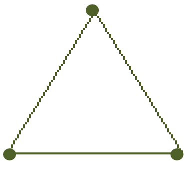
(H)

Solution: (i) Given graph is bipartite because its disjoint sets V_1={a, b, d} and

V_2={c, e, f, g} and each edge connects a vertex in one of these subsets to a vertex in the other subset.

(ii) Given graph is not bipartite because its vertex set cannot be partitioned into two subsets. So that edges do not connect two vertices from the same subset.

2. Is K_3 a bipartite?



Solution: No, the complete graph K_3 is not bipartite. If we divide the vertex set K_3 into two disjoint sets, one of the two sets must contain two vertices. If graph is bipartite, these two vertices should not be connected by an edge, but in K_3 each vertex is connected to every other vertex by an edge.

3. Show that if G is a self complementary graph then $n \equiv 0 \; or \; 1 \pmod 4$.

(or)

Any self complementary graph has 4n or (4n+1) vertices, where n is positive integer.

Proof: Given G is a self complementary graph.

$\Rightarrow$Number of edges in G = Number of edges in G^c.

$\Rightarrow$ $|E(G)| = |E(G^c)|$ (1)

Also, we know that

Total number of edges possible with n vertices

=Number of edges in G + Number of edges in G^c.

$$\Rightarrow \frac{n(n-1)}{2} = m + m = 2m.$$

$\Rightarrow$n (n-1)= 4m.

That is, n (n-1) is multiple of 4.

$\Rightarrow$ Either n (or) (n-1) is divisible by 4.

Therefore G is a self complementary graph with n vertices then $n \equiv 0 \; or \, 1 \pmod 4$.

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Topic :5 Adjacency matrix and Incidence matrix

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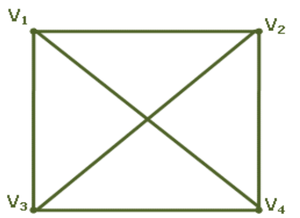
1. Define Adjacency matrix with example.

Adjacency matrix of a simple graph: Let $G = (V, E)$ be a graph with n- vertices $V_1, V_2, V_3 \dots . . V_n$. Its adjacency matrix is denoted by $A = a_{ij}$ and defined

by $A = a_{ij} = \begin{cases} 1, \text{if there exists an edge between } V_i \text{ and } V_j \\ 0, \quad \text{otherwise.} \end{cases}$

Example



	v_1	v_2	v_3	v_4
v_1	0	1	1	1
v_2	1	0	1	1
v_3	1	1	0	1
v_4	1	1	1	0

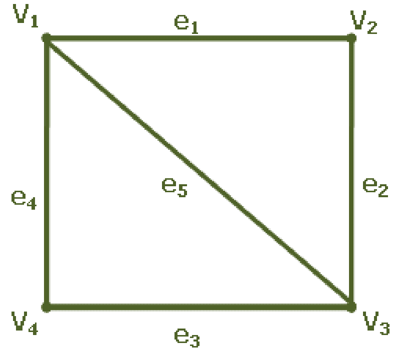
2. Define Incidence matrix with example.

Incidence matrix of a simple graph: Let $G = (V, E)$ be graph with n-vertices $V_1, V_2, V_3 \dots . . V_n$ and m-edges $e_1, e_2, e_3 \dots . . e_n$, then the matrix

$B = b_{ij} = \begin{cases} 1, \text{when edge } e_j \text{ incidence on } V_i \\ 0, \quad \text{otherwise.} \end{cases}$ is called the incidence matrix of G.

Example:

	e_1	e_2	e_3	e_4	e_5
v_1	1	0	0	1	1
v_2	1	1	0	0	0
v_3	0	1	1	0	1
v_4	0	0	1	1	0



3. Define Path matrix with Example:

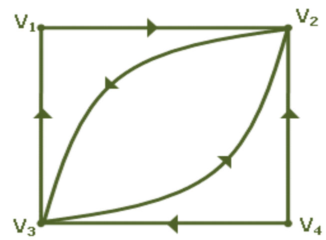
Let $G = (V, E)$ be a digraph in which $V = n$ and the nodes of G are assumed to be ordered. An $n \times n$ matrix P whose elements are given by

$$p_{ij} = \begin{cases} 1, & \text{if there exists a path from } V_i \text{ to } V_j \\ 0, & \text{otherwise.} \end{cases}$$

Problems:

1. Find the adjacency matrix of the graph given below.

(a). (b).



(c). K_4 (d). $K_{2,3}$ (e). C_4

Solution:

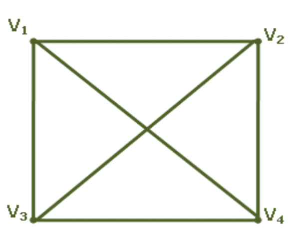
(a).

	v_1	v_2	v_3	v_4
v_1	0	1	1	1
v_2	1	0	1	0
v_3	1	1	0	0
v_4	1	0	0	0

(b).

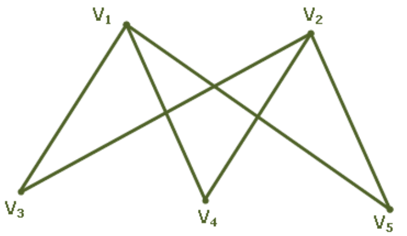
	v_1	v_2	v_3	v_4
v_1	0	1	0	0
v_2	0	0	1	0
v_3	1	1	0	0
v_4	0	1	1	0

(c) . K_4



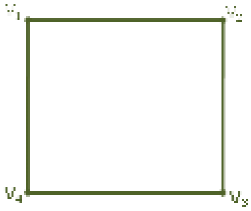
	v_1	v_2	v_3	v_4
v_1	0	1	1	1
v_2	1	0	1	1
v_3	1	1	0	1
v_4	1	1	1	0

(d). $K_{2,3}$



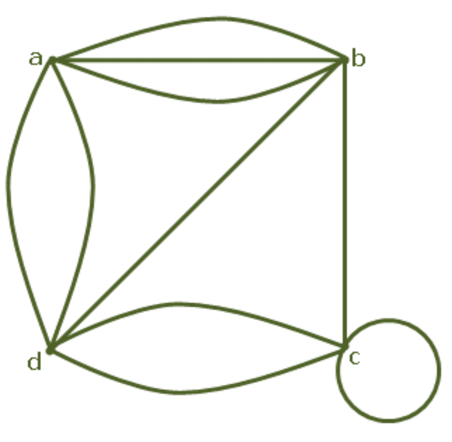
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

(e). C_4



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

2. Use an adjacency matrix to represent the pseudo graph shown below.



Solution: The adjacency matrix is

$$A = a_{ij} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{pmatrix} \end{array}$$

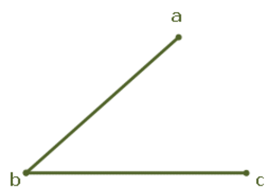
3. Draw undirected graph of the following adjacency matrix.

(i). $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ (ii). $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

Solution: (i). Let

$$\begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

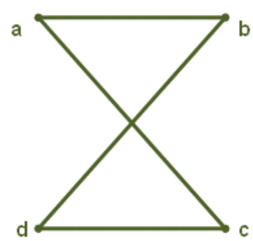
The undirected graph is



(i). Let,

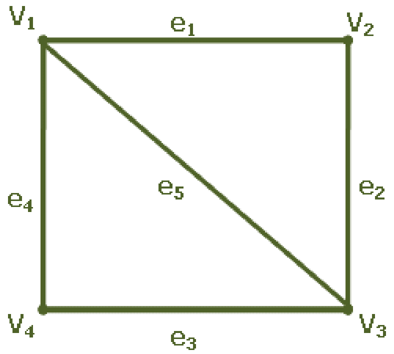
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

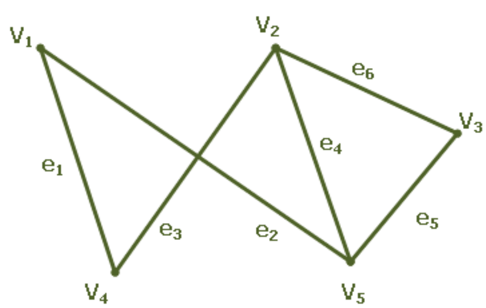
The undirected graph is



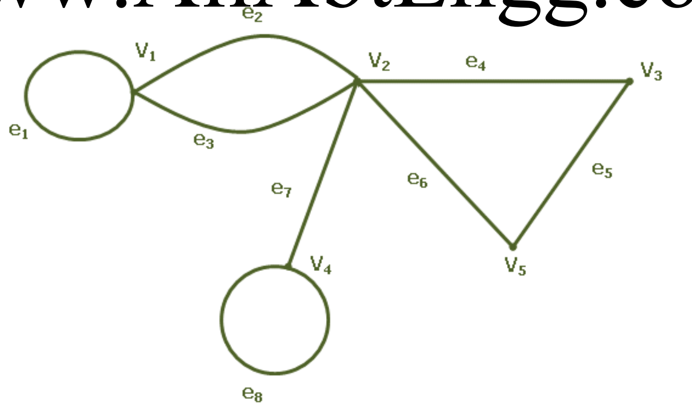
4. Find incidence matrices for the following graphs.

(i). (ii).





(iii)



Solution:

(i). The incidence matrix

$B = B_{ij} =$

	e_1	e_2	e_3	e_4	e_5
v_1	1	0	0	1	1
v_2	1	1	0	0	0
v_3	0	1	1	0	1
v_4	0	0	1	1	0

ii). The incidence matrix

$B = B_{ii} =$

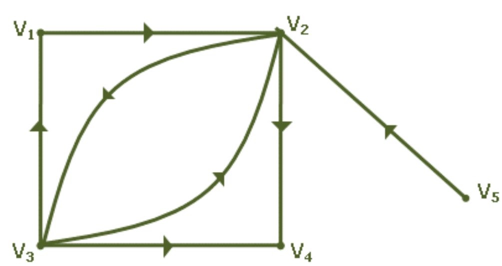
	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	1	0	0	0	0
v_2	0	0	1	1	0	1
v_3	0	0	0	0	1	1
v_4	1	0	1	0	0	0
v_5	0	1	0	1	1	0

(iii). The incidence matrix

$B = B_{ij} =$

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
v_1	1	1	1	0	0	0	0	0
v_2	0	1	1	1	0	1	1	0
v_3	0	0	0	1	1	0	0	0
v_4	0	0	0	0	0	0	1	1
v_5	0	0	0	0	1	1	0	0

5. Find the path matrix of



Solution: The path matrix is

$$P_{ij} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

{Since first row $v_1 \to v_2 \to v_3 \to v_1 \quad \therefore 1$

$v_1 \to v_2 \quad \therefore 1$

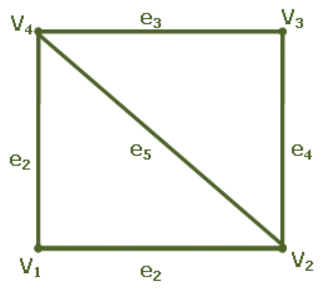
$v_1 \to v_2 \to v_3 \quad \therefore 1$

$v_1 \to v_2 \to v_4 \quad \therefore 1$

$v_1 \to v_5$ No path $\therefore 0$ }

6. Find the adjacency matrix of the following graph G. Hence find degree of each vertex. Also find A^2 and A^3. What is your observation regarding entries in A^2 and A^3.

Solution:



The adjacency matrix is

$$A = a_{ij} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

deg v_1 = sum of entries in 1^{st} row = 2

deg v_2 = sum of entries in 2^{nd} row = 3

deg v_3 = sum of entries in 3^{rd} row = 2

deg v_4 = sum of entries in 4^{th} row = 3

Now, $A^2 = A \times A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$

$$A^3 = A^2 \times A = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix} \times \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 5 & 2 & 5 \\ 5 & 4 & 5 & 5 \\ 2 & 5 & 2 & 5 \\ 5 & 5 & 5 & 4 \end{pmatrix}$$

Observation:

(i). A^2 and A^3 are symmetric matrices.

(ii). i, i^{th} entry of $A^2 = \deg v_i$.

Example: v_2, v_2 entry is $3 = \deg v_2$

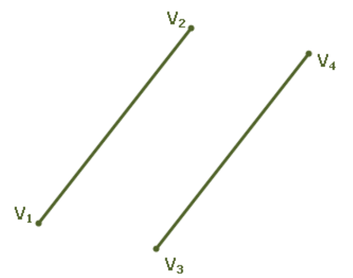
(iii). i, j^{th} entry of A^2

$= \text{Number of different paths of lengths 2 between } i^{th} \text{ and } j^{th} \text{ vertices.}$

Example: $1,3^{th}$ entry of

$A^2 = 2 =$ Number of different paths of lengths 2 between the vertices v_1 and v_3.

7. Find the adjacency matrix of the following graph G.



Solution: The adjacency matrix is

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 & v_4 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

x...x

Topic :6 & 7 Graph Isomorphism

View the video on ponjesly app

Define Graph Isomorphism:

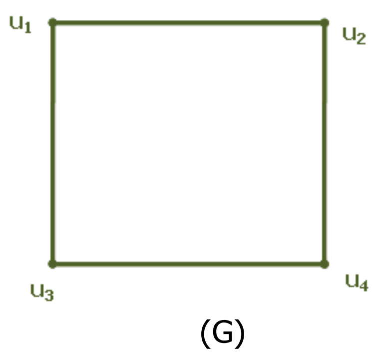
The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one to one and onto correspondence between Vertex sets which preserves adjacency of the vertices.

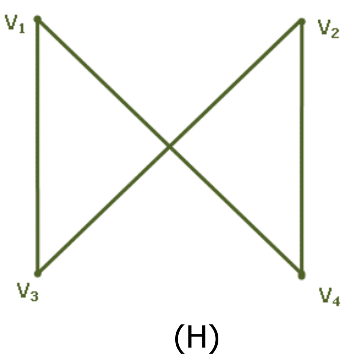
Note: If G_1 and G_2 are isomorphic then G_1 and G_2 have

1. The same number of vertices
2. The same number of edges
3. The same degree sequence.
4. The same adjacency matrices.

Problems:

1. Show that the graph G and H are displayed below are isomorphic.





Solution:

Given graphs G and H are having same number of vertices and same number of edges.

Let $u = \{u_1, u_2, u_3, u_4\}$ and $v = \{v_1, v_2, v_3, v_4\}$ are vertex sets of G and H respectively.
i.e., $u = v = 4$ and $E_1 = E_2 = 4$.

Degree of the vertices of G:
$d\, u_1 = d\, u_2 = d\, u_3 = d\, u_4 = 2$.

Degree of the vertices of H:
$d\, v_1 = d\, v_2 = d\, v_3 = d\, v_4 = 2$.

Define $f: G \rightarrow H$

Choose a vertex u_1 in G. It has degree 2 and the two adjacent vertices of 'u_1' having the degree sequence $2, 2$.

Correspondingly choose a vertex v_1 in H with same degree and same degree sequence of adjacent vertices.

$\therefore f(u_1) = v_1$

Similarly, all other vertices are having same degree and same degree sequence of the adjacent vertices from G and H.

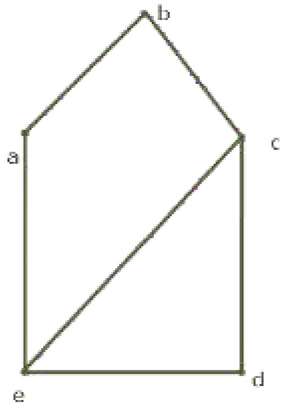
i.e.f $(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$.

Also adjacency matrices of G and H are same.

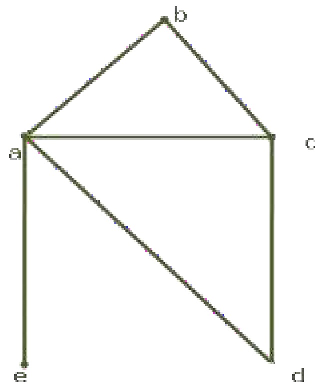
$\therefore G$ and H are isomorphic.

2.

2. Show that the graphs given below are not isomorphic.



(G)



(H)

Solution:

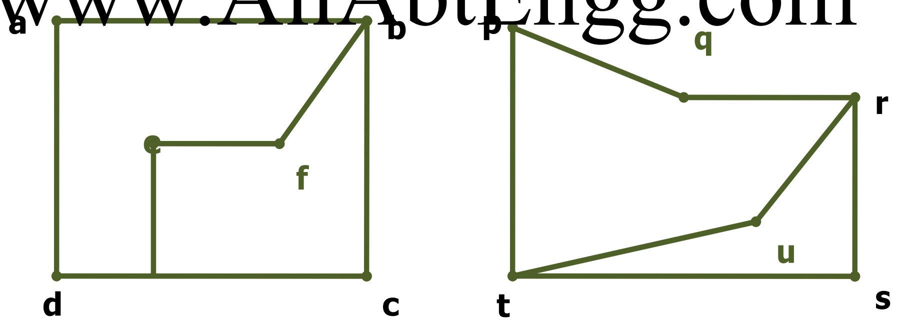
Both G and H have five vertices and six edges.

i.e., equal number of vertices and edges.

But there is no one to one correspondence between edges in G and H.
The graph G have the degree sequence $2,2,3,2,3$. But the degree sequence of H is $4,2,3,2,1$.

However H has a vertex of degree one namely 'e' whereas G has no vertices of degree one.
$\therefore G$ and H are not isomorphic.

3. Determine whether the graphs G and H given below are isomorphic.



(G) (H)

Solution:

Given graphs G and H are having same number of vertices and same number of edges.

Let $V_1 = \{a, b, c, d, e, f\}$ and $V_2 = \{p, q, r, s, t, u\}$ are vertex sets of G and H respectively.
i.e., $V_1 = V_2 = 6$ and $E_1 = E_2 = 7$.

<u>Degree of the vertices of G</u>:
$d\,a = d\,c = d\,e = d\,f = d\,d = 2$ and $d\,b = 3$.

<u>Degree of the vertices of H</u>:
$d\,p = d\,q = d\,s = d\,u = d\,t = 2$, $d\,r = 3$.

Define $f: G \to H$
Choose a vertex 'a' in G. It has degree 2 and the two adjacent vertices of 'a' having the degree sequence 3,3.

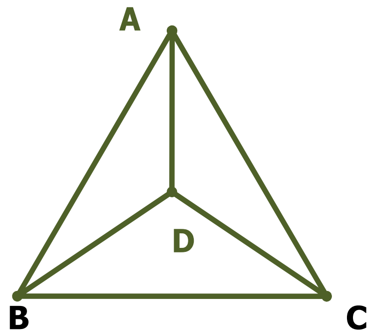
Correspondingly choose a vertex 's' in H with same degree and same degree sequence of adjacent vertices.
$\therefore f(a) = s$

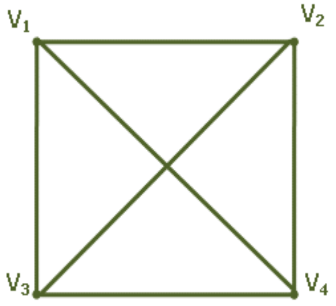
Similarly, all other vertices are having same degree and same degree sequence of the adjacent vertices from G and H.
i.e., $f(b) = r, f(c) = u, f(d) = t, f(e) = p, f(f) = q$.

Also the adjacency matrices are same.

$\therefore G$ and H are isomorphic.

4. Examine whether the following pair of graphs given below are isomorphic or not?





Solution:

Given graphs G_1 and G_2 are having same number of vertices and same number of edges.

Let $V_1 = \{A, B, C, D\}$ and $V_2 = \{v_1, v_2, v_3, v_4\}$ are vertex sets of G_1 and G_2 respectively.
i.e., $V_1 = V_2 = 4$ and $E_1 = E_2 = 6$.

Degree of the vertices of G_1:
$d\ A = d\ B = d\ C = d\ D\ = 3$.

Degree of the vertices of G_2:
$d\ v_1 = d\ v_2 = d\ v_3 = d\ v_4 = 3$.

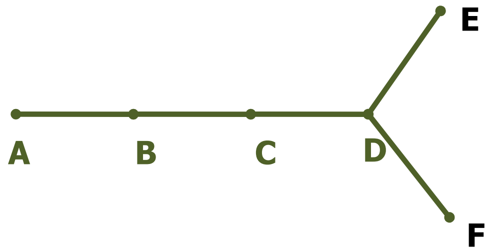
Define $f: G_1 \to G_2$
Choose a vertex 'A' in G_1. It has degree 3 and the three adjacent vertices of 'A' having the degree sequence 3,3,3.
Correspondingly choose a vertex 'v_1' in G_2 with same degree and same degree sequence of adjacent vertices.
$\therefore f\ (A) = v_1$

Similarly, all other vertices are having same degree and same degree sequence of the adjacent vertices from G_1 and G_2.
i.e.f $(B) = v_2, f(C) = v_3, f\ (D) = v_4$.

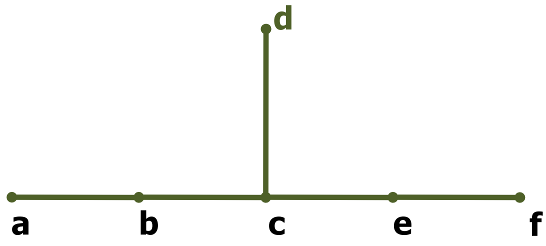
Also adjacency matrices are same

$\therefore G_1$ and G_2 are isomorphic.

5. Establish the isomorphism of the following pair of graphs.



(G)



(G′)

Solution:

Given graphs $G$and G' are having same number of vertices and same number of edges.

Let $V_1 = \{A, B, C, D, E, F\}$ and $V_2 = \{a, b, c, d, e, f\}$ are vertex sets of $G$and G' respectively.

i.e., $V_1 = V_2 = 6$ and $E_1 = E_2 = 5$.

Degree of the vertices of G:

$d\ A = d\ E = d\ F = 1, d\ B = d\ C = 2, d\ D = 3$.

Degree of the vertices of G':

$d\ a = d\ d = d\ f = 1, d\ b = d\ e = 2, d\ c = 3$.

Define $f: G \to G'$

Choose a vertex 'D' in G. It has degree 3 and the three adjacent vertices of 'D' having the degree sequence $(1,1,2)$.

But in G', a vertex 'C' has degree 3 and the three adjacent vertices of 'C' having the degree sequence $(1,2,2)$.

∴

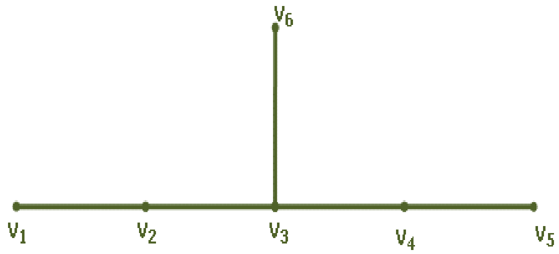
∴ The vertex 'D' in G is not mapped to any of the vertex in H.

∴

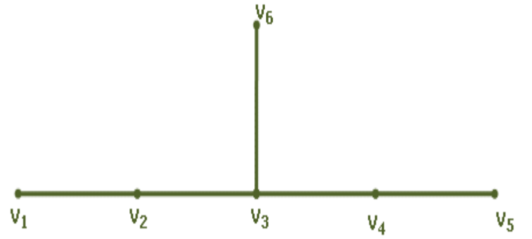
∴ Hence G and G' are not isomorphic.

∴

6. Establish the isomorphism of the following pair of graphs.



(G)



(H)

Solution: Given graphs G and H are having same number of vertices and same number of edges.

Let $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $V_2 = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ are vertex sets of G and H respectively.

i.e., $V_1 = V_2 = 6$ and $E_1 = E_2 = 5$.

Degree of the vertices of G:

$d\ v_1 = d\ v_5 = d\ v_6 = 1,\ d\ v_2 = d\ v_4 = 2, d\ v_3 = 3$.

Degree of the vertices of H:

$d\ u_1 = d\ u_5 = d\ u_6 = 1, d\ u_2 = d\ u_3 = 2,\ d\ u_4 = 3$.

Define $f: G \to H$

Choose a vertex 'v_3' in G. It has degree 3 and the three adjacent vertices of 'v_3' having the degree sequence 1,2,2.

But in 'H' a vertex 'u_4' has degree 3 and the three adjacent vertices of 'u_4' having the degree sequence 1,1,2.

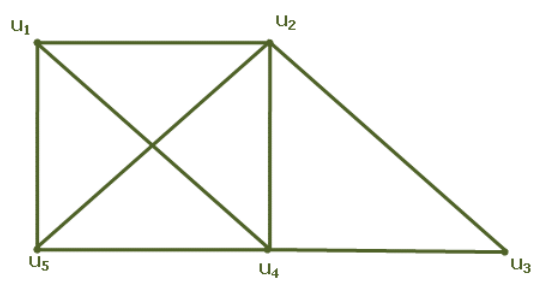
∴ The vertex 'v_3' in G is not mapped to any of the vertex in H.

∴ Hence G and H are not isomorphic.

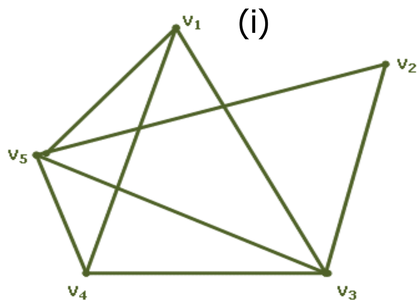
∴

7. Examine whether the following pair of graphs isomorphic or not?

(i)

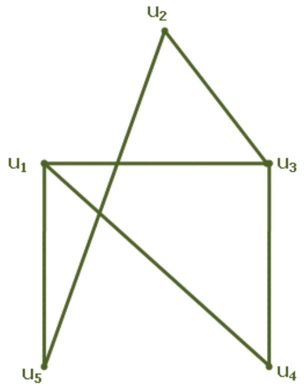


(G)

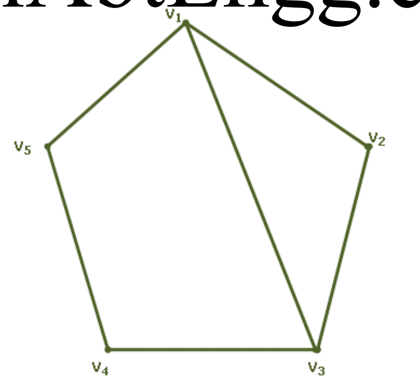


(H)

(ii)



(G)



(H)

Solution:

i) $V_1 = V_2 = 5$
$E_1 = E_2 = 8$

$d\ u_1 = d\ u_5 = 3, d\ u_2 = d\ u_4 = 4, d\ u_3 = 2$
$d\ v_1 = d\ v_4 = 3, d\ v_2 = 2, d\ v_3 = d\ v_5 = 4$

Choose $f(u_1) = v_1, f(u_2) = v_3, f(u_3) = v_2, f(u_4) = v_5, f(u_5) = v_4$
and so the adjacency matrices are same.

$\therefore G$ and H are isomorphic.

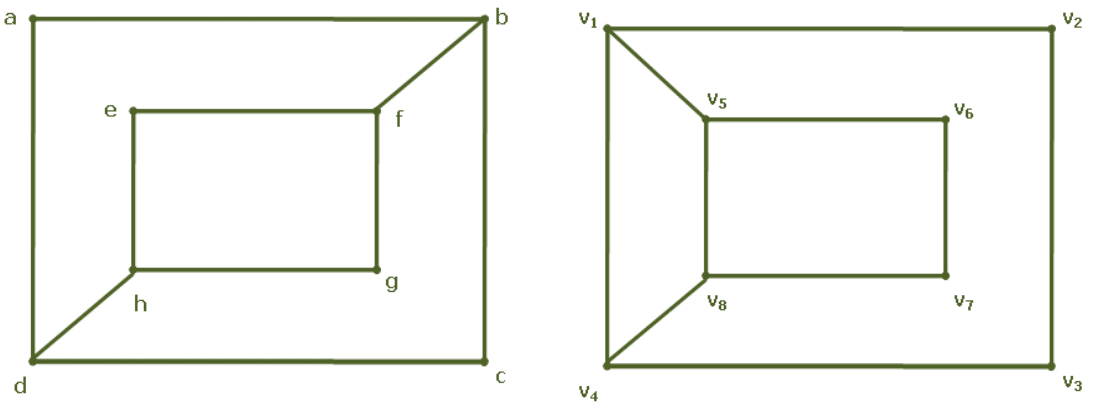
ii) $V_1 = V_2 = 5$
$E_1 = E_2 = 6$

$d\ u_1 = d\ u_3 = 3, d\ u_2 = d\ u_4 = d\ u_5 = 2$
$d\ v_1 = d\ v_3 = 3, d\ v_2 = d\ v_4 = d\ v_5 = 2$

Choose $f(u_1) = v_1, f(u_2) = v_4, f(u_3) = v_3, f(u_4) = v_2, f(u_5) = v_5$
and so the adjacency matrices are same.

$\therefore G$ and H are isomorphic.

8. Determine whether the graphs given below isomorphic or not?



(G) (H)

Solution:

The graphs G and H both have 8 vertices and 10 edges.

They also both have four vertices of degree two and four vertices of degree three.

In G , $\deg a = 2$, 'a' must correspond to either V_2, V_3, V_6 and V_7 in H because these are the vertices of degree two in H.

Each of these four vertices V_2, V_3, V_6, V_7 in H is adjacent to another vertex of degree two in H which is not true for 'a' in G.

Therefore G and H are not isomorphic.

9. Show that the graphs with the following adjacency matrices are isomorphic.

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Solution: Let us apply a series of interchange of pairs of rows and the corresponding pairs of columns.

Consider,

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_1 \Leftrightarrow R_3$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad C_1 \Leftrightarrow C_3$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \Leftrightarrow R_3$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad C_2 \Leftrightarrow C_3$$

$= A_2$.

$\therefore$ The graph represented by the adjacency matrix A_1 and A_2 are isomorphic.

10. Are the simple graphs with following adjacency matrix are isomorphic?

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Solution: Let us apply a series of interchange of pairs of rows and the corresponding pairs of columns.

Let $A_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

$$= \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad R_1 \Leftrightarrow R_4$$

$$= \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \qquad C_1 \Leftrightarrow C_4$$

$$\neq A_2, \text{ where } A_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Since first row, second row and first column of this matrix are same as in A_2.

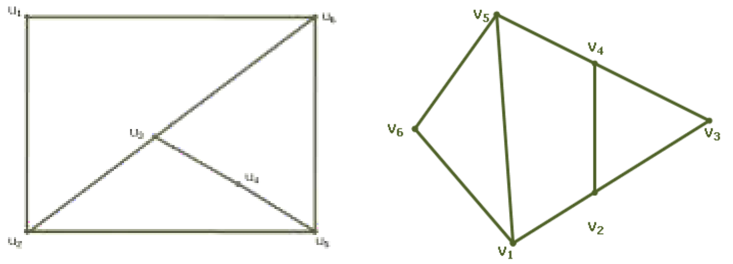
It cannot be brought to A_2 by interchanging rows and columns.

$\therefore A_1$ and A_2 cannot be similar.
Hence the corresponding graphs are not isomorphic.

11. Are the simple graphs with the following adjacency matrices isomorphic?

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Solution:



The graphs of the given adjacency matrices are given below.

The graphs G and H are having same number of vertices and same number of edges.
Let $V_1 = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ and $V_2 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ are vertex sets of G and H respectively.

i.e., $V_1 = V_2 = 6$ and $E_1 = E_2 = 8$.

Degree of the vertices of G:

$d\, u_1 = d\, u_4 = 2, d\, u_2 = d\, u_3 = d\, u_5 = d\, u_6 = 3.$

Degree of the vertices of H:

$d\, v_1 = d\, v_4 = 2, d\, v_2 = d\, v_3 = d\, v_5 = d\, v_6 = 3.$

Both the graphs G and H are having two vertices of degree 2 and four vertices of degree 3.

Define $f: G \to H$

Choose a vertex 'u_1' in G. It has degree 2 and the two adjacent vertices of 'u_1' having the degree sequence 3,3.

Correspondingly choose a vertex 'v_1' with the same degree and same degree sequence of adjacent vertices.

$\therefore f\,(u_1) = v_1$

Similarly all other vertices are having same degree and same degree sequence of the adjacent vertices from G to H.

$\therefore f\,(u_2) = v_2\,, f\,(u_3) = v_3\,, f\,(u_4) = v_4, f(u_5) = v_5, f\,(u_6) = v_6\,.$
and so the adjacency matrices are same.

Hence G and H are isomorphic.

12. Examine whether the two graphs G and H associated with the following adjacency matrices are isomorphic.

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Solution: Let G =

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$\text{and } H = \begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & 0 & 1 & 0 & 0 & 1 & 0 \\ v_2 & 1 & 0 & 1 & 0 & 0 & 0 \\ v_3 & 0 & 1 & 0 & 1 & 0 & 1 \\ v_4 & 0 & 0 & 1 & 0 & 1 & 0 \\ v_5 & 1 & 0 & 0 & 1 & 0 & 1 \\ v_6 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}$$

Here $v_1 = v_2 = 6$, $E_1 = E_2 = 7$

Degree of the vertices of G:

$d\,u_1 = d\,u_3 = d\,u_5 = d\,u_6 = 2, d\,u_2 = d\,u_4 = 3.$

Degree of the vertices of H:

$d\,v_1 = d\,v_2 = d\,v_4 = d\,v_6 = 2,\ d\,v_3 = d\,v_5 = 3.$

Define $f: G \to H$,

$\therefore f(u_1) = v_4,\ f(u_2) = v_3,\ f(u_3) = v_6,\ f(u_4) = v_5,\ f(u_5) = v_2,\ f(u_6) = v_1$.

and so the adjacency matrices are same.

Hence G and H are isomorphic.

x..x

Topic :8 PATHS and CIRCUITS

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1. Define: Path with Example:

A Path of length n from the vertex v_0 to vertex v_n is a sequence of the form v_0, e_1, v_2, e_2….,v_{n-1},e_n,v_n where $e_i = v_{i-1}v_i$, i = 1,2,3 …., n.

The vertices v_0 and v_n are called the end points of the paths, v_0 is the intial point and v_n is the terminal point of the path.

Note: (1) the number of edges appearing in the path is its length.

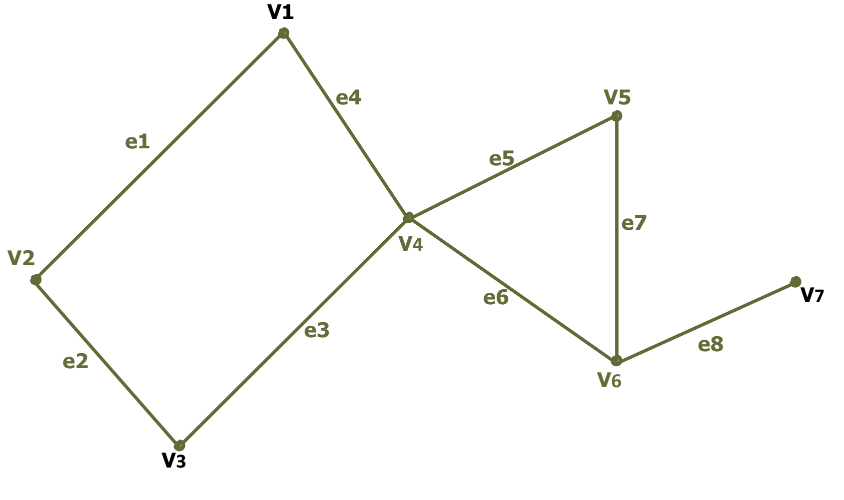
(2) A path from v_0 to v_n which does not contain repeated vertices is called a simple path.

(3) A path of length 0 i.e., It contains only one vertex is called a trivial path.

(4)

2. Define: Circuit with example:

A non-trivial path is called a circuit or cycle if it starts and ends with the same vertex

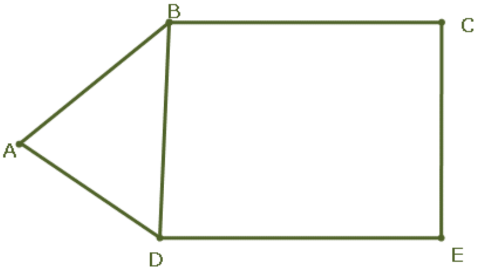


A path from v_1 to v_5 is P_1: $v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4\ e_5\ v_5$
A path from v_2 to v_7 is P_2: $v_2\ e_2\ v_3\ e_3\ v_4\ e_5\ v_5\ v_6\ e_8\ v_7$
A cycle is $v_1\ e_1\ v_2\ e_2\ v_3\ e_3\ v_4\ e_4\ v_1$

Note: (1) A loop is not a path. A loop is considered as a cycle of unit length.

(2) If the length of a cycle is k it is called a k-cycle.
Example: A triangle is a 3-cycle.

3. Find all the simple paths from A to E and all cycles with respect to vertex A of the given graph.

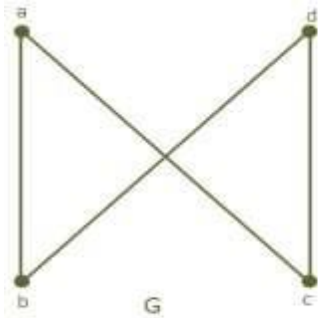


Solution: Simple paths from A to E are

(i). $A \to B \to C \to E$
(ii). $A \to B \to D \to E$
(iii). $A \to D \to E$
(iv). $A \to D \to B \to C \to E$

The cycles are
(i). $A \to B \to C \to E \to D \to A$
(ii). $A \to D \to E \to C \to B \to A$

4. For the graph G given by the figure.



Find (i) the number of paths of length 4 from a to d. (ii) the number of paths of length 3 from a to c and a to d.

Solution: The adjacency matrix of G is

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

(i) Since a is the first vertex and d is the 4th vertex, the number of paths of length 4 from a to d is $(1, 4)^{th}$ element A^4

$$\text{Now } A^2 = A.A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

$$A^3 = A.A^2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{pmatrix}$$

$$A^4 = A^2.A^2 = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{pmatrix}$$

∴the number of paths of length 4 from a to d is 8

(ii) The number of paths of length 3 from a to c is the $(1, 3)^{th}$ elements in A^3

∴the number of paths length 3 from a to c is 4.

The number of paths of length 3 from a to d is the $(1, 4)^{th}$ element in A^3.

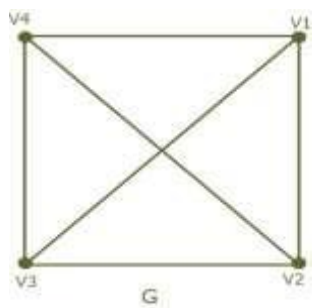
∴the number of length 3 from a to d is 0

i.e. there is no path of length 3 from a to d

5. Find the number of paths of length n between two different vertices in K_4 if n is (i) 2 (ii) 3 (iii) 4

Solution:

K_4 is the complete graph on 4 vertices.



G

The adjacency matrix of G is

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\text{Now } A^2 = A.A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$$

$$A^3 = A.A^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{pmatrix}$$

$$A^4 = A^2 . A^2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 6 & 7 & 7 & 7 \\ 7 & 6 & 7 & 7 \\ 7 & 7 & 6 & 7 \\ 7 & 7 & 7 & 6 \end{pmatrix} = \begin{pmatrix} 21 & 20 & 20 & 20 \\ 20 & 21 & 20 & 20 \\ 20 & 20 & 21 & 20 \\ 20 & 20 & 20 & 21 \end{pmatrix}$$

(i) Thenumber of paths of length 2 between any two different vertices = 2

(ii) The number of paths of length 3 between two different vertices = 7

(iii) The number of paths of length 4 between two different vertices = 20

6. **Theorem :** Let G = (V, E) be a graph with adjacency matrix A with respect to the ordering of vertices $v_1, v_2, \ldots, v_n$. The number of different paths of length r from v_i to v_j equals the $(i,j)^{th}$ entry of A^r.

Proof: Given the graph G = (V,E) with vertex set $V=\{v_1, v_2, \ldots, v_n\}$.

The adjacency matrix A =(aij) is an nxm square matrix given by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not adjacent} \end{cases}$$

We prove the theorem by induction on r.

Basis Step:

If r=1, then the number of paths from v_i to v_j of length one is the $(i, j)^{th}$ entry of A, because this entry represents the number of paths from v_i to v_j. So the result is true for r =1

Inductive Step :

We now assume that the result is true for any k(k>1)

That is in A^k the $(i, j)^{th}$ entry $a_{ij}^{(k)}$ is the number of paths from v_i to v_j of length k in G.

To prove it is true for r= k+1

i.e. To prove $(i, j)^{th}$ entry $a_{ij}^{(k+1)}$ in A^{k+1} is the number of paths from v_i to v_j of length k+1

Now $A^{k+1} = A^k.A = a_{ij}^{(k)} a_{ij}$

Hence the $(i, j)^{th}$ entry of A^{k+1} = $a_{is}^{k}\, a_{sj}$,where v_s is an intermediate vertex.

But a_{is}^{k} is the $(i, s)^{th}$ entry of A^k. By induction hypothesis a_{is}^{k} is a path from v_i to v_s of length k in G

A path of length k+1 from v_i to v_j is made up of a path of length k from v_i to v_s and an edge from v_s to v_j.

which is the $(i, j)^{th}$ entry of A^{k+1}

Hence by the principle of induction the result is true for all integers r.

x..x

Topic :9 & 10 Connected Graph

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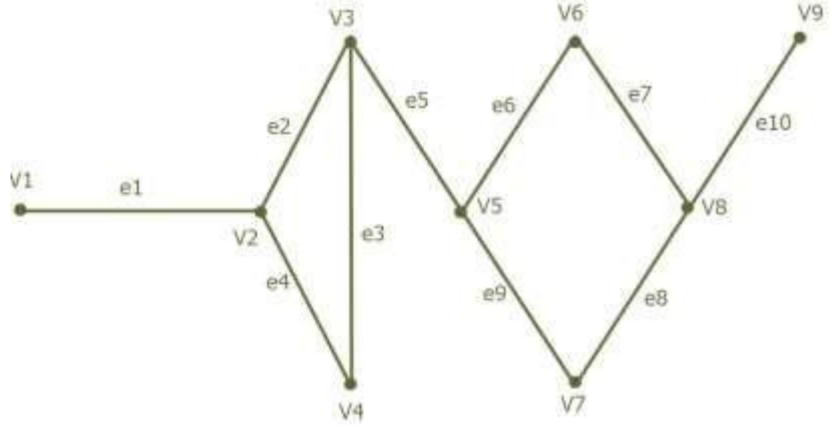
Define Connected Graph: A graph is connected if there is a path between every pair of distinct vertices of the graph. Otherwise it is disconnected.

Note: (1) Any graph with isolated vertices is a disconnected graph.
(2) Null graph is totally disconnected.

Example: Consider the graph G,



Clearly v_2,v_3,v_5,v_8 are cut vertices of G, because their removal disconnects the graph.

Note: 1. The vertex adjacent to a pendent vertex is cut vertex.

2. An isolated vertex is never a cut vertex.The edges e_1,e_5,e_{10} are cut edges of the graph G, because their removal disconnects the graph G.

Problems:

1. Prove that If a graph, connected or disconnected has exactly two vertices of odd degrees, then there must be a path joining these two vertices.

Proof:

Let G be a graph with exactly two vertices u, v of odd degree and all other vertices are of even degrees.

If the graph G is connected, there is a path joining u, v by definition of connected graph.

If the graph G is disconnected, then there are connected components of G. Each component itself is a graph.

We know that in a graph the number of odd vertices is even.

So the two odd vertices u and v must belong to the same connected component.

Hence there is a path joining u and v.

2. Prove that a simple graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Solution:
Let G be a disconnected graph with n vertices and k components say $G_1, G_2, \ldots\ldots\ldots G_k$.

$$\therefore n_1 + n_2 + \ldots\ldots\ldots + n_k = n$$

$$\text{i.e } \sum_{i=1}^{k} n_i = n \quad \ldots\ldots\ldots\ldots(1)$$

To Find $\sum_{i=1}^{k}(n_i - 1)$

We know that
$(n_1 - 1) + (n_2 - 1) + \ldots\ldots\ldots + (n_k - 1) = n_1 + n_2 + \ldots\ldots\ldots + n_k - (1 + 1 + \ldots\ldots\ldots + 1)$

$$\sum_{i=1}^{k}(n_i - 1) = n - k$$

On squaring,

$$\left(\sum_{i=1}^{k}(n_i - 1) \right)^2 = (n - k)^2$$

By Schwartz inequality ,

$$\sum_{i=1}^{k}(n_i - 1)^2 \leq (n - k)^2$$

$$(n_1 - 1)^2 + (n_2 - 1)^2 + \ldots\ldots\ldots + (n_k - 1)^2 \leq (n - k)^2$$

$$(n_1^2 - 2n_1 + 1) + (n_2^2 - 2n_2 + 1) + \ldots\ldots\ldots\ldots\ (n_k^2 - 2n_k + 1) \leq (n - k)^2$$

$$(n_1^2 + n_2^2 + \ldots\ldots\ldots\ldots + n_k^2) - 2\,(n_1 + n_2 + \ldots\ldots\ldots\ldots + n_k)$$
$$+ (1 + 1 + \ldots\ldots + 1) \leq \frac{1}{2}\sum_{i=1}^{k} n_i^{\,2} - n_i$$

$$\sum_{i=1}^{k} n_i^{\,2} - 2n + k \leq (n - k)^2$$

$$\sum_{i=1}^{k} n_i^{\,2} \leq (n - k)^2 + 2n - k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

L. H.S: Maximum no. of edges $= \frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} + \ldots\ldots\ldots\ldots\ldots + \frac{n_k(n_k-1)}{2}$

$$= \frac{1}{2}\sum_{i=1}^{k}(n_i^{\,2} - n_i)$$
$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^{\,2} - \sum_{i=1}^{k} n_i\right]$$
$$= \frac{1}{2}\left[(n - k)^2 + 2n - k - n\right]$$
$$= \frac{1}{2}\left[(n - k)^2 + (n - k)\right]$$
$$= \frac{1}{2}\left[(n - k)\ (n - k + 1)\right]$$
$$= \frac{(n-k)(n-k+1)}{2}.$$

Maximum no. of edges in G is $\frac{(n-k)(n-k+1)}{2}$ edges.

3. Prove that a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Proof:

Let G be a simple graph with n vertices and more than $\frac{(n-1)(n-2)}{2}$ edges.

Suppose if G is not connected, then G must have atleast two components. Let it be G_1 and G_2

Let v_1 be the vertex set of G_1 with $v_1 = m$.
If v_2 is the vertex set of G_2, then $v_2 = n - m$

$$\text{Maximum no. of edges} \leq \frac{m\,(m-1)}{2} + \frac{(n-m\,)(n-m-1)}{2}$$

$$\leq \frac{m^2-m}{2} + \frac{n^2-nm-n-nm+m^2+m}{2}$$

$$\leq \frac{1}{2}[m^2 - m + n^2 - nm - n - nm + m^2 + m]$$

$$\leq \frac{1}{2}[2m^2 - 2nm + n^2 - n]$$

Add and subtract 2n -2

$$\leq \frac{1}{2}[2m^2 - 2nm + n^2 - n + 2n - 2 - 2n + 2]$$

$$\leq \frac{1}{2}[2\,(m^2 - 1) - 2n(m-1) + n^2 - n - 2n + 2\,]$$

$$\leq \frac{1}{2}[\,2(m-1)(m+1) - 2n(m-1) + n(n-1) - 2\,(n-1)\,]$$

$$\leq \frac{1}{2}[\,2(m-1)(m+1-n) + (n-2)\,(n-1)\,]$$

$$\leq (m-1)(m+1-n) + \frac{(n-1)(n-2)}{2}$$

Omit the first term

$$\leq \frac{(n-1)(n-2)}{2} \text{ edges.}$$

which is a contradiction to G that has more than $\frac{(n-1)(n-2)}{2}$ edges.

Hence G is a connected graph.

Topic :11 Eulerian & Hamilton Graph

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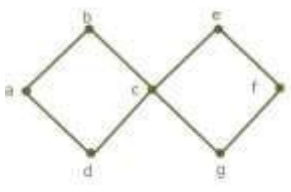
1. Define Eulerian Graph with Example:

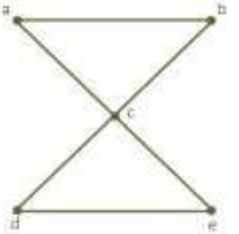
An Euler circuit (or cycle) in a graph G is a simple circuit containing every edge of G.

An Euler path in a graph G is a simple path containing every edge of G.
A connected graph with an Euler circuit is called an Euler graph (or) Eulerian graph.

Examples:



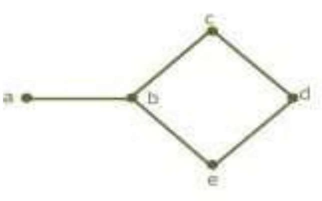
G1



G2

The graph G_1 is Eulerian, because it contains an Euler circuit a, b, c, e, f, g, c, d. a; it contains each edge only once.

The graph G_2 is Eulerian, because it contains an Euler circuit a, c, e, d, c, b, a.It contains each edge only once.



G3

G4

The graph G_3 is not Eulerian, because it does not contain Euler circuit, but it contain an Euler path a, b, c, d, e, b containing every edge only once.

The graph G_4 has no Euler path or Euler circuit.

2. Define Hamilton Graph :

Definition: A simple circuit of connected graph G is called a Hamilton circuit or Hamilton cycle if it contains every vertex of G exactly once.

A connected graph that contains a Hamilton circuit is called Hamilton graph or Hamiltonian graph.

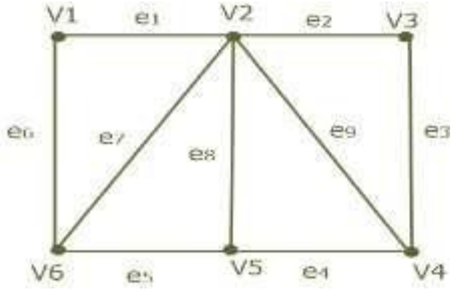
Problems:

3. Give example for the graphs which are

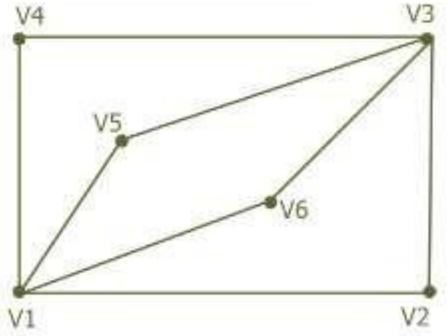
(i) Not Eulerian and Hamiltonian
(ii) Eulerian and not Hamiltonian
(iii) Neither Eulerian nor Hamiltonian
(iv) Eulerian and Hamiltonian

Solution:

(i) The graph contains Hamiltonian circuit $v_1,v_2,v_3,v_4,v_5,v_6,v_1$
Hence G is a Hamilton graph. Since the degree of each vertex is not even, it is not Eulerian.

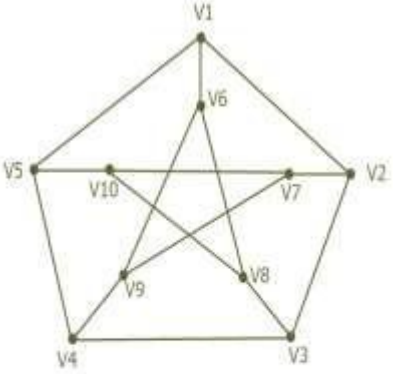


(ii)



Since every vertex is of even degree, G has an Euler circuit. But it is not Hamiltonian, because it can be seen that every circuit containing every vertex contains a vertex twice. For example $v_1, v_2, v_3, v_5, v_1, v_6, v_3, v_4, v_1$.So G is Eulerian but not Hamiltonian.

(iii)



The Peterson graph given above is not Eulerian and not Hamiltonian since every vertex is of odd degree. The Peterson graph is not Hamiltonian because it can be seen that every circuit containing every vertex contains a vertex twice.

(iv)

The circuit V_1, V_2, V_3, V_4, V_1 in the graph consists of all edges and all vertices each exactly only once.

Therefore, The above graph is contains a circuit i.e., both Eulerian as well as Hamiltonian.

4. **Theorem:** A connected graph G is Eulerian if and only if every vertex of G is of even degree.

Proof: Let G be an Eulerian graph. We have to prove all vertices are of even degree.

Since G is Eulerian, G contains an euler circuit, say,

$$v_0, e_1, v_1, e_2, \dots v_{n-1}, e_n, v_0.$$

Both the edges e_1 and e_n contribute one to the degree of v_o and so deg v_o is atleast two.

In tracing this circuit we find an edge enters a vertex and another edge leaves the vertex contributing 2 to the degree of the vertex.

This is true for all vertices and so each vertex is of degree 2, an even integer.

Conversely,

let the graph G be such that all its vertices are of even degrees.

We have to prove G is an Euler graph.

We shall construct an Euler circuit and prove. Let v be an arbitrary vertex in G. Beginning with v form a circuit C: $v, v_1, v_2, \dots v_{n-1}, v$

This is possible because every vertex are of even degree. We can leave a vertex $(\neq v)$ along an edge not used to enter it. This tracing clearly stops only at the vertex v because v is also of even degree and we started from v. Thus, we get circuit or cycle C

If C includes all the edges of G, then C is an Euler circuit and so G is Eulerian.

If C does not contain all the edges of G, consider the sub graph H of G obtained by deleting all the edges of C from G and vertices not incident with remaining edges. Note that all vertices of H have even degree. Since G is connected, h and C must have a common vertex u. Beginning with u construct a circuit C_1 for H.

Now combine C and C_1 to form a larger circuit C_2.If it is Eulerian ie. If it contains all the edges of G, the G is Eulerian.

Otherwise, continue this process until we get an Eulerian circuit.

Since G is finite this procedure must come to an end with a Eulerian circuit. Hence G is Eulerian.

5. **Theorem:** A connected graph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Proof: Given G is a connected graph.

Suppose it has an Euler path from $v_0$to v_n , say, v_o , e_1 , v_1, e_2 , ... v_{n-1}, e_n , v_n.

The edges e_1 and e_n contribute 1 to the degrees of v_0 and v_n respectively. Every time the path passes through a vertex, it contributes 2 to its degree. It is true for v_0 and v_n also.

So, the degrees of v_0 and v_n are always odd and the degrees of each internal vertices remain even. Thus the graph contains exactly two vertices of odd degree.

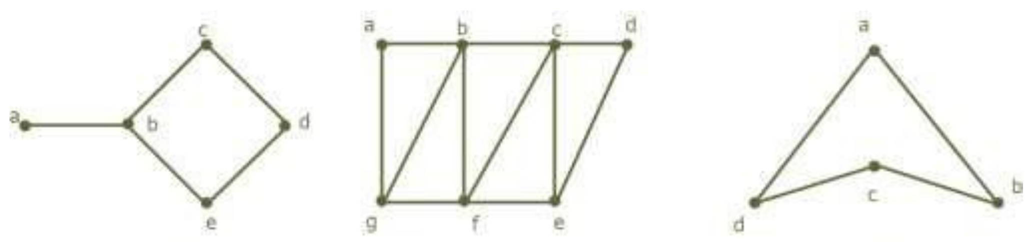
Conversely,

let the connected graph G contains two vertices of odd degree, say v_0 and v_n .

Adding a new edge $e = v_0v_n$ to G we get a graph G_1 with all even degree vertices. Therefore by previous theorem G_1 is Eulerian.

Removing $e = v_0v_n$ from G_1, we get G containing an Euler path from v_0 to v_n .

6. Determine whether Euler path exist in the following graph.



G1 G2 G3

Solution:

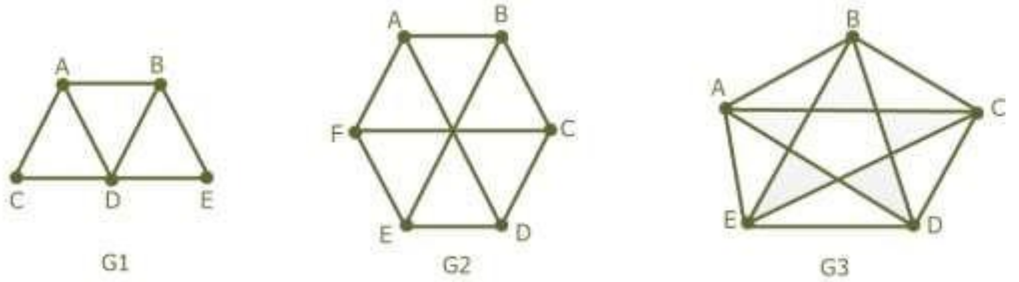
G_1 contains exactly two vertices a and b of odd degree. So, there is a Euler path a, b, c, d, e, b starting with one odd vertex and ending with other odd vertex.

G_2 contains exactly two vertices e,g of odd degree. So it contains a Euler path starting with g and ending with e namely g, a, b, c, d, e, f, b, g, f, c, e.

G_3 contains two vertices of odd degree a and c. So it contains a Euler path starting with a and ending with c namely a, b, c, d, a, c.

7. Find an Euler path or an Euler circuit, if it exists in each of the graphs

given below. If it does not exist, explain why?

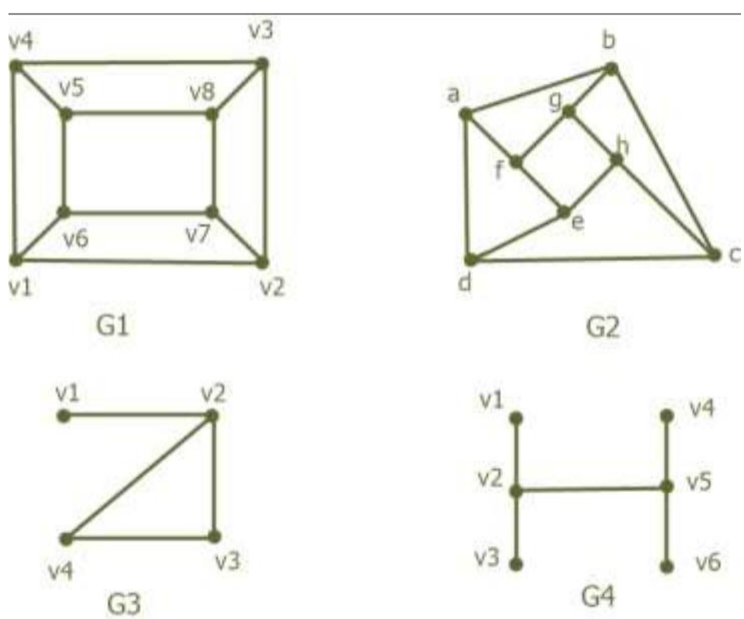


Solution: In G_1, there are two vertices namely A, B of odd degree 3 and other vertices are of even degrees 2 and 4. So G_1 has Euler path and has no Euler circuit. The Euler path between A and B is given by A, B, C, D, A, C, D, B.

In G_2 all the 6 vertices are of odd degree 3 ad so it contains neither an Euler path (A graph G is said to have a Euler path if it has exactly two vertices of odd degree) nor an Euler circuit (A necessary and sufficient condition for a graph G to have Euler circuit iff all the vertices are of even degree).

In G_3 all the 5 vertices are of even degree 4. So, A necessary and sufficient condition for a graph G to have Euler circuit iff all the vertices are of even degree) G_3 has Euler circuit A,B,C,D,E, A, C, E,B, D,A.

7. Identify Hamilton path, Hamilton circuit in the following graphs:



Solution:

In G_1, $v_1, v_2, v_3, v_4, v_5, v_8, v_7, v_6, v_1$ is a Hamilton cycle and so it is a Hamilton graph.

$v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ is a Hamilton path.

In G_2, a, b, c, d, e, h, g, f, a is a Hamilton cycle. So G_2 is a Hamilton graph. It contains a Hamilton path also. a, b, c, d, e, f, g, h is a Hamilton path.

In G_3, v_1, v_2, v_3, v_4 is a Hamilton path. But there is no Hamilton cycle, since v_1, v_2 occurs twice in every circuit. So v_2 is repeated.

In G_4, any cycle will have v_2, v_5 atleast twice. So it is not a Hamilton graph. It has no Hamilton path also.

x..x