UNIT- IV

Topic : 1 Algebraic Structures

1)Define Binary Operation with Example:

Let A be any non-empty sets. The binary operation $*$ is a function from A x A i.e. a rule which assigns to every pair $(a, b) \in A \times A$, a unique element $a*b \in A$.

Example:
Usual addition, multiplication are binary operation defined on the set of real numbers.
Matrix addition and Matrix multiplication are binary operation on the set of 2 x 2 real matrices.

2) Define Algebraic System with Example:

A non-empty set A together with one or more n-ary operations $*$ defined on it, is called an algebraic system or algebraic structure or Algebra.

We denote it by $(A, *)$

Note: $+, -, \cdot, \times, *, \cup, \cap$ etc.., are some of binary operations.

Properties of Binary operations:

Let the binary operation be $* : A \times A \to A$.

Then we have the following properties

1) Closure Property:

$a * b \in A$ for all $a, b \in A$.

2) Associativity:

$(a * b) * c = a * (b * c)$ for all $a, b, c \in A$.

3) Identity element:

$a * e = e * a = a$, for all $a \in A$, where e is called the identity element.

4) Inverse element:

If $a * b = b * a = e$, then b is called the inverse of a and it is denoted by a^{-1}, (i.e. $b = a^{-1}$).

5) Commutative:

$a * b = b * a$ for all $a, b \in A$.

6) Distributive properties: for all a, b, c $\in$ A.

(i) $a * (b \cdot c) = (a * b) \cdot (a * c)$ [Left distributive law]

(ii) $(b \cdot c) * a = (b * a) \cdot (c * a)$ [Right distributive law]

7) Cancellation properties: for all a, b, c $\in$ A

(i) $a * b = a * c \Rightarrow b = c$ [Left cancellation law]

(ii) $b * a = c * a \Rightarrow b = c$ [Right cancellation law]

Note:

If the binary operations defined on G is + and x, then we have the following table

	Properties	For all a, b, c $\in$ (G, +)	For all a, b, c $\in$ (G, x)
1.	Closure	$a+b \in G$	$a \times b \in G$
2.	Associativity	$(a + b) + c = a + (b + c)$	$(a \times b) \times c = a \times (b \times c)$
3.	Identity element	$a+0=0+a=a$, Here 0 is Additive Identity	$a \times 1 = 1 \times a = a$, Here 1 is Multiplicative Identity
4.	Inverse element	$a+(-a) = 0$, Here $(-a)$ is Additive inverse	$a \times \frac{1}{a} = \frac{1}{a} \times a = 1$, Here $\frac{1}{a}$ is multiplicative inverse
5.	Commutative	$a + b = b + a$	$a \times b = b \times a$

Notation:

$\mathbb{Z}$ or I	The set of all integer.
$\mathbb{Q}$	The set of all rational number.
$\mathbb{R}$	The set of all real number.
$\mathbb{R}^+$	The set of all positive real number.
$\mathbb{Q}^+$	The set of all positive rational number.
$\mathbb{C}$	The set of all complex number.

Example 1:

The set of integers $\mathbb{Z}$ with the binary operations + and x is an algebraic system since it satisfies all the above properties.

Example 2:

The set of real numbers $\mathbb{R}$ with binary operations + and x is an algebraic system.

3) Define Semi group with Example:

Definition: If a non-empty set S together with the binary operation $*$ satisfying the following two properties.

(a) Closure property
(b) Associative property

is called a semigroup. It is denoted by $(S, *)$.

Example:

1. Let X be any non-empty set. Then the set of all functions from X to X is the set X^X, is a semigroup w.r.to $*$, the composition of functions.
2. $(I, +)$, $(I, \times)$ are semigroups, where $+$ is the usual addition and $\times$ is the usual multiplication.
3. $(P(A), \cap)$ and $(P(A), \cup)$ are semigroups. Where P(A) is the powerset of A (the set of all subsets of A).
4. N= {0, 1, 2 …} then (N, +), (N, x) are semigroups. N is not a semigroup w.r.to the operation subtraction.

Solved Examples:

1. Show that the set of all natural numbers N is a semigroup w.r.to operation $*$ defined by $a * b = \max\{a, b\}$.

Solution:

N is closed under the operation $*$.

For $a, b, c \in N$

$a * (b * c) = \max\{a, \max\{b, c\}\} = \max\{a, b, c\}$ (1)

$(a * b) * c = \max\{\max\{a, b\}, c\}$

$(a * b) * c = \max\{a, b, c\}$ (2)

From (1) and (2),

$(a * b) * c = a * (b * c), \forall a, b, c \in N$

$\therefore$ $*$ is associative.

$\therefore$ $(N, *)$ is a semigroup.

2. Show that the set of rational numbers Q is a semigroup for the operation $*$ defined by $a * b = a + b - ab$.

Solution: Q is closed for $*$.

$a * (b * c) = a * (b + c - bc)$

$= a + b + c - bc - a(b + c - bc)$

$= a + b + c - ab - bc - ca + abc$ (1)

$(a * b) * c = (a + b - ab) * c$

$= a + b - ab + c - (a + b - ab)c$

$= a + b + c - ab - ac - bc + abc$ (2)

From (1) and (2),

$(a * b) * c = a * (b * c), \forall a, b, c \in Q$

$\therefore$ $*$ is associative.

$\therefore$ $(Q, *)$ is a semigroup.

3. Show that the set of rational numbers Q is a semigroup for operation $*$ defined by $a * b = \frac{ab}{2}$ $\forall$ a, b $\in$ Q

Solution: Q is closed for $*$.

$a * (b * c) = a * \frac{bc}{2} = \frac{abc}{4}$

$(a * b) * c = \frac{ab}{2} * c = \frac{abc}{4}$

$(a * b) * c = a * (b * c), \forall a, b, c \in Q$

$\therefore$ $*$ is associative.

$\therefore$ (Q, $*$) is a semigroup.

4. Let (A,$*$) be a semigroup. Show that for a, b, c in A if $a * c = c * a$ and then $b * c = c * b$ then $(a * b) * c = c * (a * b)$.

Solution: L.H.S. $(a * b) * c = a * (b * c)$ [$\because$ $*$ is associative]

$= a * (c * b)$ [$\because$ $b * c = c * b$]

$= (a * c) * b$ [$\because$ $*$ is associative]

$= (c * a) * b$ [$\because$ $a * c = c * a$]

$= c * (a * b)$ [$\because$ $*$ is associative]

=R.H.S.

5. Let (S,$*$) be a commutative semigroup. If $x * x = x$, $y * y = y$, prove that $(x * y) * (x * y) = x * y$.

Solution: L.H.S.: $(x * y) * (x * y)$

$= x * (y * (x * y))$

$= x * ((y * x) * y)$

$= x * ((x * y) * y)$

$= x * (x * (y * y))$

$= x * (x * y)$

$= (x * x) * y$

$= x * y$

= R.H.S.

6. Let $(S,*)$ be a commutative semigroup. If $x * x = x$, $y * y = y$, prove that $(x * y) * (x * y) = x * y$.

Solution: L.H.S.: $(x * y) * (x * y)$

$= x * (y * (x * y))$

$= x * ((y * x) * y)$

$= x * ((x * y) * y)$

$= x * (x * (y * y))$

$= x * (x * y)$

$= (x * x) * y$

$= x * y$

$=$ R.H.S.

7. Let $\{\{x, y\}, \cdot\}$ be a semigroup where $x \cdot x = y$. Show that

(i) $x \cdot y = y \cdot x$
(ii) $y \cdot y = y$.

Solution: (i) $x \cdot (x \cdot x) = (x \cdot x) \cdot x$ (Since $\cdot$ is associative)

Given $x \cdot x = y$

$\therefore x \cdot y = y \cdot x$

(ii)To prove: $y \cdot y = y$

Since the set $\{x, y\}$ is closed for operation '$\cdot$',

$x \cdot y = x$ (or) $x \cdot y = y$

Assume $x \cdot y = x$

$y \cdot y = y \cdot (x \cdot x)$

$= (y \cdot x) \cdot x$

$= (x \cdot y) \cdot x$

$\therefore y \cdot y = y$

Next consider the case $x \cdot y = y$,

$y \cdot y = (x \cdot x) \cdot y$

$= x \cdot (x \cdot y)$

$\therefore y \cdot y = y$

8. Let $(A, *)$ be a semi group. Further more for every $a, b \in A$, if $a \neq b$ then $a * b \neq b * a$

(i) Show that for every $a \in A$, $a * a = a$

(ii) For every $a \in A$, $a * (b * a) = a$.

(iii) For every $a, b, c \in A$, $(a * b) * c = a * c$.

Solution:

(i) $a * (b * c) = (a * b) * c$

Put $b = a$ and $c = a$

$a * (a * a) = (a * a) * a$

Since $(A, *)$ is not commutative, $a * a = a$.

(ii) Let us assume that $b \in A$ then

we have for $b \in A$, $b * b = b$.

Let $a * (b * b) = a * b$ $\quad [\because b * b = b]$

$(a * b) * b = a * b$ $\quad [\because \text{associative}]$

Hence $a * b = a$ ______________ (1) (Using Right Cancellation law)

$\therefore\ a * (b * a) = (a * b) * a$ $\quad$ From $[\because \text{associative}]$

$= a * a$

$= a$

(iii) $(a * b) * c = a * c$ $\quad [\because a * b = a]$

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Monoids

1)Define Monoid with Example:

A semigroup (S, ∗) with an identity element w.r.t. '∗' is called Monoid.
It is denoted by (M, ∗).

In other words, a non-empty set 'M' with respect to ∗ is said to be a monoid, if ∗ satisfies the following properties.

(a) Closure property
(b) Associative property
(c) Identity property

Examples:

1. N= {0, 1, 2 …} then (N, +), (N, x) are monoids.
2. (Z, x), (Z, +) are monoids.
3. The set of even integers E= $\{\ldots\ldots, -4, -2, 0, 2, 4, \ldots.\}$, Then (E, +) is a monoid and (E, x) is a semigroup but not a monoid.
4. (P(A),∪) is a monoid with identity element ∅.
   (P(A),∩) is a monoid with identity element A, where A is any set.

Problem:

2) Show that the set of integers, is a monoid for the operation ∗ defined by a∗b=a+b−ab, for a, b ∈ I.

Solution:

I is closed for the operation ∗ .
Further ∗ is a associative.
The element 0∈ I is the identity Element
Since $x * 0 = x + 0 - x \cdot 0 = x$ and $0 * x = 0 + x - 0 \cdot x = x, \forall x \in I$.

∴ (I,∗) is a monoid with identity 0∈ I.

x……………………………………………………………………..x

Topic 2: GROUPS

1) **Define Group with Example :**

A non-empty set G with binary operation $*$ is called a group if the following axioms are satisfied.

1. $*$ is associative, i.e.$(a*b)*c = a*b*c \; \forall \; a, b, c \in G$.
2. There exists an element $e \in G$ such that $a*e = e*a = a, \forall \; a \in G$. ($e$ is the identity element).
3. For every $a \in G$, $\exists$ an element $a^{-1} \in G$, such that $a * a^{-1} = a^{-1} * a = e$. ($a^{-1}$ is called the inverse element of a).

2)Define Abelian Group (or) Commutative Group

A group $(G,*)$ is called abelian if $a*b=b*a$, $\forall a, b \in G$. i.e. $*$ is commutative in G.

Example:

1. (I, +) is a group called the additive group of integers.
2. M_2 (R), the set of all 2 x 2 matrices is a group w.r.to matrix addition.
3. The set of all non-singular 2 x 2 matrices is a group w.r.to matrix multiplication.
4. The set of n^{th} roots of unity $\{1, w, w^2, \ldots\ldots, w^{n-1}\}$ is a group w.r.to the operation multiplication of complex numbers.
5. G= {1, -1, i, -i}. In G, the operation '·' is defined by the following table. Then (G,·) is an abelian group.

·	**1**	**-1**	**i**	**-i**
1	1	-1	i	-i
-1	-1	1	-i	I
i	i	-i	-1	1
-i	-i	i	1	-1

Here '·' is the multiplication of complex numbers.

Examples :

1. Show that $[Z_5, +_5]$ is an abelian group.

Solution: The table for addition modulo 5 is.

$+_5$	**[0]**	**[1]**	**[2]**	**[3]**	**[4]**
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

(i) Closure property:
$[a] +_5 [b]$= remainder when the sum is divided by 5.

(ii) Associative Property:
From the table for $[a], [b], [c] \in Z_5$
$[a] +_5 ([b] +_5 [c]) = ([a] +_5 [b]) +_5 [c]$

(iii) Identity:
$[0] \in Z_5$ is the identity

(iv) Inverse:
The inverse of [1] is [4].
The inverse of [2] is [3].
The inverse of [3] is [2].
The inverse of [4] is [1].
The element $[0] \in Z_5$ has self-inverse.

(v) Commutative property:
Further $[a] +_5 [b] = [b] +_5 [a], \forall [a], [b] \in Z_5$.

$\therefore (Z_5, +_5)$ is an abelian group.

2. Show that [{1,2, 3, 4}, x_5] is an abelian group.

Solution: The table for the x_5 is as follows

Z_5 = {1,2,3,4} and x_5 is multiplication operation

$\times_5$	**1**	**2**	**3**	**4**
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

(i) Closure Property:

Here $a \in Z_5$ means $a = [a]$

$a \; x_5 \; b$ = remainder when ab is divisible by 5.

(ii) Associative Property:

For $a, b, c \in Z_5$.

$a \; x_5 \; (b \; x_5 \; c) = (a \; x_5 \; b) \; x_5 \; c$

(iii) Identity:

$1 \in Z_5$; is the identity element.

(iv) Inverse:

The inverse of 1 is 1

The inverse of 2 is 3

The inverse of 3 is 2

The inverse of4 is 4

(v) Commutative property:

Further $a \; x_5 \; b = b \; x_5 \; a, \forall a, b \in Z_5$.

$\therefore$ [{1,2, 3, 4}, x_5] is an abelian group.

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Topic :3 Properties of Group

Property 1:

The identity element in a group is unique.

Proof: If (G, $*$) be a group and e_1 and e_2 be two identity elements of G.

Let $x \in G$; $x * e_1 = x$ and $x * e_2 = x$

$\therefore x * e_1 = x * e_2,$

By using left cancellation law, we get $e_1 = e_2$

$\therefore$ Identity element is unique

Property 2:

The inverse of every element in a group is unique.

Proof: Let (G, $*$) be a group, with identity element e.

Let b and c be inverses of a element $a \in G$.

$a * b = b * a = e$

$a * c = c * a = e$

$b = b * e$

$= b * (a * c)$

$= (b * a) * c$

$= e * c$

$b = c$

Property 3:

If a is an element in a group (G, $*$) then $(a^{-1})^{-1} = a$.

Property 4:(Reversal law)

If a and b are two elements in a group (G, $*$) then

$(a * b)^{-1} = b^{-1} * a^{-1}$.

[prove $(a * b) * (b^{-1} * a^{-1}) = e$ and $(b^{-1} * a^{-1}) * (a * b) = e$]

Property 5: Cancellation laws

In a group $(G,*)$, for every $a, b, c \in G$, then

(i) $a* c = b* c$ implies $a=b$ (Right Cancellation law)

(ii) $c* a = c* b$ implies $a=b$ (Left Cancellation law)

Property 6:

In a group $(G,*)$, the equations $x * a=b$ and $a * y=b$ has unique solution.

Proof:

Consider $x * a=b$ Post multiplying by a^{-1}

$x * (a * a^{-1}) = b* a^{-1}$

i.e. $x * e = b* a^{-1}$

$\therefore x = b * a^{-1}$

Proof of Uniqueness

Let x_1 and x_2 be two solutions of $x * a= b$.

Then $x_1 * a= b$ and $x_2 * a=b$.

$\therefore x_1 * a = x_2 * a$

$\Rightarrow x_1 = x_2$ [By Right cancellation law]

$\therefore$ The solution is unique.

In a similar manner, the equation $a * y=b$ has a solution $y = a^{-1} * b$ and it has unique solution.

Examples

1. Show that a group $(G,*)$ is abelian iff $(a* b)^2= a^2 * b^2$

Solution: First we assume that $(G,*)$ is abelian,

$(a* b)^2=(a * b) * (a* b)$

$=a* (b * (a* b))$

$=a* ((b * a)* b)$

Since G is abelian, $a * b = b * a$.

$$\therefore (a * b)^2 = a * ((a * b) * b)$$
$$= (a * a) * (b * b)$$
$$= a^2 * b^2$$

Conversely,

we assume that $(a * b)^2 = a^2 * b^2$.

To prove : G is abelian

$$(a * b)^2 = a^2 * b^2$$
$$(a * b) * (a * b) = (a * a) * (b * b)$$
$$a * (b * (a * b)) = a * (a * (b * b))$$
$$b * (a * b) = a * (b * b) \quad \text{[by Left cancellation law]}$$
$$(b * a) * b = (a * b) * b$$
$$b * a = a * b \quad \text{[by Right cancellation law]}$$
$$\therefore \quad a * b = b * a, \forall a, b \in G$$

$\therefore$ G is abelian.

2. Show that $(G, *)$ is abelian iff $(a * b)^{-1} = a^{-1} * b^{-1}$.

Solution: Assume that G is Abelian.

$$\therefore (a * b) = (b * a), \forall a, b \in G$$
$$(a * b)^{-1} = (b * a)^{-1}$$
$$= a^{-1} * b^{-1}$$

Conversely

assume $(a * b)^{-1} = a^{-1} * b^{-1}$

But $a^{-1} * b^{-1} = (b * a)^{-1}$ (By Reversal law)

$\therefore (a * b)^{-1} = (b * a)^{-1}$ From given

Taking inverses both sides,

$$((a * b)^{-1})^{-1} = ((b * a)^{-1})^{-1}$$
$$\Rightarrow a * b = b * a, \forall a, b \in G$$

$\therefore (G, *)$ is abelian

3. Show that if every element in a group is its own inverse, then the group is abelian.

Solution: Let G be a group such that every element in G is its own inverse.

$\therefore$ For $a \in G$, $a^{-1}=a$

Let a, b $\in$ G, then $(a * b) \in G$ and so

$(a * b)^{-1} = a * b$ (1)

But $(a * b)^{-1} = b^{-1} * a^{-1}$

Since $b^{-1} = b$, $a^{-1} = a$.

$\Rightarrow (a * b)^{-1} = b * a$ (2)

From (1) and (2) we have $a * b = b * a \ \forall a, b \in G$

$\therefore$ G is abelian.

4. Prove that if for every element a in a group $(G, *)$, $a^2 = e$ then G is an abelian group.

Solution: Let $a, b \in G$

Then $(a * b) \in G$ and so $(a * b)^2 = e$ ______(1)

Since $a \in G$, $a^2=e \Rightarrow a * a = e$

$b \in G$, $b^2=e \Rightarrow b * b = e$

From (1) $(a * b)^2 = e$

$\Rightarrow (a * b) * (a * b) = e * e$

$= (a * a) * (b * b)$

$a * (b * (a * b)) = a * (a * (b * b))$

$b * (a * b) = a * (b * b)$ [by Left cancellation law]

i.e. $(b * a) * b = (a * b) * b$

$\therefore b * a = a * b$ [by Right cancellation law]

$\therefore$ G is abelian.

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Topic 4: Symmetric Groups (S_n)

1) Define Permutation with Example:

A permutation of a set *A* is a one-to-one and onto function from set *A* to itself.

Example.:

If *A*={1,2,3,4,5}, then a permutation is function σ where: σ (1)=4, σ (2)=2, σ (3)=5, σ(4)=3, σ (5)=1. This can be represented with permutation notation as: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 & 1 \end{pmatrix}$

2) Define Symmetric Set:

If S is a finite set having n distinct elements then we shall have n! distinct permutations of the sets. The set of all distinct permutations of degree n defined on the set S is denoted by S_n called symmetric set of permutations of degree n.

Note: $O(S_n)=n!$.

Problems:

1. List all elements of the symmetric set S_3, where S={1,2,3} and prove that (S_3, °) is a non abelian group.

   Solution: Given S={1,2,3}.

   Total number of permutation on S=3!=6.

   Elements of symmetrical set S_3= {P_1,P_2,P_3,P_4,P_5,P_6}

   where

   $$p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

   $$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\ p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

   $$p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},\ p_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

   The operation '$\circ$' product of permutations defined on the set S_3={P_1,P_2,P_3,P_4,P_5,P_6} is given in the table.

$\circ$	P_1	P_2	P_3	P_4	P_5	P_6
P_1	P_1	P_2	P_3	P_4	P_5	P_6
P_2	P_2	P_1	P_4	P_3	P_6	P_5
P_3	P_3	P_5	P_1	P_6	P_4	P_2
P_4	P_4	P_6	P_2	P_5	P_1	P_3
P_5	P_5	P_3	P_6	P_1	P_4	P_2
P_6	P_6	P_4	P_5	P_2	P_3	P_1

To prove: $(S_3, \circ)$ is a non abelian group.

(i) Closure: Since the body of the table contains only the elements of S_3.

$\therefore$ $(S_3, \circ)$ is closed.

(ii) Associativity: We know composition of function S_3 is associative and so it is true in S_3 also. $(S_3, \circ)$ is associative.

$P_1 \circ (P_3 \circ P_4) = P_1 \circ P_6 = P_6$.

$(P_1 \circ P_3) \circ P_4 = P_3 \circ P_4 = P_6$.

$\therefore P_1 \circ (P_3 \circ P_4) = (P_1 \circ P_3) \circ P_4$.

(iii) Identity: $P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ is the identity element of S_3.

(iv) Inverse: From the above table $P_1^{-1} = P_1; P_2^{-1} = P_2; P_3^{-1} = P_3; P_4^{-1} = P_5; P_5^{-1} = P_4; P_6^{-1} = P_6$. Thus inverse exists for every element. Hence inverse axiom is verified.

$\therefore$ $(S_3, \circ)$ is a group.

(v) Commutative: From the table; $P_3 \circ P_4 = P_6$ and $P_4 \circ P_3 = P_2$.

$\therefore$ $P_3 \circ P_4 \neq P_4 \circ P_3$. Hence $(S_3, \circ)$ is not commutative.

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Topic 5: SUBGROUP

1) **Define SUBGROUP with Example:**

Definition: Let $G, *$ be a group. Let e be the identity element in G and let $H \subseteq G$. If H itself is a group with the same operation $*$ and the same identity element e.

(or)

Let $, *$ be a group and $H \subseteq G$. $H, *$ is called a subgroup of $G, *$, if H itself is a group with respect to $*$.

Example: $(Q, +)$ is a subgroup of $(R, +)$.

2)Define TRIVIAL SUBGROUP OR IMPROPER SUBGROUP

Solution: For any group $G, *$, $\{e, *\}$ and $G, *$ are subgroups, called trivial subgroups.

3)Define NON TRIVIAL SUBGROUP OR PROPER SUBGROUP

Solution: All other subgroups other than $\{ , *\}$ and $(G, *)$ are called non trivial subgroup.

4)What is the CONDITION FOR A NON-EMPTY SUBSET H to be subgroup of G

$H,*$ is said to be a subgroup of $G, *$ if

(i). H is closed for the operation $*$, $\forall\ a, b \in H, a * b \in H$.

(ii). H contains the identity element e
(i.e) $e \in H$ where e is the identity of G.

(iii). For any $a \in H$, $a^{-1} \in H$.

Topic 6 : Theorems on Subgroups

1)Theorem 1: State and Prove NECESSARY AND SUFFICIENT CONDITION For a subgroup :

Statement: A non-empty subset H of a group $G, *$ is a subgroup of G if and only if $a * b^{-1} \in H$ for all $a, b \in H$.

PROOF: Necessary Condition:

Let H be a subgroup of a group G and $a, b \in H$.
To prove: $a * b^{-1} \in H$.
Since H is a subgroup and $b \in H$, b^{-1} must exist and $b^{-1} \in H$.
Now, $a \in H, b^{-1} \in H \Rightarrow a * b^{-1} \in H$. [By closure property]

Sufficient Condition:

Assume $a \in H, b \in H \Rightarrow a * b^{-1} \in H$.
To prove: H be a subgroup of a group G.

(i). **IDENTITY**:

Now, $a \in H$, $a^{-1} \in H \Rightarrow a * a^{-1} \in H \Rightarrow e \in H$.
Hence the identity element, $e \in H$.

(ii). **INVERSE:**

$e \in H$, $a \in H \Rightarrow e * a^{-1} \in H \Rightarrow a^{-1} \in H$.
$\Rightarrow$ Every element $'a'$ of H has its inverse a^{-1} is in H.

(iii). **CLOSURE:**

If $b \in H$ then $b^{-1} \in H$. $a \in H$, $b^{-1} \in H \Rightarrow a * (b^{-1})^{-1} \in H \Rightarrow a * b \in H$.

(iv). **ASSOCIATIVE:**

Now $H \subseteq G$ and the associative law hold good for $G,$ as G is a group.
Hence it is true for the element of H.

Thus all axioms for a group are satisfied for H.

Hence H is subgroup of G.

2)Prove: The intersection of two subgroups of a group $G, *$ is also a subgroup of $(G, *)$ & $\text{The Union need not be a Subgroup}$.

PROOF:

Let H and K are subgroups of$(G, *)$

To prove that: $H \cap K$ is subgroup of $(G, *)$.

We have $H \cap K \neq \emptyset$. [$\because$ atleast identity element is common to both H and K].

Let $a, b \in H \cap K \Rightarrow a \in H \cap K$ and $b \in H \cap K$

$a \in H \cap \mathrm{K} \Rightarrow a \in \mathrm{H}$ and $a \in$

$b \in H \cap \mathrm{K} \Rightarrow b \in \mathrm{H}$ and $b \in$

Now, $a \in H, b \in H \Rightarrow a * b^{-1} \in H$ [H is a subgroup , Theorem 1],

$a \in K, b \in K \Rightarrow a * b^{-1} \in K$ [K is a subgroup , Theorem 1].

Therefore, $a * b^{-1} \in H \cap K$.

Thus $a \in H \cap K$ and $b \in H \cap K \Rightarrow a * b^{-1} \in H \cap K$.

$H \cap K$ is a subgroup of G. [By Theorem 1]

ALSO, The union of two subgroups need not be a subgroup.

Example:

Let $(Z, +)$ is a group.

Let H and K are subgroup of $(Z, +)$

where $H = \{ \ldots . -4, -2, 0, 2, 4, 6 \ldots .\} = \{0, \pm 2, \pm 4, \pm 6..\}$

$K = \{\ldots . -6, -3, 0, 3, 6, 9 \ldots .\} = \{0, \pm 3, \pm 6, \pm 9..\}$

$H \cup K = \{0, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9..\}$

$3, 8 \in H \cup K$ but $3 + 8 = 11 \notin H \cup K$.

Therefore, $H \cup K$ is not closed with respect to addition.

Therefore, $H \cup K$ is not a subgroup of G.

3)Prove: The union of two subgroups of a group G iff one is contained in the other.

PROOF:

Assume H and K are subgroups of G and $H \subseteq K$ or $K \subseteq H$.
To prove that. $H \cup K$ is a subgroup.

$\because$ H and K are subgroups and $H \subseteq K \Longrightarrow H \cup K = K$.
(or) H and K are subgroups and $K \subseteq H \Longrightarrow H \cup K = H$.
Therefore, $H \cup K$ is a subgroup.

Conversely,

Suppose $H \cup K$ is a subgroup.
To prove that, one is contained in the other (i.e) $H \subseteq K$ or $K \subseteq H$.

Suppose, $H \nsubseteq K$ or $K \nsubseteq H$.
Then, $\exists$ elements $a,$ such that $a \in H$ and $a \notin K$--------------------(1)
b$\in$ K and $b \notin H$-------------------(2)
Clearly, $a, b \in H \cup K$.
Since, $H \cup K$ is a subgroup of G, $ab \in H \cup K$.
Hence, $ab \in H$ or $ab \in K$.

Case 1: Let $ab \in H$. $\because$ $a \in H, a^{-1} \in H$.
Hence, $a^{-1}\, ab = b \in H,$ which is a contradiction (2).

Case 2: Let $ab \in K$. $\because$ $b \in K, b^{-1} \in K$.
Hence, $b^{-1}\, ab = a \in K,$ which is a contradiction (1).

Therefore, Our assumption is wrong.
Thus, $H \subseteq K$ or $K \subseteq H$.

PROBLEMS:

1. Find all the non-trivial subgroup of $(Z_6, +_6)$.

   Solution: $Z_6 = \{ 0 , 1 , 2 , 3 , 4 , 5 \}$ of H is a subgroup of Z_6

   Hence, $O\ (H) = 1,2,3,$ or 6.

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Subgroups are

$=> H = [0]$

$\Longrightarrow H = [0] , [3]$

$\Longrightarrow H = [0] , [2] , [4]$

2. Find all the subgroups of $(Z_9, +_9)$.

Solution:

$Z_9 = \{0,1,2,3,4,5,6,7,8\}$

Here, $O\ (H) = 1, 3.$

$\times_9$	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8	0
2	2	3	4	5	6	7	8	0	1
3	3	4	5	6	7	8	0	1	2
4	4	5	6	7	8	0	1	2	3
5	5	6	7	8	0	1	2	3	4
6	6	7	8	0	1	2	3	4	5
7	7	8	0	1	2	3	4	5	6
8	8	0	1	2	3	4	5	6	7

Subgroups are

$\Longrightarrow H = \{0\}$

$\Longrightarrow H = \{0, 3, 6\}$

$+_9$	0	3	6
0	0	3	6
3	3	6	0
6	6	0	3

3.. Check whether $H_1 = \{0, 5,10\}$ and $H_2 =\{ 0, 4,8,12\}$ are subgroups of Z_{15} with respect to $+_{15}$.

Solution: $H_1 = \{0, 5,10\}$

$H_1 = \{0, 5,10\}$

$+_{15}$	0	5	10
0	0	5	10
5	5	10	0
10	10	0	5

$(H_1, +_{15})$

$H_2 = \{ 0, 4,8,12 \}$

$+_{15}$	0	4	8	12
0	0	4	8	12
4	4	8	12	1
8	8	12	1	5
12	12	1	5	9

$(H_2, +_{15})$

Table 1:
$(H_1, +_{15})$: All the entries in the addition table for H_1 are the elements of H_1.
Therefore, H_1 is a subgroup of Z_{15}.

Table 2:
$(H_2, +_{15})$: All the entries in the addition table for H_2 are not the elements of H_2.
Therefore, H_2 is a subgroup of Z_{15}.

x...x

Topic 7: NORMAL SUBGROUPS

1) Define NORMAL SUBGROUPS with Example.

Definition: A subgroup $(H, *)$ of $(G, *)$ is called normal subgroup of G

$if\ aH = Ha\ , \forall\ a \in G.$

2) **Theorem:** Every subgroup of an abelian group is normal

Proof: Let $(G,*)$ be a abelian group and $(H,*)$ be a subgroup of G.

Let $a \in G$ be any element.

Then $aH = \{a * h\ /h \in\ \}$

$= \{h * a\ /h\ \in H\}$ (since G is abelian) = Ha

Since a is arbitrary, $aH = Ha\ \forall\ a \in G$

Therefore H is a normal subgroup of G.

3) Theorem: ($N,*$) is a normal subgroup of $(G,*)$ iff $a * n * a^{-1} \in N$

$\forall\ n \in N$ and $\forall a \in G.$

Proof: Let $(N,*)$ is a normal subgroup of $(G,*)$. Therefore $aN = Na \quad \forall a \in G$

$\Rightarrow a * N * a^{-1} = N * a * a^{-1} = N * e = N$

Therefore for any $n \in N, a * N * a^{-1} \in N$

Conversely, if $a * N * a^{-1} \in N, n \in N, \forall a \in G,$

To prove $a * N = N * a$

Let $x \in a * N \Rightarrow \mathrm{x = a * n\ for\ some}\ n \in N$

x = a ∗ n ∗ e ⇒ x= a ∗ n ∗(a^{-1} ∗ a)

$\Rightarrow x = a * n * a^{-1} * a \in N * a$

$\Rightarrow a * N \subseteq \mathbf{N} * a$ ....(1)

Let $y \in N * a \Rightarrow y = n * a$ for some $n \in N$

Then $y = a * a^{-1} * n * a = a * (a^{-1} * n * a^{-1\,-1}) \in a * N$

Therefore $y \in N * a \Rightarrow y \in a * N$ therefore $N * a \subseteq a * N$.................(2)

Therefore from (1) and (2) we get $a * N = \mathbf{N} * a$, $\forall a \in G$.

Hence N is a normal subgroup of G.

4)Theorem: prove that intersection of two normal subgroup of $(G, *)$ is a normal subgroup of $(G, *)$.

Proof: Let $(N_1, *)$ and $(N_2, *)$ be two normal subgroups of $(G, *)$.

To Prove $(N_1 \cap N_2\ ,\ *)$ is a normal subgroup of $(G,\ *)$.

$a * n * a^{-1} \in N_1 \cap N_2$ (by previous theorem)

Since N_1 and N_2 are normal subgroup of G, they are basically subgroups.

We know $_1 \cap N_2$ is a subgroup of G.

Now we shall prove it is a normal subgroup of G.

Let n $\in N_1 \cap N_2$ be any element and $a \in G$ be any element

Then n $\in N_1$ and n $\in N_2$, Since N_1 and N_2 are normal, $a * n * a^{-1} \in N_1$ and $a * n * a^{-1} \in N_2$, Therefore $a * n * a^{-1} \in N_1 \cap N_2$.

Hence $N_1 \cap N_2$ is normal.

x..x

Topic 8: Group Homomorphism:

1) Define **Group Homomorphism.**

Let $G,*$ and $G_1,\circ$ be two groups. A mapping $g: G \rightarrow G_1$ is called group homomorphism if $g\ a * b = g\ a \circ g\ b$ for all $a, b \in G$.

2) Properties of group homomorphism:

A group homomorphism preserves identities, inverses and sub groups.

Theorem 1: Homomorphism preserves identities.

(or)

If $(\ )\ = e_1$ where e and e_1 are the identity elements of G and G_1 respectively.

Proof: Let $a \in G$, If e is the identity element G,
then $a * e = e * a = a$

$\Rightarrow f\ (a * e) = f\ a$

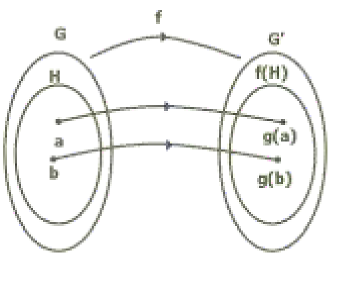
$\Rightarrow\ f(a) \circ f(e) = f\ (a)$ $\quad \because f$ is homomorphism

$\Rightarrow f\ (e) = e_1$

$\therefore f$ preserves identities.

Theorem 2: Homomorphism preserves inverse (or) $f\ (a^{-1}) = [f(a)]^{-1}$

Proof:



Let $a \in G$, $a^{-1} \in G \Rightarrow a * a^{-1} = a^{-1} * a = e$

Since $a * a^{-1} = e$

$\Rightarrow f\ (a * a^{-1}) = f\ (e)$

$\Rightarrow\ f(a) \circ f(a^{-1}) = f(e_1)$ $\quad \because f$ is homomorphism

$\therefore f\ (a^{-1}) = [f\ (a)]^{-1}$.

$\therefore f$ preserves inverse

Theorem 3: Homomorphism preserves subgroup

(or)

If H is a subgroup of ,then $f\,(H)$ is a subgroup of G_1.

Proof:

Let H be a subgroup of G ⇒ for $a, b \in H, a * b^{-1} \in H$ [$\because H$ is a subgroup]

Let $f\,(a) \in f(\,H)$ and $f\,(b) \in f(\mathrm{H})$.

To prove $f\,(a) \circ f\,(b^{\,-1}) \in f\,(\,\mathbf{H})$

Consider $\mathrm{f}\,(\mathrm{a}) \circ \mathrm{f}(\,b^{\,-1}) = f\,(\,a * b^{-1}\,) \in f\,(H)$ [$\because a * b^{-1} \in H$]

$\Rightarrow \mathrm{f}\,(\mathrm{a}) \circ \mathrm{f}(\,b^{\,-1}) \in f\,(H)$ $\forall f\,(a) \in f\,(H)$ and $f\,(b) \in f(\,H)$.

$\therefore f\,(\mathrm{H}) \subseteq G_1$ is a subgroup of G_1.

Theorem 4: Let $f: G \to G'$ be a group homomorphism and H is a subgroup of G' . Then $f^{-1}\,(\,)$is a subgroup of G.

Proof:

Clearly $f^{-1}\,(H)$ is a non empty subset of G [$\because H$ is a subgroup of G.]

Now let us consider $a = f^{-1}\,(c) \in f^{-1}\,(H)$ and b= $f^{-1}\,(d) \in f^{-1}\,(\mathrm{H})$.

For $c, d \in H$ with $f\,(\mathrm{a}) = c$ and $f(\,b) = d$.

Let a, $b \in f^{-1}\,(\mathrm{H})$ $\Rightarrow f\,(\mathrm{a}), \mathrm{f}\,(\mathrm{b}) \in H$ [$\because H$ is a subgroup.]

$\Rightarrow f\,(\mathrm{a}) * f\,(b^{\,-1}) \in H$

$\Rightarrow f\,(a * \mathrm{b}^{\,-1}) \in H$ [$\because f$ is homomorphism.]

$\Rightarrow a * b^{-1} \in f^{-1}\,(\mathrm{H})$

$\therefore$ a, $b \in f^{-1}\,(H) \Rightarrow a * b^{-1} \in f^{-1}\,(H)$

Hence $f^{-1}\,(H)$ is a subgroup of G^1.

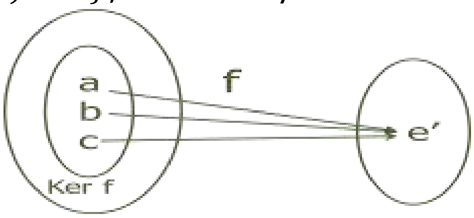
x...x

Topic 9: KERNEL OF A HOMOMORPHISM:

1)Define **KERNEL OF A HOMOMORPHISM** with Example:

Let $f: G \to G'$ be a group homomorphism. The set of elements of G which are mapped into e' (identity element in G') is called the kernel of f and it is denoted by $\ker f$.

$\ker f = \{x \in G / f(x) = e'\}$, e' is identity of G'.



then $\ker f = \{a, b, c\}$

Example: 1. $f: (Z, +) \to (Z, +)$ defined by $f(x) = 2x$ then $\ker f = \{0\}$

2. :$(R^*,\cdot) \to (R^+,\cdot)$ defined by $f(x) = |x|$, then $\ker f = \{1, -1\}$.

2) If $f: G \to G'$ is a homomorphism then $\ker f = \{e\}$ iff f is 1-1.

Proof:

Assume f is one to one Then

$f(e) = e'$

$\therefore \ker f = \{e\}$

Conversely,

Assume $\ker f = \{e\}$

Now $f(x) = f()$

$\Rightarrow f(x) * f(y^{-1}) = f(y) * f(y^{-1})$

$\Rightarrow f(xy^{-1}) = e'$

$\Rightarrow xy^{-1} \in \ker f$

$\Rightarrow xy^{-1} = e$

$\Rightarrow x = y$

$\therefore f(x) = f(y) \Rightarrow x = y$

Hence f is one to one.

3) Prove that Kernel of a homomorphism is a normal subgroup of G.

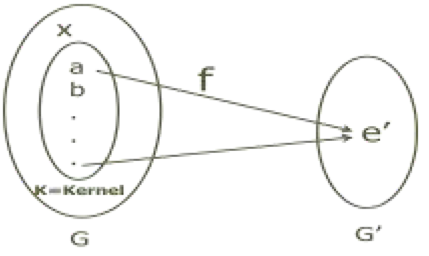
Proof:

Let $(G,*)$ and $(G',\cdot)$ be the groups and $f: G \to G'$ is a group homomorphism.

By the definition of homomorphism, $f(a * b) = f(a) \, . \, f(b) \; \forall a, b \in G$.

By the definition of kernel, $K = \{a \in G / f(a) = e'\}$

i.e., $f\,() = e' \;\; \forall a \in K$ and e' is the identity element of H.



To prove that 'K' is a normal subgroup of G.

i.e., To prove

i) K is nonempty

ii) $a * b^{-1} \in K, \forall a, b \in K$

iii) $x * h * x^{-1} \in K, \forall\, h \in K, x \in G$

i) Identity element 'e' of G is mapped to identity element of e′ of G′.

i.e., $f\,(e) = e'$

$\therefore e \in K \Longrightarrow K$ is non-empty.

ii) Let $a, b \in K \subseteq G$

$\Longrightarrow f\,(a) = f\,(b) = e'$

$f\,(a * b^{-1}) = f\,(a) \, . \, f(b^{-1})$ $\{\because f$ is homomorphism$\}$

$= e' \, . \, (e')^{-1}$

$= e' \, . \, e' = e'$

$\therefore a * b^{-1} \in K.$

Hence $\mathrm{K} = \ker \mathrm{f}$ is a subgroup

iii) Let $x \in G$ and $h \in K$ be any element.

$\Longrightarrow f\,(h) = e'$

$f\,(x * h * x^{-1}) = f\,(x) \cdot f\,(h) \cdot f(x^{-1})$

$= f\,(x) \cdot e' \cdot f\,(x^{-1}) = f\,(\mathrm{x}) \cdot f\,(x^{-1}) = e'$

$\therefore x * h * x^{-1} \in K$

Hence $\mathrm{K} = \ker \mathrm{f}$ is a normal subgroup of G.

x...x

Topic : 10 Fundamental Theorem Of Group Homomorphism

State and prove the Fundamental Theorem Of Group Homomorphism.

Statement: Let $(G,*)$ and $(G', \cdot)$ be two groups.
Let $f: G \to G'$ be a homomorphism of groups with kernel K,
then G/K is isomorphic to (G).
i.e., $G/K \cong G'$

Proof: Given that $f: G \to G'$ be a homomorphism of groups with kernel K.

Define the map $\emptyset\,(K * a) = \mathrm{f}\,(\mathrm{a})$, $\forall a \in G$

i) **<u>∅ is well defined</u>**:

Let $a, b \in G$ such that
$K * a = K * b$
$\Rightarrow a * b^{-1} \in K$ --------→ (1)
$\Rightarrow \mathrm{f}\,(\mathrm{a} * b^{-1}) = e'$ $\{\because K$ is kernel$\}$
$\Rightarrow \mathrm{f}\,(\mathrm{a}) * \mathrm{f}(b^{-1}) = e'$ $\{\because f$ is homomorphism$\}$
$\Rightarrow f\,(a) * \mathrm{f}(b)^{-1} * f(\,b) = e' * f(b)$
$\Rightarrow \mathrm{f}\,(\mathrm{a}) = \mathrm{f}(b)$
$\Rightarrow \emptyset\,(K * a) = \emptyset\,(K * b)$
$\therefore \emptyset$ is well defined.

ii) **<u>∅ is one to one</u>**:

To prove that $\emptyset\,(K * a) = \emptyset\,(K * \;) \Rightarrow K * a = K * b$
We know that $\emptyset\,(K * a) = \emptyset\,(K * b) \Rightarrow f\,(a) = \mathrm{f}(b)$
$\Rightarrow f\,(a) * \mathrm{f}(b^{-1}) = f\,(b) * \mathrm{f}(b^{-1})$
$= f\,(b * b^{-1})$
$= f\,(e)$
$\Rightarrow f\,(a) * \mathrm{f}(b^{-1}) = e'$
$\Rightarrow f\,(a * b^{-1}) = e'$
$\Rightarrow a * b^{-1} \in K$
$\Rightarrow K * a = K * b$
$\therefore \emptyset$ is one to one.

iii) **$\emptyset$ is onto**:

Let $y \in G'$

Since f is onto, there exists $a \in G$ such that f (a) $= y$

$\Rightarrow \emptyset\,(K * a) = y \qquad \{\because f\,(a) = \emptyset(K * a)\}$

Thus every element of G' has preimage in G/K

$\therefore \emptyset$ is onto.

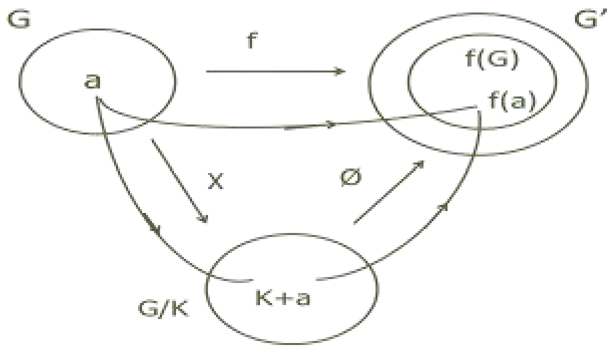
i) **$\emptyset$ is a homomorphism**:

$$\begin{aligned}\emptyset\,(K * a * K * b) &= \emptyset\,(K * a * b)\\ &= f(a * b)\\ &= f(a) * f(b)\\ &= \emptyset\,(K * a) * \emptyset\,(K * b)\end{aligned}$$

$\therefore \emptyset$ is a homomorphism.

Since $\emptyset$ is one to one, onto and homomorphism, $\emptyset$ is isomorphism between G/K and G'.

$$\therefore G/K \cong G'$$



2) State and prove the Cayley's representation theorem.

(or)

Prove that every finite group of order 'n' is isomorphic to a permutation group of order 'n'.

Proof:

To prove the theorem, we have to show the following.

a. To form a set G of permutation

b. To prove G' is a group

c. Exhibit an Isomorphism $\emptyset: G \to G'$.

a. To form a set G' of permutation:

Let G be a finite group of order 'n' and $a \in G$ be any element.
Corresponding to 'a' we define a map $f_a(x) = a * x, \forall x \in G$ then f is one to one.
$\because f_a(x) = f_a(y)$
$\Rightarrow a * x = a * y$
$\Rightarrow x = y$ (by left cancellation law)

Now $y \in G$ (Co-domain), then
$a^{-1} * y \in G$ such that
$f_a(a^{-1} * y) = a * (a^{-1} * y) = (a * a^{-1}) * y = e * y = y$
$\therefore f_a$ is onto.
Thus f_a is a one to one and onto function from $G \to G$ and so it is a permutation on G.

b. To prove G' is a group:

Let $, f_b \in G'$ be any two elements, then

$$(f_a \circ f_b)\, x = f_a(f_b(x)) = f_a(b * x) = a * (b * x) = (a * b) * x = f_{a*b}$$

$\therefore G'$ is closed.

Composition mapping is also associative.

Since 'e' is the identity element of G, $f_e \in G'$ is identity mapping.

Let $a \in G \Longrightarrow a^{-1} \in G$

$f_{a^{-1}} f_a(x) = f_{a^{-1}}(a * x) = (a^{-1} * a) * x = e * x = f_e(x)$

$\therefore f_{a^{-1}} \in G'$

Hence G' is a group.

c. Isomorphism $\emptyset: G \to G'$:

To prove G and G' are isomorphic.

Let $\emptyset: G \to G'$ be defined by $\emptyset(a) = f_a, \forall a \in G$

Now for any $a, b \in G$, $\emptyset(a * b) = f_{a*b} = f_a * f_b = \emptyset(a)\emptyset(b)$

$\therefore \emptyset$ is a homomorphism.

Suppose $\emptyset(a) = \emptyset(b)$ then

$f_a = f_b \Longrightarrow f_a(x) = f_b(x) \quad, \forall x \in G \Longrightarrow a * x = b * x \Longrightarrow a = b$ {Right Cancellation law}

$\therefore \emptyset$ is one to one

Since f_a is onto, $\emptyset$ is onto.

Thus $G \cong G'$

Topic 11: COSETS and LAGRANGE'S THEOREM:

1)Define cosets with Example:

Definition: Let $(H,*)$ be a subgroup of $(G,*)$. Let $a \in G$ be any element. Then

$aH = \{a * h \ / h \in \ \}$is called the left coset of H in G determined by a.

Sometimes aH can be written as $a * H$.

The set $Ha = \{ h * a \ / h \in H\}$ is called the right coset of H in G determined by a.

Points to remember:

1. Since $e \in H, a * e \in aH \Rightarrow a \in aH \ \ and \ \ \ e * a = a \in Ha$
2. Also $eH = e * h/h \in H = h/h \in H = H$

   and $He = h * e/h \in H = h/h \in H = H$

   So H itself is a left coset as well as right coset.
3. In general, $aH \neq Ha$.

   But if G is abelian, then $aH = Ha$ That is every left coset is a right coset.

Problems:

1.Find the left cosets of $H = (5Z, +)$ which is a subgroup of ($Z, +$)

Solution: If $H = 5Z$ then $(H, +)$ is a subgroup of (Z , +).

Then the distinct left cosets of H in Z are

$0 + H = H = 0 + 5x$ where $\in Z$

$1 + H = \ \ 1 + 5x$ where x$\in Z$

$2 + H = \ \ 2 + 5x$ where x $\in Z$

$3 + H = \ \ 3 + 5x$ where x $\in Z$

$4 + H = \ \ 4 + 5x$ where x $\in Z$

$5 + H = \ \ 5 + 5x$ where x $\in Z$

$= 5(1 + x)$ where $x \in Z = H$

$6 + H = 6 + 5x/x \in Z = 1 + 5(1 + x)$ where $x \in Z = 1 + H$ and so on. Therefore number of different left cosets of H in G is 5.

2)Theorem: Let $(H,*)$ be a subgroup of $(G,*)$. Then the set of all left cosets of H in G form a partition of G.

Proof: Let aH and bH be any two left cosets.

We shall prove either $aH = bH$

(or) $aH \cap bH = \emptyset$.

Suppose $aH \cap bH \neq \emptyset$, then there exists an element

$x \in aH \cap bH \Rightarrow x \in aH \; and \; x \in bH$

$\Rightarrow$ x= $a * h_1$ x= $b * h_2, for \; some \; h_1, h_2 \in H$ ....(1)

Therefore, $a * h_1 = b * h_2, \Rightarrow a * h_1 * h_1^{-1} = b * h_2 * h_1^{-1}$

$\Rightarrow a * (h_1 * h_1^{-1}) = b * (h_2 * h_1^{-1})$

$\Rightarrow a * e = b * (h_2 * h_1^{-1})$

a $= b * (h_2 * h_1^{-1})$ ....(2)

If x is any element in aH, then $x = a * h$

$\Rightarrow x = b * (h_2 * h_1^{-1}) * h$

$\Rightarrow$ x $= b * (h_2 * h^{-1}) * h \in$ bH

Therefore x $\in aH \Rightarrow$ x $\in bH$ therefore $aH \subseteq$bH ....(2)

Similarly we can prove $bH \subseteq$ aH ....(3)

From (2) and (3) we get $aH = aH$.

Thus any two left cosets are either equal or disjoint. Further $\bigcup_{a\in G} aH \subseteq \mathbf{G}$ since union of subsets is a subset.If x is any element in G, then $x = x * e \in xH$

Therefore x is a left coset and hence $x \in \bigcup_{a\in G} aH$. $Hence \Rightarrow x \in G \Rightarrow x \in \bigcup_{a\in G} aH \Rightarrow G \subseteq \bigcup_{a\in G} aH$. Therefore $G = \bigcup_{a\in G} aH$. Thus all the left cosets forms partition of G.

3) State and prove Lagrange's theorem:

Statement:The order of a subgroup H of a finite group G divides the order of the group. (i.e) order of H divides order of G.

Proof: Let $(G,*)$ be a group of order *n* and $(H,*)$ be a subgroup of order *m.*

Since G is a finite group, the number of left cosets of H in G is finite.

Let *r* be the number of left cosets of H in G

Let the *r* cosets be $a_1H, a_2H \dots . a_r H$.

We know that the left cosets of G forms a partition of G. (by previous theorem)

Therefore $G = a_1H \cup a_2H \cup \dots .\cup a_r H$

Therefore $o\,(\mathrm{G}) = \mathrm{o}\,(a_1H \cup a_2H \cup \dots .\cup a_r H)$

$$= \mathrm{o}\,(a_1H) + o\,(a_2\,\mathrm{H}) + \cdots \mathrm{o}\,(a_r H)$$

But o$(a_iH) = o\,(H)$ (by previous theorem)

Therefore $o\,(G) = o\,(H) + o\,(H) + \cdots \dots \mathrm{o}(H)$

r *times*

$\Rightarrow o(G) = r\,o\,(H)$

Thus O(H) divides $o(G)$

Topic 12: RINGS AND FIELDS

1)Define Ring with Example:

Definition: A non-empty set R with two binary operations + and **.** called addition and multiplication is called ring if the following axioms are satisfied.

(i) $(R, +)$ is an abelian group with 0 as identity

(ii) $(R, .)$ is a semigroup

(iii) The operation . is distributive over +

(i.e) $a. b + c = a. b + a. c$ and

$b + c \ \ . a = b. a + c. a \quad \forall\ a, b, c \in R$

2)Define commutative ring.

Definition: A ring $(R, +, .)$ is said to be commutative if $a. b = b. a \ \forall\ a, b \in R$

3)Define Ring with Identity.

Definition: A ring $(R, +, .)$ is said to be a ring with identity if there exists an element $1 \in R$ such that $1. a = a. 1 = a \quad \forall \in R$

4)Define Ring with zero divisor.

Definition: If $R, +, .$ is a commutative ring, then $a \neq 0 \in R$ is said to be a zero- divisor if there exists a non-zero $b \in R$ such that $ab = 0$.

5)Define Ring without zero divisors

Definition: If in a commutative ring $(R, +, .)$, for any $a, b \in R$ such that $a \neq 0, b \neq 0 \Rightarrow ab \neq 0$ then the ring is without zero divisors.

In a ring without zero divisors, $\mathrm{a. b} = 0 \Rightarrow \mathrm{a} = 0 \text{ or } \mathrm{b} = 0$.

6)Define Integral domain:

Definition: Integral domain: A commutative ring $(\mathrm{R}, +, .)$ with identity and without zero divisors is called an integral domain.

7)Define Field.

Definition: Field: A commutative ring $(\mathrm{R}, +, .)$ which has more than one element such that every non zero element of R has a multiplicative inverse in R is called a field.

Problems:

1. Show that $Z_5 = \{0, 1, 2, 3, 4\}$ is an integral domain under $+_5$ and $\times_5$.

Solution:

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times_5$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

We can easily verify $(Z_5, +_5, \times_5)$ is a commutative ring with identity 1. From the table for $\times_5$, we see product of non zero elements is non zero and so $(Z_5, +_5, \times_5)$ ring without zero divisors is an integral domain.

2. Prove the set $Z_4 = \{0, 1, 2, 3\}$ is a commutative ring with respect to $+_4$ and $\times_4$.

Solution:

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times_4$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

(i) All entries in both the tables $+_4, \times_4$, belongs to Z_4.
Therefore Z_4 is closed under $+_4, \times_4$.

(ii) The entries of the first row is same as those of first column.

(iii) Hence Z_4 is commutative with respect to $+_4, \times_4$

(iv) If $a, b, c \in Z_4$ we can verify

$$a +_4 b +_4 c = a +_4 b +_4 c$$

$$a \times_4 b \times_4 c = a \times_4 b \times_4 c$$

Also the law is true for $+_4$, $\times_4$.

(iv) $0 +_4 a = a +_4 0 = a \forall \in Z_4$

$1 \times_4 = a \times_4 1 = a \in {}_4$

0 is the additive identity and 1 is the multiplicative identity of Z_4 with respect to $+_4$, $\times_4$.

(v) From the table $+_4$ additive inverse of 0,1,2,3 are 0,3,2,1 respectively. And multiplicative inverse of non zero element 1,2,3 are 1,2,3 respectively.

(vi) Also we can verify distributive law

(vii) $a \times_4 (b +_4 c) = a \times_4 b +_4 (a \times_4 c)$

$b +_4 \quad \times_4 a = \quad b \times_4 a \quad + (c \times_4 a)$

Hence $(Z_4, +_4, \times_4)$ is a commutative ring with unity.

3. Prove that every field is an integral domain.

Proof: Let F be a field.

(i.e) (F,+,.) is a commutative ring with identity and non zero element has a multiplicative inverse.

To prove F is an integral domain we have to show it has no zero divisors.

Suppose $a, b \in F$ with $a . b = 0$ let $a \neq 0$, since a is a non zero element, its multiplicative invese exists (i.e) a^{-1}exists .

Therefore $a^{-1}. a. b = a^{-1}. 0 \Rightarrow a^{-1}. a . b = 0 \Rightarrow 1. b = 0$

Thus $a. b = 0 \Rightarrow a \neq 0 \Rightarrow b = 0$. Therefore F has no zero divisors.

Hence (F,+, .) is an integral domain.

4) Show that (z , $+, .$) is an integral domain where Z is set of all integers.

Proof: We know commutative ring with identity and without zero divisors is called integral domain.

If Z is set of all integers, then

(i) $(Z, +)$ is an abelian group.

(ii) $(Z, \times)$ is a semi ring.

(iii) $a \times b = b \times a \; \forall a, b, c \in Z$

(iv) $a \times b + c = a \times b + a \times c \quad \forall a, b, c \in Z$

Hence (z , $+, .$) is a commutative ring with identity.

If $a \neq 0, b \neq 0 \in Z$ then we know $ab \neq 0$. So Z is without zero divisors.

Hence (z , $+, .$) is an integral domain.

X……………………………………..X